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A New Rearrangement and the Theorem of Hardy - Littlewood - Pólya

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In this paper, we introduce a new rearrangement of functions and prove a generalization of Hardy-Littlewood-Pólya's theorem. At first we define the concept of stratus system and a concept of a new rearrngement. We study various properties of doubly-substochastic-markov operators on L^1 defined over a stratus system. At last by using Kadison's theorem and Fan's theorem, a result concerning doubly-substochastic-markov operators will be obtaind. This result is a generalization of Hardy-Littlewood-pólya's theorem.

1. Intoroduction.

It was proved by Hardy-Littlewood-Pólya[4] that non-negative vectors $x, y \in l^1{}_n$ satisfy a certain order relation $y \prec x$ if and only if there is a doubly-stochastic matrix T such that $y = Tx$. This result has been investigated and generalized in various points of views by Ryff[7][8] and others. For example, Ryff extended this theorem to the case when the vectors x, y are in $L^1[0, 1]$ and T is a doubly-stochastic operator on $L^1[0, 1]$.

In this paper, we shall obtain a generalization of the theorem of Hardy-Littlewood-Pólya.

In section 2, we shall introduce a new concept called the stratus system on σ-finite measure spaces and a generalized concept of the rearrangement on such a system. We shall rearrange functions which are measurable on the measure space. Further, some elementaly concepts and results will be shown for later sections.

In section 3, we shall study the property of doubly-stochastic operators and doubly-substochastic operators on L^1.

In section 4, Kadison's general compactness theorem and Fan's theorem which are usefull tools for our later discussion, will be introduced.

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In section 5, using these results the main theorem which is a generalization of Hardy-Littlewood-Pólya's theorem will be proved.

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2. Preliminaries.

Let X be a non-empty point set provided with a countably additive nonnegative measure μ on a σ-field Σ of subsets of X. We shall denote the measure space so defined by (X, Σ, μ).

DEFINITION 1. Let $\mathscr{F}=\{X_k\ ;\ k\in\Gamma\}$ be a subclass of Σ satisfying the following conditions :

(a) $\Gamma=\{\mu(E)\ ;\ E\in\Sigma,\ \mu(E)<\infty\}$;

(b) $\mu(X_k)=k$ for each $k\in\Gamma$;

(c) $\cup_k X_k=X$ and if $k\leq k'$ then $X_k\subset X_{k'}$.

Then, $\mathscr{F}$ will be called a *stratus* on X.

DEFINITION 2. A σ-finite measure space is said to be *homogeneous*, if for any pair of $E,\ E'\in\Sigma$ with $\mu(E)=\mu(E')<\infty$, there exists a mapping $m\ ;\ X\to X$ which is measure preserving on E' and satisfies $\mu(m^{-1}(E)\Delta E')=0$.

DEFINITION 3. A quadraplet $(X, \Sigma, \mu, \mathscr{F})$ with a homogeneous σ-finite measure space (X, Σ, μ) and a stratus $\mathscr{F}$ is called a *stratus system*.

EXAMPLES. We list below some examples of stratus system.

(1) Let, $X=\{1, 2, \cdots, n\}$ and $\Sigma=2^X$. And let μ be a counting measure on X. Putting $X_k=\{1, 2, \cdots, k\}\in\mathscr{F}$ and $\Gamma=\{1, 2, \cdots, n\}$, we see that $(X, \Sigma, \mu\ \mathscr{F})$ is a stratus system.

(2) Let $X=\{1, 2, \cdots\}$ and $\Sigma=2^X$. And let μ be a counting measure. Putting $X_k=\{1, 2, \cdots, k\}\in\mathscr{F}$ and $\Gamma=\{1, 2, \cdots\}$, we see that $(X, \Sigma, \mu, \mathscr{F})$ is a stratus system.

(3) Let $X=[0, 1]$, μ be a Lebesgue measure on X and Σ be a class of Lebesgue measurable sets. Let $X_k=[0, k]\in\mathscr{F}$ and $\Gamma=[0, 1]$, then $(X, \Sigma, \mu, \mathscr{F})$ is a stratus system.

(4) Let $X=[0, \infty)$, Σ be a class of Lebesgue measurable sets, μ be a Lebesgue measure on X, $X_k=[0, k]\in\mathscr{F}$ and $\Gamma=(0, \infty)$, then $(X, \Sigma, \mu, \mathscr{F})$ is a stratus system.

(5) Let $X=[-1, 1]$, Σ be a class of Lebesgue measurable sets, μ be a Lebesgue measure, $X_k=\left[-\frac{k}{2}, \frac{k}{2}\right]\in\mathscr{F}$ and $\Gamma=[0, 2]$, then $(X, \Sigma, \mu, \mathscr{F})$ is a stratus system.

(6) Let $X=[0, 1]\times[0, 1]$, Σ be a class of 2-dimensional Lebesgue measurable

sets, μ be a 2-dimensional Lebesgue measure, $X_k=[0, \sqrt{k}] \times [0, \sqrt{k}] \in \mathscr{F}$ and $\Gamma=[0, 1]$, then $(X, \Sigma, \mu, \mathscr{F})$ is a stratus system.

REMARK 1. If a σ-finite measure space is *non-atomic* or *discrete* then we can find a stratus on this space. From now on, the space X under consideration shall in fact be a stratus system.

DEFINITION 4. Let f be a measurable function on X. We define the distribution function d_f of f for all $t \in R$ by $d_f(t)=\mu\{x ; f(x)>t\}$, which may take a value $+\infty$.

DEFINITION 5. For every $x \in X$, The $\mathscr{F}$-distance of x is defined by $\rho(x)= sup\{k ; x \notin X_k\}$, which is not infinite as $X=\cup_k X_k$.

DEFINITION 6. Let f be a measurable function on X. We define the $\mathscr{F}$-rearrangement δ_f of f by $\delta_f(x)=\inf\{t ; d_f(t) \leq \rho(x)\}$, which takes a value $+\infty$ if the set of t's is empty, and takes $-\infty$ if the set coinsides to R.

REMARK 2. As we can see easily, if $\mu(X)=\infty$, the range of the $\mathscr{F}$-rearrangement of a measurable function f is not always equal to the range of f.

Some foundamental properties of d_f and δ_f will be shown in the following lemmas.

LEMMA 1. *The functions d_f and δ_f have the following properties* :

(1) *For all $t \in R$ such that $d_f(t)<\infty$, d_f is right continuous and nonincreasing.*

(2) $\delta_f(x) \geq \delta_f(x')$ *if $x \in X_k$ and $x' \notin X_k$ for some k.*

(3) $d_f(t)>\rho(x)$ *if and only if* $\delta_f(x)>t$.

(4) $d_f=d_{\delta_f}$.

(5) *If $f_n \uparrow f$ where symbol $\uparrow$ denotes monotone pointwise convergence almost everywhere, then $\delta_{f_n} \uparrow \delta_f$.*

Proof. (1) Suppose $t_1<t_2$ with $d_f(t_1)<\infty$. Then, $d_f(t_1)-d_f(t_2)=\mu\{x ; f(x)>t_1\}-\mu\{x ; f(x)>t_2\}=\mu\{x ; t_2 \geq f(x)>t_1\} \geq 0$. Furthermore $+\infty>\mu\{x ; t_2 \geq f(x)>t_1\} \downarrow \mu(\phi)=0$ *if* $t_2 \downarrow t_1$.

(2) The proof follows directly from the definition of δ_f.

