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Convex Functions and $\mathscr{G}$-Rearrangements on Intervals

by

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Two problems are dealt with : (1) Is a rearrangement of a convex function on $I\equiv(0, a)$ always a convex function ?; (2) Is it possible to give conditions for (1), if necessary ?

A new concept called generators on Lebesgue space is first introduced and the equimeasurable rearrangement $f^\frown$ of a function f with respect to a generator is defined, resulting generalization of the decreasing rearrangement f^* of f. Then the transmission function of a generator is defined, and the relation between the convexities of functions f and $f^\frown$ is studied. It is proved that f^* is convex whenever f is convex on I. Conditions for generators are obtained which make $f^\frown$ convex whenever f is convex.

1. Notations and Preliminaries

Throughout this paper, assume $a\in(0, \infty)$, let $X\equiv\{(0, a), \mathscr{B}, m\}$ be Lebesgue measure space on $I\equiv(0, a)$, and denote by $\mathscr{M}$ the set of all real valued measurable functions on X. A function f on an interval $K\subset\mathbb{R}\equiv(-\infty, \infty)$ is convex if and only if f satisfies the inequality

$$f(\alpha x+(1-\alpha)y)\leqq\alpha f(x)+(1-\alpha)f(y),\quad x,\ y,\ \in K,\ 0\leqq\alpha\leqq 1.$$

If f is a convex function on K, then $K_\lambda\equiv\{x: f(x)<\lambda\}\subset K$ is a convex set for each $\lambda\in\mathbb{R}$, that is, K_λ is an interval or empty set, and f is continuous on any open intervals contained in K. Moreover, g is a concave function on K if and only if $-g(x)$ is convex on K.

Generalizing the "Stratus" of Takeuchi[3] and the "Family $\mathscr{F}$" of Crowe and Zweibel,[2] we give the following definition:

DEFINITION 1. Assign a set $B(s)\in\mathscr{B}$ to each $s\in[0, a]$. If a family

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of sets

$$\mathscr{G} \equiv \{B(s): 0 \leqq s \leqq a\}$$

satisfies the following two conditions 1^0 and 2^0, then we call $\mathscr{G}$ a *generator on* X:

1^0 $B(s) \subset B(s')$ whenever $0 \leqq s < s' \leqq a$, and $B(a) = I$.

2^0 $m(B(s)) = s$ for every $0 \leqq s \leqq a$.

Moreover, to each $f \in \mathscr{M}$ assign a measurable function

$$f^{\wedge}(x) \equiv \sup\{s:\ x \in B(d_f(s))\},$$

where $d_f(t) \equiv m(\{x:\ f(x) > t\})$ is the *distribution function* of f. We call $f^{\wedge}$ the *equimeasurable rearrangement of* f *on* X *with respect to a generator* $\mathscr{G}$ (in short, *$\mathscr{G}$-rearrangement of* f).

It is well known that $d_f(t)$ is a decreasing and right continuous function of $t \in \mathbb{R}$ and that functions f and $f^{\wedge}$ satisfy $d_f = d_{f^{\wedge}}$. Such a pair of functions g and h having the same distribution function is said to be *equimeasurable* to each other (in symbols, $g \sim h$). Put $B(s) = (0,\ s)$, $0 \leqq s \leqq a$. Then it follows that

$$f^{\wedge}(x) = \sup\{t:\ d_f(t) > x\} \equiv f^*(x).$$

The above function f^* is the well known *decreasing rearrangement* of f, which is continuous from the right and decreasing on I. (See Chong and Rice[1] for details.)

2. Convex Functions and $\mathscr{G}$-Rearrangements

PROPOSITION 2. *Let* $\mathscr{G} = \{B(s):\ 0 \leqq s \leqq a\}$ *be a generator on* X. *Then,* $\mathscr{G}$*-rearrangement* $f^{\wedge}$ *of* $f \in \mathscr{M}$ *satisfiies the following* 1^0 *and* 2^0.

1^0 $\{x:\ f^{\wedge}(x) > s\} = B(d_f(s))$ *m-a.e., hence* $f^{\wedge} \sim f$.

2^0 *If* $0 \leqq s < s' \leqq a$, $y \in B(s)$ *and* $x \in B(s') \cap B(s)^c$, *then* $f^{\wedge}(x) \leqq f^{\wedge}(y)$.

