Operational Finite Element Analysis of Circular Rings Surrounded with Elastic Foundation

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1. Introduction

The present paper will treat the circular ring which is surrounded with elastic foundation and is subjected to a radial concentrated load P applied at its crown point (Fig. 1). At times, such structural systems have been treated by replacing the foundation members by ordinary flexural members subjected to given distributed loads. Since the distribution of the reaction from foundation is unknown, it is sometimes assumed to be some given applied force. In order to be compatible with oblique or horizontal loads, as well as vertical loads, it is necessary to extend the usual Winkler assumption for the lateral deflection and the corresponding normal reaction to that of tangential or longitudinal displacement. Under these two kinds of Winkler assumptions, the foundation beams or the combined structures involving such beam or beams can be treated without difficulty by means of the operational method avoiding simultaneous equations. The method of the analysis to be adopted will be the operational finite element one.

The circular ring is then divided into a series of finite elements of short span length or rectilinear beams, each of which is governed by the ordinary longitudinal and flexural behaviors. It is assumed that the reaction from foundation by Winkler assumptions applies at one point of each member to tangential and normal directions.

At first, it will be assumed that every finite element has an elastic foundation. Then, with application of the concentrated load P at the crown point, several elements in the vicinity of the point of application, say in the domain $(-\phi, +\phi)$, will have the "negative" normal reactions. Since the negative reaction

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is to be prohibited in the structure, those elements will be replaced by ordinary beam elements that have no foundation. Inverse interporation techniques for several different cases will permit the theoretical determination of the separation angle ϕ.

As the structure is divided into finer elements, the results obtained will tend to the rigorous solution.

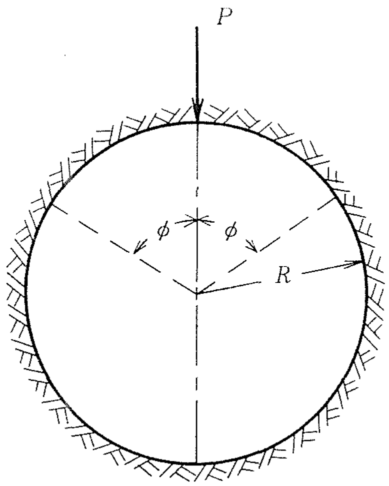

Fig. 1. Circular Ring with Elastic Foundation.

2. Complete State Vector

The complete state vector of a member is a sixth-order column matrix, consisting of the longitudinal displacement, the axial force, the lateral deflection, the flexural slope, the flexural moment, and the shearing force. Then the state vector at any point ρ is given by the equations[1)]

$$\mathbf{W}(\rho) = \mathbf{R}(\rho)\mathbf{X}, \tag{1a}$$

or

$$\mathbf{W}'(\rho) = \mathbf{R}(\rho)[\mathbf{X} + \mathbf{K}], \tag{1b}$$

in which $\mathbf{W}(\rho)$ or $\mathbf{W}'(\rho)$ holds for the normal or conjugate domain respectively. Here $\mathbf{R}(\rho)$ is the complete abscissa matrix of size 6-by-6, $\mathbf{X}$ is the sixth-order eigenmatrix which is the assemblage of integration constants,[2)] and $\mathbf{K}$ is the load-matrix which is compatible with any external loads.[3)]

It should be noticed here that the right sides of Eqs. 1 exhibit the complete classification of data, and then attention can be focused at attacking the eigenmatrix only.

3. Connection Conditions and Recurrence Formula

The connection conditions are the compatibility and equilibrium at the common point of any two adjacent segments $(r-1)$ and (r). This point is defined at point $\rho=1$ of member $(r-1)$ and also at point $\rho=0$ of member (r), so that we have the following equation:

$$\boldsymbol{P}(\phi_{r-1})\boldsymbol{W}'_{r-1}(1)=\boldsymbol{P}(\phi_r)\boldsymbol{W}_r(0), \tag{2}$$

in which $\boldsymbol{P}(\phi)$ denotes the projection matrix or briefly the "projector."

