Nonlinear Effects on the Analysis of Pin-Jointed Trusses

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Synopsis: The effects of nonlinearity caused by the material property and the geometrical change of structure in the analysis of plane pin-jointed trusses are discussed herein. Several typical systems are analyzed by both of the linear and nonlinear theories, and corresponding comments are given to the numerical results.

INTRODUCTION

On the basis of the small deformation assumption, the criterion in the design of the prevailing structures has been made, correspondingly, the member stresses generated by the specified loading condition is limited within the actual or idealized linear stress-strain realm of consisting material. Using a large or empirical safety factor, the above condition can be satisfied in the practice of structural design. As a more rational criterion for the structural design, the reliability analysis has been proposed in recent years.[1),2)] In this study, to estimate the load bearing capacity of structure or to analyze the failure mechanism of structure is one of the important problems.[3)] Consequently, the introduction from the accurate characteristic of consisting materials into the analysis is necessitated. Taking the entire history of the stress-strain behavior of materials into account, considerable quantity of strain takes place with the increase of the stress intensity, and the shape of the diagram becomes nonlinear, the former effects on the geometrical change in structure and the latter the reduction in the structural stiffness. Introducing each effect into the analysis, the governing equation becomes nonlinear, and therefore, no formulary description will be available so that the numerical treatment preferable.

In this paper, as the first examination, the effect of each nonlinear factor on the analysis of plane pin-jointed trusses is discussed individually. In the treatment of the material nonlinearity, a polygonal analogy to the stress-strain curve and the incremental-variable elasticity procedure[4)] are adopted for use, because they are the most surefooted way in the computer operation. The problem of the finite deformation in structure is analyzed by three attempts,

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namely, the method of displacement correction, the method of superposition and the method of equilibration.[5] The merits in each method are described considering the combined use with the treatment of material nonlinearity above. Note here that although the finite deformation problems in the structural analysis have been investigated about the deformable structures in the elastic stress condition such as the design of suspension bridges,[6] the same problem appears even in the undeformable structures when the entire behavior of the stress-strain relationship in materials is considered as described above, and this paper treats the latter case. In the final examination, numerical examples are given to the analysis of statical behavior of pin-jointed trusses considering the material nonlinearity together with the geometrical change of the structures.

The letter symbols and the illustrative terms adopted for use in this paper are defined where they first appear.

BASIC CONCEPTS

In the structural analysis, the prevailing infinitesimal theory gives the following equations:

member force: $$\mathbf{F} = \mathbf{SPD}. \tag{1}$$

equilibration: $$\mathbf{P}^T\mathbf{F} = \mathbf{L}. \tag{2}$$

displacement: $$\mathbf{D} = [\mathbf{P}^T\mathbf{SP}]^{-1}\mathbf{L}. \tag{3}$$

Here, $\mathbf{F}$ = the member force matrix, $\mathbf{S}$ = the stiffness matrix of structure, $\mathbf{P}$ = the projection matrix given by the structural geometry, $\mathbf{D}$ = the displacement matrix and $\mathbf{L}$ = the load matrix. They are interrelated by the Hooke's law and the statical force equilibrium condition, but are independent of each other, therefore, the solution can be obtained uniquely in any loading conditions.

When the nonlinear property of the stress-strain relationship in materials is taken into account, the stiffness matrix $\mathbf{S}$ is no longer invariant. It depends on the stress level in each member. The governing equation in this case is written

$$\mathbf{D} = [\mathbf{P}^T\mathbf{S}(\sigma)\mathbf{P}]^{-1}[\mathbf{L} - \mathbf{P}^T\mathbf{C}(\sigma)], \tag{4}$$

in which $\mathbf{S}(\sigma)$ = the stiffness matrix and $\mathbf{C}(\sigma)$ = the domain correction matrix. Both of them are the function of the stress level in each constituent member, so that the solution can not be obtained uniquely.

Treating the geometrical change of structure, the projection matrix $\mathbf{P}$ varies with the nodal displacements generated by the loading. In this case, we have

$$\mathbf{D} = [\mathbf{P}(x, y)^T\mathbf{SP}(x, y)]^{-1}[\mathbf{L} - \mathbf{P}(x, y)^T\mathbf{SB}(x, y)]. \tag{5}$$

Here, $\mathbf{P}(x, y) =$ the projection matrix evaluated by the nodal displacements under certain loading condition and $\mathbf{B}(x, y) =$ the corresponding geometry correction matrix. This equation also gives the nonlinearity in the structural analysis.