Further, any functions $f^{\wedge} \in \mathscr{M}$ *satisfying the above* 1^0 *and* 2^0 *coincide with the* $\mathscr{G}$*-rearrangement of* f *m-a.e.*

Proof. The proof of the first paragraph is straightforward from the definition of $f^{\wedge}$, while that of the second paragraph is not so hard and is therefore omitted.

PROPOSITION 3. *Assume* $f \in \mathscr{M}$ *and put* $c=\text{ess.}\inf\{f(x)\colon x \in I\}$. *Then the following statements are true:*

(i) *If* f *is convex on* I, *then* d_f *is convex on* $[c, \infty)$.

(ii) d_f *is convex on* $[c, \infty)$ *if and only if* f^* *and* f_* *are convex on* I, *where* $f_*(x)\equiv f^*(a-x)$ $(x \in I)$ *is the increasing rearrangement of* f.

Proof. Proof of (i): Suppose f is convex on I, and c in the proposition is finite. It is true, in general, that $g(x)$ is convex on I if and only if $g(x)-c$ is convex on I. Further, $d_g(t)=d_{g-c}(t-c)$ on I and $(g-c)^*=g^*-c$. Therefore, considering the function $\{f(x)-c\}$, if necessary, we may suppose without loss of generality that $f(x)\geqq 0$ on I and that c in the proposition is 0. Suppose now that $f(x)\geqq 0$ and $c=0$. Then, there exist two non-negative convex functions f_1 and f_2 such that $f=f_1+f_2$, where f_1 is non-decreasing, f_2 is non-increasing, and $\text{ess.}\inf\{f_1(x)\colon x \in I\}=\text{ess.}\inf\{f_2(x)\colon x \in I\}=0$. But then, it follows that both d_{f_1} and d_{f_2} are convex on $[0, \infty)$ since convex functions f_1 and f_2 are monotone. Therefore, $d_f(t)=d_{f_1}(t)+d_{f_2}(t)$ is convex on $[0, \infty)$, which has completed the proof.

On the other hand, the statement (ii) is evident. Thus, the proof is completed.

DEFINITION 4. Let $\mathscr{G}=\{B(s)\colon 0\leqq s\leqq a\}$ be a generator on X. Define the *transmission function* $t(x)$ *of the generator* $\mathscr{G}$ by

$$t(x)=\inf\{s\colon x \in B(s),\ 0\leqq s\leqq a\}.$$

It is easy to see that the transmission function $t(x)$ satisfies

$$0\leqq t(x)\leqq a \text{ for any } x\in I.$$

LEMMA 5. *Assume* $f \in \mathscr{M}$ *and put* $c=\text{ess.}\inf\{f(x)\colon x \in I\}$. *If* d_f *is continuous on* $[c, \infty)$, *then it is true that*

$$t(x)=d_f(f^\wedge(x)) \text{ for any } x \in I \text{ such that } f^\wedge(x) \text{ is finite.}$$

Proof. It follows from the definition of $f^\wedge$ that there exists a $t_0 \in \mathbb{R}$ such that $f^\wedge(x)-\varepsilon<t_0$ and $x \in B(d_f(t_0))$ for any $\varepsilon>0$. Then, it follows from the definition of $t(x)$ that

$$t(x)\leqq d_f(t_0)\leqq d_f(f^\wedge(x)-\varepsilon),$$

since $d_f(t)$ is non-increasing. Suppose now the continuity of d_f on $[c, \infty)$. Then, it follows from the above inequality that

$$t(x) \leqq d_f(f^\wedge(x)), \quad x \in I.$$

Here, note that the assertion is clear if $d_f(f^\wedge(x))=0$, as seen from the above inequality. Therefore, it suffices to prove the assertion when $d_f(f^\wedge(x))>0$, which condition yields $t(x)>0$.

Suppose now that $0<t(x)<d_f(f^\wedge(x))$, and $f^\wedge(x)< f^*(t(x))$, with contradiction. Then, $x \notin B(d_f(f^*(t(x)))$ by the definition of $f^\wedge$. Hence it follows that

$$(1) \qquad d_f(f^*(x))<t(x),$$

by the definition of $t(x)$. But, since d_f is continuous on $[c, \infty)$, it is true that $t(x)=d_f(f^*(t(x))$, which is contradictory to (1). Thus, the assertion is true, which has completed the proof.

