# On the Summability $|\overline{N}, p_n|$ of a Fourier Series

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## 1. Introduction

Previously T. Pati has proved a following theorem for the absolute Nörlund summability of a Fourier series at a point.

Theorem.[1] *If* $\varphi(t) \in BV(0, \pi)$, *and* $\{p_n\}$ *is a positive, monotonic non-increasing sequence such that* $P_n \to \infty$ *as* $n \to \infty$,

$$\{(n+1)p_n/P_n\} \in BV$$

*and*

$$\left\{\sum_{\nu=1}^{n} (\nu+1)^{-1} P_\nu / P_n\right\} \in BV,$$

*then the Fourier series of* $f(t)$, *at* $t = x$, *is summable* $|N, p_n|$.

Later on he[2] has proved that in the theorem, "non-increasing" can be omitted.

In this note, we shall prove an analogous theorem for the summability $|N, p_n|$ of a Fourier series.

As is easily seen, the transformations $|N, p_n|$ and $|\overline{N}, p_n|$ take symmetric forms, hence we can expect the close relation between them. However, these transfomations are not equivalent in general.[3),4)]

## 2. Definitions and Notations

Let $\sum a_n$ be a given infinite series and $\{s_n\}$ the sequence of its partial sums. Let $\{p_n\}$ be a sequence of constants, real or complex, and let us write

$$P_n \equiv p_0 + p_1 + \cdots + p_n\,; \; P_{-k} = p_{-k} \equiv 0 \qquad \text{for } k \geqq 1.$$

The sequence-to-sequence transformation :

$$t_n \equiv \frac{1}{P_n} \sum_{\nu=0}^{n} p_{n-\nu}\, s_\nu \qquad (P_n \neq 0) \tag{1}$$

defines the sequence $\{t_n\}$ of Nörlund means of the sequence $\{s_n\}$, generated by the sequence of coefficients $\{p_n\}$.

The series $\sum a_n$ is said to be summable $(N, p_n)$ to the sum $s$ if $\lim_{n\to\infty} t_n$ exists and is equal to $s$, and is said to be absolutely summable $(N, p_n)$, or summable $|N, p_n|$, if the sequence $\{t_n\}$ is of bounded variation, that is, the series $\sum |t_n - t_{n-1}|$ is convergent.[5]

Similarly, the sequence-to-sequence transformation :

$$\bar{t}_n = \frac{1}{P_n} \sum_{\nu=0}^{n} p_\nu s_\nu \qquad (P_n \neq 0) \tag{2}$$

defines the sequence $\{\bar{t}_n\}$ of discontinuous Riesz means of the sequence $\{s_n\}$, generated by the sequence of coefficients $\{p_n\}$. The series $\sum a_n$ is said to be summable $(\bar{N}, p_n)$ to the sum $s$ if $\lim_{n\to\infty} \bar{t}_n$ exists and is equal to $s$, and is said to be absolutely summable $(\bar{N}, p_n)$, or summable $|\bar{N}, p_n|$, if the sequence $\{\bar{t}_n\}$ is of bounded variation, that is, the series $\sum |\bar{t}_n - \bar{t}_{n-1}|$ is convergent.

Let $f(t)$ be a periodic function, with period $2\pi$, and integrable in the Lebesgue sense over $(-\pi, \pi)$.

We assume, without any loss of generality, that the constant term in the Fourier series of $f(t)$ is zero, so that

$$\int_{-\pi}^{\pi} f(t)\, dt = 0$$

and

$$f(t) \sim \sum_{1}^{\infty} (a_n \cos nt + b_n \sin nt) \equiv \sum_{1}^{\infty} A_n(t).$$

We write throughout

$$s_n = s_n(t) \equiv \sum_{\nu=1}^{n} A_\nu(t),$$

$$\varphi(t) = \varphi_x(t) \equiv \frac{1}{2}\Big\{ f(x+t) + f(x-t) \Big\}. \tag{3}$$

Moreover, by "$\{t_n\} \in \mathrm{BV}$" we shall mean that $\{t_n\}$ is a sequence of bounded variation. Similarly, by "$f(x) \in BV(a, b)$" we shall mean that $f(x)$ is a function of bounded variation over the interval $(a, b)$.