Eq. 2 then yields the desired recurrence formula

$$\boldsymbol{X}_r=\boldsymbol{L}_r\boldsymbol{X}_{r-1}+\boldsymbol{L}_r\boldsymbol{K}_{r-1}, \tag{3}$$

providing

$$\boldsymbol{L}_r=[\boldsymbol{P}(\phi_r)\boldsymbol{R}_r(0)]^{-1}\boldsymbol{P}(\phi_{r-1})\boldsymbol{R}_{r-1}(1). \tag{4}$$

Here the $\boldsymbol{L}_r$ matrix is the shift operator or briefly the "shiftor," with which the $\boldsymbol{X}_{r-1}$ matrix can be shifted from span $(r-1)$ to the adjacent span (r). Eq. 3 is the desired recurrence formula, with which all the eigenmatrices $\boldsymbol{X}_r$'s $(r=2, 3, \cdots\cdots, n)$, can be expressed in terms of the first eigenmatrix, $\boldsymbol{X}_1$. The recurrent application of Eq. 3 then gives

$$\boldsymbol{X}_r=\boldsymbol{Q}_r\boldsymbol{X}_1+\lfloor\boldsymbol{R}\rfloor_{r-1}\{\boldsymbol{K}\}_{r-1}, \tag{5}$$

in which

$$\boldsymbol{Q}_r=\boldsymbol{L}_r\boldsymbol{Q}_{r-1}, \tag{6a}$$

$$\lfloor\boldsymbol{R}\rfloor_{r-1}=\lfloor\boldsymbol{R}_1 \quad \boldsymbol{R}_2 \quad \cdots \quad \boldsymbol{R}_{r-1}\rfloor_{r-1}=\lfloor\boldsymbol{L}_r\lfloor\boldsymbol{R}\rfloor_{r-2} \quad \boldsymbol{L}_r\rfloor, \tag{6b}$$

$$\{\boldsymbol{K}\}_{r-1}=\{\boldsymbol{K}_1 \quad \boldsymbol{K}_2 \quad \cdots \quad \boldsymbol{K}_{r-1}\}. \tag{6c}$$

Note that the integrated shiftor $\boldsymbol{Q}_r$ is always a 6-by-6 square matrix, the integrated feeder $\lfloor\boldsymbol{R}\rfloor_{r-1}$ is a 6-by-$6(r-1)$ rectangular matrix, and the partial assemblage of load-matrices $\{\boldsymbol{K}\}_{r-1}$ is a $6(r-1)$-by-1 column matrix.

4. Final Equation

Eq. 5 indicates that all the eigenmatrices have been expressed in terms of the single eigenmatrix $\boldsymbol{X}_1$, so that we have $(r=n)$

$$\boldsymbol{X}_n=\boldsymbol{Q}_n\boldsymbol{X}_1+\lfloor\boldsymbol{R}\rfloor_{n-1}\{\boldsymbol{K}\}_{n-1}. \tag{7}$$

The last connection point which is the common point of the last segment (n) and the first segment (1) must be considered, which is expressed with Eq. 5 as

$$\boldsymbol{X}_1=\boldsymbol{Q}_1\boldsymbol{X}_1+\lfloor\boldsymbol{R}\rfloor_n\{\boldsymbol{K}\}_n, \tag{8}$$

in which

$$\mathbf{Q}_1 = \mathbf{L}_1\mathbf{Q}_n. \tag{9}$$

Eq. 8 at once permits the determination of the first eigenmatrix $\mathbf{X}_1$; i. e.,

$$\mathbf{X}_1 = -(\mathbf{Q}_1 - \mathbf{E})^{-1}\lfloor \mathbf{R} \rfloor_n \{\mathbf{K}\}_n, \tag{10}$$

which is the desired final equation.[4] Eq. 10 takes the form

$$\mathbf{X}_1 = \lfloor \mathbf{G} \rfloor \{\mathbf{K}\}_n, \tag{11}$$

of which the $\lfloor \mathbf{G} \rfloor$ matrix is of size 6-by-$6n$ and the $\{\mathbf{K}\}_n$ matrix is of size $6n$-by-1. The former, $\lfloor \mathbf{G} \rfloor$, depends on only the geometry and material properties of the structure, and hence it is called the "geometry matrix." The latter, $\{\mathbf{K}\}_n$, is the assemblage of all the load-matrices.

5. Numerical Example

As a numerical example of the present analysis, the circular ring subjected to a concentrated load acting at the crown point is taken as shown in Fig. 1. The geometry and material properties of the ring are taken to be as follows (Table 1):

Table 1. Numerical Values Adopted.

E (t/m²)	I (m⁴)	A (m²)	R (m)	j (t/m²)	k (t/m²)
2 100 000. 0	0. 010 8	0. 36	3. 0	351. 53	1 054. 6

Here E = Young's modulus, I = the moment of inertia of the cross section, A = the cross-sectional area, R = the radius of the circular ring, j = the shear modulus of foundation, and k = the usual normal modulus of foundation; all of these constants being measured with ton-meter unit. Table 2 shows the values of the complete state vector at equidistant intermittent points, when the complete ring is divided into sixty-four finite elements. Fig. 2 gives the curves of the lateral deflection, the flexural moment, and the normal reaction, which are plotted from Table 2. It is noted that scales of respective physical quantities in Fig. 2 are for convenience not the same.

