Journal of the Faculty of Engineering, Shinshu University, No. 69, 1991
信州大学工学部紀要 第69号

A Theory of Finite Topology and Image Processing

Yatsuka NAKAMURA*, Yasushi FUWA** and Hiroshi IMURA***
(Received May 31, 1991)

By borrowing the concept of a neighbourhood from the theory of topological space in continuous cases and extending it to a discrete case such as a space of lattice points, we have defined such concepts as boundaries, closures, interiors, isolated points, and connected points as in the case of continuity. By associating each of these concepts with the various processes in image processing, we have shown examples of them using actual data. Also, if we consider these processes as transformations, we can obtain a number of topological transformations. By examining combinations of these transformations, we show how to reduce them into a single transformation.

We have also introduced concepts for speeding up neighbourhood calculations using a new concept called SDM.

1. Introduction

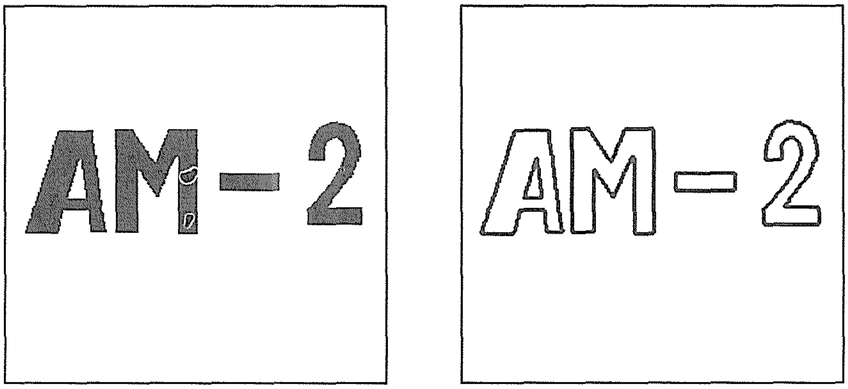
Topological geometry and the more generalized topological space theory or the theory of general topologies based on set theory can be said to be geometries for continuous spaces. For example, the continuity of a function can be defined by using the concept of a neighbourhood as described in the theory of general topologies.

However, in recent years, it is common for a space to be divided into a lattice or grid and treated as a discrete space to facilitate processing with a computer. In former theories of topology, discrete spaces were merely taken as sets of points. Therefore, all points were considered isolated points. However, even though spaces are divided, there is an adjacency relationship between points and there can be many types of entire space structures.

In the field of image processing, the concept of a neighbourhood, or processing template, is gaining use in the cases of lattice point spaces as well. However, this concept of a neighbourhood is different from that mentioned above in regards to a topology and as of yet has not been given any mathematical basis.

The authors have constructed a topology theory for the cases of discrete spaces and have given it the name of Finite Topology. Along with the presentation of a

* Professor, Department of Information Engineering.
** Assistant, Department of Information Engineering.
*** Student of master course.



A A^{∂}

Figure 1: Example of a boundary.

few additional concepts, we would like to test the validity of this theory based on actual image data. Finally, we examine the relationship between neighbourhood processing and logical operations as well as the relationship with calculation time.

2. Finite Topology Concepts and Neighbourhoods

Let X be a general set. If a subset $U(x)$ of X is determined for each element x in X, we call the pair $(X, U(\cdot))$ a finite topological space. The subset $U(x)$ is called a neighbourhood of x. In most cases, we assume $x \in U(x)$ (in this case, we call X *filled*), but it is not necessary in all cases. $U(x)$ is interpreted as a set of points neighbouring to x.

Using this notion of a neighbourhood, we can define many concepts.

Definition (Boundary). A *boundary* A^{∂} of a subset A of X is defined as:

$$A^{\partial} = \{x : U(x) \cap A \neq \phi \textit{ and } U(x) \cap A^{c} \neq \phi\},$$

where A^{c} is the complement of A and ϕ is the empty set.

In the field of image processing, the process of obtaining A^{∂} from A is called an extraction of the outline of A, which is shown in Figure 1.

