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On the Homotopy Groups of Rotation Groups R_n

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1 Introduction

On the homotopy groups of rotation groups R_n, considerably many results have been obtained: In [7] and others, MIMURA determined groups $\pi_i(R_n)$ and their generators for $i \leqq 14$, and in [5] KERVAIRE determined the groups $\pi_i(R_n)$ for $i \geqq n+4$.

In the present paper, we shall determine the 2-primary components of groups $\pi_i(R_n)$, $i=15$, 16 and 17, together with their generators. For this purpose, we consider the homotopy exact sequence

$$\longrightarrow \pi_i(R_n) \xrightarrow{i_*} \pi_i(R_{n+1}) \xrightarrow{p_*} \pi_i(S^n) \longrightarrow \pi_{i-1}(R_n) \longrightarrow$$

of bundle (R_{n+1}, p, S^n), and J-homomorphism

$$J: \pi_i(R_n) \longrightarrow \pi_{n+i}(S^n).$$

Starting with R_3 which is homeomorphic to real projective 3-space, we obtain our results inductively. The auther wishes to thank to S. Saito for his advice throughout the preparation of the paper.

2 Preliminaries

For any fibre space (X, p, B), we have the following homotopy exact sequence

(2.1) $$\cdots\cdots \longrightarrow \pi_i(F) \xrightarrow{i_*} \pi_i(X) \xrightarrow{p_*} \pi_i(B) \xrightarrow{\Delta} \pi_{i-1}(F) \longrightarrow \cdots\cdots,$$

where F is the fibre $p^{-1}(x_0)$ on a base point x_0 of B, $i: F \longrightarrow X$ is the inclusion map and Δ is the boundary homomorphism. Homomorphisms i_*, p_* and Δ of (2.1) satisfy the following relation

(2.2)
$$\begin{aligned} i_*(\alpha\beta) &= i_*(\alpha)\beta && \text{for } \alpha \in \pi_j(F),\ \beta \in \pi_i(S^j), \\ p_*(\alpha\beta) &= p_*(\alpha)\beta && \text{for } \alpha \in \pi_j(X),\ \beta \in \pi_i(S^j), \\ \Delta(\alpha E\beta) &= \Delta(\alpha)\beta && \text{for } \alpha \in \pi_j(B),\ \beta \in \pi_{i-1}(S^{j-1}), \end{aligned}$$

where $E : \pi_i(S^j) \longrightarrow \pi_{i+1}(S^{j+1})$ is the suspension homomorphism.

Let R_{n+1} be the rotation group of euclidean (n+1)-space, and $i : R_n \longrightarrow R_{n+1}$ be the inclusion map. Then (R_{n+1}, p, S^n) is a fibre space with fibre R_n. Since the group R_3 is topologically equivalent to real projective 3-space P^3, $\pi_3(R_3) \approx Z$ with a generator $[\eta_2]$ (*cf.* [12]), and the correspondence $[\eta_2]\ \alpha \longrightarrow \eta_2\alpha$, $\alpha \in \pi_i(S^3)$, induces the homomorphism

$$p_* : \pi_i(R_3) \longrightarrow \pi_i(S^2), \tag{2.3}$$

which is an isomorphism for $i > 2$.

For $n = 3$ or 7, the bundle (R_{n+1}, p, S^n) is equivalent to product bundle $S^n \times R_n$ over S^n, and if we denote the homotopy class of the cross section of this bundle by $[\iota_n]$, the correspondence $(\alpha, \beta) \longrightarrow i_*\alpha + [\iota_n]\beta$, $\alpha \in \pi_i(R_n)$, $\beta \in \pi_i(S^n)$, yields an isomorphism

$$\pi_i(R_n) + \pi_i(S^n) \approx \pi_i(R_{n+1}). \tag{2.4}$$

Now, the notations of this paper conform to those of [13]; in particular, $\pi_i(X:2)$ denotes the 2-primary component of the group $\pi_i(X)$, and a subgroup π_i^n of $\pi_i(S^n)$ is defined by setting

$$\pi_i^n = \begin{cases} \pi_n(S^n) & \text{if } i = n, \\ E^{-1}\pi_{2n}(S^{n+1}:2)) & \text{if } i = 2n-1, \\ \pi_i(S^n:2) & \text{if } i \neq n,\ 2n-1. \end{cases} \tag{2.5}$$

Groups π_i^n and their generators are given in Table 1.