The point from which the foundation begins to experience the positive reaction is obtained by inverse interporation techniques, from which we have $\phi = 74°12'$.

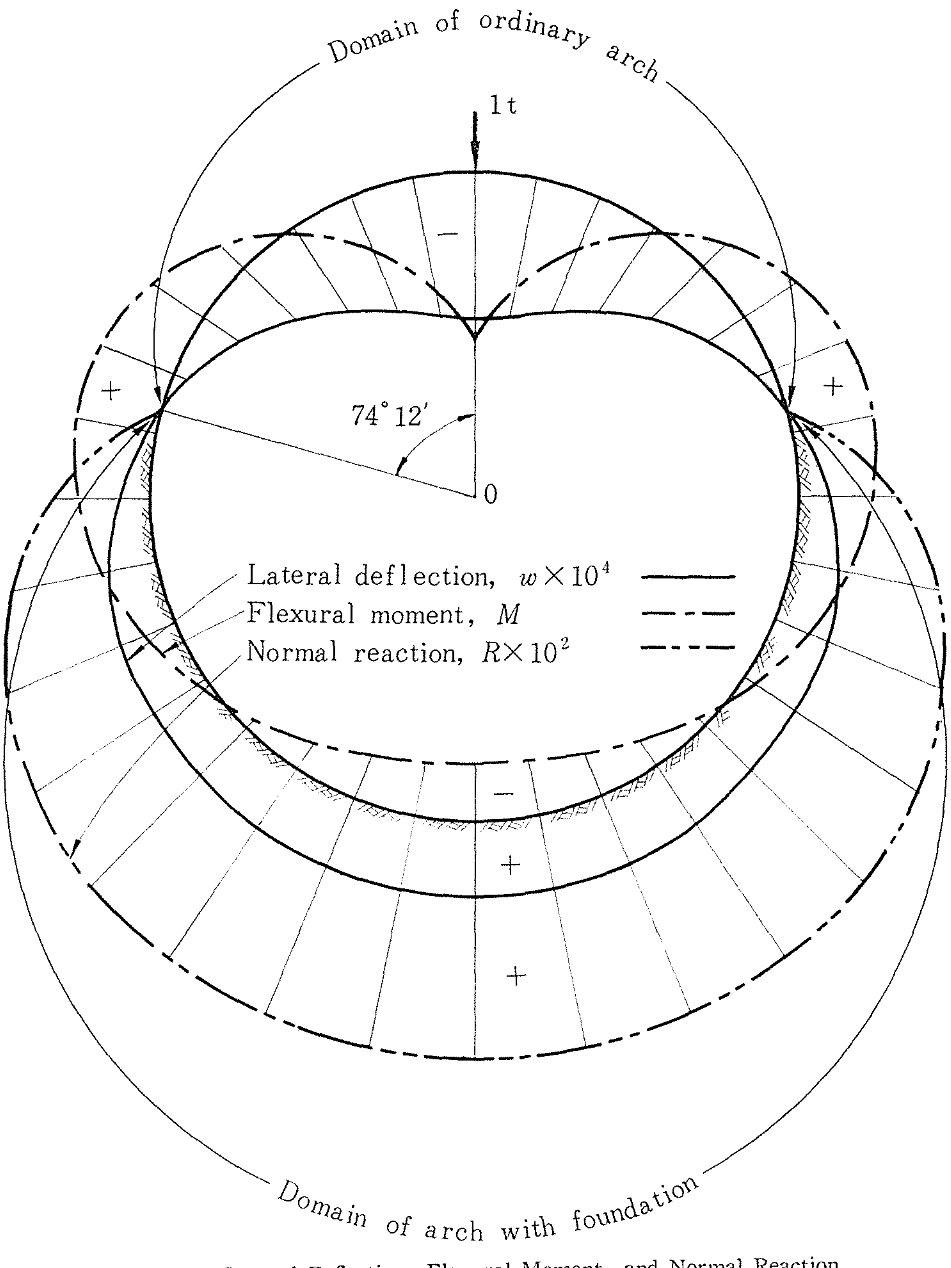

Fig. 2. Lateral Deflection, Flexural Moment, and Normal Reaction.

Table 2. Evaluation of State Vectors.