We can divide A^{∂} into the following two parts - an inner boundary A^{∂_i} and an outer boundary A^{∂_o} which are defined as:

$$A^{\partial_i} = A \cap A^{\partial} \textit{ and } A^{\partial_o} = A^{c} \cap A^{\partial}.$$

Of course, we see from these definitions that $A^{\partial} = A^{\partial_i} \cup A^{\partial_o}$.

Definition (Interior). An *interior* A^i of a subset A of X is defined as:

$$A^i = \{x : x \in A \Rightarrow U(x) \subseteq A\}.$$

In image processing, the process of obtaining A^i is called a contraction of A.

Definition (Closure). A *closure* A^b of a subset A of X is defined as:

$$A^b = \{x : U(x) \cap A \neq \phi\}.$$

This concept is referred to as "expansion" in image processing.

Definition (Isolated Points). A set of *isolated points* A^s of a subset A of X is defined as:

$$A^s = \{x : x \in A \text{ and } (U(x) \setminus x) \cap A = \phi\}.$$

A point in A^s is referred to as "noise" in image processing. Conversely, we define a continuous part A^n of A as $A^n = A \setminus A^s$, which can also be written as:

$$A^n = \{x : x \in A \text{ and } (U(x) \setminus x) \cap A \neq \phi\}.$$

A^n is called a "noise elimination" of A in image processing.

Definition (Inflation). An *inflation* A^f of a subset A of X is defined as:

$$A^f = \{x : (\exists y)(y \in A, x \in U(y))\} = \bigcup_{y \in A} U(y).$$

This concept seems similar to "closure", but they do not always coincide. We present the following definition.

Definition (Symmetry). A neighbourhood $U(\cdot)$ of a finite topological space $(X, U(\cdot))$ is called *symmetric* if and only if:

$$\text{For all } x, y \in X, y \in U(x) \text{ implies } x \in U(y).$$

The neighbourhoods (a), (b), and (c) shown in Figure 2 are symmetric, but neighbourhoods (d), (e), and (f) are not.

By using the concept of symmetricity, we can give the following lemma.

Lemma 2.1. If a neighbourhood $U(\cdot)$ is symmetric, then for all subsets A:

$$A^b = A^f.$$

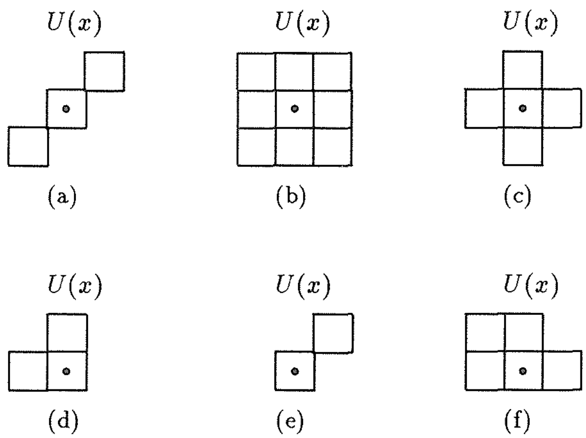


Figure 2: Symmetricity of Neighbourhoods.

Proof. Let us look at the following formulae.

$$\begin{aligned} x \in A^f &\Leftrightarrow \; for\ some\ y \in A, x \in U(y) \\ &\Leftrightarrow \; for\ some\ y \in A, y \in U(x)\ (by\ symmetricity) \\ &\Leftrightarrow \; U(x) \cap A \neq \phi \\ &\Leftrightarrow \; x \in A^b. \end{aligned}$$

Thus, we get the result.

Q.E.D.

The following two definitions are analogies of very popular concepts of openness and closedness in continuous topologies.

Definition (Open Set). A subset G of X is *open* if and only if:

$$G = G^i.$$

Definition (Closed Set). A subset F of X is *closed* if and only if:

$$F = F^b.$$

We present here a second type of closure and interior.

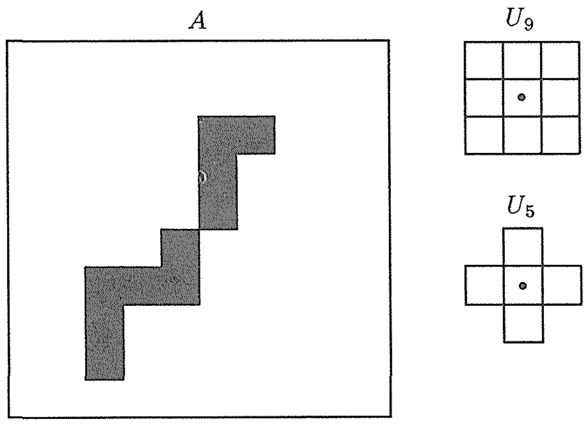


Figure 3: A is connected by neighbourhood U_9, but not by U_5.

