Embedding Paths and Circuits into the Hypercube

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The importance of the Boolean n-dimensional hypercube as a basis for the architecture of future highly parallel computers is now widely recognized among the computer architects. It is very important to know how to map communication and computation trees on the hypercube in order to minimize the communication overhead during parallel computations. This problem seems to be rather difficult, but it is possible to solve it for some special kinds of trees. Here we will consider the simplest kind of tree, the path. The hypercube is a Hamiltonian graph and there are many ways how to construct its Hamiltonian paths or circuits. In this paper, several new algorithms for constructing the Hamiltonian paths and circuits in the n-dimensional Boolean hypercube under given constraints are presented. Our main result can be stated as follows: Given a path p, 2^n-1 in length, with end vertices u and v and with two inner vertices x and y such that the distance between u and x is odd and the distance between v and y is even, it is possible to embed this path into an n-cube so that the vertices x and v become neighbors of u in the n-cube and the distance between v and y becomes two. An analogous result is proved for another path, defined in the same way, except that the distance between u and x is also even. Since the end vertices of embedded paths are neighbors in the hypercube, these results have corollaries on embedding Hamiltonian circuits. Another corollary of the main result is that the balanced 3-quasistar with 2^n vertices is a spanning tree of the n-cube.

1. Introduction

Among the proposed architectures for parallel computers, those based on the Boolean n-dimensional hypercube (usually called only hypercube or, when the dimensionality is important, n-cube) are considered to be the most promising. The n-cube is a graph with 2^n vertices labelled 0, 1, ..., 2^n-1 and with an edge joining two vertices whenever their binary representations differ in a single coordinate. Hypercube multiprocessors are the first highly parallel computers produced commercially and by them, researchers have been given a real possibility to experiment with parallel programming. In contrast to this technological progress, mathematical theories of the methods

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how to exploit effectively this massive parallelism are far from well developed. One of the most crucial problems arising on all levels of hypercube architectures is that of the optimal mapping the tree structures into the hypercube. Its difficulty is given by the high regularity of hypercube on the one hand and by the arbitrary irregularity of general trees on the other hand. Especially, the characterization of spanning trees of hypercubes is important. But this problem is one of those still unsolved problems. Ref. 1 contains a comprehensive survey of the state of art in this field and a detailed bibliography.

The most simple trees are paths. A hypercube is a Hamiltonian graph, hence its Hamiltonian path is at the same time its simple spanning tree. If the end vertices of a Hamiltonian path are adjacent vertices in the hypercube, the path becomes a Hamiltonian circuit. Problems of embedding paths and circuits into the hypercube under various constraints and conditions are interesting in themselves. But more importantly, they arise naturally when we solve embedding problems for more complicated trees. Embedding a balanced n-quasistar into the n-cube can serve as a good example. The n-quasistar is a graph homeomorphic to a star $K(1, n)$. A graph is balanced if it has $2\times k$ vertices and there exists a 2-coloring of its vertices such that k vertices are colored with one color and k vertices are colored with the other color. It was conjectured[2] that any balanced n-quasistar with 2^n vertices is embeddable into the n-cube, but the proof was given only for $n=$ 3, 4, 5[3]. Solving this problem for any n leads to a series of path embedding problems.

Very few results on embedding paths have been published until now (see, e. g. Ref. 2). Here, we present several new results on embedding Hamiltonian paths and circuits. The relations between paths and other trees will be illustrated by a case of balanced 3-quasistar. Its embeddability into the n-cube appears to be a corollary of one of our theorems.

2. Definitions and terminology

$G=(V, E)$ is a general graph with vertices $V=V(G)$ and edges $E=E(G)$. Edge $\langle u, v\rangle$ is an edge joining vertices u and v of G. A graph H is a *subgraph* of G if $V(H)\subseteq V(G)$ and $E(H)\subseteq E(G)$. If $V(H)=V(G)$, H is a *spanning subgraph* of G. A *neighbor* of vertex $u\in V(G)$ is any $v\in V(G)$ such that $\langle u, v\rangle\in E(G)$. $nb(v)$ will denote the set of all the neighbors of v in G. *The degree* of a vertex $v\in V(G)$, $deg(v)$, is the number of its neighbors in the graph G, hence $deg(v)=|nb(v)|$. ΔG is the degree of the graph G, defined as $\max\{deg(v);\ v\in V(G)\}$.

The path of length m with end vertices u and v, $p< u,\ v: m >$, is a graph with $m+1$ vertices $u=u_0,\ u_1,\ldots,\ u_m=v$ and for all $i\in\{1,\ldots,m\}$, vertex u_{i-1} is joined by an edge with vertex u_i (hence $deg(u)=deg(v)=1$ and the degree of the remaining vertices is 2). If $m=1$, we write simply $< u,\ v >$; an edge of a graph can be considered to be a path with unit length. Path $p< u,\ x,\ y,\ldots,v: m >$ is a path $p< u,\ v: m >$ with the vertices $x,\ y,\ldots$ between u and v in this order. If we join the end vertices of a path by an edge, we get a *circuit.* Our notation for circuits is compatible with that for paths. For example, $p< u,\ x,\ y,\ldots,\ v: m > \cup < v,\ u > = c< u,\ x,\ y,\ldots,v: m+1 >$, $c< u: m >$ specifies that one of vertices of the circuit is u, $c< m >$ denotes just a circuit with length m, and $c< u,\ x,\ v: m >$ and $c< v,\ u,\ x: m >$ are equivalent specifications.

The path of G is a path which is a subgraph of graph G. A graph G is said to be *connected* if there is at least one path between any two vertices of G. *The Hamiltonian path of* G is any path of G which is a spanning subgraph of G. $hp< u,\ v: G >$ denotes a Hamiltonian path of graph G with end vertices u, v, $hp< u,\ x,\ y,\ldots,v: G >$ is $hp< u,\ v: G >$ passing through the vertices $x,\ y,\ldots$ in this order. If the vertices u, v in $hp< u,\ v: G >$ are neighbors in G, the graph $hp< u,\ v: G > \cup < u,\ v >$ is called *a Hamiltonian circuit* of G, $hc< G >$. If the order of vertices along the Hamiltonian circuit is important, we can write again, for example, $hc< u,\ v,\ x,\ldots, :G >$.

A *distance*, $dist_G(u,\ v)$, between any two vertices u, v of a connected graph G is the length of the shortest path between u and v. Since there are two distances between any two vertices along the circuit, the value of $dist_C(u,\ v)$, where c is a circuit, is a set of two values. If $p=hp< u,\ w,\ldots,\ v : G >$ is a Hamiltonian path of G such that $dist_p(u,\ w)=dist_G(u,\ w)$ (p connects vertices u and w in the shortest possible way), we use notation $p=hp< u\rightarrow w,\ldots,\ v : G >$.

