Complex Eigenfunction Method for Bending Analysls of Rectangular Plates

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Synopsis

The present article will give a Fadle eigenfunction analysis of rectangular plates in flexure by means of complex matrix algebra. Rectangular plates have many structural applications, and their flexural analysis is thus of considerable importance.

The solution of the flexural problems of classical elasticity generally involves the satisfaction of the homogeneous biharmonic equation and the imposed boundary conditions. Although these boundary value problems have been the subject of many investigators and the literature is replete with numerous solutions, many problems of practical interest have not been solved with respect to the actual imposed boundary conditions.

Fadle and Papkovitch were the first to present a method for solving rectangular plate problems by the use of complex biharmonic eigenfunction. The utility of a representation in terms of a Fadle eigenfunction series is contingent on the ability to express arbitrary functions in terms of the series. Each term of a series of these functions satisfies the governing differential equation $\nabla^4 w = 0$ and certain homogeneous boundary conditions on two parallel edges identically. In addition, each term of the general eigenfunction series, when written for finite rectangular plates, contains two arbitrary complex constants which can be used to satisfy arbitrary boundary conditions on the remaining two edges. Thus, the use of these eigenfunction permits the simultaneous satisfaction of the boundary conditions on all four sides of the rectangular plate. An approximate expansion formula is developed and applied to the flexural rectangular plate problem.

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The analysis can be made for complex quantitiesas as these appear, and needs not to separate real parts from imaginary ones, because these can be evaluated numerically with digital computers.

I. First Solution for Clamped Rectangular Plates

1. Introduction.

The analysis of such plates with two opposing edges simply supported may be easily made by a Fourier series analysis which is called Lévy's solution. When this technique is applied to a plate that is clamped on all four edges, then the resulting matrix has an infinite number of terms. The usual procedure is truncation of this matrix; however, this leads to poor convergence of the solution near the corners of the plate. The infinite matrix results from the fact that the trigonometric functions satisfy only one of the two boundary conditions at each of the plate boundaries. The enforcement of the second boundary condition then gives rise to the infinite system of simultaneous equations. In the forming of the infinite matrix, it is necessary to expand hyperbolic fuctions in terms of Fourier series. It is the slow convergence of those series that lead to poor convergence of the solution near the corners. In order to avoid this infinite matrix, considerable effort has been expanted in developing eigenfunctions that satisfy all the boundary conditions at the plate boundaries.

In the following analysis, the Fadle eigenfunction method is used in the development of a general solution method for clamped rectangular plate problems.

2. Homogeneous Deflection.

The coordinate system used in describing the flexure problem considered is shown in Fig. 1; for brevity, only loading functions that are symmetric about both the x and y axes will be considered.

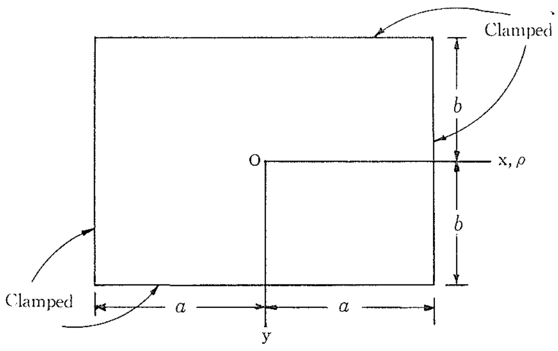

Fig. 1. Clamped Rectangular Plate.

The flexural deflection w of a plate is in general governed by the differential equation

$$\nabla^4 w = \frac{q}{D}. \tag{1}$$

It is convenient to write the complete solution to Eq. 1 as a sum of two functions; i. e.,

$$w = w_h + w_p, \tag{2a}$$

in which w_h = the homogeneous solution meant to satisfy the prescribed boundary conditions and w_p = the particular soltution compatible with the loading condition. The second function w_p is a particular solution to Eq. 1, satisfying

$$\nabla^4 w_p = \frac{q}{D}. \tag{2b}$$

Combining Eq. 2b with Eqs. 1 and 2a, the following governing equation for w_h is obtained:

$$\nabla^4 w_h = 0. \tag{2c}$$

The homogeneous functions satisfying Eq. 2c take from symmetry the form (Fig. 1)

$$w_h = \sum_n \lfloor \cos\lambda\rho,\quad \lambda\rho\sin\lambda\rho \rfloor \mathbf{N}\,\mathrm{ch}\frac{\lambda y}{a} \quad \left(\rho = \frac{x}{a}\right), \tag{3}$$

in which the approach eigenmatrix $\mathbf{N}$ is a 2–by–1 column matrix, each element of which will be a complex constant.

3. Complex Homogeneous State Vectors.

The plate flexure will consist of ten elements, and it will be for convenience to classify into the two state vectors

$$\mathbf{W}_h = \begin{bmatrix} w \\ \theta_x \\ \theta_y \\ M_x \\ M_y \end{bmatrix} = \begin{bmatrix} 1 \\ \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \\ -D\left(\dfrac{\partial^2}{\partial x^2} + \nu\dfrac{\partial^2}{\partial y^2}\right) \\ -D\left(\dfrac{\partial^2}{\partial y^2} + \nu\dfrac{\partial^2}{\partial x^2}\right) \end{bmatrix} w_h, \tag{4a}$$

$$\begin{bmatrix} M_{xy} \\ S_x \\ S_y \end{bmatrix}_h \quad \begin{bmatrix} -D(1-\nu)\dfrac{\partial^2}{\partial x \partial y} \\ -D\left(\dfrac{\partial^3}{\partial x^3} + \dfrac{\partial^3}{\partial x \partial y^2}\right) \\ -D\left(\dfrac{\partial^3}{\partial y^3} + \dfrac{\partial^3}{\partial y \partial x^2}\right) \end{bmatrix}$$

$$\bar{\boldsymbol{S}}_h = \begin{bmatrix} \bar{S}_x \\ \bar{S}_y \end{bmatrix}_h = -D \begin{bmatrix} \dfrac{\partial^3}{\partial x^3} + (2-\nu)\dfrac{\partial^3}{\partial x \partial y^2} \\ \dfrac{\partial^3}{\partial y^3} + (2-\nu)\dfrac{\partial^3}{\partial y \partial x^2} \end{bmatrix} w_h. \tag{4b}$$