The combination of the above effects results in the following formula, wherein all factors except $\mathbf{L}$ on the right side depend on the displacement matrix $\mathbf{D}$ on the left side of the equation.

$$\mathbf{D} = [\mathbf{P}(x, y)^T \mathbf{S}(\sigma)\mathbf{P}(x, y)]^{-1}[\mathbf{L} - \mathbf{P}(x, y)^T[\mathbf{C}(\sigma) + \mathbf{S}(\sigma)\mathbf{B}(x, y)]]. \tag{6}$$

MATERIAL NONLINEARITY

As a fundamental concept, it can be stated that the stress intensity in each constituent member in a structure, which is generated by a given loading condition, must be determined uniquely by virtue of the inherent stiffness of the structure even if the stress-strain property of the material is linear or not. However, introducing the nonlinear property of materials into the analysis, it is impossible to find the stress condition directly because the stiffness matrix $\mathbf{S}(\sigma)$ in Eq. 4 becomes to the function of the member stress. To solve this problem, the iterative-direct loading procedure, the incremental-initial strain procedure and the incremental-variable elasticity procedure may be used. To the author's examination, the third procedure is suitable to the treatment of complicated stress-strain curves such as structural steel, and the first procedure to a little idealized flat curves. The second procedure is not so recommended. [7)]

In the statically determinate truss systems made of a nonlinear material, the relationship between the incremental load and the nodal displacements is analogous to the property of the stress-strain curve, but the incremental load-member force relationship is always linear regardless of the material property. On the other hand, in the case of the statically indeterminate systems, different aspects are experienced accompanied with the increment of nodal loads. However, it may be noted that the linear relationship between the incremental load and the nodal displacements or the member forces is hold as far as the statically indeterminate condition of the system is not lost by the appearance of large strain of a member due to its stiffness reduction in the loading process.

As a practical application of this study, examining the load bearing capacity of pin-jointed trusses made of structural steel, the strain hardening realm in the material considerably effects on the capacity, and the statically indeterminate system has superior capacity to the determinate one. [7),8)]

CHANGE OF GEOMETRY

The direct determination of the rigorous change of geometry in a structure under certain loading condition is impossible. Because, in Eq. 5, both matrices $\boldsymbol{P}(x, y)$ and $\boldsymbol{B}(x, y)$ are the function of the displacement matrix $\boldsymbol{D}$ on the left side. Therefore, the avenue to reach the final shape of the structural deformation must be left to the iterative procedure. The author tried the following three methods in this treatment.

Method of Displacement Correction

The displacement of a nodal point in the pin-jointed truss systems is represented by the horizontal component u and the vertical component v in this paper. Fig. 1 shows the state of a nodal point before and after loading. The system is deformed by the loading and the displacement components at this point are considered u and v as shown in the figure.

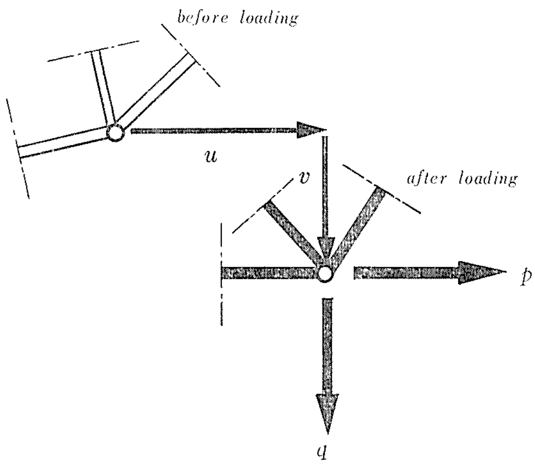


Fig. 1. Method of Displacement Correction.

In this manner, the elongation of a member in the system is represented by the displacements of two nodal points whereto the member is pin-connected, and the corresponding formula becomes the Taylor or binomial expansion of the nodal displacements. We consider the first order terms of the displacements in the formula as unknown factors and the higher order terms of them as known quantities which are obtained from the iterative computation in one step before. The equilibrium condition must be treated by the deformed geometry of the system, then the projection matrix $\boldsymbol{P}_j$ used in the (j)-th iterative computation is renewed by the nodal displacements obtained in the $(j-1)$-th computation. The governing equation in the present method is given as follows:

$$\mathbf{D}_j = [\mathbf{P}_j^T \mathbf{S}_o \mathbf{P}_o]^{-1} [\mathbf{L} - \mathbf{P}_j^T \mathbf{S}_o \mathbf{B}_j]. \quad (7)$$

Here, $\mathbf{D}_j$= the displacement matrix obtained in the (j)-th iteration, $\mathbf{P}_j^T$= the transpose of $\mathbf{P}_j$, $\mathbf{S}_o$= the stiffness matrix (constant), $\mathbf{P}_o$= the projection matrix before loading and $\mathbf{B}_j$= the geometry correction matrix renewed by $\mathbf{D}_{j-1}$.