2-*Coloring* of vertices of graph G is an assignment of 2 colors, e. g. black and white, to the vertices in such a way that no two neighbors have got the same color. *The bipartite graph* is a graph for which there exists a 2-coloring. 2-Coloring means that the set of vertices V can be partitioned into two subsets V_1 and V_2 in such a way that every edge of the graph joins V_1 with V_2. If every vertex of V_1 is joined with every vertex of V_2, the bipartite graph is *complete.* $K(m,\ n)$ denotes a complete bipartite graph with $|V_1|=m$ and $|V_2|=n$. Graph $K(1,\ n)$ is called *a star* S_n. A bipartite graph is *balanced* whenever $|V_1|=|V_2|$. Hence, if G is a balanced graph, $|V(G)|$ is even. If $|V_1|\neq|V_2|$, the graph is *imbalanced* and the *imbalancement of* G, $imb(G)$, will be defined as $\mathrm{abs}(|V_1|-|V_2|)$.

Two graphs are *homeomorphic* if they can be reduced to the same graph by omitting some or all vertices of degree 2. For example, any two paths are homeomorphic. *The n-quasistar*, R_n, is a graph homeomorphic to a star $K(1, n)$. The n-quasistar consists of a central vertex, *center*, and n *rays*. *Even* (*odd*) *ray* is a ray with even (odd) length. The n-quasistar is balanced if and only if it has exactly one odd ray.

The *n-dimensional Boolean hypercube* Q_n is a graph with 2^n vertices labelled 0, 1,.., 2^n-1 and with an edge joining two vertices whenever their binary representations differ in a single coordinate. There are $n\times 2^{n-1}$ edges in Q_n, every vertex has n neighbors, and ΔQ_n is n. The distance between two vertices u, v of Q_n is the *Hamming distance* $\rho(u, v)$ between the binary representations of u and v. If $u\in V(Q_n)$, $nb(u)=\{v\in V(Q_n);\ \rho(u, v)=1\}$. The hypercube is a connected balanced bipartite graph. For u, $v\in V(Q_n)$, $\delta(u, v)$ will be the set of all dimensions $i\in\{1,..,n\}$, in which the binary representations of u, v differ. $\varepsilon(u, v)$ is on the contrary the set of all dimensions, in which the coordinates of u and v are the same. Hence $\varepsilon(u, v)=\{1,..,n\}-\delta(u, v)$ and $\delta(u, v)=|\rho(u, v)|$. The definition of function ε can be extended to any subset of $V(Q_n)$. If $V'\subseteq V(Q_n)$, then $\varepsilon(V')=\cap\ \{\varepsilon(u, v);\ u,\ v\in V'\}$. The 2-cube is usually called *a square*. The square is isomorphic to a circuit $c\langle 4\rangle$.

The basic property of the hypercube is its recursively defined structure. Given any $i\in\{1,..,n\}$, Q_n can be decomposed into two copies of (n-1)-cubes $Q^1{}_{n-1}$ and $Q^2{}_{n-1}$ in such a way that all the vertices in each copy have the same value of the i-th bit in their binary representation. This decomposition, called *the i-canonical decomposition*, will be written: $Q^1{}_{n-1}\,\|_i\,Q^2{}_{n-1}$. The indices n-1 can be omitted if they can be understood from the context. Similarly, i can be omitted if it does not matter along which coordinate the decomposition is made. Every vertex u of one subcube matches exactly one vertex $\bar{u}$ in the other subcube. If e.g. $u\in V(Q^1)$, we write $\bar{u}=u(Q^2)$ and say that $\bar{u}$ is an *image* of $u\in V(Q^1)$ in Q^2. $\rho(u, \bar{u})=1$ and $\delta(u, \bar{u})=\{i\}$ and there are 2^{n-1} edges joining vertices in one subcube with their images in the other subcube.

The embedding of a graph G into Q_n is a mapping Ψ: $V(G)\rightarrow V(Q_n)$ such that if $\langle u, v\rangle\in E(G)$, then $\rho(\Psi(u), \Psi(v))=1$. To simplify the notation we will identify the names of vertices of an embedded graph with the labels of their images in the hypercube, whenever it will not lead to ambiguity.

3. Previous results

In Ref. 2, the following two basic results were proved. For the completeness of our paper, we will repeat the proofs here using our notation.

Lemma 3.1: *Let* $n \geq 2$, *let* u, v *be two different vertices of* Q_n, *and let* $k \equiv \rho(u, v)$ (mod2) *be an integer such that* $\rho(u, v) \leq k \leq 2 - \rho(u, v)$. *Then there exists* $c = hc < u, v : Q_n >$ *such that* $dist_c(u, v) = \{k, 2^n - k\}$.

Proof: The case of $n=2$ is obvious. Let $n > 2$ and assume the lemma holds for all $n' < n$. In other words, we can construct a Hamiltonian circuit $c = hc < u, v : Q_{n'} >$ such that $dist_c(u, v) = \{k, m\}$, whenever $k + m = 2^{n'}$ and $k \equiv \rho(u, v)$ (mod2). Let k, m be two integers, $k + m = 2^n$, $k \equiv \rho(u, v)$(mod2).

1. $\rho(u, v) = 1$ (Fig. 3. 1). Then k is odd and $k \neq m$. Let $k < m$ and $m' = m - 2^{n-1}$. Then $m' >, 0$ $k + m' = 2^{n-1}$, and $m' \equiv \rho(u, v)$(mod2). Let $Q_n = Q^1 \parallel Q^2$ such that $u, v \in V(Q^1)$. Let $c1 = hc < u, v : Q^1 >$ such that $dist_{c1}(u, v) = \{k, m'\}$ (by induction). Let w be a vertex of $c1$ such that $1 \in dist_{c1}(u, w)$, $m' - 1 \in dist_{c1}(v, w)$. Let $\bar{u}$, $\bar{w}$ be images of u, w in Q^2 and let $c2 = hc < \bar{u}, \bar{w} : Q^2 >$, $dist_{c2}(\bar{u}, \bar{w}) = \{1, 2^{n-1} - 1\}$. Let $c = c1 \cup c2 \cup < w, \bar{w} > \cup < u, \bar{u} > - < u, w > - < \bar{u}, \bar{w} >$. Then $c = hc < u, v : Q_n >$ with $dist_c(u, v) = \{k, m\}$.

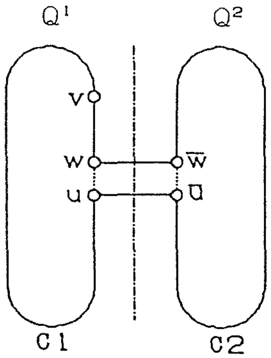

Fig. 3. 1: Hamiltonian circuit between vertices of Q_n with Hamming distance 1

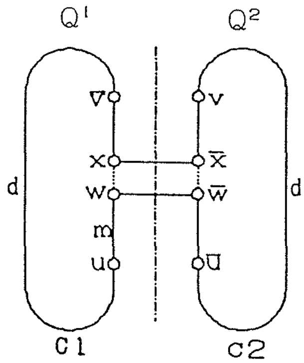

Fig. 3. 2: Hamiltonian circuit between vertices of Q_n with odd Hamming distance greater than 1.