Applying (2.1) to the bundle (R_{n+1}, p, S^n), we have

$$\longrightarrow \pi_i(R_n:2) \xrightarrow{i_*} \pi_i(R_{n+1}:2) \xrightarrow{p_*} \pi_i^n \xrightarrow{\Delta} \pi_{i-1}(R_n:2) \longrightarrow \quad . \tag{2.6}$$

Let $[\alpha]$ denote an element of $\pi_i(R_{n+1}:2)$ such that $p_*([\alpha]) = \alpha \in \pi_i^n$, and let $j : R_{n+1} \longrightarrow R_m$, $m > n+1$, be the inclusion map, and define $[\alpha]_m \in \pi_i(R_m:2)$ by setting $[\alpha]_m = j_*([\alpha])$.

The groups $\pi_i(R_n:2)$ and their generators are known for $i \leqq 14$. We need them in the subsequent calculation, so we give them in the Table 2.

For the image of the boundary homomorphism $\Delta : \pi_i^n \longrightarrow \pi_{i-1}(R_n:2)$, we have the results given in the Table 3.

The homomorphism

$$\mathrm{J} : \pi_i(R_n) \longrightarrow \pi_{i+n}(S^n)$$

of G. W. WHITEHEAD was defined as follows :

Table 1 : π^n_{n+i}

for $n > i+1$

$i=$	0	1	2	3	4	5	6	7	8
π^n_{n+i}	Z	Z_2	Z_2	Z_8	0	0	Z_2	Z_{16}	Z_2 Z_2
generators	ι_n	η_n	η_n^2	ν_n			ν_n^2	σ_n	$\overline{\nu_n}$, ε_n

$n=$	2	3	4	5
π^n_{14}	Z_4 Z_2 Z_2	Z_4 Z_2 Z_2	Z_8 Z_4 Z_2	Z_2 Z_2 Z_2
generators	$\eta_2\mu'$, $\eta_2\nu'\overline{\nu_6}$, $\eta_2\nu'\varepsilon_6$	μ', $\varepsilon_3\nu_{11}$, $\nu'\varepsilon_6$	$\nu_4\sigma'$, $E\varepsilon'$, $\eta_4\varepsilon_5$	ν_5^3, μ_5, $\eta_5\varepsilon_6$
π^n_{15}	Z_2 Z_2	Z_2 Z_2	Z_2 Z_2 Z_2 Z_2 Z_2 Z_2	Z_8 Z_2
generators	$\eta_2\nu'\mu_6$, $\eta_2\nu'\eta_6\varepsilon_7$,	$\nu'\mu_6$, $\nu'\eta_6\varepsilon_7$	$\nu_4\sigma'\eta_{14}$, $\nu_4\varepsilon_7$, $\nu_4\overline{\nu_7}$, $\varepsilon_4\nu_{12}$, $(E\nu)'\varepsilon_7$, $E\mu'$	$\nu_5\sigma_8$, $\eta_5\mu_6$
π^n_{16}	Z_2	Z_2	Z_8 Z_2 Z_2 Z_2 Z_2 Z_2	Z_2 Z_2 Z_2
generators	$\eta_2\nu'\eta_6\mu_7$,	$\nu'\eta_6\mu_7$	$\nu_4\sigma'\eta_{14}^2$, ν_4^4, $\nu_4\mu_7$, $\nu_4\eta_7\varepsilon_8$, $(E\nu')\mu_7$, $E\nu'\eta_7\varepsilon_8$	ζ_5, $\nu_5\sigma_8$, $\nu_5\varepsilon_8$
π^n_{17}	Z_2	Z_2	Z_8 Z_2 Z_2	Z_2 Z_2 Z_2
generators	$\eta_2\varepsilon_3\nu_{11}^2$	$\varepsilon_3\nu_{11}^2$	$\nu_4^2\sigma_{10}$, $\nu_4\eta_7\mu_8$, $(E\nu)'\eta_7\mu_8$	ν_5^4, $\nu_5\mu_8$, $\nu_5\eta_8\varepsilon_9$
π^n_{18}	Z_2	Z_2	Z_8 Z_2 Z_2	Z_2 Z_2
generators	$\eta_2\overline{\varepsilon_3}$	$\overline{\varepsilon_3}$	$\nu_4\zeta_7$, $\varepsilon_4\nu_{12}$, $\nu_4\overline{\nu_7}\nu_{15}$	$\nu_5\sigma_8\nu_{15}$, $\nu_5\eta_8\mu_9$