At the start of the computation, we take $\mathbf{P}_j = \mathbf{P}_o$ and $\mathbf{B}_j = 0$, then Eq. 7 coincides with Eq. 3 (small displacement theory). The convergent characteristic of this method is shown in Fig. 4.

In this method, by the author's examination, it is of no use to take many higher order terms in the expansion, because the method itself depends on the iterative procedure. Then, how to take the convergent accuracy on the results obtained in each iterative step is more important.

As a simple example, numerical results in the analysis of two member system shown in Fig. 4 are given in the following. The nodal load Q =10000 kg is applied directly in this case. Here, a= the limit of convergency between the nodal displacements obtained in the $(j-1)$-th and (j)-th iteration, and t= the number of higher order terms of the displacements used in the binomial expansion in the formula.

I. Influence of higher order terms in expansion (a = 0.001)

t	u(cm)	v(cm)	F_1(kg)	F_2(kg)
2	0.101 599	2.274 864	41 043	40 798
3	0.101 599	2.274 861	41 043	40 798
4	0.101 599	2.274 861	41 043	40 798

II. Influence of convergent accuracy (t = 4)

a	u(cm)	v(cm)	F_1(kg)	F_2(kg)
0.001	0.101 599	2.274 861	41 043	40 798
0.000 1	0.101 588	2.274 282	41 039	40 793
0.000 01	0.101 588	2.274 304	41 039	40 793
0.000 001	0.101 588	2.274 307	41 039	40 793

When the nonlinear characteristic of materials is treated with the geometrical change of the structure, the incremental loading procedure is usually introduced. In this case, the external loads are divided into small parts and accumulated step by step until the initial value of the loads. In each loading step, the stiffness of the structure and the effect of the geometrical change are checked, and corresponding corrections are made. In such integrative method, how to take the convergent limit has important influence on the final result.

The following results obtained from the analysis of two member system (E = constant) in Fig. 4 show the feature of this problem.

IIIa Influence of step number n (a = 0.001, t = 4)

n	u(cm)	v(cm)	F_1(kg)	F_2(kg)
1	0.101 599	2.274 861	41 043	40 798
5	0.101 593	2.274 529	41 041	40 795
10	0.101 582	2.273 996	41 036	40 791
20	0.101 597	2.274 762	41 042	40 797
50	0.101 578	2.273 779	41 035	40 789
100	0.101 583	2.274 045	41 037	40 791
200	0.101 596	2.274 675	41 042	40 796

IIIb Influence of step number n (a = 0.000 001, t = 4)

n	u(cm)	v(cm)	F_1(kg)	F_2(kg)
1	0.101 588	2.274 307	41 039	40 793
5	0.101 588	2.274 307	41 039	40 793
10	0.101 588	2.274 306	41 039	40 793
20	0.101 588	2.274 307	41 039	40 793
50	0.101 588	2.274 307	41 039	40 793
100	0.101 588	2.274 306	41 039	40 793
200	0.101 588	2.274 307	41 039	40 793
SM	0.119 995	2.925 136	48 374	48 166

(Values on the row SM are the results by the small displacement theory)

Referring to Eq. 7, the present method has two kinds of projection matrices $\boldsymbol{P}_0$ (invariable) and $\boldsymbol{P}_j$ (variable). Besides, the element in the geometry correction matrix $\boldsymbol{B}_j$ has complicated form of binomial expansion. Then, from the viewpoint of the computer programing, this method is tiresome comparing with the methods described in the following, and the convergency is inferior to the method of equilibration as shown in Fig. 4.

Method of Superposition

The difference between the results obtained by the small and finite displacement theories is negligible when the external load applied to the structure is very small comparing with the entire stiffness of the structure. In other words, under such loading condition, the nodal displacements obtained by the small displacement theory can be considered as those obtained by the finite displacement theory.