(3) Suppose $d_f(t_0)>\delta(x_0)$ for arbitrarily fixed $t_0 \in R$ and $x_0 \in X$. We can find an $\varepsilon>0$ such that $\rho(x_0)<d_f(t_0+\varepsilon)$ as d_f is right continuous, then $d_f(t)>\rho(x_0)$ for all $t<t_0+\varepsilon$, as d_f is non-increasing. Hence $d_f(t) \leq \rho(x_0)$ implies $t \geq t_0+\varepsilon$, so that $\delta_f(t_0)=\inf\{t ; d_f(t) \leq \rho(x_0)\} \geq t_0+\varepsilon>t_0$. To prove the converse assume $d_f(t_0) \leq \rho(x_0)$, then $t_0 \in\{t ; d_f(t) \leq \rho(x_0)\}$. This implies that $\delta_f(x_0)=\inf\{t ; d_f(t) \leq \rho(x_0)\} \leq t_0$.

(4) By using the result of (3) we have $d_{\delta_f}(t_0)=\mu\{x\,;\,\delta_f(x)>t_0\}=\mu\{x\,;\,d_f(t_0)>\rho(x)\}$. Now we shall show $\mu(A)=d_f(t_0)$ where $A=\{x\,;\,d_f(t_0)>\rho(x)\}$. *If* $d_f(t_0)=\infty$, then $A=X$ and so $\mu(A)=d_f(t_0)$. Thus we may assume $d_f(t_0)<\infty$. It is easy to see that $x\in A$ implies $x\in X_{d_f(t_0)}$. This fact shows that $\mu(A)\leq\mu(X_{d_f(t_0)})=d_f(t_0)$. We will prove the converse inequality. In the case that there is an $\varepsilon\geq o$ such that $k\leq d_f(t_0)-\varepsilon$ for all $k\in\Gamma$ satisfying $X_k\subsetneqq X_{d_f(t_0)}$, it is true that $\rho(x)\leq d_f(t_0)-\varepsilon<d_f(t_0)$ for every $x\in X_{d_f(t_0)}$. From this, we obtain $X_{d_f(t_0)}\subset A$ and therefore $\mu(A)\leq\mu(X_{d_f(t_0)})=d_f(t_0)$. In another case, for any $\varepsilon>0$ we can find $k\in\Gamma$ satisfying $d_f(t_0)-\varepsilon\leq k<d_f(t_0)$. Then, from the fact that $X_k\subset A$, we obtain that $d_f(t_0)-\varepsilon\leq k=\mu(X_k)\leq\mu(A)$. Since ε is arbitrary, we have $d_f(t_0)\leq\mu(A)$.

(5) Suppose $f_n\uparrow f$ we can easily see that $\mu\{x\,;\,f_n(x)>t\}\uparrow\mu\{x\,;\,f(x)>t\}$ for each $t\in R$; that is $d_{f_n}\uparrow d_f$. Consequently, we have $\delta_{f_{n+1}}(x)=inf\{t\,;\,d_{f_{n+1}}(t)\leq\rho(x)\}\geq inf\{t\,;\,d_{f_n}(t)\leq\rho(x)\}=\delta_{f_n}(x)$. Since $\cup_n\{x\,;\,\delta_{f_n}(x)>t\}=\cup_n\{x\,;\,d_{f_n}(t)>\rho(x)\}=\{x\,;\,d_f(t)>\rho(x)\}=\{x\,;\,\delta_f(x)>t\}$, we obtain $\delta_{f_n}\uparrow\delta_f$.

In the following lemma we denote max $\{f,0\}$ by f^+ and $-\min\{f,0\}$ by f^-.

LEMMA 2. (1) *$\delta_{f+\alpha}=\delta_f+\alpha$ for all $\alpha\in R$.*

(2) *$d_{f^+}(t)=df(t)$, for all $t\geq 0$.*

(3) *$\delta_f{}^+=\delta_{f^+}$, for all $x\in X$ such that $\rho(x)\neq\mu(X)$.*

(4) *Let $\mu(X)<\infty$. If $\rho(y)=\mu(X)-\rho(x)$, then $-\delta_{-f}(x)=\delta_f(y)$.*

(5) *Let $\mu(X)=\infty$. If $\int_X f^-<\infty$, then $\delta_f\geq 0$.*

Proof. (1) The proof follows from

$$\begin{aligned}\delta_{f+\alpha}(x)&=\inf\{t\,;\,d_{f+\alpha}(t)\leq\rho(x)\}=\inf\{t\,;\,d_f(t-\alpha)\leq\rho(x)\}\\&=\inf\{t+\alpha\,;\,d_f(t)\leq\rho(x)\}=\inf\{t\,;\,d_f(t)\leq\rho(x)\}+\alpha\\&=\delta_f(x)+\alpha.\end{aligned}$$

(2) If $t\geq 0$, then $d_{f^+}(t)=\mu\{x\,;\,f^+(x)>t\}=\mu\{x\,;\,f(x)>t\}=d_f(t)$.

(3) We can easily see thet $\inf\{t\,;\,d_{f^+}(t)\leq\rho(x)\}\geq 0$. And we see also that $\{t\,;\,d_f(t)\leq\rho(x)\}=\phi$ if and only if $\{t\geq 0\,;\,d_f(t)\leq\rho(x)\}=\phi$. Then, by using (2), we obtain that

$$\begin{aligned}\delta_f{}^+(x)&=[\inf\{t\,;\,d_f(t)\leq\rho(x)\}]^+=\inf\{t\geq 0\,;\,d_f(t)\leq\rho(x)\}\\&=\inf\{t\geq 0\,;\,d_{f^+}(t)\leq\rho(x)\}=\inf\{t\,;\,d_{f^+}(t)\leq\rho(x)\}\\&=\delta_{f^+}(x).\end{aligned}$$

(4) Since $\mu(X)<\infty$ and $\rho(y)=\mu(X)-\rho(x)$, we have

$$\begin{aligned}d_{-f}(t)&=\mu\{x\,;\,(-f)x>t\}=\mu\{x\,;\,f(x)<-t\}\\&=\mu(X)-\mu\{x\,;\,f(x)\geq -t\},\end{aligned}$$

and

$$\begin{aligned}-\delta_{-f}(x) &= -\inf\{t\ ;\ \mu(X)-\mu\{x\ ;\ f(x)\geq -t\}\leq \rho(x)\}\\ &= -\inf\{t\ ;\ \mu\{x\ ;\ f(x)\geq -t\geq \mu(X)-\rho(x)=\rho(y)\}\\ &= \sup\{-t\ ;\ \mu\{x\ ;\ f(x)\geq -t\}\leq \rho(y)\}\\ &=\inf\{t\ ;\ \mu\{x\ ;\ f(x)\geq t\}\leq \rho(y)\}.\end{aligned}$$

Let $t_0=-\delta_{-f}(x)$ and $t_1=\delta_f(y)$, then as $t>t_0$ implies $\mu\{x\ ;\ f(x)\geq t\}\leq\rho(y)$, $\rho(y)\geq\lim_{t\downarrow t_0}\mu\{x\ ;\ f(x)\geq t\}=\mu\{x\ ;\ f(x)>t_0\}=d_f(t_0)$. Therefore, by LEMMA 1–(3), we have $t_1=\delta_f(y)\leq t_0=-\delta_{-f}(x)$. As $\delta_f(y)\leq t_1$, we have $\rho(y)\geq d_f(t_1)=\mu\{x\ ;\ f(x)>t_1\}\geq\mu\{x\ ;\ f(x)\geq t\}$ for $t>t_1$. This fact shows that $t_0=-\delta_{-f}(x)\leq t$ for all $t>t_1$ and therefore $t_0=-\delta_{-f}(x)\leq t_1=\delta(y)$.