Definition (f-closure and f-interior). An *f-closure* A^{f_b} and an *f-interior* A^{f_i} of a subset A of X is defined as:

$$A^{f_b} = \bigcap\{F : A \subseteq F, F \text{ is closed}\},$$

$$A^{f_i} = \bigcup\{G : G \subseteq A, G \text{ is open}\}.$$

We can show that an f-closure is gotten by a repetition of an ordinary closure (see Lemma 2.1 in Reference [1]) and an f-interior is gotten by a repetition of an ordinary interior.

Definition (Connected Set). A subset A of X is connected, if and only if, for any B, C in X:

$$A = B \cup C, B \neq \phi, C \neq \phi, \text{ and } B \cap C = \phi \text{ implies } B^b \cap C \neq \phi.$$

This concept coincides with the usual intuitive concept of connectivity. In Figure 3, the image A is connected by the neighbourhood U_9, but not by the neighbourhood U_5.

For connectivity, we present the following theorem.

Theorem 2.1. Let X be filled and finite (i.e., containing only a finite number of points). Then, a subset A of X is connected if and only if for every $x \in A$:

$$(\cdots((\{x\}^b \cap A)^b \cap A)^b \cdots)^b \supseteq A.$$

That is, by a finite process of taking closures of A, A is covered.

Proof. $\Rightarrow$) We assume that for some $x \in A$:

$$(\cdots((\{x\}^b \cap A)^b \cap A)^b \cdots)^b \not\supseteq A.$$

For convenience, we denote an intersection of A and an n-th closure as $P_n (= P_{n-1}^b \cap A, P_1 = \{x\}^b \cap A)$. As X is finite, we can assume that $P_{n+1} = P_n$. If we let

$$B = P_n, C = A \setminus B (= A \setminus P_n),$$

then $B \cup C = B \cup (A \setminus B) = A \cup B = A$ and $C \neq \phi$ (as $P_n \not\supseteq A$). Since X is filled, $x \in P_n$. Hence, $x \in B = P_n \neq \phi$. $B \cap C = \phi$ is clear. But,

$$B^b \cap C = P_n^b \cap C = P_n \cap (A \setminus P_n) = \phi.$$

Therefore, X is not connected.

$\Leftarrow$) Let B and C be non-void subsets of A such that $B \cap C = \phi$ and $B^b \cap C = \phi$. Then, there exists an element x in B, and we can construct a set P_n as a procedure described previously and $P_{n+1} = P_n$.

Let us show that $(B^b \cap A)^b \cap C = \phi$. If the left hand side is not empty, there exists some element z in it such that:

$$z \in C \text{ and } z \in (B^b \cap A)^b.$$

Thus, $U(z) \cap B^b \cap A \neq \phi$, where $B^b \cap A = B^b \cap (B \cup C) = (B^b \cap B) \cup (B^b \cap C) = B \cup B^b \cap C$. Hence, $U(z) \cap B^b \cap C \neq \phi$ (as $B \cap U(z) = \phi$ is clear), which contradicts with $B^b \cap C = \phi$. Thus, we can set $B = B^b \cap A$. If $P_i \subseteq B^b \cap A$, then $P_{i+1} \subseteq B^b \cap A$ because:

$$P_{i+1} = P_i^b \cap A \subseteq (B^b \cap A)^b \cap A = B^b \cap A.$$

Therefore, we obtain the result:

$$P_n \subseteq B^b \cap A = B^b \cap (B \cup C) = B,$$

i.e.,

$$P_n \not\supseteq A.$$

Q.E.D.

The following facts are easily derived:

1. $((A^c)^i)^c = A^b, ((A^c)^b)^c = A^i$.

2. $A^\partial = A^b \cap (A^c)^b, (A^c)^\partial = A^\partial$.