THEOREM 6. *Let $\mathscr{G}=\{B(s)$: $0 \leqq s \leqq a\}$ be a generator on X such that $B(s) \in \mathscr{G}$ is concave for any $0 \leqq s \leqq a$ and that the transmission function $t(x)$ is concave on I. Then,*

$f^\wedge(x)$ is convex on I whenever $f(x)$ is convex on I.

Moreover, it is true that

$f^\wedge(x)=f^*(t(x))$, $x \in I$, with a convention $f^*(a)=f^*(a-0)$.

Proof. First we claim that $f^\wedge(x)<\infty$ $(x \in I)$. Suppose $f^\wedge(x)=\infty$ for some $x \in I$, with contradiction. Then, there exists a sequence $\{s_n\}$ such that $s_n \uparrow \infty$ and $x \in B(d_f(s_n))$ for any natural numbers n. But, then $m(B(d_f(s_n)))=d_f(s_n) \to 0$, on our letting $n \to \infty$, which is contradictory to the assumption that $B(s)$ is concave for any $0 \leqq s \leqq a$.

Suppose now that f is convex. Then it follows that d_f is convex and hence continuous on (ess. inf$\{f(x)$: $x \in I\}$, $\infty)$, by Proposition 3. Therefore, it follows from Lemma 5 that

$$(2) \qquad t(x)=d_f(f^\wedge(x)) \text{ for any } x \in I.$$

Hence $f^*(t(x))=f^*(d_f(f^\wedge(x))=f^\wedge(x)$, that is,

$$(3) \qquad f^\wedge(x)=f^*(t(x)) \text{ for any } x \in I.$$

Then, (3) yields the assertion that $f^\wedge(x)$ is a convex function on I, since $t(x)$ is a concave function on I and $f^*(x)$ is a non-increasing convex function on I, by Proposition 3. Thus the proof is completed.

THEOREM 7. *Let $\mathscr{G}=\{B(s)$: $0 \leqq s \leqq a\}$ be a generator on X such that $B(s)$ is concave for every $0 \leqq s \leqq a$, and denote by $t(x)$ the transmission function of $\mathscr{G}$.*

Then, $t(x)$ *is concave on* I *if and only if* $B(s)=(0,\ ps] \cup (a-(1-p)s,\ a)$ m-*a.e. for some* $p \in [0,\ 1]$. *Moreover, if* $t(x)$ *is concave, then we can write down* $t(x)$ *as follows*:

(i) *If* $p=0$, *then* $t(x)=a-x$.

(ii) *If* $p=1$, *then* $t(x)=x$.

(iii) *If* $p\neq 0,\ 1$, *then*

$$t(x)=\begin{cases} x/p & (0<x<pa) \\ (x-a)/(p-1) & (pa\leqq x<a). \end{cases}$$

Proof. The proof is so easy to be omitted.

THEOREM 8. *Let* $\mathscr{G}=\{B(s): 0\leqq s\leqq a\}$ *be a generator on* X. *Then the following statements* (i) *and* (ii) *are equivalent to each other.*

(i) $f^{\wedge}(x)$ *is convex on* I *whenever* $f(x)$ *is convex on* I.

(ii) $B(s)=(0,\ ps] \cup (a-(1-p)s,\ a)$ m-*a.e. for some* $0\leqq p\leqq 1$.

Proof. Put $f(x)=x$. Then $f(x)$ is a convex function on I such that $f^{\wedge}(x) \sim x$. Suppose (i), and then it results in only the following three cases:

C-1 $$f^{\wedge}(x)=\begin{cases} a-x/p & (0<x\leqq pa) \\ (x-pa)/(1-p) & (pa<x<a) \end{cases}$$

for some $0\leqq p\leqq 1$.

C-2 $f^{\wedge}(x)=x$ on I.

C-3 $f^{\wedge}(x)=a-x$ on I.

Then, it follows that $B(s)=(0,\ ps] \cup (a-(1-p)s,\ a)$ m-a.e. for some $0\leqq p\leqq 1$, by Proposition 2.

Further, if $B(s)$ is defined as above, then $t(x)$ is concave, and then, by Theorem 7, it follows that $f^{\wedge}(x)$ is convex on I whenever $f(x)$ is convex on I. Thus the proof is completed.

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