(5) By our assumption, it is clear that for arbitrarily chosen $\varepsilon>0$, $d_f(-\varepsilon)=\mu\{x\ ;\ f(x)>-\varepsilon\}=\infty$. Since $X=\cup_k X_k$, there exists a $k_0\in\Gamma$ such that $\delta_f(x)=\inf\{t\ ;\ d_f(t)\leq\rho(x)\}\geq\inf\{t\ ;\ d_f(t)\leq k_0\}$ for each $x\in X$. If the right hand side of the above inequality is negative, then for some $\varepsilon>0$, $d_f(-\varepsilon)\leq k_0<\infty$. This is a contradiction.

The results of LEMMA 1 and LEMMA 2 will be used without warning in the rest of this paper,

DEFINITION 7. Let $E\in\Sigma$ with $\mu(E)<\infty$. We denote $X_{\mu(E)}$ by E^* and denote the characteristic function of $E\in\Sigma$ by 1_E.

LEMMA 3. *Let* $E\in\Sigma$ *with* $\mu(E)<\infty$. *Then,*

(1) $\int_{X_k}1_E\leq\int_{X_k}\delta_{1_E}$, *for all* $k\in\Gamma$;

(2) $\int_X 1_E=\int_X\delta_{1_E}$.

Proof. (1) We can easily show that

$$\begin{aligned}d_{1_E}(t) &= \mu\{x\ ;\ 1_E(x)>t\}\\ &= \begin{cases}0, & \text{if } 1\geq t\ ;\\ \mu(E), & \text{if } 0\leq t<1\ ;\\ \mu(X), & \text{if } t<0.\end{cases}\end{aligned}$$

Therefore, we have

$$\begin{aligned}\delta_{1_E}(x) &= \inf\{t\ ;\ d_{1_E}(t)\leq\rho(x)\}\\ &= 1,\ \text{if } \rho(x)<\mu(E)\ ;\\ &= 0,\ \text{if } \rho(x)\geq\mu(E),\end{aligned}$$

as $\mu(X)\geq\rho(x)$ always consist. We obtain $\delta_{1_E}=1_{E^*}$ where the notation $=$ means "*almost everywhere equal*". Because, it is known in the proof of LEMMA 1–(4) that

$\mu\{x\,;\,\rho(x)<\mu(E)\}=\mu(E)$ and $\{x\,;\,\rho(x)<\mu(E)\}\subset X_{\mu E}=E^*$, we have

$$\begin{aligned}\int_{X_k} 1_E &= \mu(X_k\cap E)\leq \min\{\mu(X_k),\ \mu(E)\}\\ &= \min\{\mu(X_k),\ \mu(E^*)\}=\mu(X_k\cap E^*)\\ &= \int_{X_k} 1_{E^*}=\int_{X_k}\delta_{1_E}.\end{aligned}$$

(2) Since $\mu(X\cap E)=\mu(X\cap E^*)$, (2) follows from (1).

We can show LEMMA 4 and LEMMA 5 by the essentially same way in the proof of LEMMA 3.

LEMMA 4. *Let $E_1,\ E_2\in\Sigma$ with the properties that $E_1\supset E_2$ and $\mu(E_1)<\infty$. If $\alpha_1,\ \alpha_2\leq 0$, then*

$$\delta_{(\alpha_1 1_{E_1}+\alpha_2 1_{E_2})}=\alpha_1\delta_{1_{E_1}}+\alpha_2\delta_{1_{E_2}}.$$

LEMMA 5. *Let $E_1,\ E_2\in\Sigma$ satisfying the conditions $\mu(E_1)<\infty,\ \mu(E_2)<\infty$. Then, if $\alpha_1,\ \alpha_2\geq 0$*

(1) $\int_{X_k}\delta_{(\alpha_1 1_{E_1}+\alpha_2 1_{E_2})}\leq\int_{X_k}(\alpha_1\delta_{1_{E_1}}+\alpha_2\delta_{1_{E_2}})$, *for every $k\in\Gamma$;*

(2) $\int_X\delta_{(\alpha_1 1_{E_1}+\alpha_2 1_{E_2})}=\int_X(\alpha_1\delta_{1_{E_1}}+\alpha_2\delta_{1_{E_2}})$.

LEMMA 6. *Let f be a non-negative measurable function. Then,*

(1) $\int_{X_k} f\leq\int_{X_k}\delta_f$, *for every $k\in\Gamma$;*

(2) $\int_X f=\int_X\delta_f$.

The value of the integral may be ∞.

Proof. (1) By our assumption, we can find a sequence $\{f_n\,;\,f_n=\sum^{m_n}\alpha_i 1_{E_i}\}$ of simple functions with the properties :

(a) $\alpha_i\geq 0$ $(i=1,2,\cdots,m_n)$, for each n ;

(b) $E_1\supset E_2\supset\cdots\supset E_{m_n}$ and $\mu(E_1)<\infty$, for each n ;

(c) $f_n\uparrow f$.

By LEMMA 4, LEMMA 5 and Lebesgue's monotone convergence theorem, we obtain, for every $k\in\Gamma$,

$$\begin{aligned}\int_{X_k} f &= \lim_n\int_{X_k}\sum^{m_n}\alpha_i 1_{E_i}\leq\lim_n\int_{X_k}\sum^{m_n}\alpha_i\delta_{1_{E_i}}\\ &= \lim_n\int_{X_k}\delta_{\sum^{m_n}\alpha_i 1_{E_i}}=\int_{X_k}\delta_f,\end{aligned}$$

as the condition (c) implies $\delta^{m_n}_{\sum \alpha_i 1 E_i} \uparrow \delta_f$.

(2) The proof is obtained by changing X_k and $\leq$ to X and $=$ in (1).

LEMMA 7. *Let $\mu(X) < \infty$. Suppose $\int_X f^+ < \infty$ or $\int_X f^- < \infty$, then*

$$\int_X f = \int_X \delta_f.$$

The value of the integral may be $\pm\infty$.

Proof. If $\int_X f^- = \infty$, then it is true that $\int_X \delta_f^+ = \int_X \delta_{f^+} = \int_X f^+ < \infty$. Thus, by using LEMMA 6 and LEMMA 2–(4) we obtain

$$-\infty = \int_X f = -\int_X f^- = -\int_X \delta^-_f = \int_X \delta_f.$$

If $\int_X f^- < \infty$, we have similarly that

$$\int_X \delta^-_f = \int_X \delta^-_f = \int_X f^- < \infty.$$

Thus, if $\int_X f^+ = \infty$, then

$$+\infty = \int_X f = \int_X f^+ = \int_X \delta_{f^+} = \int_X \delta_f^+ = \int_X \delta_f.$$

Finally, if $\int_X f^+ < \infty$ and $\int_X f^- < \infty$, then

$$\int_X f = \int_X f^+ - \int_X f^- = \int_X \delta_f^+ - \int_X \delta_f^- = \int_X \delta_f.$$

REMARK 3. Let $(X, \Sigma, \mu, \mathscr{F})$ be a stratus system. For an arbitrarily but fixed $k_o \in \Gamma$, let $\Sigma_{k_o} = \{X_{k_o} \cap E\ ;\ E \in \Sigma\}$, $\mathscr{F}_{k_o} = \{X_k\ ;\ k \leq k_o\}$ and μ_{k_o} be a restriction of μ on X. Then, $(X_{k_o}, \Sigma_{k_o}, \mu_{k_o}, \mathscr{F}_{k_o})$ is a stratus system. Furthermore, $\mu_{k_o}(X_{k_o}) < \infty$. Let f be a measurable function on X. Now we can consider the rearrngement $\delta_{f^{k_o}}$ of f^{k_o} which is a restriction of f on X_{k_o}. It is easy to see that $\delta_f \geq \delta_{f^{k_o}}$ on X_{k_o}.