Eq. 4b may be referred to as the edge-shear vector. Since the complex deflection w_h is given by Eq. 3, the two state vectors yield the approach equations

$$\boldsymbol{W}_h = \begin{bmatrix} w \\ \theta_x \\ \theta_y \\ M_x \\ M_y \\ M_{xy} \\ S_x \\ S_y \end{bmatrix}_h = \begin{bmatrix} \mathrm{ch}\dfrac{\lambda y}{a} \\ \dfrac{\lambda}{a}\mathrm{ch}\dfrac{\lambda y}{a} \\ \dfrac{\lambda}{a}\mathrm{sh}\dfrac{\lambda y}{a} \\ -D\dfrac{\lambda^2}{a^2}\mathrm{ch}\dfrac{\lambda y}{a} \\ -D\dfrac{\lambda^2}{a^2}\mathrm{ch}\dfrac{\lambda y}{a} \\ -D(1-\nu)\dfrac{\lambda^2}{a^2}\mathrm{sh}\dfrac{\lambda y}{a} \\ -D\dfrac{\lambda^3}{a^3}\mathrm{ch}\dfrac{\lambda y}{a} \\ -D\dfrac{\lambda^3}{a^3}\mathrm{sh}\dfrac{\lambda y}{a} \end{bmatrix}^D \begin{bmatrix} \cos\lambda\rho, & \lambda\rho\sin\lambda\rho \\ -\sin\lambda\rho, & \sin\lambda\rho - \lambda\rho\cos\lambda\rho \\ \cos\lambda\rho, & \lambda\rho\sin\lambda\rho \\ -(1-\nu)\cos\lambda\rho, & 2\cos\lambda\rho - (1-\nu)\lambda\rho\sin\lambda\rho \\ (1-\nu)\cos\lambda\rho, & 2\nu\cos\lambda\rho + (1-\nu)\lambda\rho\sin\lambda\rho \\ -\sin\lambda\rho, & \sin\lambda\rho + \lambda\rho\cos\lambda\rho \\ 0, & -2\sin\lambda\rho \\ 0, & 2\cos\lambda\rho \end{bmatrix} \boldsymbol{N}, \tag{5a}$$

$$\bar{\boldsymbol{S}}_h = \begin{bmatrix} \bar{S}_x \\ \bar{S}_y \end{bmatrix}_h = -D\frac{\lambda^3}{a^3} \begin{bmatrix} \mathrm{sh}\dfrac{\lambda y}{a} \\ \mathrm{sh}\dfrac{\lambda y}{a} \end{bmatrix}^D \begin{bmatrix} -(1-\nu)\sin\lambda\rho, & (1+\nu)\sin\lambda\rho + (1-\nu)\lambda\rho\cos\lambda\rho \\ -(1-\nu)\cos\lambda\rho, & 2(2-\nu)\cos\lambda\rho - (1-\nu)\lambda\rho\sin\lambda\rho \end{bmatrix} \boldsymbol{N}. \tag{5b}$$

4. Boundary Condition of Rectangular Plate.

The boundary conditions of being clamped along all edges are expressed by the equations

$$\begin{bmatrix} w \\ \theta_x \end{bmatrix}_{\rho=\pm 1} = 0, \quad \text{and} \quad \begin{bmatrix} w \\ \theta_y \end{bmatrix}_{y=\pm b} = 0. \tag{6}$$

The boundary conditions defined by Eq. 2a are then expressed as

$$\begin{bmatrix} w \\ \theta_x \end{bmatrix}_{h,\,\rho=\pm 1} = 0, \quad \begin{bmatrix} w \\ \theta_x \end{bmatrix}_{p,\,\rho=\pm 1} = 0, \quad \text{and} \quad \begin{bmatrix} w \\ \theta_y \end{bmatrix}_{h,\,y=\pm b} + \begin{bmatrix} w \\ \theta_y \end{bmatrix}_{p,\,y=\pm b} = 0. \tag{7}$$

In view of Eq. 5, the boundary equation defined by Eqs. 7a yields the characteristic equation

$$\begin{bmatrix} \cos\lambda\rho, & \lambda\rho\sin\lambda\rho \\ -\sin\lambda\rho, & \sin\lambda\rho - \lambda\rho\cos\lambda\rho \end{bmatrix} \boldsymbol{N} = 0. \tag{8}$$

Eq. 8 holds only when the determinant vanishes, from which we obtain the eigenvalue equation

$$\lambda + \sin\lambda\cos\lambda = 0, \quad \text{or} \quad 2\lambda + \sin 2\lambda = 0. \tag{9}$$

The solution of Eq. 9 yields only complex roots which are written as $\lambda = \alpha + i\beta$, provided only positive values of α and β need to be considered. Eq. 9 has been solved with digital computers, and its first sevral zeros are given in Table 1.

Table 1. Zeros of $2\lambda + \sin 2\lambda = 0$.

n	$2\alpha_n$	$2\beta_n$
1	4.212 392 230 5	2.250 728 611 6
2	10.712 537 397 3	3.103 148 745 8
3	17.073 364 853 2	3.551 087 347 0
4	23.398 355 225 7	3.858 808 993 1
5	29.708 119 825 3	4.093 704 924 8
6	36.009 866 016 4	4.283 781 587 8
7	42.306 826 717 6	4.443 445 830 3
8	48.600 684 124 1	4.581 104 573 5

Eq. 8 has then only one significant component equation, so that the homogeneous eigenmatrix $\boldsymbol{N}$ reduces to the form

$$\boldsymbol{N} = \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} \lambda\sin\lambda \\ -\cos\lambda \end{bmatrix} K, \quad K = \frac{1}{\lambda\sin\lambda} A. \tag{10}$$

Accordingly, the approach homogeneous deflection w_h reduces to the equation

$$w_h = \sum_n \lfloor \cos\lambda\rho, \quad \lambda\rho\sin\lambda\rho \rfloor \begin{bmatrix} \lambda\sin\lambda \\ -\cos\lambda \end{bmatrix} \mathrm{ch}\frac{\lambda y}{a} \boldsymbol{i}\boldsymbol{K}, \tag{11a}$$

in which

$$\boldsymbol{i} = \lfloor 1 \ \ i \rfloor, \quad \boldsymbol{K} = \begin{bmatrix} K_1 \\ K_2 \end{bmatrix}, \tag{11b}$$

in which $\boldsymbol{K}$ is called the real eigenmatrix.

5. Representation for Particular Solution.

The particular deflection w_p satisfies Eq. 2b. In view of symmetry about both the x and y axes, it takes the form

$$w_p = \lfloor 1 \quad \rho^2 \quad \rho^4 \rfloor \boldsymbol{N}_p \{1 \quad \eta^2 \quad \eta^4\}, \tag{12a}$$

in which

$$\boldsymbol{N}_p = \begin{bmatrix} a_0 & a_2 & a_4 \\ b_0 & b_2 & 0 \\ c_0 & 0 & 0 \end{bmatrix}, \qquad \eta = \frac{y}{a}, \tag{12b}$$

provided that

$$b_2 = -3a_4 - 3c_0 + \frac{qa^4}{8D}. \tag{13}$$

Noticicing that Eq. 12a must satisfy Eqs. 2b and 7b, it will after a little manipulation be found that the particular eigenmatrix $\boldsymbol{N}_p$ results in the desired value

$$\boldsymbol{N}_p = \frac{qa^4}{24D}\begin{bmatrix} 1 & 0 & 0 \\ -2 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \tag{14}$$

and then, the particular state vector becomes

$$\boldsymbol{W}_p = \begin{bmatrix} w \\ \theta_x \\ \theta_y \\ M_x \\ M_y \\ M_{xy} \\ S_x \\ S_y \end{bmatrix}_p = \begin{bmatrix} \dfrac{qa^4}{24D} \\ \dfrac{qa^3}{12D} \\ 0 \\ -\dfrac{qa^2}{12} \\ -\dfrac{\nu qa^2}{12} \\ 0 \\ -qa \\ 0 \end{bmatrix}^D \begin{bmatrix} 1 & \rho^2 & \rho^4 \\ 0 & \rho & 2\rho^3 \\ 0 & 0 & 0 \\ 0 & 1 & 6\rho^2 \\ 0 & 1 & 6\rho^2 \\ 0 & 0 & 0 \\ 0 & 0 & \rho \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}. \tag{15}$$