2. $\rho(u, v) > 1$ (Fig. 3. 2). Let $Q_n = Q^1 \parallel Q^2$ such that $u \in V(Q^1)$ and $v \in V(Q^2)$. Let $\bar{v} = v(Q^1)$, $\bar{u} = u(Q^2)$. Then $\bar{u} \neq v$ and $\bar{v} \neq u$. Suppose $k \leq m$. Let $d = \rho(u, v) - 1$ and $c1 = hc < u, \bar{v} : Q^1 >$ with $dist_{c1}(u, \bar{v}) = \{d, 2^{n-1} - d\}$. Let $c2$ be a mirror image of $c1$ in Q^2. Since $k > d$ and $k \equiv (d+1)$(mod2), there exists an integer $q \geq 0$, $q = (k - d - 1)/2$. Let x, w be two vertices of the part of $c1$ with length $2^{n-1} - d$ in the distance q and $q+1$ from u. Let $\bar{x}, \bar{w}$ be their images in Q^2 and let $c = c1 \cup c2 \cup < x, \bar{x} > \cup < w, \bar{w} > - < x, w > - < \bar{x}, \bar{w} >$. It is easy to see that $dist_c(u, v) = \{k, m\}$ and hence c is the solution. □

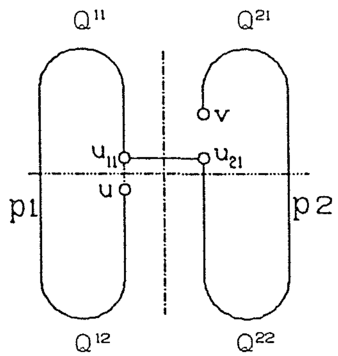

Fig. 3.3: Hamiltonian path between vertices of Q_n with odd Hamming distance.

Lemma 3.2: *Let $n \geq 2$ and let u, v be vertices of Q_n such that $\rho(u, v)$ is odd. Then there exists $p = hp\langle u, v : Q_n\rangle$.*

Proof: The case $n=2$ is obvious. Suppose $n>2$.

1. $\rho(u, v)=1$. Let $c=hc\langle u, v: Q_n\rangle$ with $dist_c(u, v)=\{1, 2^n-1\}$ (by Lemma 3.1). Let $p=c-\langle u, v\rangle$. Then $p=hp\langle u, v: Q_n\rangle$.

2. $\rho(u, v) \geq 3$. Then there exists a decomposition $Q_n=Q^1 \| Q^2$, $Q^1=Q^{11} \| Q^{12}$, $Q^2=Q^{21} \| Q^{22}$ such that $u \in V(Q^{12})$ and $v \in V(Q^{21})$ (Fig. 3.3). Let $u_{11}=u(Q^{11})$ and $u_{21}=u_{11}(Q^{21})$. Then $u_{21} \neq v$ and $\rho(u_{21}, v)$ is odd. Let $p1=hp\langle u, u_{11}: Q^1\rangle$ and $p2=hp\langle u_{21}, v: Q^2\rangle$ (by induction). Let $p=p1 \cup p2 \cup \langle u_{11}, u_{21}\rangle$. Then $p=hp\langle u, v: Q_n\rangle$. □

We will need the two following corollaries of Lemma 3.1.

Corollary 3.1: *Let $n \geq 2$. Let $u, v, x \in V(Q_n)$, $\rho(u, v)=1$. Let k be an integer such that $k \equiv \rho(u, x) (\mathrm{mod}2)$ and $\rho(u, x) \leq k \leq 2^n-1-\rho(v, x)$. Then there exists a Hamiltonian path $p=hp\langle u, x, v: Q_n\rangle$ such that $dist_p(u, x)=k$.*

Corollary 3.2: *Let $n \geq 2$. Let u, $v \in V(Q_n)$, $\rho(u\ v)=1$. Let k be an integer, $1 \leq k < 2^n-1$. Then there exist a vertex $x \in V(Q_n)$ and a Hamiltonian path $p=hp\langle u, x, v: Q_n\rangle$ such that $dist_p(u, x)=k$, $\rho(u, x) \in \{1, 2\}$ and $k \equiv \rho(u, x)$ (mod2).*

4. Embedding paths and circuits

Remark: The technique used in the case 1 of the proof of Lemma 3.1 will be applied very often in the following proofs, since the situation when we can construct a solution in one subcube by induction and then add a Hamiltonian path or circuit of the other subcube, is common. This type of solution will be called the solution *by induction and expansion* in the following text.

Lemma 4.1: *Let $n \geq 2$, and let u, v, w be three different vertices of Q_n such that $\rho(u, v)$ is odd, Then there exists $p=hp\langle u \rightarrow w, v: Q_n\rangle$.*

Proof: The case of $n=2$ is obvious. Let $n>2$ and assume the lemma holds for all $n'<n$. Let $M=\varepsilon\{u, v, w\}$.

1. $M \neq \phi$. Then there exists a decomposition of Q_n such that all three vertices u, v, w belong to the same subcube and the solution is by induction and expansion.

2. $M=\phi$.

2. 1. $\varepsilon(u, w)=\phi$. Then $\varepsilon(u, v)\neq\phi$. Let $i\in\varepsilon(u, v)$, $Q_n=Q^1 \parallel_i Q^2$, $u, v\in V(Q^1)$, $w\in V(Q^2)$, $\bar{u}=u(Q^2)$ (Fig. 4. 1). Let $x\in V(Q^2)\cap nb(\bar{u})-\{v(Q^2)\}$ and $\bar{x}=x(Q^1)$. Then $\bar{x}\neq v$. Let $p1=hp\langle u\rightarrow\bar{x}, v:Q^1\rangle$, $p2=hp\langle\bar{u}\rightarrow w, x:Q^2\rangle$ (by induction). Let $p=p1\cup p2\cup\langle x,\bar{x}\rangle\cup\langle u,\bar{u}\rangle-\langle\bar{x}, u\rangle$. Then $p=hp\langle u\rightarrow w, v:Q_n\rangle$.

2. 2. n is odd and $\varepsilon(u, v)=\phi$. Then $\varepsilon(u, w)\neq\phi$. Let $i\in\varepsilon(u, w)$, $Q_n=Q^1 \|_i Q^2$, $u, w\in V(Q^1)$, $v\in V(Q^2)$ (Fig. 4. 2). Let $x\in V(Q^1)\cap nb(u)-\{w\}$ and $\bar{x}=x(Q^2)$. Then $\rho(v,\bar{x})=n-2$ is odd. Let $p1=hp\langle u\rightarrow w, x:Q^1\rangle$ (by induction), $p2=hp\langle v, \bar{x}:Q^2\rangle$ (by Lemma 3. 2). Let $p=p1\cup p2\cup\langle x,\bar{x}\rangle$. Then $p=hp\langle u\rightarrow w, v:Q_n\rangle$.