We use this remark in the proof of LEMMA 8.

LEMMA 8. *Let f be a function with $\int_{X_{k_o}} f^+ < \infty$ or $\int_{X_{k_o}} f^- < \infty$ for some $k_o \in \Gamma$. Then,*

$$\int_{X_k} f \leq \int_{X_k} \delta_f, \textit{ for every } k \in \Gamma \textit{ with } k \leq k_0.$$

The value of the integral may be $\pm\infty$.

Proof. By the assumption, $\int_{X_k} f^+ < \infty$ or $\int_{X_k} f^- < \infty$ for every $k \in \Gamma$ satisfying $k \leq k_0$. We clearly have

$$\int_{X_k} f = \int_{X_k} \delta_{f^k} \leq \int_{X_k} \delta_f,$$

from LEMMA 7.

LEMMA 9. *Let f be a function with $\int_X f^+ < \infty$ or $\int_X f^- < \infty$. Then,*

(1) $\int_{X_k} f \leq \int_{X_k} \delta_f$, *for every* $k \in \Gamma$;

(2) $\int_X f \leq \int_X \delta_f$.

Proof. We need only to prove (2) in case $\mu(X) = \infty$. If $\int_x f^- < \infty$, then by LEMMA 2–(5), $\delta_f \geq 0$. Thus,

$$\int_X f = \int_X f^+ - \int_X f^- \leq \int_X f^+ = \int_X \delta_{f^+} = \int_X \delta_f{}^+ = \int_X \delta_f.$$

If $\int_X f^- = \infty$, then

$$-\infty = -\int_X f^- = \int_X f \leq \int_X \delta_f.$$

REMARK 4. Let $\mathscr{F} = \{X_k \,;\, k \in \Gamma\}$ and $\mathscr{F}' = \{X_k' \,;\, k \in \Gamma\}$ be two strati for a homogeneous σ-finite measure space (X, Σ, μ). We let δ_{1_E} be the $\mathscr{F}$-rearrangement of 1_E and δ_{1_E}' be the $\mathscr{F}'$-rearrangement of $1_E'$, then $\delta_{1_E} = 1_{X_{\mu(E)}}$ and $\delta_{1_E}' = 1_{X'_{\mu(E)}}$. We can easily show that for every $k \in \Gamma$,

$$\int_{X_k} \delta_{1_E} = \mu(X_k \cap X_{\mu(E)}) = \mu(X_k' \cap X_{\mu}'(E)) = \int_{X_k'} \delta_{1_E}'.$$

Furthermore, if δ_f is the $\mathscr{F}$-rearrangement of $f \in L^1$ and δ_f' is the $\mathscr{F}'$-rearrangement of f, then it is easy to see that

$$\int_{X_k} \delta_f = \int_{X_k'} \delta_f', \text{ for all } k \in \Gamma.$$

3. Doubly-stochastic operators and doubly-substochastic operators.

In this section we introduce important relations between integrable functions on a stratus system $(X, \Sigma, \mu, \mathscr{F})$. We shall denote the set of all integrable functions on X by L^1 and the set of all essentially bounded functions on X by L^∞.

DEFINITION 1. Let $f, g \in L^1$. We denote $f \ll g$ whenever $\int_{X_k} \delta_f \leq \int_{X_k} \delta_g$ for all $k \in \Gamma$, and $f \prec g$ if $f \ll g$ and $\int_X \delta_f = \int_X \delta_g$.

Hence, these relations are free from the individual stratus, by REMARK 4 of §2. On the other hand the rearrangement depends on the individual stratus. We can also define them by the way of Luxemburg.[6] [see the note of Chong and Rice].[1]

LEMMA 1. *Let $f, g \in L^1$ with $f \prec g$. Then for any non-negative $u \in L^\infty(L^1)$*

$$\int_X f\delta_u \leq \int_X \delta_f \delta_u \leq \int_X \delta_g \delta_u.$$

Proof. There exists a sequence of simple functions $\left\{\sum^{mn} \alpha_i 1_{X_{k_i}} ; \alpha_i \geq 0, \ k_i \in \Gamma\right\}$ with $\sum^{mn} \alpha_i 1_{X_{k_i}} \uparrow \delta_u$. By LEMMA 9 of §2 and *Lebesgue's dominated convergence theorem*, we have

$$\int_X f\delta_u = \int_X \lim_n f\cdot\left(\sum^{mn} \alpha_i 1_{X_{k_i}}\right) = \lim_n \int_X f\cdot\left(\sum^{mn} \alpha_i 1_{X_{k_i}}\right)$$

$$= \lim_n \sum^{mn} \alpha_i \int_X f\cdot 1_{X_{k_i}} = \lim_n \sum^{mn} \alpha_i \int_{X_{k_i}} f$$

$$\leq \lim_n \sum^{mn} \alpha_i \int_{X_{k_i}} \delta_f$$

$$\left[= \lim_n \sum^{mn} \alpha_i \int_X \delta_f 1_{X_{k_i}} = \int_X \lim_n \delta_f\cdot\left(\sum^{mn} \alpha_i 1_{X_{k_i}}\right) = \int_X \delta_f \delta_u\right].$$

$$\leq \lim_n \sum^{mn} \int_{X_{k_i}} \delta_g = \int_X \delta_g \delta_u.$$

DEFINITION 2. We denote by $[L^1]$ (or $[L^\infty]$) the set of all linear operators on L^1 (L^∞). An operator $T \in [L^1]$ is said to be *doubly-substochastlc* if it satisfies the following conditions :

(a) $T \geq 0$;

(b) $Tf \ll f$ for non-negative $f \in L^1$.

A *doubly-stochastic* operator T is a doubly-substochastic operator which satisfies $Tf \prec f$ for any non-negative $f \in L^1$. We denote the set of all doubly-substochastic operators by $\mathscr{S}$ and the set of all doubly-stochastic operators by $\mathscr{D}$.

LEMMA 2. *Let* $T \in \mathscr{S}$. *Then*

$$\int_X |Tf| \leq \int_X |f| \text{ for every } f \in L^1.$$

Proof. It is easy to see that $Tf^+ - Tf^- = Tf = (Tf)^+ - (Tf)^-$. From the fact that $T \geq 0$ and $f^+ - f \geq 0$, it follows $T(f^+ - f) = Tf^+ - Tf^- \geq 0$. This shows that $Tf^+ \geq \max\{Tf, 0\} = (Tf)^+$. Since $(Tf)^+ - (Tf)^- = Tf^+ - Tf^- \geq (Tf)^+ - Tf^-$, it follows that $Tf^- \geq (Tf)^-$. We now have

$$\begin{aligned}\int_X |Tf| &= \int_X ((Tf)^+ + (Tf)^-) \leq \int_X (Tf^+ + Tf^-) \\ &= \int_X T(f^+ + f^-) = \int_X T|f| = \int_X \delta_{T|f|} \\ &\leq \int_X \delta_{|f|} = \int_X |f|,\end{aligned}$$

from LEMMA 6 of §2 and the definition of $T \in \mathscr{S}$.