6. Fourier Series Representation for Eigenfunction.

The four element ρ-eigenfunctions occurring on the right side of Eq. 5 are expanded into Fourier series as follows :

$$\begin{bmatrix} \cos\lambda\rho \\ \rho\sin\lambda\rho \\ \sin\lambda\rho \\ \rho\cos\lambda\rho \end{bmatrix} = \sum_{m=0}^{\infty} \cos m\pi\rho \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix}, \tag{16}$$

in which $(I_1 \sim I_4)$ are the complex Fourier coefficients, which are to be computed. Eq. 5 then becomes

$$\boldsymbol{W}_h = \sum_{m=0}^{\infty} \cos m\pi\rho \,[I]_{mn} \boldsymbol{Y}_n(y) \boldsymbol{i} \boldsymbol{K}, \tag{17}$$

which is the desired Fourier series representaion of the complex homogeneous state vector $\boldsymbol{W}_h$.

7. Fourier Series Expansion for Particular Solution.

The element particular solutions will be represented by Fourier series as follows :

$$\begin{bmatrix} 1 \\ \rho \\ \rho^2 \\ \rho^3 \\ \rho^4 \end{bmatrix} = \sum_{m=0}^{\infty} \cos m\pi\rho \begin{bmatrix} J_0 \\ J_1 \\ J_2 \\ J_3 \\ J_4 \end{bmatrix}, \tag{18}$$

and then, the particular state vector, Eq. 15, will be defined by

$$\boldsymbol{W}_p = \sum_{m=0}^{\infty} \cos m\pi\rho \,[\boldsymbol{p}_m]. \tag{19}$$

8. Final Equation for Real Eigenmatrices.

The complete state vector, $\boldsymbol{W}$, is given by the sum of homogeneous and particular state vectors from Eq. 2a, so that we obtain

$$\boldsymbol{W} = \boldsymbol{W}_h + \boldsymbol{W}_p. \tag{20}$$

From Eqs. 17 and 19, this equatioin should be expressed in terme of Fourier series, and then we can write

$$\mathbf{W} = \sum_{m}^{\infty} \cos m\pi\rho \, [\lfloor \mathbf{X}_{mn}(y) \rfloor \{\mathbf{K}_n\} + \mathbf{p}_m], \tag{21a}$$

in which

$$\mathbf{X}_{mn}(y) = [I]_{mn} \mathbf{Y}_n(y) \mathbf{i}. \tag{21b}$$

The boundary conditions at the second opposing edges $y = \pm b$ are then expressed by the equations

$$\mathbf{S}\mathbf{W}_{y=\pm b} = 0, \tag{22}$$

in which $\mathbf{S}$ is a selector and is a 2-by-8 rectangular matarix. If for example edges $y = \pm b$ are clamped, then Eq. 22 becomes

$$\begin{bmatrix} w \\ \theta_y \end{bmatrix}_{y=\pm b} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \mathbf{W}_{y=\pm b} = 0. \tag{23}$$

Eq. 22 yields in view of Eq. 21a the equation

$$\lfloor \mathbf{Z}_{mn} \rfloor \{\mathbf{K}_n\} + \mathbf{P}_m = 0, \tag{24a}$$

in which

$$\lfloor \mathbf{Z}_{mn} \rfloor = \mathbf{S} \lfloor \mathbf{X}_{mn}(\pm b) \rfloor, \quad \mathbf{P}_m = \mathbf{S}\mathbf{p}_m. \tag{24b}$$

The matrix $\mathbf{Z}_{mn}$ is a 2-by-2 square matrix and may be called the element stiffness matrix of orders m and n. The matrix $\mathbf{P}_m$ is a 2-by-1 column matrix and is the element load matrix of order m.

The element stiffness and load matrices for the clamped condition on all the four edges are expressed by

$$\lfloor \mathbf{Z}_{mn} \rfloor = \frac{4\lambda^4(-1)^m}{[\lambda^2 - (m\pi)^2]^2} \begin{bmatrix} \operatorname{ch}\dfrac{\lambda y}{a} \\ \dfrac{\lambda}{a}\operatorname{sh}\dfrac{\lambda y}{a} \end{bmatrix}_{y=\pm b} \mathbf{i}, \quad \mathbf{P}_m = \frac{qa^4}{24D} \begin{bmatrix} -\dfrac{48(-1)^m}{(m\pi)^4} \\ 0 \end{bmatrix}. \tag{25a}$$

For the first special case in which $m = 0$, Eq. 25a will become

$$\mathbf{Z}_{0n} = 4 \begin{bmatrix} \operatorname{ch}\dfrac{\lambda y}{a} \\ \dfrac{\lambda}{a}\operatorname{sh}\dfrac{\lambda y}{a} \end{bmatrix}_{y=\pm b} \mathbf{i}, \quad \mathbf{P}_0 = \frac{qa^4}{24D} \begin{bmatrix} \dfrac{16}{15} \\ 0 \end{bmatrix}. \tag{25b}$$

Eq. 24a will hold for $m = 0, 1, 2, 3, 4, \cdots$, so that the column assemblage for these integers yields the desired system of simultaneous equations. Since Eq. 24a has an infinite number of rows and columus, it is necessary to truncate them, so that the complet stiffness matrix can be a square one. We then take

$$m=0,\ 1,\ 2,\ 3,\ \cdots\cdots,\ N-1, \quad \text{and} \quad n=1,\ 2,\ 3,\cdots\cdots,\ N. \tag{26}$$

Eq. 24a may then be expressed in the final explanatory form

$$\begin{bmatrix} \boldsymbol{Z}_{0,1} & \boldsymbol{Z}_{0,2} & \boldsymbol{Z}_{0,3} & \boldsymbol{Z}_{0,4} & \cdots & \boldsymbol{Z}_{0,N} \\ \boldsymbol{Z}_{1,1} & \boldsymbol{Z}_{1,2} & \boldsymbol{Z}_{1,3} & \boldsymbol{Z}_{1,4} & \cdots & \boldsymbol{Z}_{1,N} \\ \boldsymbol{Z}_{2,1} & \boldsymbol{Z}_{2,2} & \boldsymbol{Z}_{2,3} & \boldsymbol{Z}_{2,4} & \cdots & \boldsymbol{Z}_{2,N} \\ \boldsymbol{Z}_{3,1} & \boldsymbol{Z}_{3,2} & \boldsymbol{Z}_{3,3} & \boldsymbol{Z}_{3,4} & \cdots & \boldsymbol{Z}_{3,N} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \boldsymbol{Z}_{N-1,1} & \boldsymbol{Z}_{N-1,2} & \boldsymbol{Z}_{N-1,3} & \boldsymbol{Z}_{N-1,4} & \cdots & \boldsymbol{Z}_{N-1,N} \end{bmatrix} \begin{bmatrix} \boldsymbol{K}_1 \\ \boldsymbol{K}_2 \\ \boldsymbol{K}_3 \\ \boldsymbol{K}_4 \\ \cdots \\ \boldsymbol{K}_N \end{bmatrix} + \begin{bmatrix} \boldsymbol{P}_0 \\ \boldsymbol{P}_1 \\ \boldsymbol{P}_2 \\ \boldsymbol{P}_3 \\ \cdots \\ \boldsymbol{P}_{N-1} \end{bmatrix} = 0, \tag{27}$$

or in abbreviated notation,

$$[\boldsymbol{Z}_{mn}]\{\boldsymbol{K}_n\} + \{\boldsymbol{P}_m\} = 0. \tag{28}$$

9. Numerical Examples.

The proposed matrix eigenfunction method is very suitable for computer programming. As numerical example of the preceding analysis, a rectangular plate, which is uniformly loaded over its entire surface and is clamped along the four edges, is taken.