3. If $x \in A^s$, then $x \notin (A \setminus \{x\})^b$.
4. If $A^s \neq \phi$, then A is not connected.
5. If X is filled, then $A \subseteq A^b, A^i \subseteq A$.
6. If A is open, then A^c is closed. Conversely, if A is closed, then A^c is open.

3. Logic Functions and Neighbourhoods

Let us assume that the nature or concept of a point x in a finite topological space $(X, U(\cdot))$ is represented by a combination of predicative functions of the following type:

$$P(x, y_1, y_2, \cdots, y_n),$$

using qualifiers and logical operations, where $x \in X$ and variables $y_1, y_2, \cdots, y_n$ move over $U(x)(\ni y_1, y_2, \cdots, y_n)$. In such a case, we call the concept *local.*

For example, the definition of the boundary of A is written as:

$$x \in A^\partial \Leftrightarrow (\exists y_1)(P_1(x, y_1)) \wedge (\exists y_2)(P_2(x, y_2)),$$

where $P_1(x, y_1) = [y_1 \in U(x) \wedge y_1 \in A]$ and $P_2(x, y_2) = [y_2 \in U(x) \wedge y_2 \notin A]$.

The definition of the interior of A is written as:

$$x \in A^i \Leftrightarrow (\forall y_1)(P_0(x, y_1) \Rightarrow P_1(x, y_1)),$$

where $P_0(x, y_1) = [y_1 \in U(x)]$. The closure of A is written as:

$$x \in A^b \Leftrightarrow (\exists y_1)(P_1(x, y_1)).$$

Similarly,

$$x \in A^s \Leftrightarrow P_A(x) \wedge \neg(\exists y)(P_1(x, y) \wedge \neg P_e(x, y)),$$

where $P_A(x) = [x \in A]$ and $P_e(x, y) = [x = y]$.

Thus, "boundary", "interior", "closure", and "isolated points" are local concepts.

If $U(x)$ is a finite set for all $x \in X$, then the local concept is represented by a propositional logic over $U(x)$, i.e., by a formula of logical combinations of:

$$[x \in A], [y_1 \in A], [y_2 \in A], \cdots, [y_n \in A],$$

where y_i ranges over $U(x)$. Writing:

$$X = [x \in A] \text{ and } Y_i = [y_i \in A],$$

we see:

$$x \in A^{\partial} \Leftrightarrow (Y_1 \vee Y_2 \vee \cdots \vee Y_n) \wedge (\neg Y_1 \vee \neg Y_2 \vee \cdots \vee \neg Y_n).$$

In such a manner, we can represent a local concept by a formula of propositional logic over $U(x)$.

We can consider a topological concept more broadly by using propositional logic formulae over $U(x)$. For example, let us define:

$$N_i(x) = [Card(U(x)) = i],$$

where i is a number $1, 2, 3, \cdots$. Clearly, $N_i(x)$ can be represented by a propositional logic formula if $U(x)$ is finite. Let us define A^l by

$$x \in A^l \Leftrightarrow [x \in A] \wedge N_4(x) \vee N_3(x).$$

Then, A^l is a transformation of A, a so-called "life game", where the neighbourhood $U(x)$ is taken as U_9 in Figure 2(b). Thus, the life game transformation is also a general topological concept.

4. K-formulas and Neighbourhood Combinations

By using operations of set theory and transformations of topological concepts, we can make complicated modifications to some given sets $A, B, C, D, \cdots$, in following manner:

$$(A^b \cup (B^c)^i)^b \cap C^{\partial} \setminus D^s.$$

This type of formula was introduced by Kuratowski in a continuous case, so we call such a formula a K-formula.

A K-formula is useful in representing a process of modification in image processing in a simple way.

Topological concepts depend on a neighbourhood $U(x)$, so we write:

$$A^{b<U>}, A^{\partial<U>}, A^{i<U>},$$

for A^b, A^{∂}, and A^i, respectively. Other concepts are also written in this manner.

We can change a neighbourhood as follows. For example, if we consider $U_1(x)$, $U_2(x)$, $U_3(x)$, and $U_4(x)$ as shown in Figure 4, then the set of extreme points A^e of A is given as:

$$A^e = A^{\partial_i<U_1>} \cap A^{\partial_i<U_2>} \cap A^{\partial_i<U_3>} \cap A^{\partial_i<U_4>}.$$

Figure 5 is an example of image processing used to obtain a set of extreme points, which are called "corner points" in image processing.