	$u\times10^4$ (m)	$w\times10^4$ (m)	$\theta\times10^4$	F (t)	S (t)	M (t)	$V\times10^2$ (t/m)	$R\times10^2$ (t/m)
1	0.0	−2.273	0.0	−0.126	0.500	−0.791	—	—
2	0.438	−2.177	0.168	−0.221	0.466	−0.506	—	—
3	0.840	−1.920	0.265	−0.308	0.414	−0.246	—	—
4	1.180	−1.551	0.299	−0.382	0.346	−0.021	—	—
5	1.439	−1.120	0.281	−0.442	0.265	0.159	—	—
6	1.611	−0.669	0.222	−0.486	0.173	0.288	—	—
7	1.696	−0.237	0.136	−0.510	0.075	0.362	—	—
8	1.700	0.150	0.040	−0.524	−0.028	0.376	1.759	0.466
9	1.634	0.474	−0.053	−0.475	−0.107	0.335	1.691	1.471
10	1.511	0.728	−0.130	−0.416	−0.157	0.255	1.564	2.261
11	1.346	0.915	−0.184	−0.353	−0.181	0.154	1.393	2.842
12	1.151	1.044	−0.210	−0.291	−0.183	0.046	1.191	3.240
13	0.935	1.124	−0.208	−0.235	−0.167	−0.058	0.968	3.490
14	0.702	1.170	−0.181	−0.187	−0.137	−0.149	0.732	3.632
15	0.474	1.193	−0.133	−0.152	−0.097	−0.218	0.491	3.703
16	0.237	1.202	−0.070	−0.130	−0.050	−0.262	0.246	3.732
17	0.0	1.204	0.0	−0.123	0.0	−0.276	0.0	3.739
18	−0.237	1.202	0.070	−0.130	0.050	−0.262	−0.246	3.732
	...	...	...	...	...	...	...	...

6. Further Developments

The present investigation can be extended to box frames surrounded with elastic foundation (Fig. 3).

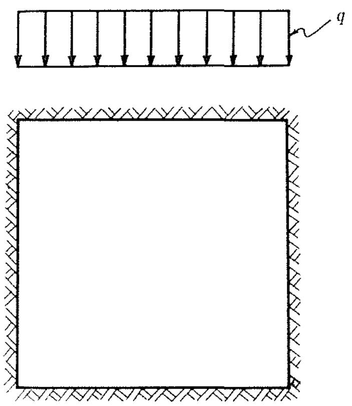

Fig. 3. Box Frame with Elastic Foundation.