Input data of the numerical example are given in Table 2.

Table 2. Plate Parameters and Loading.

	Example 1	Example 2
Size of plate	2 m × 2 m	1.5 ft × 2 ft
Thickness of plate	0.02 m	0.25 in
Modulus of elasticity	1.2×10^{7} t/m^2	0.42×10^{6} psi
Poisson's ratio	0.3	0.35
Uniformly distributed load	0.1 t/m^2	10 psf

Example 1. Figs. 2 and 3 show the convergence of the real-eigenmatrices and the element load-matrices of order n in the infinite series.

Tables 3 and 4 show the comparison of respective physical quantities on the Ox and Oy axes, when the infinite series is truncated with the first four and eight terms.

Figs. 4 to 6 show the curves of the deflection, the flexural moments and the shearing forces, when the first eight terms are retained.

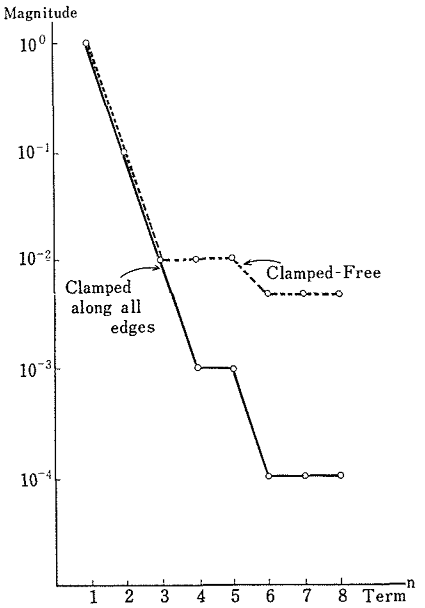

Fig. 2. Magnitude of Real Eigenmatrix.

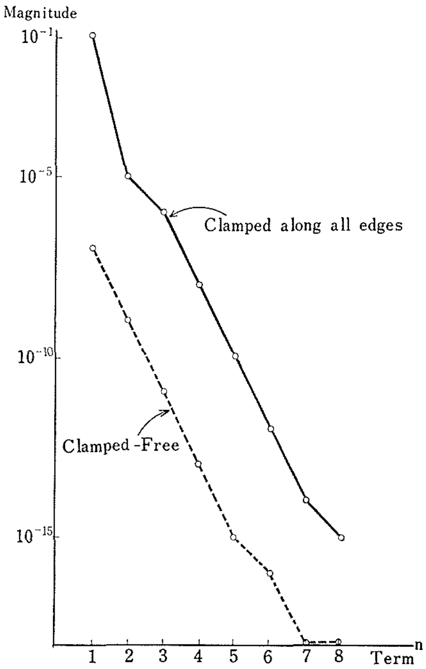

Fig. 3. Magnitude of Element Load-matrix.

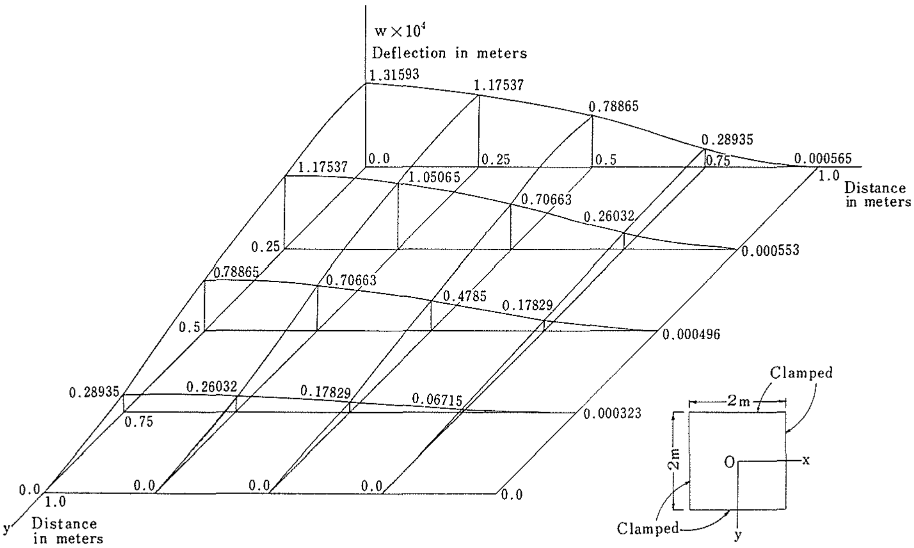

Fig. 4. Deflection for Clamped Plate.

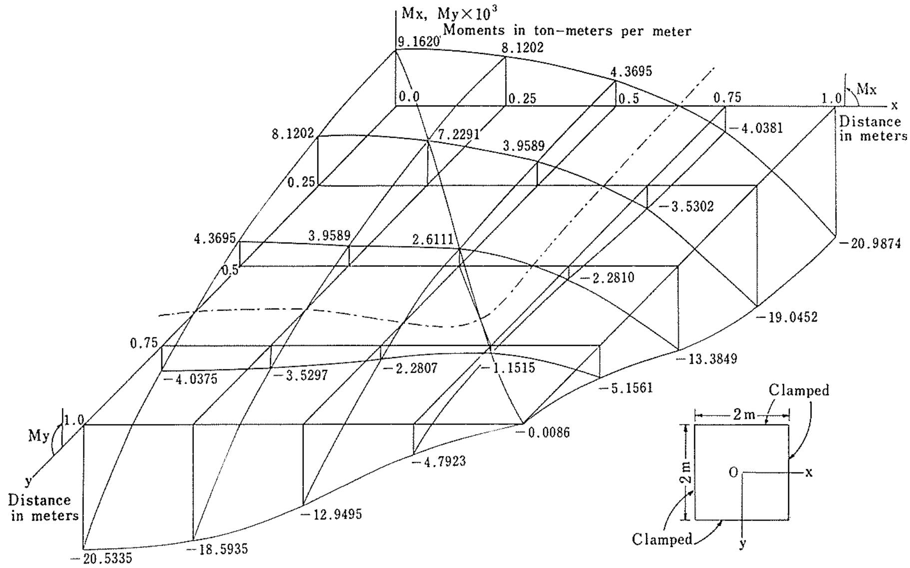

Fig. 5. Flexural Moments, M_x, M_y, for Clamped Plate.