DEFINITION 3. We denote by T^* the adjoint of $T \in [L^1]$ which acts on L^∞ : that is $T^* \in [L^\infty]$. By LEMMA 2, if $T \in \mathscr{S}$, then as we can see easily T^* is a positive contraction on L^∞. We define $\mathscr{S}^* = \{T^* ; T \in \mathscr{S}\}$.

LEMMA 3. *Let* $T \in \mathscr{S}$. *If* $f \in L^1 \cap L^\infty$, *then*

$$\text{ess sup } |Tf| \leq \text{ess sup } |f|.$$

Proof. Suppose ess sup $|Tf| >$ ess sup $|f|$ for some $f \in L^1 \cap L^\infty$. We can find $E \in \Sigma$ with $0 < \mu(E) < \infty$ on which $|Tf| >$ ess sup $|f|$. From this we have

$$\int_{E^*} \delta_{T|f|} \geq \int_{E^*} \delta_{|Tf|} > \int_{E^*} \text{ess sup } |f| \leq \int_{E^*} \delta_{|f|}.$$

But, $T|f| \ll |f|$ implies $\int_{E^*} \delta_{T|f|} \leq \int_{E^*} \delta_{|f|}$, Which is a contradiction.

LEMMA 4. *Let* $T \in \mathscr{S}$. *Then, for any* $1_E \in L^1$,

$$0 \leq T1_E \leq 1 \text{ and } \int_X T1_E \leq \mu(E).$$

Proof. The result follows directly from

$$0 \leq T1_E = |T1_E| \leq \text{ess sup } |1_E| \leq 1$$

and

$$\int_X T1_E = \int_X \delta_{T1_E} \leq \int_X \delta_{1_E} = \int_X 1_{E^*} = \mu(E).$$

LEMMA 5. *Let $T \in \mathscr{D}$. Then,*

$$\int_X T1_E = \mu(E) \text{ for any } E \in \Sigma.$$

Proof. We obtain the result changing $\leq$ to $=$ in the proof of LEMMA 4.

LEMMA 6. *Let $T^* \in \mathscr{S}^*$. Then for any $E \in \Sigma$*

$$0 \leq T^*1_E \leq 1 \text{ and } \int_X T^*1_E \leq \mu(E).$$

The value of the integral may be ∞.

Proof. Let $E \in \Sigma$. It is clear that $0 \leq T^*1_E = |T^*1_E| \leq \text{ess sup } |1_E| \leq 1$. There exists a sequence $\{1_{X_{k_n}} T^*1_E \,;\, k_n \in \Gamma\}$ with the property that if $n \to \infty$, $1_{X_{k_n}} T^*1_E \uparrow T^*1_E$. Now we have for each n,

$$\int_X 1_{X_{k_n}} T^*1_E = \int_X T1_{X_{k_n}} \cdot 1 \leq \int_X 1 \cdot 1_E = \mu(E),$$

and therefore

$$\int_X T^*1_E = \lim_n \int_X 1_{X_{k_n}} T^*1_E \leq \mu(E).$$

LEMMA 7. *If $T \in \mathscr{D}$, Then $T^*1 = 1$.*

Proof. For any $1_E \in L^1$, by LEMMA 4 we have

$$\int_E 1 = \int_X 1_E = \mu(E) = \int_X T1_E = \int_X T1_E \cdot 1 = \int_X 1_E \cdot T^*1 = \int_E T^*1.$$

Then, $T^*1 = 1$.

DEFINITION 4. Let $S \in [L^1 \cap L^\infty]$ and assume that satisfies the following conditions :

(a) $0 \leq S1_E \leq 1$ for $1_E \in L^1 \cap L^\infty$;

(b) $\int_X S1_E \leq \mu(E)$ for $1_E \in L^1 \cap L^\infty$.

Let $f \in L^1$ (L^∞) with $f \geq 0$ and let $\{f_n\,;\, f_n \in L^1 \cap L^\infty\}$ be a sequence of nonnegative functions such that $f_n \uparrow f$. Then we define $\hat{S}f$ by $\hat{S}f = \lim_n Sf_n$. For arbitrarily $f \in L^1$ we define $\hat{S}f$ by $\hat{S}f = \hat{S}f^+ - \hat{S}f^-$.

REMARK 1. We can find that $\hat{S}f$ is well defined for any $f \in L^1$ (L^∞) with $f \geq 0$.[3] We shall show that if $f \in L^1$ (L^∞) with $f \geq 0$, then $\hat{S}f \in L^1$ (L^∞) in the following lemmas. This fact also shows that for arbitrarily $f \in L^1$ (L^∞), $\hat{S}f \in L^1$ (L^∞) by the definition of $\hat{S}f$.

LEMMA 8. *Let $f \in L^1$ with $f \geq 0$. Then $\hat{S}f \in L^1$.*

Proof. It follows directly from the definition of $\hat{S}$ that if $f \geq 0$, then $\hat{S}f \geq 0$ and that $\hat{S}$ is linear. We can find a sequence $\left\{\sum^{m_n} \alpha_i 1_{E_i}\,;\, 1_{E_i} \in L^1 \cap L^\infty,\ \ \alpha_i \geq 0\right\}$ with $\sum^{m_n} \alpha_i 1_{E_i} \uparrow f$. Then, we have

$$0 \leq S\left(\sum^{m_n} \alpha_i 1_{E_i}\right) \uparrow \hat{S}f \text{ and } S\left(\sum^{m_n} \alpha_i 1_{E_i}\right) \in L^1 \cap L^\infty.$$

This shows, by *Lebesgue's monotone convergence theorem*,

$$\begin{aligned}\int_X \hat{S}f &= \int_X \lim_n S\left(\sum^{m_n} \alpha_i 1_{E_i}\right) = \lim_n \int_X S\left(\sum^{m_n} \alpha_i 1_{E_i}\right) \\ &= \lim_n \sum^{m_n} \alpha_i \int_X S1_{E_i} \leq \lim_n \sum^{m_n} \alpha_i \int_X 1_{E_i} \\ &= \lim_n \int_X \sum^{m_n} \alpha_i 1_{E_i} = \int_X \lim_n \left(\sum^{m_n} \alpha_i 1_{E_i}\right) \\ &= \int_X f < \infty.\end{aligned}$$

The proof is complete.