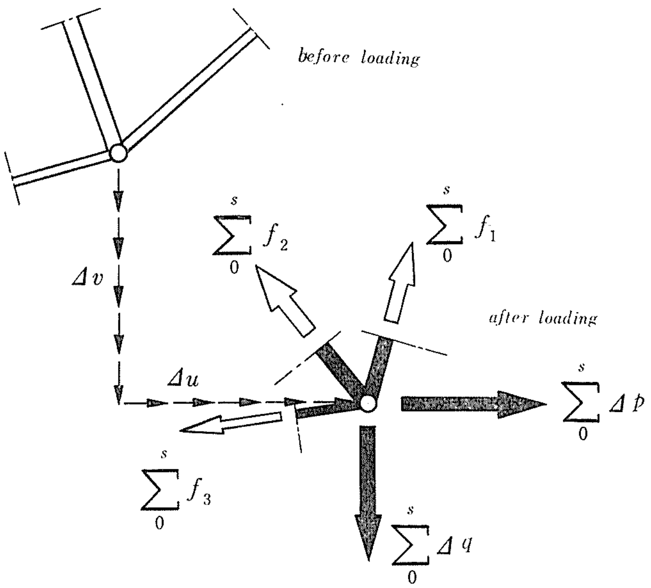


Fig. 2. Method of Superposition.

On the basis of the above empirical fact, the present attempt has been made. Fig. 2 shows the concept of this method. The external load is divided into finer elemental load so as to satisfy the above condition. Applying the elemental load to the structure, corresponding nodal displacements of the system are evaluated by the small displacement theory. After that, the geometry of the system and the projection matrix are renewed by these nodal displacements. This procedure is accumulated until the summation of the elemental load reaches to the value of the actual load applied to the structure. In Fig. 2 is shown the (s)-th step of this treatment, wherein the nodal point has been displaced by the accumulated displacements $\sum \Delta u$ and $\sum \Delta v$. In virtue of the accumulated deformation of the structure, member forces have been generated as shown in the figure. They effect on the equilibrium condition in this instance. The structural displacement matrix in the (s)-th step is given as follows:

$$\Delta \mathbf{D}_s = [\mathbf{P}_s{}^T \mathbf{S}_o \mathbf{P}_s]^{-1}[\sum_o^s \Delta \mathbf{L} - \mathbf{P}_s{}^T \mathbf{S}_o \Delta_s]. \tag{8}$$

Here, $\Delta \mathbf{D}_s =$ the incremental displacement matrix in the (s)-th loading step, $\mathbf{P}_s{}^T, \mathbf{P}_s =$ the projection matrices made by the $(s-1)$-th accumulated deformation of the structure, $\mathbf{S}_o =$ the stiffness matrix, $\sum_o^s \Delta \mathbf{L} =$ the accumulation of the elemental load matrix and $\Delta_s =$ the member elongation matrix made by the $(s-1)$-th accumulated deformation of structure. In this method, the convergency of the numerical result is satisfied by taking the finer division of the applied load.

The convergent condition of this method is shown in Fig. 4, which has different feature from other methods. In this example, a smooth convergency of the results can be observed with the increase of the accumulative step. About the horizontal displacement u, taking the number of load division $n=50$, the error compared with the result by $n=1600$ is 1.54%, and thus, $n=100$(0.76%), 200(0.38%), 400(0.18%), 800(0.07%).

The present method of superposition needs considerable time and labour in the computation, but the operational style is quite same as the incremental loading procedure in the treatment of material nonlinearity. Therefore, the combinative use of them is available to the treatment of material nonlinearity considering the geometrical change of the structure. In this case, finer division of the applied load is necessitated, and the stress-strain diagram treated in the computation must be confined to the continuous and smooth shape.

Method of Equilibration

The structure has an inherent geometry of deformation against the external load. In virtue of the geometrical change from the initial state of the structure, the internal forces are generated in the constituent members. The condition of equilibration between the external and the internal forces is perfectly held at