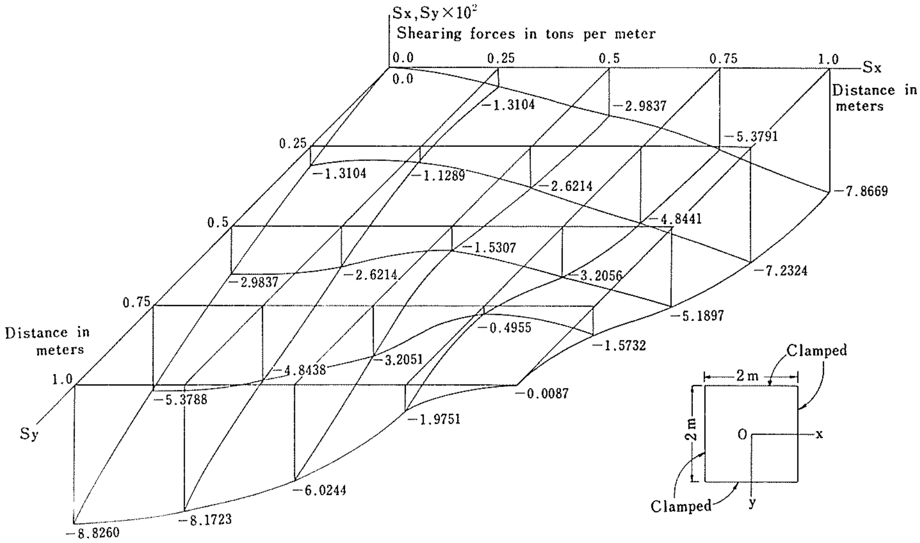

Fig. 6. Shearing Forces, S_x, S_y, for Clamped Plate.

Table 3. State Vector Components for Clamped Plate.

y (Distance in meter)	Component	x = 0.0	x = 0.25	x = 0.5	x = 0.75	x = 1.0
0.0	w	1.315 928	1.175 372	0.788 651	0.289 355	0.0
	θ_y	0.0	−1.103 025	−1.907 597	−1.882 434	0.0
	θ_x	0.0	0.0	0.0	0.0	0.0
	M_y	9.162 022	8.12 0217	4.369 537	−4.037 582	−20.533 434
	M_x	9.162 019	8.12 3422	5.043 250	0.104 271	−6.160 030
	S_y	0.0	−1.31 0493	−2.983 732	−5.378 800	−8.825 998
	S_x	0.0	0.0	0.0	0.0	0.0
0.25	w	1.175 372	1.050 648	0.706 628	0.260 325	0.0
	θ_y	0.0	−0.979 416	−1.700 223	−1.688 248	0.0
	θ_x	−1.103 024	−0.979 415	−0.645 450	−0.229 389	0.0
	M_y	8.123 430	7.229 076	3.958 974	−3.529 739	−18.593 501
	M_x	8.120 213	7.229 082	4.552 532	0.158 147	−5.578 050
	S_y	0.0	−1.128 995	−2.721 448	−4.843 800	−8.172 236
	S_x	−1.310 489	−1.128 998	−0.584 475	0.319 236	1.543 689
0.5	w	0.788 652	0.706 628	0.478 561	0.178 292	0.0
	θ_y	0.0	−0.645 446	−1.134 041	−1.146 669	0.0
	θ_x	−1.907 591	−1.700 227	−1.134 029	−0.412 920	0.0
	M_y	5.043 392	4.552 404	2.611 137	−2.280 618	−12.949 757
	M_x	4.369 524	3.959 018	2.611 045	0.053 135	−3.884 927
	S_y	0.0	−0.584 376	−1.530 893	−3.205 110	−6.024 242
	S_x	−2.983 585	−2.621 554	−1.530 709	0.293 308	2.909 290
0.75	w	0.289 348	0.260 323	0.178 302	0.067 163	0.0
	θ_y	0.0	−0.229 542	−0.412 968	−0.430 544	0.0
	θ_x	−1.882 655	−1.688 202	−1.146 485	−0.430 903	0.0
	M_y	0.102 811	0.156 695	0.057 820	−1.150 267	−4.800 962
	M_x	−4.030 502	−3.532 962	−2.284 100	−1.147 746	−1.440 288
	S_y	0.0	0.323 352	0.285 208	−0.487 063	−1.981 979
	S_x	−5.372 267	−4.849 440	−3.203 406	−0.489 059	3.295 963
1.0	w	0.001 804	0.000 924	0.000 759	−0.001 071	0.0
	θ_y	0.0	−0.025 095	0.022 642	0.005 423	0.0
	θ_x	−0.033 868	0.018 724	0.012 124	−0.022 558	0.0
	M_y	−6.644 296	−5.363 479	−3.579 056	−1.747 744	−0.179 650
	M_x	−19.819 708	−19.033 307	−13.117 493	−4.248 802	−0.053 895
	S_y	0.0	1.670 156	2.549 225	3.896 267	1.088 374
	S_x	−9.556 789	−7.902 479	−5.401 256	−2.486 488	−1.905 244

$w = 10^{-4}$, $\theta_y = 10^{-4}$, $\theta_x = 10^{-4}$, $M_y = 10^{-3}$, $M_x = 10^{-3}$, $S_y = 10^{-2}$, $S_x = 10^{-2}$

Note : size of plate = 2 m × 2 m ;
thickness of plate = 0.02 m ;
modulus of elasticity = 1.2 × 10^7 t/m^2 ;
Poisson's ratio = 0.3 ;
and loading = 0.1 t/m^2 uniformly distributed.

Table 4. State Vector Components along 0x Axis for Clamred Plate.

x	Flexural deflection w 10^{-4} m		Flexural moment M_x 10^{-3} tm		Flexural moment M_y 10^{-3} tm		Shearing force S_x 10^{-2} t	
	Four eigen-values	Eight eigen-values	Four eigenvalues	Eight eigenvalues	Four eigen-values	Eight eigen-values	Four eigen-values	Eight eigen-values
0.0	1.315 922	1.315 922	9.161 978	9.161 978	9.161 986	9.161 986	0.0	0.0
0.25	1.175 367	1.175 367	8.120 191	8.120 191	8.123 374	8.123 374	−1.310 495	−1.310 495
0.5	0.788 647	0.788 647	4.369 494	4.369 494	5.043 213	5.043 213	−2.983 728	−2.983 728
0.75	0.289 353	0.289 353	−4.037 575	−4.037 575	0.104 241	0.104 241	−5.378 766	−5.378 766
1.0	0.0	0.0	−20.533 290	−20.533 290	−6.159 989	−6.159 989	−8.825 910	−8.825 910

Table 5. State Vector Components along $0y$ axis for Clamped Plate.

y	Flexural deflection w 10^{-4} m		Flexural moment M_x 10^{-3} tm		Flexural moment M_y 10^{-3} tm		Shearing force S_y 10^{-2} t	
	Four eigen-values	Eight eigen-values	Four eigenvalues	Eight eigenvalues	Four eigenvalues	Eight eigenvalues	Four eigen-values	Eight eigen-values
0.0	1.315 922	1.315 922	9.161 978	9.161 978	9.161 986	9.161 986	0.0	0.0
0.25	1.175 366	1.175 366	8.123 312	8.123 312	8.120 223	8.120 223	−1.310 523	−1.310 523
0.5	0.788 642	0.788 642	5.042 216	5.042 216	4.369 716	4.369 716	−2.984 498	−2.984 498
0.75	0.289 342	0.289 342	0.100 819	0.100 819	−4.051 955	−4.051 955	−5.395 571	−5.395 571
1.0	0.002 338	0.002 338	−57.674 960	−57.674 960	−21.287 350	−21.287 350	−8.379 058	−8.379 058

Table 6. Deflection along Axis Ox for Clamped Plate.