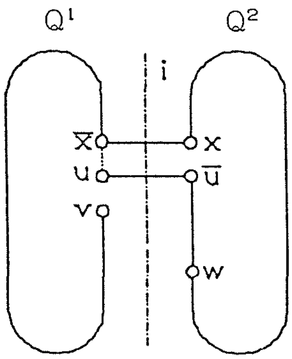

Fig. 4. 1: Hamiltonian path $p=hp\langle u\rightarrow w,\ v:\ Q_n\rangle$ when u and w are opposite vertices of Q_n.

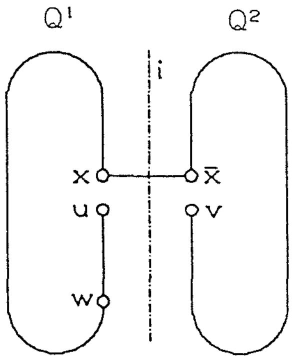

Fig. 4. 2: Hamiltonian path $p=hp\langle u\rightarrow w,\ v:\ Q_n\rangle$ when u and v are opposite vertices of Q_n.

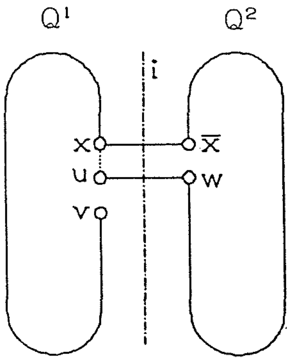

Fig. 4. 3: Hamiltonian path $p=hp\langle u\rightarrow w, v:Q_n\rangle$ when u and w are neighboring vertices of Q_n.

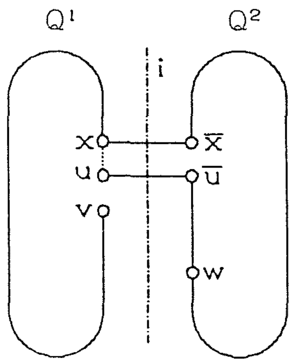

Fig. 4. 4: Hamiltonian path $p=hp\langle u\rightarrow w,\ v:\ Qn\rangle$ in a general case.

2. 3. $\varepsilon(u, v)\neq\phi$, $\varepsilon(u, w)\neq\phi$, $\varepsilon\{u, v, w\}=\phi$. Let $i\in\varepsilon(u, v)$, $Q_n=Q^1 \parallel_i Q^2$, $u, v\in V(Q^1)$, $w\in V(Q^2)$.

2. 3. 1. $\varepsilon(u,v)=\{i\}$ and $\varepsilon(v,w)=\phi$. Then $u(Q^2)=w$ (Fig. 4. 3). Let $x\in V(Q^1)\cap nb(u)$ and $\bar{x}=x(Q^2)$. Then $x\neq v$. Let $p1=hp<u\to x, v:Q^1>$ (by induction) and $p2=hp<w,\bar{x}:Q^2>$ (by Lemma 3. 2). Let $p=p1\cup p2\cup<x,\bar{x}>\cup<u,w>-<x,u>$. Then $p=hp<u\to w, v:Q_n>$.

2. 3. 2. $|\varepsilon(u,v)|>1$ or $\varepsilon(v,w)\neq\phi$. Assume there exists at least one neighbor x of u in Q^1 different from v and from $w(Q^1)$ (Fig. 4. 4). Let $\bar{x}=x(Q^2)$, $\bar{u}=u(Q^2)$. Then $\bar{x}\neq w$. Let $p1=hp<u\to x, v:Q^1>$ and $p2=hp<\bar{u}\to w,\bar{x}:Q^2>$ (by induction). Let $p=p1\cup p2\cup<x,\bar{x}>\cup<u,\bar{u}>-<x,u>$. Then $p=hp<u\to w, v:Q_n>$. The case that there is no such neighbor can happen only for $n=3$ and the solution is trivial. □

Lemma 4. 2: *Let $n\geq 3$, let $m1$, $m2$ be positive even integers such that $m1+m2=2^n-2$, and let $p1$ and $p2$ be two paths $p1=p<x1,x3:m1>$, $p2=p<x2,x4:m2>$. Then there exists an embedding of $p1$ and $p2$ into Q_n such that $\rho(x1,x2)=\rho(x2,x3)=\rho(x3,x4)=1$ and $\rho(x1,x4)=3$.*

Proof: The case of $n=3$ is easy. Let $n>3$ and assume the lemma holds for all $n'<n$. Without loss of generality, assume $m1<m2$.

1. $m2>2^{n-1}$. Then the induction and expansion can be applied.

2. $m1=2^{n-1}-2$, $m2=2^{n-1}$. Let $Q_n=Q^1\parallel Q^2$. Let $x1, x2, x3\in V(Q^1)$ and $x4\in V(Q^2)$ be mapped on vertices of Q_n as implied by Fig. 4. 5. Let $\bar{x}2=x2(Q^2)$. Let $p'=hp<x2\to x1, x3:Q^1>$ (by Lemma 4. 1) and $p''=hp<x4,\bar{x}2:Q^2>$ (by Lemma 3. 2). Then $p'-<x1,x2>$ and $p''\cup<x2,\bar{x}2>$ are the required embeddings. □

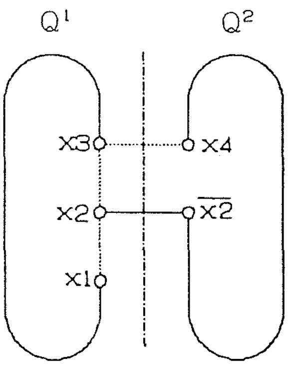

Fig. 4. 5: Two vertex disjoint paths in a Q_n.

Theorem 4. 1: *Let $n\geq 2$, p be a path $p<u1, v1:2^n-1>$ and $u2$, $v2$ be vertices on p such that $dist_p(u1,u2)>0$ is even and $dist_p(v1,v2)<2^n-1$ is odd. Then there exists an embedding of p into Q_n such that $\rho(u1,u2)=2$ and $\rho(v1,v2)=\rho(u1,v1)=1$.*

Proof: The case of $n=2$ is trivial. Let $n\geq 3$ and assume the lemma holds for all $n'<n$.

1. $dist_p(u1,u2)+dist_p(v1,v2)=2^n-1$. Immediately from Corollary 3. 2, since $u2=v2$.

2. $dist_p(u1,u2)+dist_p(v1,v2)<2^n-1$. Let $m1=dist_p(u1,u2)$, $m3=dist_p(v1,v2)$, and $m2=2^n-1-m1-m3$ (Fig. 4. 6). Then $m1>0$, $m2>0$ are even, $m3>0$ is odd.

2. 1. $m1>2^{n-1}$ or $m3>2^{n-1}$ or $m2>2^{n-1}$. The induction and expansion can be applied.

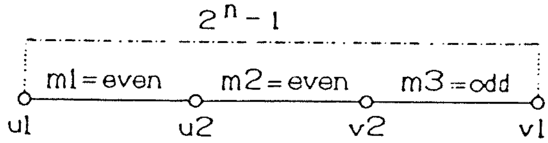

Fig. 4.6: Path $p=p\langle u1, u2, v2, v1: 2^n-1\rangle$ with $dist_p(u1, u2)$ even and $dist_p(v1, v2)$ odd.