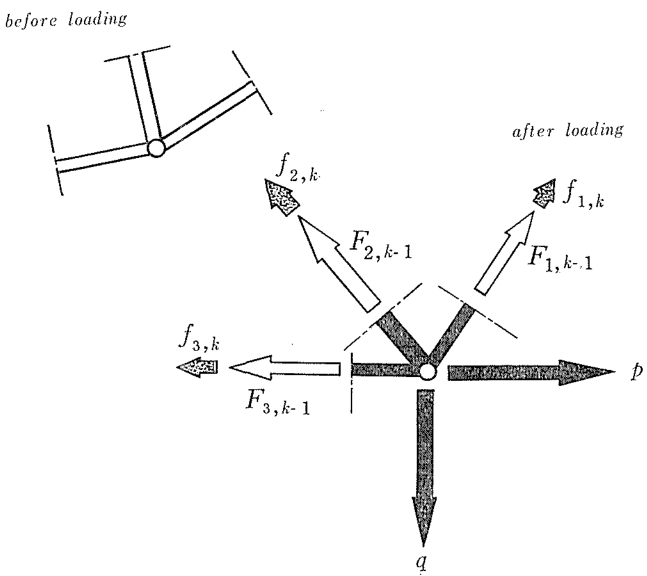


Fig. 3. Method of Equilibration.

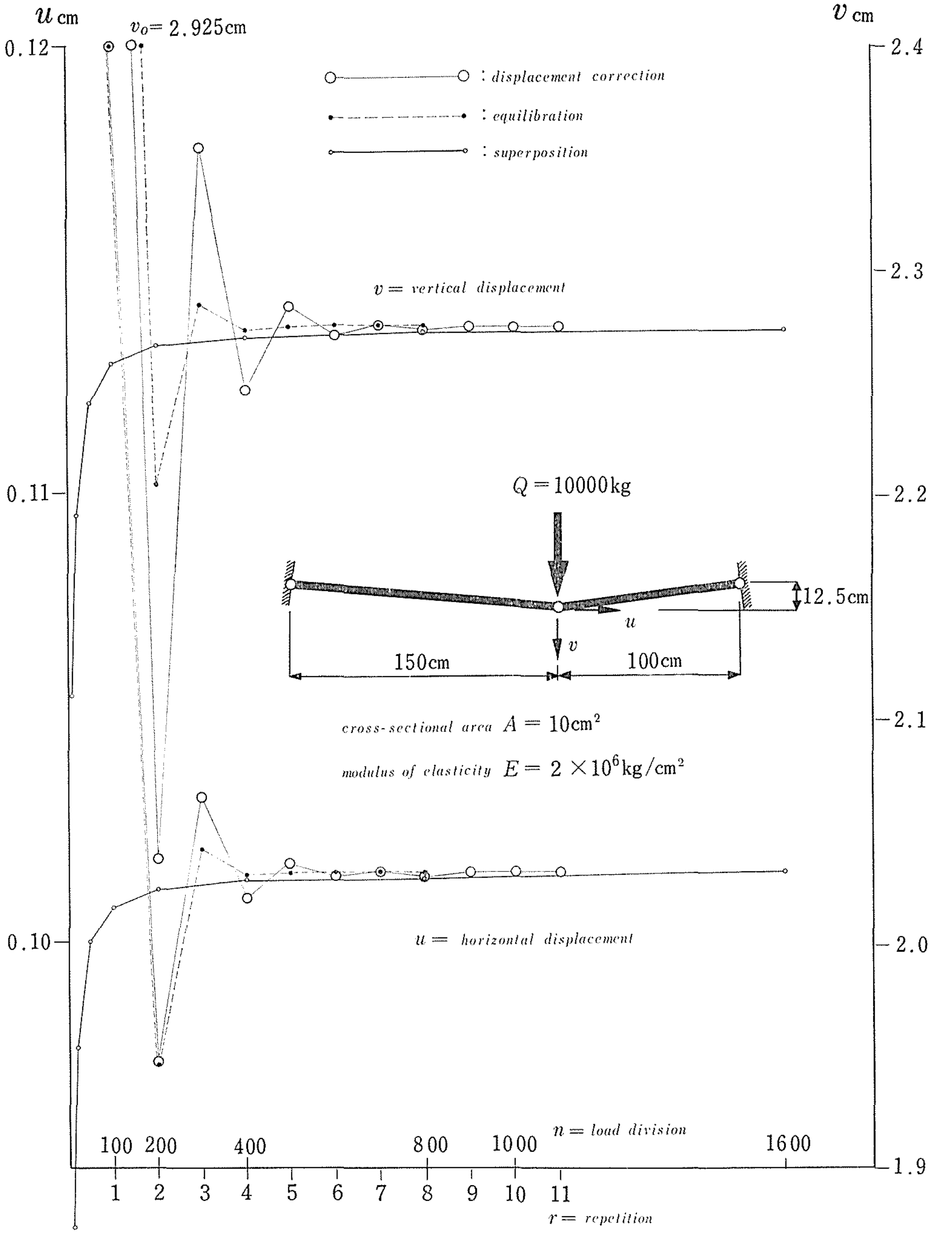


Fig. 4. Feature of Convergency in Three Methods of Analysis.

the deformed state of the structure.