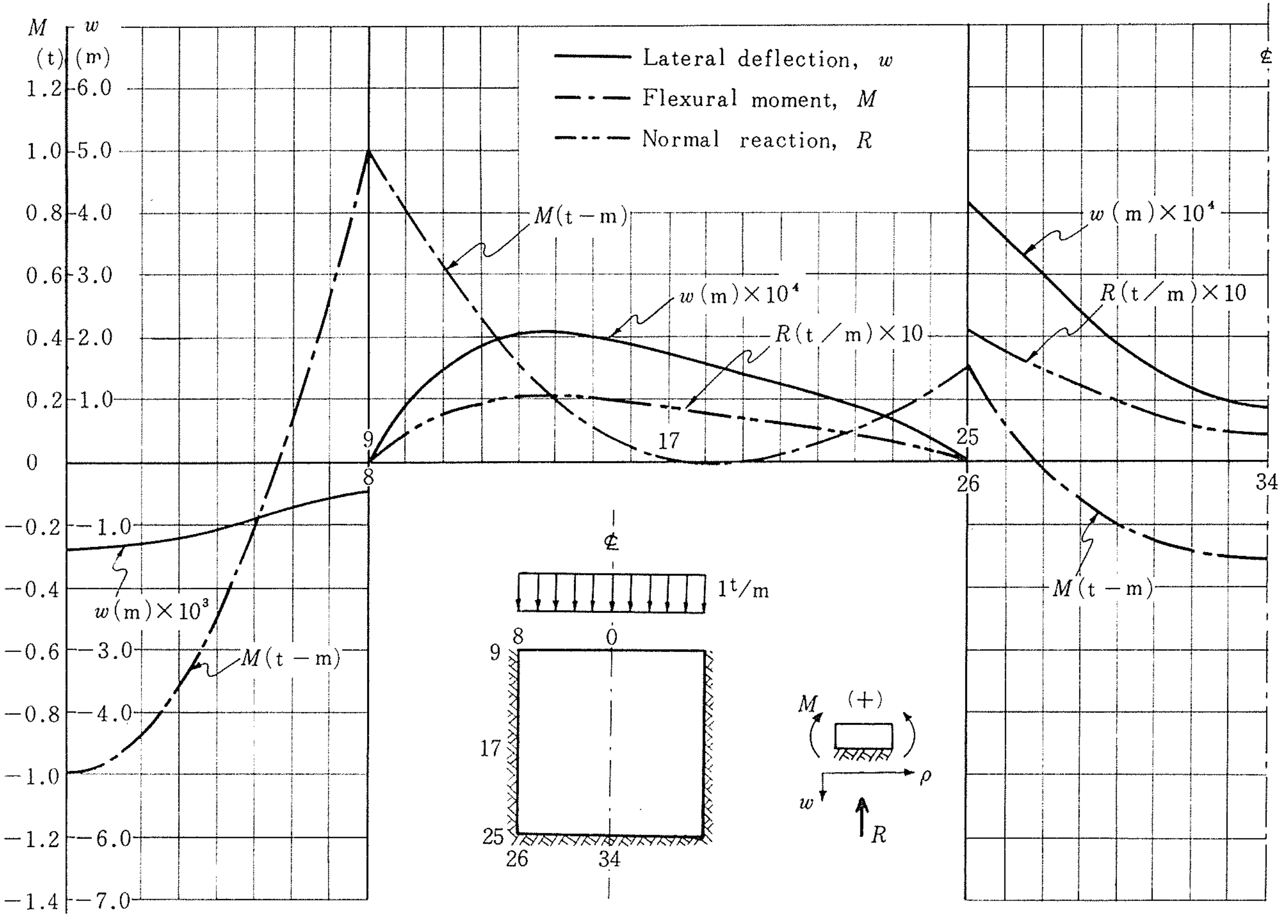

Fig. 4. Lateral Deflection, Flexural Moment, and Normal Reaction.

The geometry and material properties of this box frame are given by Table 3. For brevity, a 4.0 m × 4.0 m square box frame is taken, the four members being assumed to be of the same cross section and material properties. Fig. 4

Table 3. Numerical Values Adopted.

E (t/m^2)	I (m^4)	A (m^2)	j (t/m^2)	k (t/m^2)
2 100 000.0	0.000 675	0.09	700.0	2 000.0

gives the lateral deflection, the flexural moment, and the normal reaction at several intermittent points.

7. Conclusions

This paper presents the operational finite element method, which is applicable to the analysis of circular ring structures. It permits recursion avoiding large-size simultaneous equations.

In virtue of the perfectly classified data of configulations and external loading conditions, the analysis can be made without any assumption of distributed reactions. The point at which the positive reaction begins to act can be determined analytically.

References

1) S. NATSUME, N. YOSHIZAWA, H. HAMANO, and B. TANIMOTO, "Direct Analysis of Continuous Beam-Columns," Journal of the Faculty of Engineering, Shinshu University, Nagano, Japan, Vol. 23, Feb., 1967, pp. 2–3.

2) H. HAMANO and B. TANIMOTO, "Finite Element Arch Analysis by the Operational Method," Journal of the Faculty of Engineering, Shinshu University, Nagano, Japan, Vol. 25, Dec., 1968, pp. 2–3 and 9.

3) N. YOSHIZAWA and B. TANIMOTO, "Operational Method for Continuous Beams, Second Report, Generalized Continuous Beams," Journal of the Faculty of Engineering, Shinshu University, Nagano, Japan, Vol. 20, June, 1966, p. 33.

4) H. HAMANO and B. TANIMOTO, "Tunnel Structure Analysis by the Operational Method, First Report," Journal of the Faculty of Engineering, Shinshu University, Nagano, Japan, Vol. 25, Dec., 1968, p. 15.

Appendix. —Notation

The following symbols are used in this paper:

A = cross-sectional area;

EI = flexural rigidity;

F = axial force;
$\lfloor \mathbf{G} \rfloor$ = geometry matrix; Eq. 11;
j = shear modulus of foundation;
k = normal modulus of foundation;
$\mathbf{K}$ = load-matrix; Eq. 1b;
$\{\mathbf{K}\}$ = load-matrix assemblage; Eq. 11;
$\mathbf{L}$ = shiftor; Eq. 4;
M = bending moment;
$\mathbf{P}$ = projector; Eq. 2;
R = normal reaction;
$\mathbf{R}(\rho)$ = abscissa matrix; Eq. 1;
S = shearing force;
u = axial displacement;
V = tangential reaction;
$\mathbf{W}(\rho)$, $\mathbf{W}'(\rho)$ = state vector for normal and conjugate domain respectively; Eq. 1;
w = lateral deflection;
$\mathbf{X}$ = 6-by-1 eigenmatrix; Eq. 1;
θ = flexural slope;
ρ = x/L, dimensionless curent abscissa; Eq. 1;
$\lfloor \quad \rfloor$ = row vector; and
$\{ \quad \}$ = column vector.