$U_1(x)$ $U_2(x)$ $U_3(x)$ $U_4(x)$

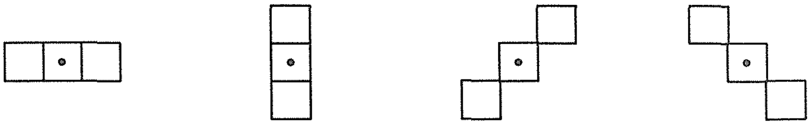
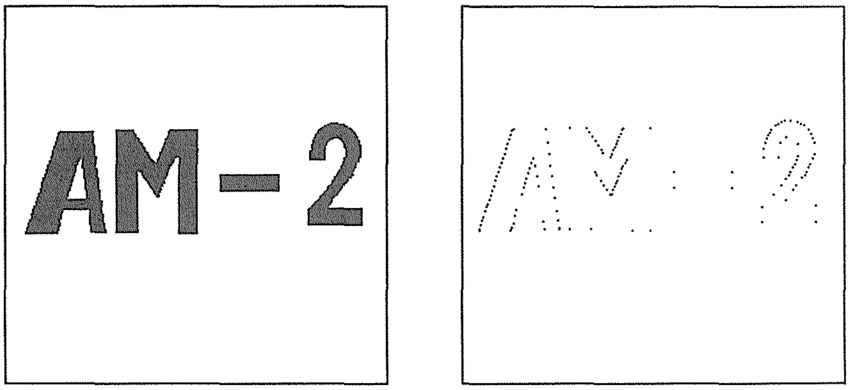


Figure 4: Neighbourhoods used to define extreme points.



A A^e

Figure 5: Extreme points A^e of original picture A.

Let us look at the relationship between a logic formula and a K-formula. Let φ_1 and φ_2 be logic formulae over the neighbourhoods U_1 and U_2, respectively. The topological concept $A^{\varphi_1<U_1>}$ is given by:

$$A^{\varphi_1<U_1>} = \{x : \varphi_1(x, y_1, y_2, \cdots, y_n)\,,\, y_i \in U_1(x) \;:\; i = 1, 2, \cdots, n\}.$$

Thus, by denoting $U = U_1 \cup U_2$, we obtain:

$$A^{\varphi_1<U_1>} \cup A^{\varphi_2<U_2>} = A^{(\varphi_1 \vee \varphi_2)<U>},$$

where $\varphi_1 \vee \varphi_2$ is a logic function generated by connecting φ_1 and φ_2 with the logical or-operation.

Other operations of set theory can be transformed to logical combinations of logic functions:

$$A^{\varphi_1<U_1>} \cap A^{\varphi_2<U_2>} = A^{(\varphi_1 \wedge \varphi_2)<U>},$$

$$(A^{\varphi_1<U_1>})^c = A^{\neg\varphi_1<U_1>},$$

$$A^{\varphi_1<U_1>} \setminus A^{\varphi_2<U_2>} = A^{(\varphi_1 \wedge \neg\varphi_2)<U>}.$$

A repetition of topological transformations becomes more complicated. Let us consider the following case:

$$(A^{\varphi_1<U_1>})^{\varphi_2<U_2>}.$$

This we can represent as:

$$A^{\varphi<U>}.$$

What is the relationship between φ, φ_1, and φ_2? U, U_1, and U_2? We can easily see that:

$$For\ all\ x \in X,\ \ U(x) = U_2(x)^{f<U_1>},$$

i.e., U is an inflation of U_2 with respect to the neighbourhood U_1. To see the form of φ, let us assume that:

$$x \in (A^{\varphi_1<U_1>})^{\varphi_2<U_2>}.$$

Then:

$$(\varphi_2([y_1 \in A^{\varphi_1<U_1>}], [y_2 \in A^{\varphi_1<U_1>}], \cdots, [y_n \in A^{\varphi_1<U_1>}])$$

is true, where $U_1(x) = \{y_1, y_2, \cdots, y_n\}$. The following is then clear:

$$[y_i \in A^{\varphi_1<U_1>}] \Leftrightarrow \varphi_1([z_1^{(i)} \in A], [z_2^{(i)} \in A], \cdots, [z_{m_i}^{(i)} \in A])$$

is true, where $U_2(y_i) = \{z_1^{(i)}, z_2^{(i)}, \cdots, z_{m_i}^{(i)}\}$. Clearly, $U(x) = \bigcup_{y \in U_1(x)} U_2(y) = \{z_1^{(1)}, z_2^{(1)}, \cdots, z_1^{(2)}, z_2^{(2)}, \cdots, z_{m_n}^{(n)}\}$. Then, $x \in (A^{\varphi_1<U_1>})^{\varphi_2<U_2>}$ is equivalent to:

$$For\ all\ y_i \in U_1(x),\ \ putting\ U_2(y_i) = \{z_k^{(i)} : k = 1, 2, \cdots, m_i\}$$

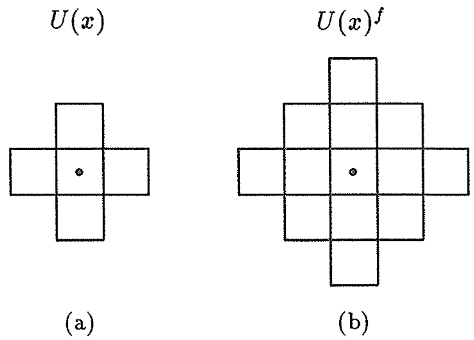


Figure 6: An inflation of $U(x)$ given in (a) is shown as $U(x)^f$ in (b). Taking a closure twice by (a) is the same as taking a closure once by (b).

$$\textit{and } u_i = \varphi_1([z_1^{(i)} \in A], [z_2^{(i)} \in A], \cdots, [z_{m_i}^{(i)} \in A]),$$
$$\varphi_2(u_1, u_2, \cdots, u_n) \textit{ consists.}$$

The above statement is also a logic function over $U(x)$, so we write it as $\varphi = \varphi_2 \otimes \varphi_1$, and we call it a convolution of φ_1 and φ_2.

Summing up the above, we get:

Theorem 4.1. For any two logic functions φ_1 and φ_2 over U_1 and U_2, respectively, and for any subset A:
$$(A^{\varphi_1<U_1>})^{\varphi_2<U_2>} = A^{\varphi<U>},$$
where $\varphi = \varphi_2 \otimes \varphi_1$ *and* $U(x) = U_2(x)^{f<U_1>}$ *(for all* $x \in X$*).*

Hence, we can easily see that:
$$(A^{b<U>})^{b<U>} = A^{b<U^f>},$$
where U^f is a neighbourhood given by each inflation of $U(x)$. This situation is explained in Figure 6.

5. Subspace Topology

Let $(X, U(\cdot))$ be a finite topological space, and let Y be a subset of X. If we set
$$U_Y(x) = U(x) \cap Y,$$
for $x \in Y$, then $(Y, U_Y(\cdot))$ is also a finite topological space. We call such a space, a subspace of X, a neighbourhood of which is a restriction of an original neighbourhood to Y. A finite space of 2-dimensional lattice points $\{1, 2, \cdots, m\} \times$

$\{1, 2, \cdots, n\}$ is a subspace of an infinite space consisting of 2-dimensional lattice points $\{\cdots, -1, 0, 1, 2, \cdots\}^2$ of some given neighbourhoods.

For subspaces, the validity of the following lemma is clear.

Lemma 5.1. Let $(X, U(\cdot))$ be a finite topological space and let A and Y be subsets of X such that $A \subseteq Y \subseteq X$. Then, A is open in X implies that A is also open in Y by the topology of a subspace.

Proof.

$$For\ all\ x \in A, U_Y(x) = U(x) \cap Y \subseteq A$$

is clear, as $U(x) \subseteq A$.

Q.E.D.

For boundaries, the converse is true:

Lemma 5.2. Let $(X, U(\cdot))$ be a finite topological space and let A and Y be subsets of X such that $A \subseteq Y \subseteq X$. Then, for $x \in Y, x \in A^{\partial<U_Y>}$ in Y (by the topology of a subspace) implies $x \in A^{\partial<U>}$.