LEMMA 9. *If $f \in L^\infty$ with $f \geq 0$, then $\hat{S}f \in L^\infty$.*

Proof. For any $1_E \in L^1 \cap L^\infty$, we have $\int_{X_k} \delta_{T1_E} \leq \int_{X_k} \delta_{1_E}$ for each $k \in \Gamma$. Because $S1_E \leq 1$ implies $\delta_{T1_E} \leq 1$ and $\delta_{1_E} = 1$ on E, we can show that

$$\int_{X_k} \delta_{S1_E} \leq \int_{X_k} \delta_{1_E} \quad \text{whenever } k \leq \mu(E)\,;$$

$$\int_{X_k} \delta_{S1_E} \leq \int_X \delta_{S1_E} = \int_X S1_E \leq \mu(E) = \int_{X_k} \delta_{1_E} \quad \text{whenever } k > \mu(E).$$

While, there exists a sequence $\left\{\sum^{m_n}\alpha_i 1_{E_i}\,;\, 1_{E_i}\in L^1\cap L^\infty,\ \alpha_i\geq 0\right\}$ such that $E_1\supset \cdots\supset E_{m_n}$ and $\sum^{m_n}\alpha_i 1_{E_i}\uparrow f$. Suppose for some n_0, there exists a set $E\in\Sigma$ with $0<\mu(E)<\infty$ on which $S\left(\sum^{m_{n_0}}\alpha_i 1_{E_i}\right)>\text{ess sup}\sum^{m_{n_0}}\alpha_i 1_{E_i}$. Then we obtain

$$\int_{E^*}\delta_{S(\sum^{m_{n_0}}\alpha_i 1_{E_i})}>\int_{E^*}\text{ess sup}\sum^{m_{n_0}}\alpha_i 1_{E_i}\geq\int_{E^*}\delta_{\sum^{m_{n_0}}\alpha_i 1_{E_i}}.$$

But we see that

$$\int_{E^*}\delta_{S(\sum^{m_{n_0}}\alpha_i 1_{E_i})}\leq\sum^{m_{n_0}}\alpha_i\int_{E^*}\delta_{S1_{E_i}}$$
$$\leq\sum^{m_{n_0}}\alpha_i\int_{E^*}\delta_{1_{E_i}}=\int_{E^*}\delta_{\sum^{m_{n_0}}\alpha_i 1_{E_i}}.$$

This is a contradiction. Hence we obtain that for all n,

$$S\left(\sum^{m_n}\alpha_i 1_{E_i}\right)\leq\text{ess sup}\sum^{m_n}\alpha_i 1_{E_i}\leq\text{ess sup}\ f,$$

and therefore

$$\hat{S}f=\lim_n S\left(\sum^{m_n}\alpha_i 1_{E_i}\right)\leq\text{ess sup}\ f.$$

REMARK 2. By LEMMA 8 (9), we can consider that $\hat{S}$ is an operator on L^1 (L^∞).

LEMMA 10. *$\hat{S}\in\mathscr{S}$ as an operator on L^1.*

Proof. Let $f\in L^1$ with $f\geq 0$, then there exists a sequence $\left\{\sum^{m_n}\alpha_i 1_{E_i}\,;\, 1_{E_i}\in L^1\cap L^\infty,\ \alpha_i\geq 0\right\}$ with $E_1\supset\cdots\supset E_{m_n}$ and $\sum^{m_n}\alpha_i 1_{E_i}\uparrow f$. Then for each $k\in\Gamma$,

$$\int_{X_k}\delta_{\hat{S}f}=\lim_n\int_{X_k}\delta_{S(\sum^{m_n}\alpha_i 1_{E_i})}=\lim_n\sum^{m_n}\alpha_i\int_{X_k}\delta_{S1_{E_i}}$$
$$\leq\lim_n\sum^{m_n}\alpha_i\int_{X_k}\delta_{1_{E_i}}=\lim_n\int_{X_k}\delta_{(\sum^{m_n}\alpha_i 1_{E_i})}$$
$$=\int_{X_k}\delta_f.$$

LEMMA 11. *For any* $E \in \Sigma$,

$$0 \leq \hat{S}1_E \leq 1 \text{ and } \int_X \hat{S}1_E \leq \mu(E).$$

The value of the integral may be ∞.

Proof. It follows from Lemma 9 that $0 \leq \hat{S}1_E \leq 1$. There exists $\{X_{k_n}\} \subset \{X_k ; k \in \Gamma\}$ with $1_{X_{kn}} 1_E \uparrow 1_E$. It is easy to see that $1_{X_{kn}} 1_E = 1_{X_{kn} \cap E} \in L^1 \cap L^\infty$. Since $\int_X S1_{X_{kn} \cap E} \leq \mu(X_{k_n} \cap E) \leq \mu(E)$ for all k_n, we have

$$\int_X \hat{S}1_E = \int_X \lim_n S1_{X_{kn} \cap E} = \lim_n \int_X S1_{X_{kn} \cap E} \leq \mu(E).$$

REMARK 3. By LEMMA 2 LEMMA 11, we see that

$$\begin{aligned} \mathscr{S}^* &= \{T^* ; T \in \mathscr{S}\} \\ &= \{T \in [L^\infty] ; 0 \leq T1_E \leq 1, \int_X T1_E \leq \mu(E) \text{ for any } E \in \Sigma, \\ &\quad 0 \leq f_n \uparrow f (f_n \in L^1 \cap L^\infty) \text{ implies } Tf_n \uparrow Tf\}. \end{aligned}$$

The following lemma and remark are essentially due to Ryff.[7)]

LEMMA 12. *Let* $\mu(X) < \infty$. *If an operator* T *on* L^∞ *satisfies the condition*: $0 \leq T1_E \leq 1$ *and* $\int_X T1_E = \mu(E)$ *for any* $E \in \Sigma$, *then* $\hat{T} \in \mathscr{D}$.

REMARK 4. Similarly to REMARK 3, we see that

$$\begin{aligned} \mathscr{D}^* &= \{T^* ; T \in \mathscr{D}\} \\ &= \{T \in [L^\infty] ; 0 \leq T1_E \leq 1, \int_X T1_E = \mu(E) \quad \text{for} \quad E \in \Sigma\} \end{aligned}$$

by LEMMA 12.

4. Kadison's thorem and Fan's theorem.

DEFINITION 1. The topology λ on $[L^\infty]$ is said to be the *weak*-operator topology* if a subbasic neighbourhood of the null operator in this topology is given by

$$N(f, g, \varepsilon) = \{T \in [L^\infty] ; |\int_X fTg| < \varepsilon\},$$

where $f \in L^1$, $g \in L^\infty$ and $\varepsilon > 0$.

DEFINITION 2. We denote the all of positive contraction operators on L^{∞} by P_c.

The following theorem which was given by Kadison[5] is very powerfull.

THEOREM 1. (Kadison's general compactness theorem)
P_c is compact in weak-operator topology.*

The following theorem is due to Fan[2] and is also of considerable importance in functional analysis. Recently Takahashi[10] gave a simple proof of this this theorem and some applications.

THEOREM 2. (Fan's theorem)
Let K be a compact convex subset of a topologycal vector space E. Let $\{f_i ; i \in I\}$ be a class of real valued lower semicontinuous convex functions defined on K. Then, the system of inequalities

$$f_i(x) \leq 0 \quad (i \in I)$$

is consistent on K [i. e. there exists a point $x \in K$ satisfying $f_i(x) \leq 0$ $(i \in I)$] if and only if, for any finite subclass $\{f_{i_1}, \cdots, f_{i_n}\} \subset \{f_i ; i \in I\}$ and for any n non-negative numbers $\{\alpha_1, \cdots, \alpha_n\}$, there exists a point $x \in K$ such that

$$\sum_{k=1}^{n} \alpha_k f_{i_k}(x) \leq 0.$$

5. A generalization of Hardy-Littlewood-Pólya's theorem.

LEMMA 1. *$\mathscr{S}^*$ is compact in weak*-operator topology.*

Proof. Let $\{T_\alpha ; T_\alpha \in \mathscr{S}^*\}$ be a net which converges to T_o in weak*-operator topology. For any $E \in \Sigma$ and for any $k \in \Gamma$, $\int_X 1_{X_k} T_\alpha 1_E - \int_X 1_{X_k} T_o 1_E = \int_X 1_{X_k} (T_\alpha - T_o) 1_E$ converges to 0 if $T_\alpha \to T_o$. While for any $k \in \Gamma$, $\int_X 1_{X_k} T_\alpha 1_E \leq \int_X T_\alpha 1_E \leq \mu(E)$. Then we have $\int_X 1_{X_k} T_o 1_E \leq \mu(E)$ for any $k \in \Gamma$. This fact implies that $\int_X T_o 1_E \leq \mu(E)$. On the other hand, since P_c is compact and $\mathscr{S}^* \subset P_c$, we obtain that $0 \leq T_o 1_E \leq 1$ and therefore $T_o \in \mathscr{S}$. This completes the proof.