Finally, as usual $[\tau]$ denotes the greatest integer not greater than $\tau$.

### 3. Theorem and Proof

We state our result as follows :

Theorem. *If* $\varphi(t) \in BV(0, \pi)$, *and* $\{p_n\}$ *is a positive, monotonic sequence* *such that* $P_n \to \infty$ *as* $n \to \infty$,

$$\{(n+1)p_n/P_n\} \in BV \tag{4}$$

*and*

$$\left\{\sum_{\nu=1}^{n} (\nu+1)^{-1} P_\nu/P_n\right\} \in BV, \tag{5}$$

*then the Fourier series of* $f(t)$, *at* $t = x$, *is summable* $|\overline{N}, p_n|$.

We require the following lemmas for the proof of our theorem.

Lemma I. *If* $q_n$ *is non-negative and non-increasing, then, for* $0 \leqq a \leqq b \leqq \infty$, $0 \leqq t \leqq \pi$, *and any* $n$,

$$\left|\sum_{k=a}^{b} q_k e^{i(n-k)t}\right| \leqq Q_\tau, \tag{6}$$

*where* $\tau \equiv [1/t]$ *and* $Q_m \equiv q_0 + q_1 + \cdots + q_m$.

The result is originally due to Hill and Tamarkin.

Lemma 2. *For* $\nu \geqq 0$,

$$\sum_{n=\nu+1}^{\infty} \frac{p_n}{P_n P_{n-1}} = \frac{1}{P_\nu}. \tag{7}$$

This is evident, since $p_n = P_n - P_{n-1}$, and $P_n \to \infty$ with $n$.

Lemma 3. *Uniformly in* $0 < t \leqq \pi$,

$$\left|\sum_{k=0}^{\nu} \sin(k+1)t\right| \leqq \pi t^{-1}.$$

The proof of this is easy.

Proof of the theorem. We have, by (2)

$$\bar{t}_n - \bar{t}_{n-1} = \sum_{\nu=0}^{n} \frac{p_\nu}{P_n} s_\nu - \sum_{\nu=0}^{n-1} \frac{p_\nu}{P_{n-1}} s_\nu$$

$$= \sum_{\nu=0}^{n} \left(\frac{1}{P_n} - \frac{1}{P_{n-1}}\right) p_\nu s_\nu + \frac{p_n}{P_{n-1}} s_n.$$

For the Fourier series of $f(t)$, at $t = x$

$$s_\nu = s_\nu(x) = \frac{2}{\pi}\int_0^\pi \frac{\sin(\nu + \frac{1}{2})t}{2\sin\frac{1}{2}t} \varphi(t)\, dt$$

$$\equiv \frac{2}{\pi}\int_0^\pi \varphi(t)D_\nu(t)dt,$$

so that,

$$\bar{t}_n - \bar{t}_{n-1} = \frac{2}{\pi}\int_0^\pi \varphi(t)\sum_{\nu=0}^{n}\left(\frac{1}{P_n} - \frac{1}{P_{n-1}}\right)p_\nu D_\nu(t)dt + \frac{p_n}{P_{n-1}}\frac{2}{\pi}\int_0^\pi \varphi(t)D_n(t)dt, \tag{8}$$

where

$$D_\nu(t) \equiv \frac{\sin(\nu+\frac{1}{2})t}{2\sin\frac{1}{2}t} = \frac{1}{2} + \cos t + \cdots + \cos \nu t.$$

Now, by Abel's transformation,

$$\sum_{\nu=0}^{n}\left(\frac{1}{P_n} - \frac{1}{P_{n-1}}\right)p_\nu D_\nu(t)$$

$$= \sum_{\nu=0}^{n-1}\left(\frac{1}{P_n} - \frac{1}{P_{n-1}}\right)P_\nu \Delta D_\nu(t) + \left(\frac{1}{P_n} - \frac{1}{P_{n-1}}\right)P_n D_n(t)$$

$$= \frac{p_n}{P_n P_{n-1}}\sum_{\nu=0}^{n-1}P_\nu \cos(\nu+1)t - \frac{p_n P_n}{P_n P_{n-1}}D_n(t), \tag{9}$$