2.2. $m1<2^{n-1}$ and $m3<2^{n-1}-1$. Let $Q_n=Q^1 \| Q^2$, $u1\in V(Q^1)$, and $v1=u1(Q^2)$. Let $x\in V(Q^1)$ be any neighbor of $u1$, and $\bar{x}=x(Q^2)$ (Fig. 4.7). Let $p1=hp\langle u1, u2, x: Q^1\rangle$ such that $dist_{p1}(u1, u2)=m1$ and $\rho(u1, u2)=2$ (by Corollary 3.2.). Let $p2=hp\langle v1, v2, \bar{x}: Q^2\rangle$ such that $dist_{p2}(v1, v2)=m3$ and $\rho(v1, v2)=1$ (by Corollary 3.2.). Then $p=p1\cup p2\cup\langle x, \bar{x}\rangle$ is the required embedding.

2.3. $m1=2^{n-1}$. The construction is the same, but in this case, $u2$ is identical with $x(Q^2)$ (Fig. 4.8).

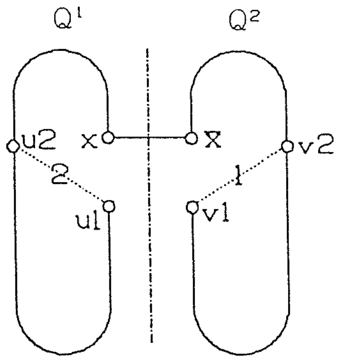

Fig. 4.7: Embedding the path $p=p\langle u1, u2, v2, v1: 2^n-1\rangle$ with $dist_p(u1, u2)<2^{n-1}$ even and $dist_p(v1, v2)<2^{n-1}-1$ odd.

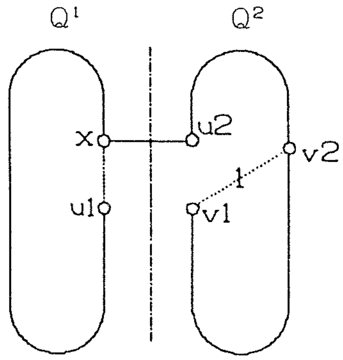

Fig. 4.8: Embedding the path $p=p\langle u1, u2, v2, v1: 2^n-1\rangle$ with $dist_p(u1, u2)=2^{n-1}$ and $dist_p(v1, v2)<2^{n-1}-1$ odd.

2.4. $m3=2^{n-1}-1$. The construction is the same, but in this case, $v2$ is identical with $x(Q^2)$ (Fig. 4.9).

3. $dist_p(u1, u2)+dist_p(v1, v2)>2^n-1$. Let $m1=2^n-1-dist_p(v1, v2)$, $m3=2^n-1-dist_p(u1, u2)$, and $m2=2^n-1-m1-m3=dist_p(u1, u2)+dist_p(v1, v2)-2^n+1$ (Fig. 4.10). Then $m1>0$, $m2>0$ are even and $m3>0$ is odd.

3.1. $m1>2^{n-1}$ or $m3>2^{n-1}$ or $m2>2^{n-1}$. The induction and expansion can be applied.

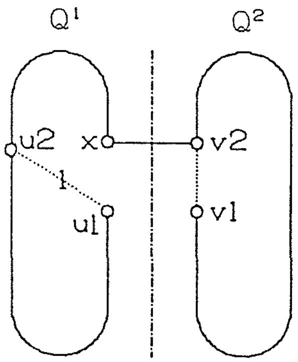

Fig. 4.9: Embedding the path $p=p\langle u1,\ u2,\ v2,\ v1:\ 2^n-1\rangle$ with $dist_p\ (u1,\ n2)<2^{n-1}$ even and $dist_p\ (v1,\ v2)=2^{n-1}-1$.

3.2. $m1\ =2^{n-1}$. Then $dist_p(u1,\ u2)>2^{n-1}$, $dist_p(v1, v2)=2^{n-1}-1$, and $m2<2^{n-1}-1$. Let $Q_n=Q^1\|Q^2, v1\in V(Q^2)$, and let $u2, v2\in V(Q^2)$ be any two neighbors of $v1$. Let $u1,\ y, x$ be images of $v1,\ u2,\ v2$ in Q^1, respectively (Fig. 4. 11). Let $p1=hp\langle u1\rightarrow y, x:Q^1\rangle$ (by Lemma 4. 1). Let $p2=hp\langle v2, u2,\ v1:Q^2\rangle$ such that $dist_{p2}(v2,\ u2)=m2$ (by Corollary 3. 1). Then $p=p1\cup p2\cup\langle x, v2\rangle$ is the required embedding.

3.3. $m3=2^{n-1}-1$. Then $dist_p(u1, u2)=2^{n-1}$, $dist_p(v1, v2)>2^{n-1}$, and $m2<2^{n-1}$. Let $Q_n=Q^1\,||\,Q^2$, $u1\in V(Q^1)$, and let $v1, x\in V(Q^1)$ be any two neighbors of $u1$. Let $u2, v2, y$ be images of $x, v1$, $u1$ in Q^2, respectively (Fig. 4. 12). Let $p1=hp$

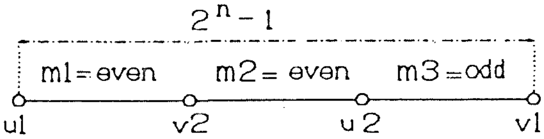

Fig. 4.10: Path $p=p\langle u1,\ v2,\ u2,\ v1:\ 2^n-1\rangle$ with $dist_p\,(u1,\ u2)$ even and $dist_p\,(v1,\ v2)$ odd.

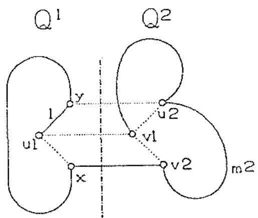

Fig. 4.11: Embedding the path $p=p\langle u1,\ v2,\ v1:\ 2^n-1\rangle$ with $dist_p\ (u1,\ u2)<2^{n-1}$ even and $dist_p\ (v1,\ v2)=2^{n-1}-1$.

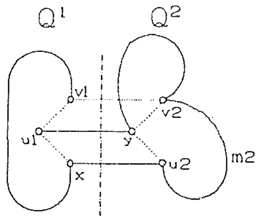

Fig. 4.12: Embedding the path $p=p\langle u1,\ v2,\ u2,\ v1:\ 2^n-1\rangle$ with $dist_p\ (u1,\ u2)=2^{n-1}$ and $dist_p\ (v1,\ v2)<2^{n-1}-1$ odd.

$\langle u1\rightarrow v1, x:Q^1\rangle$ (by Lemma 4. 1). Let $p2=hp\langle u2, v2,\ y:Q^2\rangle$ such that $dist_{p2}(v2,\ u2)=m2$ (by Corollary 3. 1). Then $p=p1\cup p2\cup\langle x,\ u2\rangle\cup\langle u1,\ y\rangle-\langle u1,\ v1\rangle$ is the required embedding.