LEMMA 2. *If $\mu(x) < \infty$, then $\mathscr{D}^*$ is compact in weak*-operator topology.*

DEFINITION 1. If an operator $T \in \mathscr{S}$ satisfies $T^* 1 = 1$, then T is said to be a *doubly-substochastic-markov* operator, which is introduced by Sakai.[9] We denote by $\mathscr{S}$-m the set of all doubly-substochastic-markov operators. We define $\mathscr{S}$-m^*

by $\mathscr{S}\text{-}m^* = \{T^* ; T \in \mathscr{S}\text{-}m\}$. It is clear, by LEMMA 7 of § 3, that $\mathscr{D}^* \subset \mathscr{S}^*$.

REMARK 1. It is easy to see that $\mathscr{D} = \mathscr{S}\text{-}m$. If $\mu(X) < \infty$, $T1 = 1$ follows from $T \in \mathscr{D}$. But, if $\mu(X) = \infty$, $T \in \mathscr{D}$ not implies $T1 = 1$. Then, our definition of $\mathscr{D}$ not equal the ordinaly definition of doubly-stochastic operators [$T1 = 1$ and $T^*1 = 1$]. Therefore, if $\mu(X) = \infty$, we use the notation $\mathscr{S}\text{-}m$.

LEMMA 3. *$\mathscr{S}\text{-}m^*$ is compact in weak*-operator topology.*

Proof. Let $\{T_\alpha ; T_\alpha \in \mathscr{S}\text{-}m^*\}$ be a net which converges to T_o in weak*-operator topology. We need only to prove $T_o 1 = 1$. Since $T_\alpha \in \mathscr{S}\text{-}m^*$, $T_\alpha 1 = 1$ for all α. For any $f \in L^1$, we have

$$\int_X f1 - \int_X f T_o 1 = \int_X f T_\alpha 1 - \int_X f T_o 1 = \int_X f(T_\alpha - T_o)1.$$

If $T_\alpha \to T_o$, then $\int_X f(T_\alpha - T_o)1 \to 0$. Thus we obtain $T_o 1 = 1$.

LEMMA 4. *$\mathscr{S}\text{-}m^*$ is convex.*

LEMMA 5. *Let $\{A_1, \cdots, A_n\}$ and $\{B_1, \cdots, B_n\}$ be subclass of Σ which are pairwise disjoint. If $\mu(A_i) = \mu(B_i) < \infty$ for each i $(1 \leq i \leq n)$, then there exists a measure preserving mapping $\sigma ; X \to X$ which satisfies the condition : $\mu(\sigma^{-1}(A_i) \Delta B_i) = 0$ for each i.*

Proof. By the definition of stratus system, there exists a mapping $\sigma_i ; X \to X$ which are measure preserving on B_i and satisfying $\mu(\sigma_i^{-1}(A_i)\Delta B_i) = 0$ for each i. Further, if we put

$$A = A_1 \cup \cdots \cup A_n \cup B_1 \cup \cdots \cup B_n - A_1 \cup \cdots \cup A_n ;$$
$$B = A_1 \cup \cdots \cup A_n \cup B_1 \cup \cdots \cup B_n - B_1 \cup \cdots \cup B_n,$$

then $\mu(A) = \mu(B) < \infty$ and therefore there exists a measure preserving mapping $\sigma' ; X \to X$ with $\mu(\sigma'^{-1}(A)\Delta B) = 0$. Now, we define $\sigma : X$ onto X by

$$\sigma = \begin{cases} \sigma_i \text{ on } \sigma_i^{-1}(A_i) \cap B_i ; \\ \sigma' \textit{ on } \sigma'^{-1}(A) \cap B ; \\ I \text{ (identity mapping) on } (A_1 \cup \cdots \cup A_n \cup \cdots \cup B_n)^c. \end{cases}$$

Then, σ has the desired properties.

DEFINITION 2. Let σ be a measure preserving mapping X onto X. We define an operator $T_\sigma \in [L^1]([L^\infty])$ by, for any $f \in L^1(L^\infty)$, $T_\sigma f = f \circ \sigma$.

LEMMA 6. $T_\sigma \in \mathscr{S}\text{-}m$.

Proof. Since

$$\begin{aligned} d_{T_\sigma f}(t) &= \mu\{x\ ;\ T_\sigma f > t\} = \mu\{x\ ;\ f\circ\sigma > t\} \\ &= \mu(\sigma^{-1}\{x\ ;\ f(x) > t\}) = \mu\{x\ ;\ f(x) > t\} \\ &= d_f(t), \end{aligned}$$

for any $f \in L^1$, we have $\delta_{T_\sigma f} = \delta_f$ and have $T_\sigma \in \mathscr{S}\text{-}m$.

LEMMA 7. *Let $\{A_1, \cdots, A_n\}$ and $\{B_1, \cdots, B_n\}$ be two subclass of pairwise disjoint sets from Σ. Then, there exists a measure preserving mapping $\sigma\ ;\ X \to X$ such that, for n nonnegative α_i,*

$$T_\sigma\Big(\sum^n \alpha_i 1_{A_i}\Big) = \sum \alpha_i 1_{B_i} \quad \text{and} \quad T_\sigma{}^*\Big(\sum^n \alpha_i 1_{B_i}\Big) = \sum^n \alpha_i 1_{A_i}.$$

Proof. There exists σ which satisfies the conditions of LEMMA 5. Then

$$\begin{aligned} T_\sigma\Big(\sum^n \alpha_i 1_{A_i}\Big) &= \Big(\sum^n \alpha_i 1_{A_i}\Big)\circ\sigma = \sum^n \alpha_i (1_{A_i}\circ\sigma) \\ &= \sum^n \alpha_i 1_{\sigma^{-1}(A_i)} = \sum^n \alpha_i 1_B\ . \end{aligned}$$

On the other hand, for any $1_E \in L^1$, we have

$$\begin{aligned} \int_E T_\sigma{}^* 1_{B_i} &= \int_X 1_E T_\sigma{}^* 1_{B_i} = \int_X (T_\sigma 1_E)\cdot 1_{B_i} \\ &= \int_X (1_E\circ\sigma)(1_{A_i}\circ\sigma) = \int_X (1_E 1_{A_i})\circ\sigma \\ &= \int_E 1_{A_i}. \end{aligned}$$

DEFINITION 3. Let $f, g \in L^1$. For any $E \in \Sigma$ with $\mu(E) < \infty$, we define the mapping $F_E\ ;\ \mathscr{S}\text{-}m^* \to R$ by

$$F_E(T^*) = \int_E (\delta_f - T\delta_g).$$

LEMMA 8. *For any $E \in \Sigma$ with $\mu(E) < \infty$, F_E is convex and weak*-operator continuous.*

Proof. Since

$$F_E(T^*) - F_E(T_o{}^*) = |\int_E (T_o - T)\delta_g| = |\int_X \delta_g (T_o{}^* - T^*) 1_E|,$$

F_E is continuous in weak*-operator topology. If α, β are non-negative real numbers with $\alpha + \beta = 1$, then

$$F_E(\alpha T^* + \beta T_o{}^*) = \int_E (\delta_f - (\alpha T + \beta T_o)\delta_g)$$

$$= \int_E \alpha(\delta_f - T\delta_g) + \int_E \beta(\delta_f - T\delta_g)$$

$$= \alpha F_E(T^*) + \beta F_E(T_o{}^*).$$

From this fact, it follows that F_E is convex.