Distance in meter	Eigenfunction method	Energy method
0.0	$1.315\,932 \times 10^{-4}$	$1.316\,427 \times 10^{-4}$
0.25	$1.175\,375 \times 10^{-4}$	$1.165\,104 \times 10^{-4}$
0.50	$7.886\,535 \times 10^{-5}$	$7.846\,249 \times 10^{-5}$
0.75	$2.893\,560 \times 10^{-5}$	$2.777\,345 \times 10^{-5}$
1.00	0.0	$3.685\,676 \times 10^{-22}$

Note : size of plate = 2 m × 2 m; thickness of plate = 0.02 m; modulus of elasticity = 1.2×10 t/m²; Poisson's ratio = 0.3; and loading = 0.1 t/m² uniformly distributed.

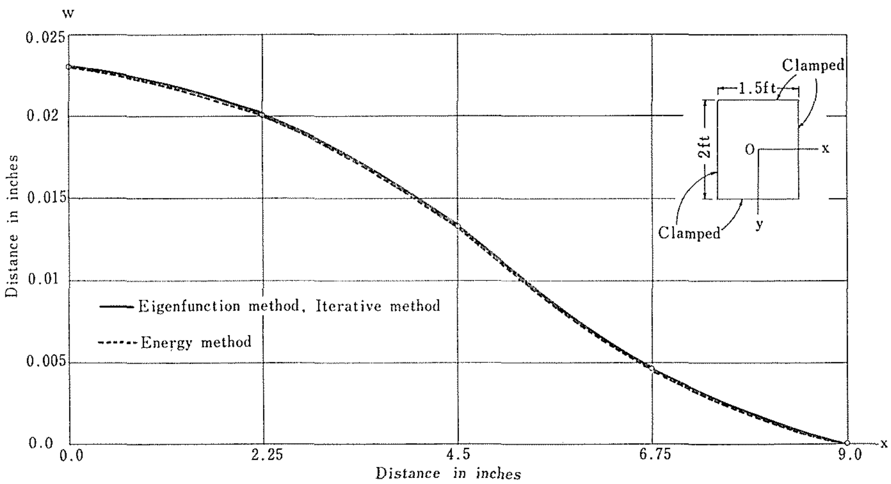

Fig. 7. Deflection along Axis of Symmetry $0x$ for Plate Clamped along All Edges.

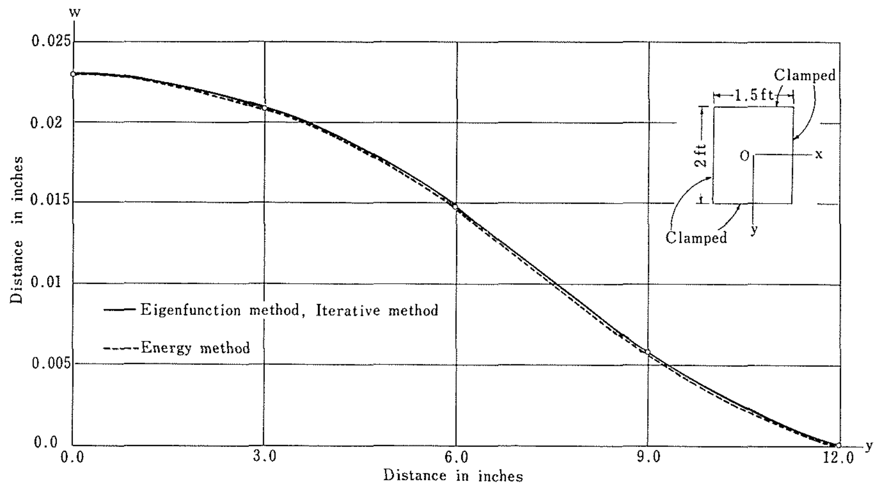

Fig. 8. Deflection along Axis $0y$ for Plate Clamped along All Edges.

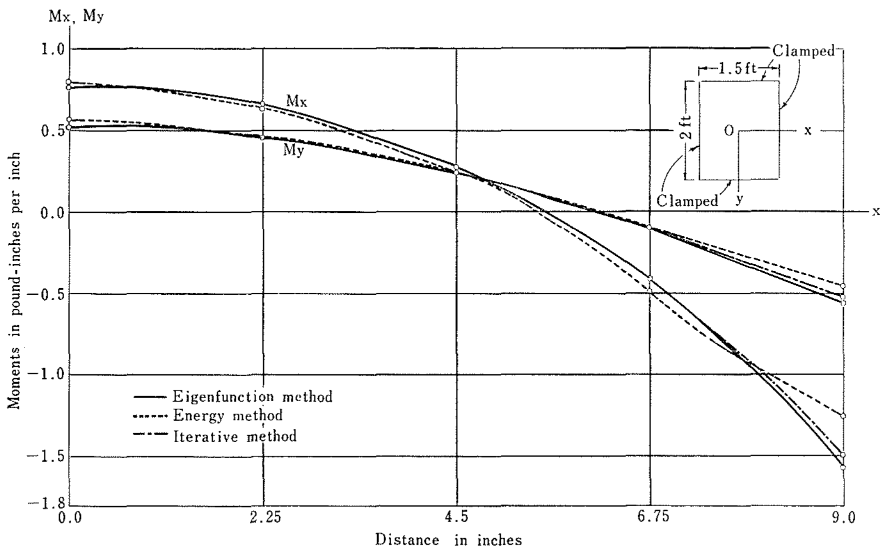

Fig. 9. Flexural Moments along Axis $0x$ for Plate Clamped along All Edges.

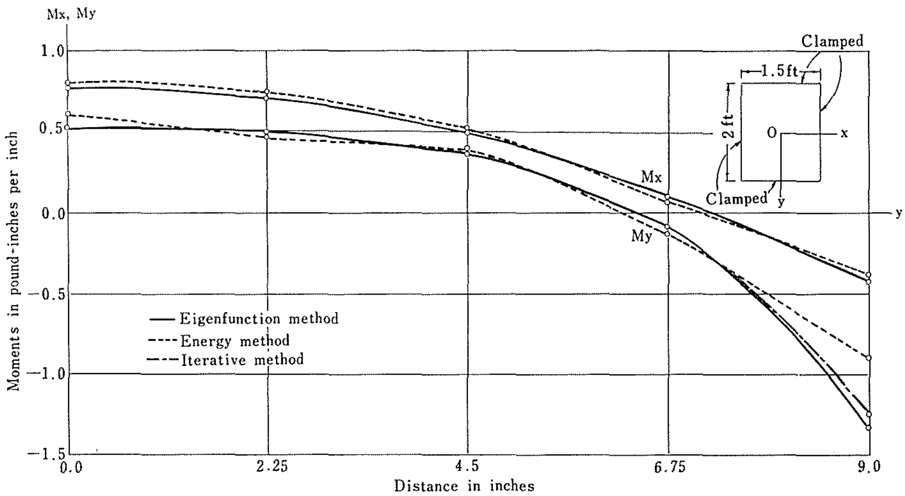

Fig. 10. Flexural Moments along Axis $0y$ for Plate Clamped All Edges.