Fig. 3 shows the displaced state of nodal point in a pin-jointed truss system by the external loading. The geometry of the system changes from the initial shape, and consequently, the member forces, $F_{1,k-1}$, $F_{2,k-1}$, $F_{3,k-1}$, are generated. In this deformed geometry, the nodal equilibrium condition must be satisfied between the external load and the member forces. But, this condition can not be satisfied perfectly, therefore, the complementary forces, $f_{1,k}$, $f_{2,k}$, $f_{3,k}$, must be introduced to supply the above unbalance as shown in the figure.

To produce the complementary forces, the system must yield an additional deformation, which is given by the small displacement theory. In this manner, the computation is continued until the sufficient equilibration of the system is secured. The additional displacement matrix $\mathbf{d}_k$ in the (k)-th repetition is given by the following formula, wherein the matrices on the right hand side are referred to those defined in Eq. 8 rewriting the subscript k to s.

$$\mathbf{d}_k = [\mathbf{P}_k{}^T \mathbf{S}_o \mathbf{P}_k]^{-1} [\mathbf{L} - \mathbf{P}_k{}^T \mathbf{S}_o \Delta_k]. \tag{9}$$

The convergent feature of this method is shown in Fig. 4, which is analogous to that of the method of displacement correction and has superior convergency to it. To the individual finite deformation analysis of structures or to the inclusive analysis of the material nonlinearity, the present method of equilibration can be effectively applied because of its convergent property and the simplicity in the computer operation.

The table (p. 11) shows the comparison between the results obtained by the small and the finite displacement theories. The system treated is the Warren truss shown in Fig. 5, which is made of linear material ($E = 2\,000\,000$ kg/cm^2 ...... constant). Each member has cross-sectional area 10 cm^2 and length 200 cm. The nodal points and the members are designated as shown in the figure.

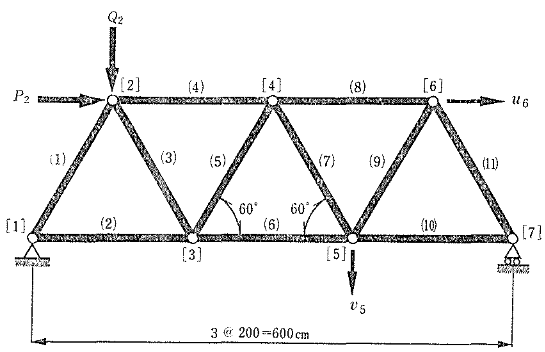
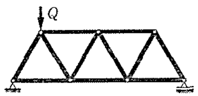
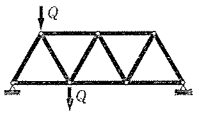
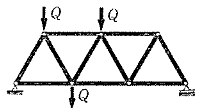
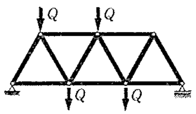
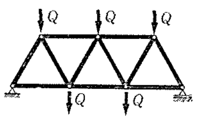


Fig. 5. Three Panel Warren Truss.

Analysis of Warren Truss by Small and Finite Displacement Theories.