where

$$\Delta D_\nu(t) \equiv D_\nu(t) - D_{\nu+1}(t).$$

From (8) and (9),

$$\bar{t}_n - \bar{t}_{n-1} = \frac{2}{\pi}\int_0^\pi \varphi(t)\left\{\frac{p_n}{P_n P_{n-1}}\sum_{\nu=0}^{n-1} P_\nu \cos(\nu+1)t - \frac{p_n}{P_{n-1}}D_n(t) + \frac{p_n}{P_{n-1}}D_n(t)\right\}dt$$

$$\equiv \frac{2}{\pi}\int_0^\pi \varphi(t)\Omega(n,t)dt,$$

where

$$\Omega(n,t) \equiv \frac{p_n}{P_n P_{n-1}}\sum_{\nu=0}^{n-1}P_\nu \cos(\nu+1)t.$$

Thus, in order to prove the theorem, we have to establish that

$$\sum_n |\bar{t}_n - \bar{t}_{n-1}| = \frac{2}{\pi}\sum_n \left|\int_0^\pi \varphi(t)\Omega(n, t)dt\right| \leqq K,$$

where $K$ is used throughout to denote an absolute positive constant, but it is not necessarily the same at each occurrence.

We observe that

$$\int_0^\pi \varphi(t)\Omega(n, t)dt = \left[\left(\int_0^\pi \Omega(n, u)du\right)\varphi(t)\right]_0^\pi - \int_0^\pi \left(\int_0^t \Omega(n, u)du\right)d\varphi(t)$$

$$= -\int_0^\pi \left(\int_0^t \Omega(n, u)du\right)d\varphi(t),$$

so that,

$$\sum_n |\bar{t}_n - \bar{t}_{n-1}| \leqq \frac{2}{\pi}\sum_n \left|\int_0^\pi \left(\int_0^t \Omega(n, u)du\right)d\varphi(t)\right|$$

$$\leqq \frac{2}{\pi}\sum_n \left|\int_0^\pi d\varphi(t)\right| \left|\int_0^t \Omega(n, u)du\right|.$$

Since, by hypothesis,

$$\left|\int_0^\pi d\varphi(t)\right| \leqq K,$$

it suffices for our purpose to show that, uniformly for $0 < t \leqq \pi$,

$$\sum_n \left|\int_0^\pi \Omega(n, u)du\right| \leqq K,$$

or what is the same thing, uniformly for $0 < t \leqq \pi$,

$$J \equiv \sum_n \left|\frac{p_n}{P_n P_{n-1}} \sum_{\nu=0}^{n-1} P_\nu \frac{\sin(\nu+1)t}{\nu+1} dt\right| \leqq K.$$

In order to deal with $J$, we consider two cases separately.

Case (i) Let $\{p_n\}$ be a posiitve, monotonic non-increasing sequence.

Then,

$$J \leqq \left(\sum_{n=1}^{\tau} + \sum_{n=\tau+1}^{\infty}\right) \frac{p_n}{P_n P_{n-1}} \left| \sum_{\nu=0}^{n-1} \frac{P_\nu}{\nu+1} \sin(\nu+1)t \right|$$

$$\equiv L_1 + L_2, \text{ say,}$$

where $\tau \equiv [1/t]$.

Since,

$$|\sin(\nu+1)t| \leqq (\nu+1)t \leqq nt,$$

and by hypothesis,

$$0 < \frac{np_n}{P_n} < 1, \qquad \frac{1}{P_{n-1}} \sum_{\nu=0}^{n-1} \frac{P_\nu}{\nu+1} < \infty,$$

we have

$$L_1 \leqq t \sum_{n=1}^{\tau} \frac{np_n}{P_n P_{n-1}} \sum_{\nu=0}^{n-1} \frac{P_\nu}{\nu+1}$$

$$\leqq Kt \sum_{n=1}^{\tau} \frac{np_n}{P_n} \leqq Kt\tau \leqq K.$$

But, since $\{p_n\}$ is positive monotonic non-increasing, $\left\{\frac{P_\nu}{\nu+1}\right\}$ is so, too. Hence, we have, by Lemma I,

$$\left| \sum_{\nu=0}^{n-1} \frac{P_\nu}{\nu+1} \sin(\nu+1)t \right| \leqq \sum_{\nu=0}^{\tau} \frac{P_\nu}{\nu+1},$$

$$L_2 \leqq \sum_{n=\tau+1}^{\infty} \frac{p_n}{P_n P_{n-1}} \sum_{\nu=0}^{\tau} \frac{P_\nu}{\nu+1}.$$

Also, we have, by Lemma 2,

$$L_2 \leqq \frac{1}{P_\tau} \sum_{\nu=0}^{\tau} \frac{P_\nu}{\nu+1} \leqq K.$$

Thus, we obtain that the Fourier series of $f(t)$, at $t = x$, is summable $|\overline{N}, p_n|$.