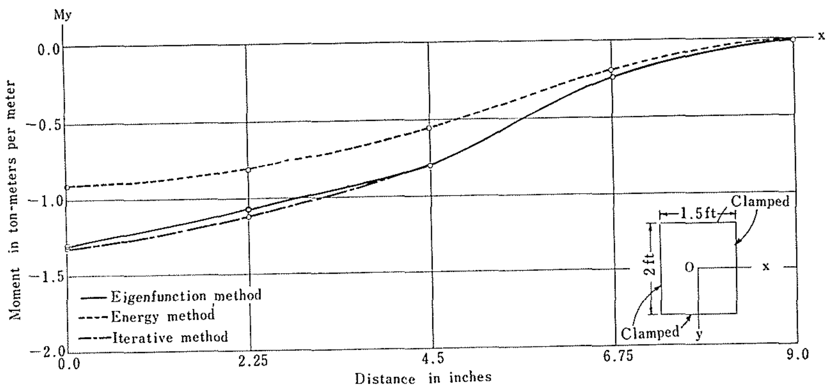

Fig. 11. Moment M_y along Edge Parallel to x-Axis for Plate Clamped along All Edges.

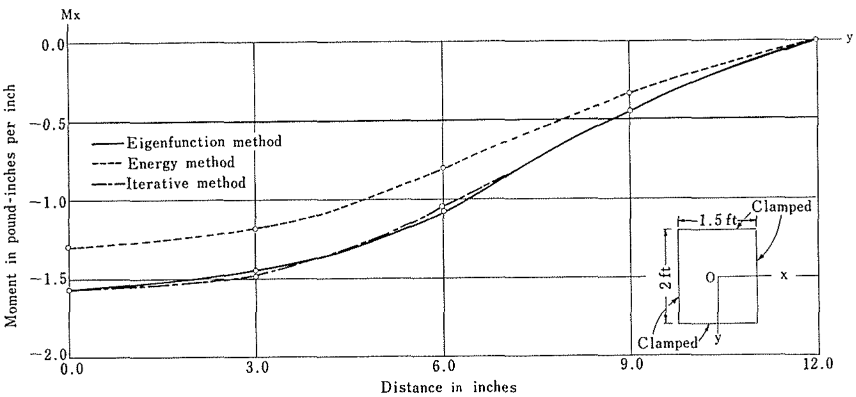

Fig. 12. Moment M_x along Edge Parallel to y-Axis for Plate Clamped along All Edges.

Example 2. Figs. 7 to 12 show the deflections, the flexural moments and the edge moments due to the present eigenfunction method, the iterative method by B. Sen (4) and the energy method by C. T. Wang (5).

II. Second Solution for Clamped along Two Opposite Edges and Free along the Other Two Edges

10. Introductory Remarks.

It will be assumed for example that the first two opposing edges $x = \pm a$ are both clamped, and the other two opposing edges $y = \pm b$ are free from traction (Fig. 13).

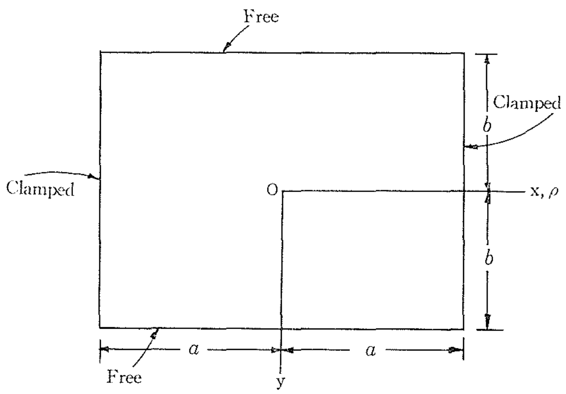

Fig. 13. Clamped-Free Rectangular Plate.

11. Boundary Conditions at First Opposing Edges.

The boundary conditions at the first opposing edges $\rho = \pm 1$ are both clamped, and the other opposing edges $y \pm b$ are free from traction, which are given by the equations

$$\begin{bmatrix} w \\ \theta_x \end{bmatrix}_{\rho=\pm 1} = 0, \quad \begin{bmatrix} M_y \\ \bar{S}_y \end{bmatrix}_{y=\pm b} = 0. \tag{29}$$

On account of the symmetry about the Oy-axis (Fig. 13), the approach homogeneous and particular deflections are defined by

$$w_h = \sum_n [\cos \lambda\rho, \quad \lambda\rho \sin \lambda\rho] \begin{bmatrix} \lambda \sin \lambda \\ -\cos \lambda \end{bmatrix} \mathrm{ch} \frac{\lambda y}{a} iK, \tag{30a}$$

$$w_p = \frac{qa^4}{24D} \begin{bmatrix} 1 & \rho^2 & \rho^4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}. \tag{30b}$$

12. Final Equation for Clamped-Free Plate.

As the second bounding edges $y = \pm b$ are assumed to be free from traction, Eq. 29b will yield the element stiffness matrix, $\mathbf{Z}_{mn}$, given by the equation

$$\mathbf{Z}_{mn} = -\frac{4D\lambda^4(-1)^m}{a^2[\lambda^2 - (m\pi)^2]^2} \begin{bmatrix} (\lambda^2 - \nu m^2\pi^2)\mathrm{ch}\frac{\lambda y}{a} \\ \frac{\lambda}{a}[\lambda^2 - (2-\nu)(m\pi)^2]\mathrm{sh}\frac{\lambda y}{a} \end{bmatrix}_{y=\pm b} \mathbf{i}, \tag{31a}$$

$$\mathbf{P}_m = \nu q a^2 \begin{bmatrix} -\frac{2(-1)^m}{(m\pi)^2} \\ 0 \end{bmatrix}, \tag{31b}$$

and the selector $\mathbf{s}$ represents

$$\mathbf{s} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{31c}$$

13. Numerical Example of Clamped-Free Plate.

The rectangular plate, clamped along two opposite edges and free along the other two edges, will be referred to as the clamped-free plate.

Input data for the numerical example of the clamped-free plate are given in Table.

Table 7. Input Data of Clamped-Free Plate.

Size of plate = 2 m ×2 m;
thickness of plate = 0.1 m;
modulus of elasticity = 2.1 ×10^7 tm;
poisson's ratio = 0.3; and
loading = 0.1 t/m^2 uniformly distributed.

Figs. 15 and 16 show the curves of the deflection w, the flexural moment M_y, when the first eight terms in the infinite series are retained. Tables 7 and 8 show the comparison of respective quantities, or state vector components on the Ox- and Oy-axes, when the infintite series are truncated with the first four and eight terms.