loading condition		horizontal displacements						vertical displacements					member forces		
		u_2	u_3	u_4	u_5	u_6	u_7	v_2	v_3	v_4	v_5	v_6	F_3	F_6	F_8
	S	.0882	.0481	.0497	.0769	.0304	.0866	.1620	.1629	.1416	.1037	.0546	−1 924	+2 888	−1 923
	F	.0882	.0481	.0497	.0769	.0304	.0865	.1621	.1630	.1417	.1037	.0546	−1 931	+2 889	−1 926
	p	−0.00	−0.00	−0.00	−0.00	−0.00	−0.12	+0.06	+0.06	+0.07	+0.00	+0.00	+0.36	+0.03	+0.16
	S	.2165	.0866	.1010	.1732	.0433	.2020	.3250	.4666	.4083	.3000	.1583	+5 782	+8 667	−5 758
	F	.2164	.0862	.1008	.1729	.0429	.2016	.3251	.4666	.4084	.3001	.1584	+5 749	+8 671	−5 778
	p	−0.05	−0.46	−0.20	−0.17	−0.92	−0.20	+0.03	+0.00	+0.02	+0.03	+0.06	−0.57	+0.05	+0.35
	S	.3464	.1154	.1732	.2886	.0577	.3464	.4666	.7333	.7666	.5666	.3000	+11 575	+17 328	−11 493
	F	.3461	.1145	.1726	.2880	.0565	.3451	.4671	.7334	.7674	.5671	.3004	+11 511	+17 351	−11 554
	p	−0.09	−0.78	−0.35	−0.21	−2.08	−0.38	+0.09	+0.01	+0.10	+0.09	+0.13	−0.55	+0.13	+0.53
	S	.4490	.1347	.2373	.3656	.0449	.4618	.5703	.9296	1.033	.8703	.4629	+15 447	+23 094	−19 164
	F	.4486	.1332	.2362	.3645	.0429	.4593	.5711	.9299	1.035	.8707	.4635	+15 352	+23 138	−19 249
	p	−0.09	−1.11	−0.46	−0.30	−4.45	−0.54	+1.40	+0.03	+0.12	+0.05	+0.13	−0.62	+0.19	+0.44
	S	.5051	.1443	.2742	.4041	.0433	.5484	.6250	1.033	1.175	1.033	.6250	+17 387	+25 980	−23 018
	F	.5046	.1424	.2727	.4029	.0407	.5454	.6260	1.034	1.177	1.034	.6260	+17 281	+26 047	−23 118
	p	−0.11	−1.32	−0.55	−0.30	−6.00	−0.55	+0.16	+0.06	+0.14	+0.06	+0.16	−0.61	+0.26	+0.43

The finite displacement analysis is carried out by the method of equilibration. Taking the convergent limit $a = 1/100\,000$, the computation was finished within three repetitions in each case. In the table, the row symbols S and F denote the results obtained by the small and the finite displacement theories, respectively, and p is the ratio between them, which is given by $100 \times (F-S)/S$ percent. The column symbols u_i and v_i are the horizontal and vertical displacements at nodal point $[i]$ and F_i is the internal force of member (i). The vertical load Q equals to 10 000 kg at each node.

Observing the results obtained by the finite deformation theory, it can be said that, under the given loading condition shown in the table, the horizontal displacements yield smaller, while the vertical ones larger than them obtained by the small displacement theory. Similar aspects can be observed in the column of the member forces. With the increase of the applied loads, the system becomes deformable, consequently, the ratio p defined above becomes larger. Thus, the importance of the effect produced by the geometrical change of the structure increases under such deformable loading condition of the structure, even though the stress condition of the consisting members is kept within the linear (elastic) domain.

COMBINED EFFECT

The combined nonlinear effect on the analysis of pin-jointed truss structures is caused by the nonlinear characteristic of consisting materials and the geometrical change of the structure. All the analytical equations in this instance are reduced to Eq. 6, and it is advisable to combine the method of equilibration with the incremental-variable elasticity procedure. The following examples are treated by this combination.

Fig. 6 shows the behavior of three member truss analyzed by the small and finite displacement theories. Each member has cross-sectional area 10 cm^2, and its stress-strain characteristic is shown in the figure. Horizontal load P is applied to the interconnected joint and increased up to 10 000 kg. In this figure, f_1, f_2, f_3 = the member forces by the small displacement theory, F_1, F_2, F_3 = the member forces by the finite displacement theory, and U, V = the nodal displacements by the finite displacement theory.

In the small displacement theory, the truss maintains its original shape regardless the external loading, so that, under the present loading condition, the load P is carried by the members (1) and (3) only, and therefore, $f_2 = 0$ and no vertical displacement appears.