Proof. The result is shown as follows:

$$U(x) \cap A \supseteq U(x) \cap Y \cap A = U_Y(x) \cap A \neq \phi$$

and

$$U(x) \cap A^c \supseteq U(x) \cap Y \cap A^c = U_Y(x) \cap A^c \neq \phi.$$

Q.E.D.

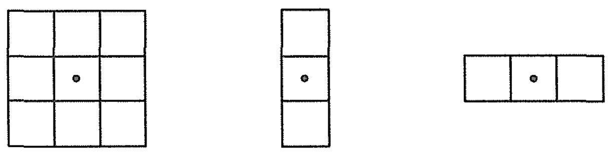
Thus, some concepts are hereditary from X to Y, and some from Y to X, while others do not exhibit a hereditary nature.

6. Computation

In general, neighbourhood processing is a time-consuming task. For example, we let the number of points in a neighbourhood $U(x)$ be $n(U)$ (each point x being the same) and the number of points in set A be $n(A)$. Then, if we calculate as usual whether or not subset A is open, the number of times we must access each pixel is:

$$n(A) \times n(U).$$

U U_1 U_2



(a) (b) (c)

Figure 7: U is an inflation of U_1 and U_2.

To obtain the closure of A, if we let U be a 3×3 neighbourhood as shown in Figure 7(a), U_1 a 1×3 neighbourhood as shown in Figure 7(b), and U_2 a 3×1 neighbourhood as shown in Figure 7(c), we can treat it as two levels:

$$A^{b<U>} = (A^{b<U_1>})^{b<U_2>}. \tag{1}$$

Then, the processing on the right-hand side becomes:

$$n(A^{b<U_1>}) \times n(U_2) + n(A) \times n(U_1) \simeq n(A) \times 3 + n(A) \times 3 = 6n(A),$$

and the processing on the left-hand side becomes:

$$n(A^{b<U>}) = n(A) \times n(U) = 9n(A).$$

In the case of (1) calculating with the right-hand side results in a fewer number of access times.

Estimated calculation loads vary according to the restrictions of hardware and software. If the hardware is well designed, parallel processing becomes possible using pipeline processing. As a concept for economizing calculation loads, we introduce the following Stacking and Driving Machine (SDM). In SDM, if we have some neighbourhood $U(x) = \{y_1, y_2, \cdots, y_n\}$, we let $u_i = [z_i \in A]$ and assign the following two types of mapping:

$$\begin{aligned} \gamma : (u_1, u_2, \cdots, u_n) &\longmapsto y_i \textit{ or } 0, \\ \delta : (u_1, u_2, \cdots, u_n) &\longmapsto y_{i1}, y_{i2}, \cdots, y_{im} \subseteq U(x). \end{aligned}$$

We call δ a stack mapping and γ a drive mapping. 0 is a special character for indicating a stop condition.

The neighbourhood prcoessing using (δ, γ) is given as follows.

1. Perform processing pertaining to $U(x_0)$.

2. Place the points $\delta(U(x_0))$ in a LIFO (stack).

3. Set $x_0 := \gamma(U(x_0))$. If x_0 is not 0, go to step 1.

4. Pull point y_i from the LIFO, set $x_0 := y_i$ and go to step 1.
If the LIFO is empty, stop.

If (δ, γ) are selected carefully, the task of obtaining A^{∂} or A^{fb} or determining whether or not A is connected becomes easy.

7. Conclusion

We have extended the concept of a neighbourhood so that it can be applied to discrete cases such as a space of lattice points. Based on this work, we have formalized the methods of image processing used thus far. We also introduced a new concept called SDM for neighbourhood calculations.

Finally, we would like to express gratitude to Ms. P. Kawamoto for her assistance with the English translation.

References

1) Y. Nakamura: Finite Topology Concept for Discrete Spaces:11th Seminar on Applied Functional Analysis(edited by H.Umegaki), 1988.

2) J. L. Kelly: General Topology, van Nostrand, 1955.

3) O.Ore: Theory of Graphs, American Math. Society, 1962.

4) A.Weil: Sur les espaces à structure uniforme et la topologie générale, Act. Sci. Indus.,551,1938.