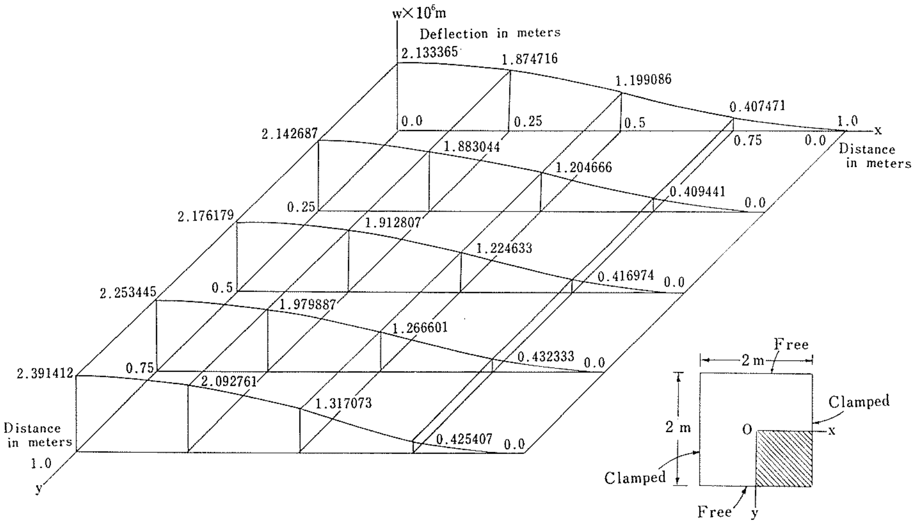

Fig. 14. Deflection for Clamped-Free Plate.

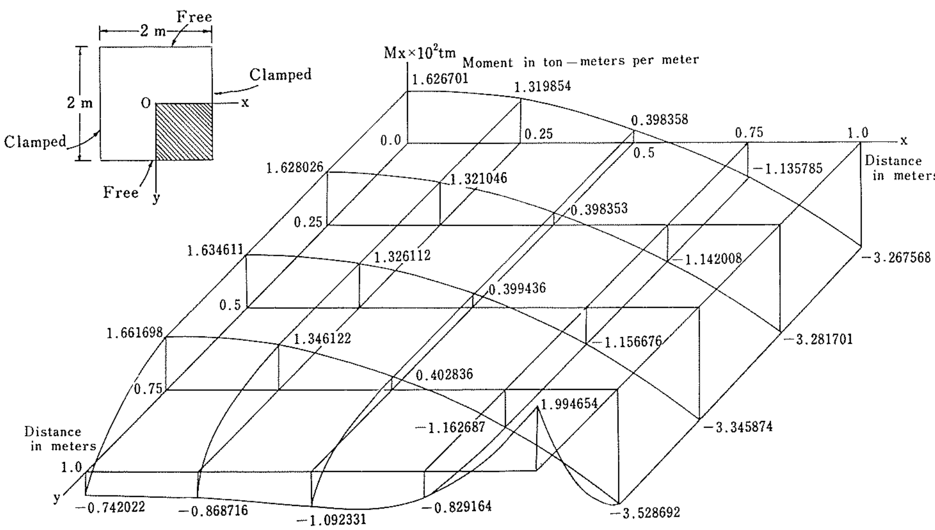

Fig. 15. Flexural Moment M_x for Clamped-Free Plate.

Table 8. State Vector Components along $0x$ Axis for Clamped-Free Plate.

x	Flexural deflection w 10^{-6} m		Flexural moment M_x 10^{-2} tm		Flexural moment M_y 10^{-3} tm		Shearing force S_x 10^{-2} t	
	Four eigenvalues	Eight eigenvalues	Four eigenvalues	Eight eigenvalues	Four eigenvalues	Eight eigenvalues	Four eigenvalues	Eight eigenvalues
0.0	2.139 856	2.133 365	1.632 149	1.626 701	4.461 411	4.382 419	0.0	0.0
0.25	1.880 656	1.874 716	1.325 154	1.319 854	3.585 730	3.514 889	−2.430 061	−2.423 614
0.5	1.203 341	1.199 086	0.402 512	0.398 358	0.940 185	0.898 525	−4.879 516	−4.858 981
0.75	0.409 143	0.407 471	−1.137 748	−1.135 785	−3.511 640	−3.509 796	−7.336 063	−7.291 390
1.0	0.0	0.0	−3.282 543	−3.267 568	−9.847 630	−9.802 705	−9.752 315	−9.696 058

Table 9. State Vector Components along $0y$ Axis for Clamped-Free Plate.

y	Flexural deflection w 10^{-6} m		Flexural moment M_x 10^{-2} tm		Flexural moment M_y 10^{-3} tm		Shearing force S_y 10^{-3} t	
	Four eigenvalues	Eight eigenvalues	Four eigenvalues	Eight eigenvalues	Four eigenvalues	Eight eigenvalues	Four eigenvalues	Eight eigenvalues
0.0	2.139 856	2.133 365	1.632 149	1.626 701	4.461 411	4.382 419	0.0	0.0
0.25	2.148 025	2.142 687	1.633 396	1.628 026	4.325 537	4.237 351	−0.779 186	−0.829 345
0.5	2.177 872	2.176 179	1.645 107	1.634 611	3.807 971	3.735 614	−1.673 529	−1.932 257
0.75	2.251 188	2.253 440	1.745 030	1.661 698	2.428 970	2.588 031	1.367 728	−3.261 465
1.0	2.423 734	2.391 412	2.214 526	−0.742 223	6.806 408	38.158 670	79.128 490	-579.639 50

14. Conclusions.

The flexural analysis of a uniformly loaded plate may be expressed in terms of two functions, a homogeneous solution and a particular solution. The homogenous solution may be expressed as a Fadle eigenfunction series. The arbitrary complex eigenmatrix $\{\boldsymbol{K}_n\}$, that appear in this series may be determined with a high degree of accuracy from approximate expansion formulas.

15. References.

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Plates." Journal of the Structural Division. ASCE, Vol. 98, No. ST 1, Proc. Paper 8644, January, 1972, pp. 135 151.

(5) Wang, C. T., "Applied Elasticity." McGraw-Hill Book Co., Inc., New York, 1953, pp. 286.

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16. Acknowledgments.

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17. Notation.

The following symbols are used in this paper:

λ = complex eigenvalue;
n = order of eigenfunctions;
m = order of Fourier series numbers;
ρ = ratio between width and length of plate;
w_h = homogeneous solution to plate equation;
w_p = particular solution to plate equation;
x, y = rectangular coordinates;
D = flexural rigidity of plate;
$\mathbf{W}_h$ = complex homogeneous state vector;
$\mathbf{W}_p$ = particular state vector;
$\mathbf{N}$ = complex eigenmatrix (eigenfunction coefficents);
$\mathbf{K}$ = real-eigenmatrix;
q = normal surface load;
$\mathbf{Z}_{mn}$ = complex element stiffness matrix;
$\mathbf{P}_m$ = element load-matrix;
α, β = real and imaginary parts of complex eigenvalue λ respectively;
$K = \lfloor 1 \ i \rfloor \begin{bmatrix} K_1 \\ K_2 \end{bmatrix}$;
$\nabla^4 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2$; and
$\lfloor \rfloor, \{\}$ = row and column matrices respectively.