3.4. $m1<2^{n-1}$ and $m3<2^{n-1}$ and $m2\leq 2^{n-1}$. Then $dist_p(u1,\ u2)>2^{n-1}$ and $dist_p(v1, v2)>2^{n-1}$. Let $Q_n=Q^1\,\|\,Q^2$, $u1\in V(Q^1)$, and let $x\in V(Q^1)$ be any neighbor

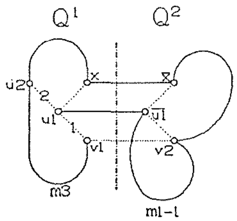

Fig. 4.13: Embedding the path $p=p<u1, v2, u2, v1: 2^n-1>$ with $dist_p(u1, u2)>2^{n-1}$ even and $dist_p(v1, v2)>2^{n-1}-1$ odd.

of $u1$. Let $p1=hp<u1, u2, x:Q^1>$ such that $dist_{p1}(u1, u2)=m3+1$ and $\rho(u1, u2)=2$ (by Corollary 3.2). Let $v1 \in V(Q^1)$ be the neighbor of $u1$ such that $dist_{p1}(u1, v1)=1$. Let $\bar{x}, \bar{u}1, v2$ be images of x, $u1$, $v1$ in Q^2, respectively (Fig. 4.13). Let $p2=hp<\bar{u}1, u2, \bar{x}:Q^2>$ such that $dist_{p2}(\bar{u}1, v2)=m1-1$ (by Corollary 3.1). Then $p=p1 \cup p2 \cup <x, \bar{x}> \cup <u1, \bar{u}1> - <u1, v1>$ is the required embedding. □

Lemma 4. 3: *Let $n \geq 2$, and p be a path $p<u1, u2, v2, v1:2^n-1>$ such that $m1=dist_p(u1, u2)>0$ and $m3=dist_p(v1, v2)>0$ are even and $m1+m3<2^n-2$. Then there exists an embedding of p into Q_n such that $\rho(u1, u2)=\rho(v1, v2)=2$ and $\rho(u1, v1)=\rho(u1, v2)=\rho(v1, u2)=1$.*

Proof: The case of $n=2$ is trivial. Let $n \geq 3$ and assume the lemma holds for all $n'<n$. Let $m2=2^n-1-m1-m3$ (Fig. 4.14). Then $m2>1$ is odd. Without loss of generality, assume $m1 \geq m3$.

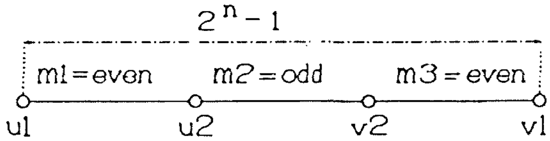

Fig. 4.14: Path $p=p<u1, u2, v2, v1: 2^n-1>$ with $m1=dist_p(u1, u2)$ and $m3=dist_p(v1, v2)$ even, $m1+m3<2^n-2$.

1. $m1>2^{n-1}$. The induction and expansion can be applied.

2. $m1=2^{n-1}$. Then $m3 \leq 2^{n-1}-4$ and $m2 \leq 2^{n-1}-3$. This case can happen only for $n>3$ Let $Q_n=Q^1 \| Q^2$. Let $u1, v1, v2, y \in V(Q^1)$ be four different vertices of Q^1 such that $\rho(v1, u1)=\rho(u1, v2)=\rho(v2, y)=1$ and $\rho(v1, y)=3$ (Fig. 4.15). Then there exist two vertex disjoint paths $p1$, $p2$ in Q^1, $p1=p<v1, v2:m3>$ and $p2=p<u1, y:2^{n-1}-m3-2>$ (by Lemma 4.2). Let $u2=v1(Q^2)$, $x=v2(Q^2)$, and $\bar{y}=y(Q^2)$ Let $p3=hp<x, u2, \bar{y}:Q^2>$ such that $dist_{p3}(x, u2)=m2-1$ (by Corollary 3.1). Then $p=p1 \cup p2 \cup p3 \cup <x, v2> \cup <y, \bar{y}>$ is the required embedding.

3. $m1<2^{n-1}$. Then $dist_p(u1, v2)=m1+m2$ is odd. Since $m1 \geq m3$, $dist_p(u1, v2) \geq 2^{n-1}+1$. Let $Q_n=Q^1||Q^2$, $u1 \in V(Q^1)$, and let $v1, v2 \in V(Q^1)$ be any two neighbors of $u1$ (Fig. 4.16). Let $p1=hp<u1, v2, v1:Q^1>$ such that $dist_{p1}(u1, v2)=m1+m2-2^{n-1}$ (by Corollary 3.1). Then $dist_{p1}(v1, v2)=m3$. Let y be the neighbor of $u1$ such that $dist_{p1}(u1, y)=1$ (if $m1+m2=2^{n-1}+1$, then y is identical with $v2$). Let $u2, x, \bar{y}$ be images of $v1, u1, y$ in Q^2, respectively. Let $p2=hp$

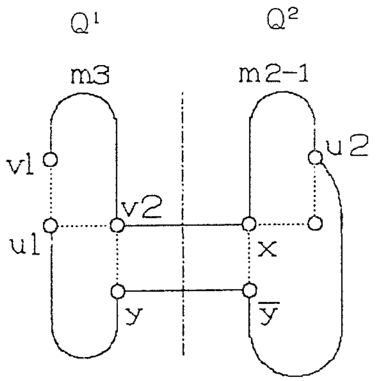

Fig. 4.15: Embedding the path $p=p\langle u1,\ u2,\ v2,\ v1\colon 2^n-1\rangle$ with $dist_p\ (u1,\ u2)=2^{n-1}$ and $dist_p\ (v1,\ v2)\leq 2^{n-1}-4$ even.

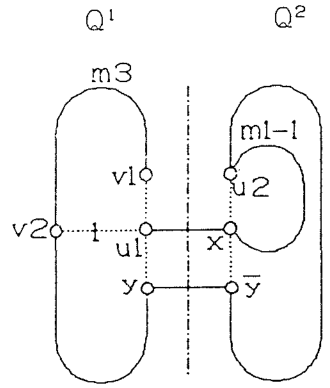

Fig. 4.16: Embedding the path $p=p\langle u1,\ u2,\ v2,\ v1\colon 2^n-1\rangle$ with $dist_p\ (u1,\ u2)<2^{n-1}$ even and $dist_p\ (v1,\ v2)$ even.