Considering the geometrical change in the system, a slight upward dis-

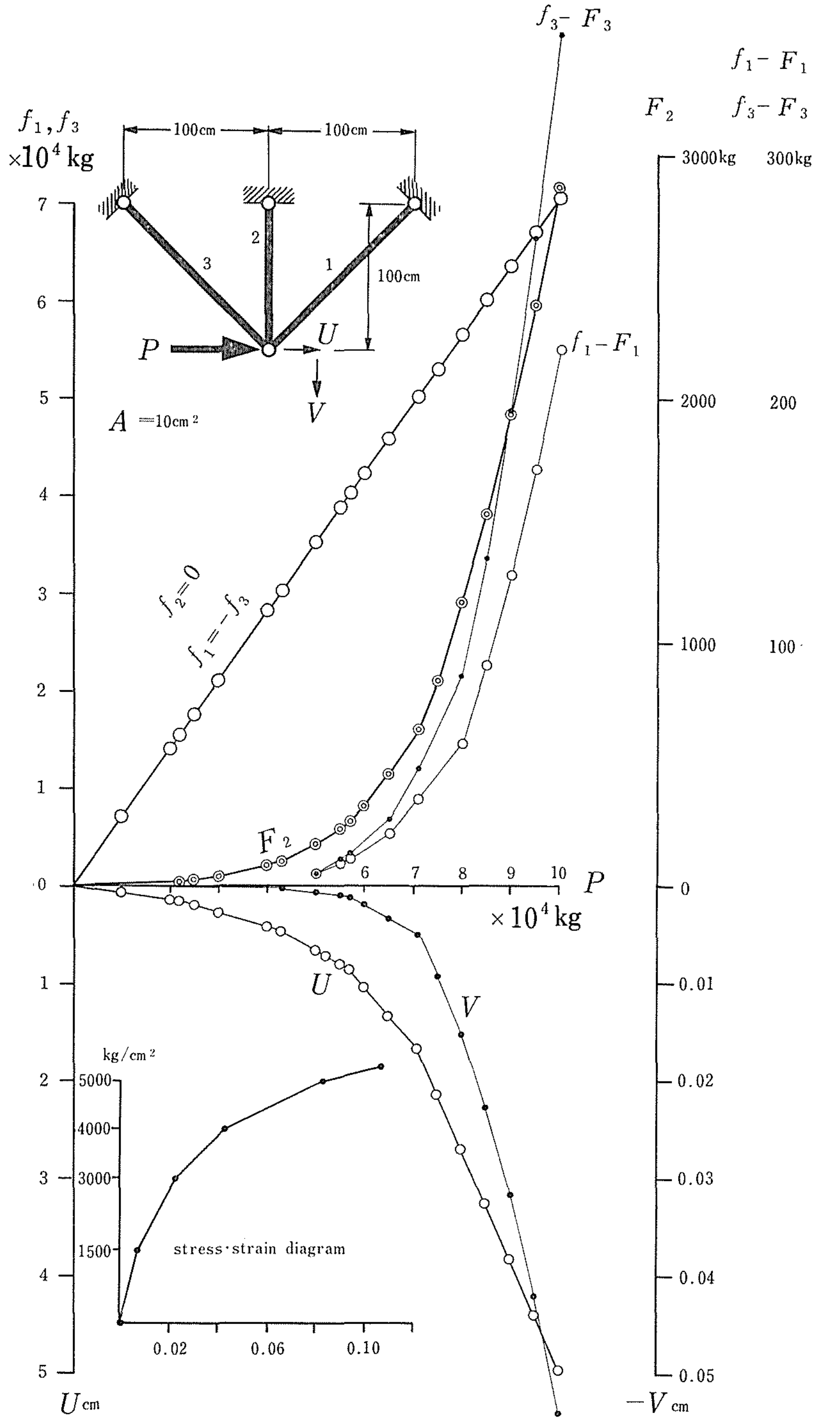


Fig. 6. Combined Effect.

placement appears at the interconnection joint. In virtue of this behavior, the member force F_2 varies as shown in the figure. Variation of other member forces is also nonlinear, but very small, then the difference between both theories is shown in the figure. The behavior of the nodal displacements U, V is analogous to the stress-strain diagram of consisting material. In the figure, no visible difference can be observed between the horizontal displacements obtained by both theories, because the difference is only 0.3 percent at the state of the final loading condition.

CONCLUSION

Numerical approaches to the treatment of nonlinear problems in the pin-jointed truss structures have been presented in this paper. On the viewpoint of the author's examination, it is stated that the incremental-variable elasticity method is the steady approach to the treatment of material nonlinearity, and the method of equilibration is effective for the analysis of geometrical nonlinearity. The combination of these methods paves the way to the detailed research for the structual behavior.

The effect of nonlinearity compared to the prevailing infinitesimal theory of structural analysis is not so seriously observed in the examples shown in this paper. Because the stress-strain diagram used for the computation is smooth and continuous flat one as shown in Fig. 6. Treating the characteristic of structural steel having the elastic, the yielding and the strain hardening domains, the abrupt reduction of stiffness in the yielding domain of a member effects seriously on the change of geometry of the structure.[7] To trace the structural behavior under such state, it consumes too much time and labor, and therefore, this problem will be reported with valid devices for programing in the near future.

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