Case (ii) Let $\{p_n\}$ be a positive, monotonic increasing sequence.

Then, we have

$$L_1 = \sum_{n=1}^{\tau} \frac{p_n}{P_n P_{n-1}} \left| \sum_{\nu=0}^{n-1} \frac{P_\nu}{\nu+1} \sin(\nu+1)t \right|$$

$$\leqq t\sum_{n=1}^{\tau} \frac{np_n}{P_nP_{n-1}} \sum_{\nu=0}^{n-1} \frac{P_\nu}{\nu+1}$$

$$= t\sum_{n=1}^{\tau} \frac{np_n}{P_n}\left(\frac{1}{P_{n-1}} \sum_{\nu=0}^{n-1} \frac{P_\nu}{\nu+1}\right)$$

$$\leqq Kt\sum_{n=1}^{\tau} \frac{np_n}{P_n}$$

$$< Kt\sum_{n=1}^{\tau} \frac{(n+1)p_n}{P_n}$$

$$\leqq Kt\tau$$

$$\leqq K,$$

by virtue of the hypothesis (4) and (5) of the theorem.

Also, $\left\{\frac{P_\nu}{\nu+1}\right\}$ is positive monotonic increasing sequence, by hypothesis. Hence, by Abel's transformation,

$$\sum_{\nu=0}^{n-1} \frac{P_\nu}{\nu+1}\sin(\nu+1)t$$

$$= \sum_{\nu=0}^{n-2}\left\{\sum_{k=0}^{\nu} \sin(k+1)t\right\}\left(\frac{P_\nu}{\nu+1} - \frac{P_{\nu+1}}{\nu+2}\right) + \frac{P_{n-1}}{n}\sum_{k=0}^{n-1} \sin(k+1)t$$

$$= -\sum_{\nu=0}^{n-2}\left\{\sum_{k=0}^{\nu} \sin(k+1)t\right\}\left(\frac{P_{\nu+1}}{\nu+2} - \frac{P_\nu}{\nu+1}\right) + \frac{P_{n-1}}{n}\sum_{k=0}^{n-1} \sin(k+1)t,$$

whence applying Lemma 3,

$$\left|\sum_{\nu=0}^{n-1} \frac{P_\nu}{\nu+1}\sin(\nu+1)t\right| \leqq \frac{\pi}{t}\left(\frac{P_{n-1}}{n} - P_0 + \frac{P_{n-1}}{n}\right)$$

$$\leqq K(\tau+1)\frac{P_{n-1}}{n},$$

where

$$\tau \equiv [1/t] \leqq 1/t \leqq \tau + 1.$$

Thus, we have

$$L_2 \equiv \sum_{n=\tau+1}^{\infty} \frac{p_n}{P_nP_{n-1}}\left|\sum_{\nu=0}^{n-1} \frac{P_\nu}{\nu+1}\sin(\nu+1)t\right|$$

$$\leqq \sum_{n=\tau+1}^{\infty} \frac{p_n}{P_n P_{n-1}} \frac{P_{n-1}}{n} K(\tau+1)$$

$$= K(\tau+1) \sum_{n=\tau+1}^{\infty} \frac{p_n}{nP_n}$$

$$= K(\tau+1) \sum_{n=\tau+1}^{\infty} \frac{(n+1)p_n}{P_n} \frac{1}{n(n+1)}$$

$$\leqq K(\tau+1) \sum_{n=\tau+1}^{\infty} \frac{1}{n(n+1)}$$

$$= K.$$

Therefore,

$$J \leqq L_1 + L_2 = K.$$

Thus, we obtain that the Fourier series of $f(t)$, at $t = x$, is summable $|\overline{N}, p_n|$.

This terminates the proof of our theorem.

**References**

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