$\langle x, u2, \bar{y}\colon Q^2\rangle$ such that $dist_{p2}(x, u2)=m1-1$ (by Corollary 3. 1). Then $p=p1\cup p2\cup\langle u1, x\rangle\cup\langle y, \bar{y}\rangle-\langle u1, y\rangle$ is the required embedding. □

Lemma 4. 4: *Let $n\geq 3$, and let p be a path $p\langle u1, u2, v2, v1\colon 2^n-1\rangle$ such that $m1=dist_p(u1, u2)>0$, and $m3=dist_p(v1, v2)>0$ are even, $m1\geq m3$ and $m1+m3=2^n-2$. If $m3\neq 2$, then there exists an embedding of p into Q_n such that $\rho(u1, u2)=\rho(v1,\ v2)=2$ and $\rho(u1,\ v1)=\rho(u1,\ v2)=\rho(v1,\ u2)=1$. If $m3=2$ and $m1=2^n-4$, p can be embedded only in such a way that $\rho(u1,\ u2)=\rho(v1, v2)=2$, $\rho(u1,\ v1)=1$, and either $\rho(u1,\ v2)=1$ and $\rho(v1,\ u2)=3$ or $\rho(u1,\ v2)=3$ and $\rho(v1,\ u2)=1$.*

Proof: For $n=3$, $m1=4$, and $m3=2$, the lemma holds. Let $n>3$.

1. $m3=2, m1=2^n-4$. Then all the three vertices $u2$, $v2$, and $v1$ have to belong to one subcube Q_3 of Q_n, so the situation is the same as for $n=3$.

2. $m3>2$. For $n=4$, the only two possibilities are $m1=10$ and $m3=4$ or $m1=8$ and $m3=6$. In both the cases, there exists an embedding with $\rho(u1, u2)=\rho(v1, v2)=2$ and $\rho(u1,\ v1)=\rho(u1,\ v2)=\rho(v1,\ u2)=1$. Let $n>4$.

2. 1. $m1>2^{n-1}$. The induction and expansion can be applied.

2. 2. $m1=2^{n-1}$ and $m3=2^{n-1}-2$. The construction is trivial by double induction and expansion (Fig. 4. 17). □

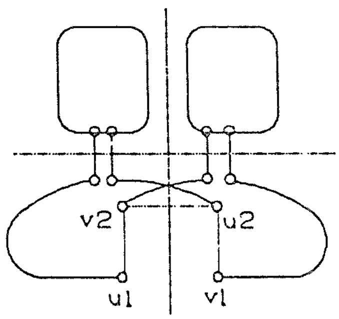

Fig. 4.17: Embedding the path $p=p\langle u1,\ u2,\ v2,\ v1\colon 2^n-1\rangle$ with $dist_p\ (u1,\ u2)=2^{n-1}$ and $dist_p\ (v1,\ v2)=2^{n-1}-2$.

Lemma 4.5: *Let $n\geq 2$, and let p be a* *path* $p\langle u1, u2, v2, v1{:}2^n-1\rangle$ *such that distances* $m1=dist_p(u1, u2)$ *and* $m3=dist_p(v1, v2)$ *are odd, and* $m1+m3<2^n$. *Then there exists an embedding of* p *into* Q_n *such that* $\rho(u1, u2)=\rho(v1, v2)=\rho(u1, v1)=1$.

Proof: The case of $n=2$ is trivial. Let $n\geq 3$ and assume the lemma holds for all $n'<n$. Let $m2=2^n-1-m1-m3$ (Fig. 4.18). Then $m2>0$ is odd. Without loss of generality, assume $m1\geq m3$.

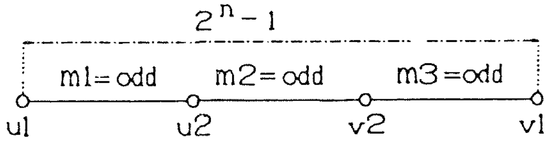

Fig. 4.18: Path $p=p\langle u1, u2, v2, v1{:}\ 2^n-1\rangle$ with $dist_p(u1, u2)$ and $dist_p(v1, v2)$ odd.

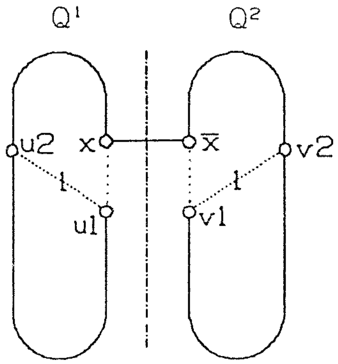

Fig. 4.19: Embedding the path $p=p\langle u1, u2, v2, v1{:}\ 2^n-1\rangle$ with $dist_p(u1, u2)$ and $dist_p(v1, v2)$ odd.

1. $m1>2^{n-1}$. The induction and expansion can be applied.

2. $m1\leq 2^{n-1}$. The construction is trivial by Corollary 3.2. Let $Q_n=Q^1||Q^2$, $u1\in V(Q^1)$, $v1=u1(Q^2)$, $x\in V(Q^1)$ be any neighbor of $u1$, $\bar{x}=x(Q^2)$ (Fig. 4.19). Let $p1=hp\langle u1, u2, x{:}Q^1\rangle$ such that $dist_{p1}(u1, u2)=m1$ and $\rho(u1, u2)=1$ (by Corollary 3.2.). Let $p2=hp\langle v1, v2, \bar{x}{:}Q^2\rangle$ such that $dist_{p2}(v1, v2)=m3$ and $\rho(v1, v2)=1$ (by Corollary 3.2.). Then $p=p1\cup p2\cup\langle x, \bar{x}\rangle$ is the required embedding. If $m1=2^{n-1}-1$, the construction is the same, only $u2=x$. Similarly, $m3=2^{n-1}-1$ implies $v2=\bar{x}$. □

Theorem 4.2: *Let* $n\geq 2$, *and let* p *be a path* $p\langle u1, v1{:}2^n-1\rangle$. *Let* $u2, v2$ *be vertices of* p *such that* $m1=dist_p(u1, u2)>0$ *and* $m3=dist_p(v1, v2)>0$ *are even,* $m1\geq m3$. *Then there exists an embedding of* p *into* Q_n *such that* $\rho(u1, u2)=\rho(v1, v2)=2$ *and* $\rho(u1, v1)=\rho(u1, v2)=\rho(v1, u2)=1$ *with the only exception for* $n\geq 3$, $m3=2$, *and* $m1=2^n-4$, *when* p *can be embedded only in such a way that* $\rho(u1, u2)=\rho(v1, v2)=2$, $\rho(u1, v1)=1$, *and either* $\rho(u1, v2)=1$ *and* $\rho(v1, u2)=3$ *or* $\rho(u1, v2)=3$ *and* $\rho(v1, u2)=1$.

Proof:

1. $m1+m3\leq 2^n-2$. Immediately from Lemma 4.3 and Lemma 4.4.