Now, we have had suficient tools to prove the following theorem, which is a goal of our work.

THEOREM 1. *Let* $f, g \in L^1$ *satisfying the following conditions* :

(a) $\delta_f \geq 0$, $\delta_g \geq 0$;

(b) $\int_X \delta_f = \int_X \delta_g$.

Then, for every $k \in \Gamma$,

$$\int_{Xk} \delta_f \leq \int_{Xk} \delta_g$$

if and only if there exists a $T \in \mathscr{S}\text{-}m$ *such that*

$$\delta_f = T\delta_g.$$

Proof. It is easy to check that $\delta_f = T\delta_g$ if and only if, for any $E \in \Sigma$ with $\mu(E) < \infty$,

$$F_E(T^*) = \int_E (\delta_f - T\delta_g) = 0.$$

We first show that there exists a $T \in \mathscr{S}\text{-}m$ such that $F_E(T^*) \leq 0$ for any $E \in \Sigma$ with $\mu(E) < \infty$. By Fan's theorem, we need only to show that for any fiinite class $\{E_1, \cdots, E_n\}$ and for n non-negative numbers α_i, there exists $T \in \mathscr{S}\text{-}m$ *such that*

$$\sum^n \alpha_i F_{E\,i}(T^*) = \int_X \sum^n \alpha_i 1_{E\,i}(\delta_f - T\delta_g) \leq 0.$$

Without loss of generality, we assume that $\{E_1, \cdots, E_n\}$ is pairwise disjoint. It follows from LEMMA 7 that there exists a measure preserving mapping σ such that $T\sigma^* u = \delta_u$ and $T\sigma\delta_u = u$, where $u = \sum^n \alpha_i 1_E$. Then, by LEMMA 1 of §3,

$$\int_X u\delta_f \leq \int_X \delta_u \delta_f \leq \int_X \delta_u \delta_g = \int_E (T_\sigma{}^* u)\delta_g = \int_X u\cdot(T_\sigma \delta_g).$$

Thus we obtain $T_\sigma \in \mathscr{S}\text{-}m$ which has the desired properties. Therefore, there exists $T \in \mathscr{S}\text{-}m$ such that $\delta_f \leq T\delta_g$. suppose $\mu(A) > 0$ with $A = \{x\,;\,\delta_f < T\delta_g\}$. Then

$$\int_X \delta_f = \int_A \delta_f + \int_{A_c} \delta_f < \int_A T\delta_g + \int_{A_c} T\delta_g = \int_X T\delta_g \leq \int_X \delta_g,$$

which is a contradiction. Thus $\delta_f = T\delta_g$.

Let us prove the converse. If there exists a $T \in \mathscr{S}\text{-}m$ with $\delta_f = T\delta_g$, then, for any $k \in \Gamma$,

$$\int_{X_k} \delta_f = \int_{X_k} T\delta_g \leq \int_{X_k} \delta_{T\delta_g} \leq \int_{X_k} \delta_g.$$

If $\mu(X) < \infty$, we obtain the following theorem, which was essentially proved by Ryff. [8] [see REMARK 1.]

THEOREM 2. *Let* $\mu(X) < \infty$ *and* $f, g \in L^1$ *which satisfy the condition :*

(a) $\delta_f \geq 0$, $\delta_g \geq 0$;

(b) $\int_X \delta_f = \int_X \delta_g$.

Then, for every $k \in \Gamma$

$$\int_{X_k} \delta_f \leq \int_{X_k} \delta_g.$$

if and only if there exists a $T \in \mathscr{D}$ *such that*

$$\delta_f = T\delta_g.$$

Examples.

(1) Let $X = \{1, 2, \ \}$, $\Sigma = 2^X$, μ be a counting measure. Let $X_k = \{1, 2, \ , k\}$ and $\Gamma = \{1, 2, \cdots\}$. Let $(a_1, a_2, \cdots)$ and $(b_1, b_2, \cdots)$ have the following properties :

(a) $a_1 \geq a_2 \geq \cdots \geq 0$, $b_1 \geq b_2 \geq \cdots \geq 0$;

(b) $\sum_i a_i = \sum_i b_i$.

Then, for each k, $\sum_{i=1}^k a_i \leq \sum_{i=1}^k b_i$ if and only if there exists a matrix $\mathbf{P} = (P_{ij})_{i,j=1,2,\cdots}$ such that $\sum_i P_{ij} = 1$, $\sum_j P_{ij} \geq 1$, $P_{ij} \geq 0$ and $a_i = \sum_j P_{ij} b_j$ for each i.

(2) Let $X = \{1, 2, \ , n\}$, $\Sigma = 2^X$, μ be a counting measure. Let $X_k = \{1, 2, \cdots, k\}$ and $\Gamma = \{1, 2, \cdots, n\}$. Let $(a_1, \cdots, a_n)$ and $(b_1, \cdots, b_n)$ have the following proerties :

(a) $a_1 \geq \cdots \geq a_n \geq 0$, $b_1 \geq \cdots \geq b_n \geq 0$;

(b) $\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i$.

Then for each k, $\sum_{i=1}^{k} a_i \leq \sum_{i=1}^{k} b_i$ if and only if there exists a $n \times n$-matrix $\mathbf{P} = (P_{ij})$ such that $P_{ij} \geq 0$, $\sum_i P_{ij} = 1$, $\sum_j P_{ij} = 1$ and $a_i = \sum_j P_{ij} b_j$ for each i.

(3) Let $X = [0, \infty)$, Σ be a class of Lebesgue measurable sets and μ be a Lebesgue measure. Let $X_k = [0, k]$ and $\Gamma = [0, \infty)$. Let $f, g \in L^1\ [0, \infty)$ with the following conditions :

(a) $\delta_f \geq 0$, $\delta_g \geq 0$;

(b) $\int_0^\infty \delta_f = \int_0^\infty \delta_g$.

Then, for each k $\int_0^k \delta_f \leq \int_0^k \delta_g$ if and only if there exists a $T \in \mathscr{S}$-m *such that* $\delta_f = T\delta_g$.

(4) Let $X = [0, 1]$, Σ be a class of Lebesgue measurable sets and μ be a Lebesgue measure. Let $X_k = [0, k]$ and $\Gamma = [0, 1]$. Let $f, g \in L^1 [0, 1]$ satisfying the following properties :

(a) $\delta_f \geq 0$, $\delta_g \geq 0$;

(b) $\int_0^1 \delta_f = \int_0^1 \delta_g$.

Then for each $[k \in 0, 1]$, $\int_0^k \delta_f \leq \int_0^k \delta_g$ if and only if there exists a $T \in \mathscr{D}$ such that $\delta_f = T\delta_g$.

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