2. $m1+m3>2^n-2$. Immediately from Lemma 4.5, since any path $p=p\langle u1, v2, u2, v1{:}2^n-1\rangle$ with $m1=dist_p(u1, u2)>0$ and $m3=dist_p(v1, v2)>0$ such

that $m1$ and $m3$ are even and $m1+m3>2^n-2$, can be considered in the same time to be a path $p=p<u1, v2, u2, v1:2^n-1>$ with $m1'=dist_p(u1, v2)=2^n-1-m3$ and $m3'=dist_p(v1, u2)=2^n-1-m1$, where $m1'$ and $m3'$ are odd and $m1'+m3'<2^n$. □

Corollary 4.1: *Let $n\geq 2$, and let p be a path $p<u1, v1:2^n-1>$. Let $u2, v2$* be *vertices of p such that $m1=dist_p(u1, u2)>0$ and $m3=dist_p(v1, v2)>0$ are odd, $m1\geq m3$. Then there exists an embedding of p into Q_n such that $\rho(u1, v1)=\rho(u1, u2)=\rho(v1, v2)=1$, hence also $\rho(u1, v2)=\rho(v1, u2)=2$, with the only exception for $n\geq 3$, $m3=3$, and $m1=2^n-3$, when p can be embedded so that either $\rho(u1, v1)=\rho(v1, v2)=1$ and $\rho(u1, u2)=3$ or $\rho(u1, v1)=\rho(u1, u2)=1$ and $\rho(v1, v2)=3$.*

Because the Hamming dis tances between the end vertices of Hamiltonian paths of Q_n in Theorems 4. 1 and 4. 2 were always 1, these two theorems have the following corollaries:

Corollary 4. 2: *Let $n\geq 3$. Let c be any circuit $c<u, u1, u2:2^n>$ such that distances between u, $u1$ and $u2$ are even* (Fig. 4. 20). *Then there exists an embedding of c into Q_n such that $\rho(u, u1)=\rho(u, u2)=2$.*

Proof: Immediately from Theorem 4. 1.

Corollary 4.3: *Let $n\geq 2$. Let c be any circuit $c<u, u1, u2:2^n>$ such that distances between u and $u1$ are even and distances between u and $u2$ are odd* (Fig. 4.21). *Then there exists an embedding of c into Q_n such that $\rho(u, u1)=2$ and $\rho(u, u2)=1$.*

Proof: Immediately from Theorem 4.2.

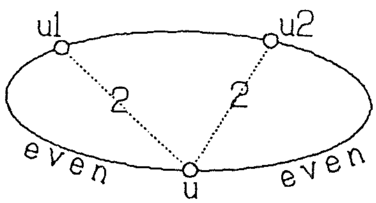

Fig. 4.20: Circuit $c<u, u1, u2: 2^n>$ with even distances between u, $u1$ and $u2$.

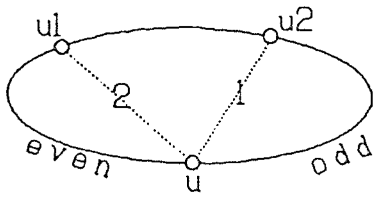

Fig. 4.21: Circuit $c<u, u1, u2: 2^n>$ with even distances between u and $u1$ and odd distances between u and $u2$.

Another corollary of Theorem 4.1 relates to the 3-quasistar. This result is not a new one, it was proved by Havel[2)] recently, but his proof was based on another idea.

Corollary 4.4: *Balanced 3-quasistar with 2^n vertices is a spanning tree of Q_n. Moreover, it can be embedded into Q_n so that the end vertices of even*

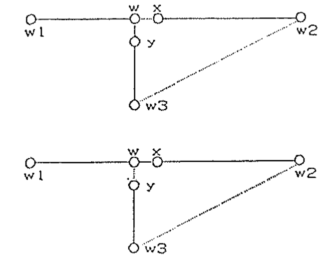

Fig. 4.22: Embedding the 3-quasistar.

rays have Hamming distance 2 *in* Q_n.

Proof: Let R_3 be a balanced 3-quasistar with 2^n vertices. Let w be its center, and $r1$, $r2$, $r3$ its two even rays and the odd ray, respectively. Let $w1, w2, w3$ be the end vertices of rays $r1, r2, r3$, respectively. Let x, y be the neighbors of center on the rays $r2$, $r3$ (Fig. 4.22.). Then the graph $R_3 \cup <w2, w3> - <w, x>$ is a path $p1 = p<w1, w, w3, w2, x: 2^n - 1>$ with odd distance between w and x and it can be embedded in Q_n so that $\rho(w, x) = 1$ (by Corollary 3.2). By removing $<w2, w3>$ and by adding $<w, x>$, we get the original quasistar. But there is another way, how to transform R_3 into a path. Let $p2 = R_3 \cup <w2, w3> - <w, y>$. Then $p2$ is a path $p<w1, w, w2, w3, y: 2^n - 1>$ with even $dist_{p2}(w1, w2)$ and odd $dist_{p2}(w, y)$ and by Theorem 4.1, there exists an embedding such that $\rho(w1, w2) = 2$ and $\rho(w, y) = 1$. □

5. Conclusions and further work

Our results did not exploit all the possibilities how to embed paths and circuits into the hypercubes. We can strengthen the constraints on the embeddings so that our results can be considered to be simpler cases of more complex theorems which are to be proved yet and can be formulated only as conjectures now.

Conjecture 1: *Let* $n \geq 4$, *and let* c *be a circuit* $c<u_0, u_1.., u_{n-2}: 2^n>$ *such that* $dist_c(u_0, u_1) > 1$ *and* $dist_c(u_0, u_{n-2}) > 1$ *are odd, and* $dist_c(u_i, u_{i+1}) > 1$ *are even for all* $i = 1, 2, .., n-3$. *Then* c *can be embedded in* Q_n *so that* $\rho(u_0, u_i) = 1$ *for all* $i = 1, 2, .., n-2$ (Fig. 5.1).

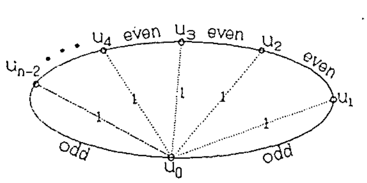

Fig. 5.1: Circuit $c<u_0, u_1, .., u_{n-2}: 2^n>$.

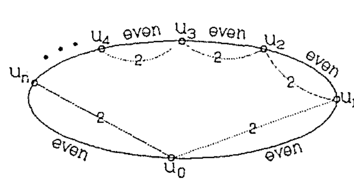

Fig. 5.2: Circuit $c<u_0, u_1, .., u_n: 2^n>$.

If proved, this conjecture has a corollary generalizing Corollary 4.2.

Conjecture 2: *Let* $n \geq 3$, *and let* c *be a circuit* $c<u_0, u_1, .., u_n: 2^n>$ *such that* *the distances between vertices* $u_0, u_1, .., u_n$ *are even. Then c can be embedded in* Q_n *so that* $\rho(u_i, u_{i+1})=2$ *for all* $i=1,2,.., n-1$ *and* $\rho(u_0, u_n)=2$ (Fig. 5. 2).

The solution of this problem seems to be a key to the proof of the conjecture[2] that a balanced n-quasistar with 2^n vertices is a spanning tree of Q_n.

References

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2) I. Havel, On Hamiltonian circuits and spanning trees of hypercubes, Casopis pro pestovani matematiky, 109, 135-154 (1984).

3) L. Nebesky, On quasistars in n-cubes, Casopis pro pestovani matematiky, 109, 153-156 (1984).