$n=$	6	7	8	9
π^n_{14}	Z_8 Z_2	Z_8		
generators	$\overline{\nu_6}$, ε_6	σ'		
π^n_{15}	Z_2 Z_2 Z_2	Z_2 Z_2 Z_2	Z Z_8	
generators	ν_6^3, μ_6, $\eta_6\varepsilon_7$	$\sigma'\eta_{14}$, $\overline{\nu_7}$, ε_8	σ_8 $E\sigma'$	
π^n_{16}	Z_8 Z_2	Z_2 Z_2 Z_2 Z_2	Z_2 Z_2 Z_2 Z_2	
generators	$\nu_6\sigma_9$, $\eta_6\mu_7$	$\sigma'\eta_{14}^2$, ν_7^3, μ_7, $\eta_7\varepsilon_8$	$\sigma_8\eta_{15}$, $(E\sigma')\eta_{15}$, $\overline{\nu_8}$, ε_8	
π^n_{17}	Z_8 Z_2	Z_8 Z_2	Z_2 Z_2 Z_2 Z_2 Z_2	Z_2 Z_2 Z_2
generators	ζ_6, $\nu_6\nu_{14}$	$\nu_7\sigma_{10}$, $\eta_7\mu_8$	$\sigma_8\eta_{15}^2$, $(E\sigma')\eta_{15}^2$, ν_8^3, μ_8, $\eta_8\varepsilon_9$	$\sigma_9\eta_{16}$, ν_9, ε_8
π^n_{18}	Z_{16}	Z_8 Z_2	Z_8 Z_8 Z_2	Z_2 Z_2 Z_2 Z_2
generators	$\Delta(\sigma_{13})$	ζ_7, $\overline{\nu_7}\nu_{15}$	$\sigma_8\nu_{15}$, $\nu_8\sigma_{11}$, $\eta_8\mu_9$	$\sigma_9\eta_{16}^2$, ν_9^3, μ_9, $\eta_9\varepsilon_{10}$

Let $f: S^i \longrightarrow R_n$ be a representative of an element $\pi_i(R_n)$. Define a mapping $F: S^i \times S^{n-1} \longrightarrow S^{n-1}$ by setting

$$F(x, y) = f(x)y \qquad \text{for any } x \in S^i \text{ and } y \in S^{n-1}.$$

Let $G(F): S^{n+i} \approx S^i * S^{n-1} \longrightarrow S^n$ be the Hopf-construction of F, where $A * B$ denotes the join of A and B. Then, $G(F)$ represents an element $\mathrm{J}(\alpha) \in \pi_{i+n}(S^n)$.

We have a diagram

(2.7)
$$\begin{array}{ccccccccc}
\cdots\cdots\longrightarrow & \pi_i(R_n:2) & \xrightarrow{i_*} & \pi_i(R_{n+1}:2) & \xrightarrow{p_*} & \pi_i^n & \xrightarrow{\Delta} & \pi_{i-1}(R_n:2) & \longrightarrow\cdots\cdots \\
 & \downarrow \mathrm{J} & & \downarrow \mathrm{J} & & \downarrow \mathrm{E}^{n+1} & & \downarrow \mathrm{J} & \\
\cdots\cdots\longrightarrow & \pi^n_{i+n} & \xrightarrow{E} & \pi^{n+1}_{i+n+1} & \xrightarrow{H} & \pi^{2n+1}_{i+n+1} & \xrightarrow{\Delta} & \pi^n_{i+n+1} & \longrightarrow\cdots\cdots
\end{array}$$

which is commutative up to sign, and its lower sequence is exact ([11], Proposition 4.2). Moreover, the homomorphism $\mathrm{J}: \pi_i(R_n) \longrightarrow \pi_{i+n}(S^n)$ satisfies

(2.8) $$\mathrm{J}(\alpha\beta) = \mathrm{J}(\alpha) E^n \beta \qquad \text{for any } \alpha \in \pi_j(R_n) \text{ and } \beta \in \pi_i(S^j).$$

Recall that

(2.9) $$\mathrm{J}([\iota_7]_n) = \sigma_n \qquad \text{for } n > 8.$$
$$\mathrm{J}([\nu_5]_n) = \nu_n \qquad \text{for } n > 8.$$

3 Groups $\pi_{15}(R_n)$ and their generators

In this section, we shall determine the generators of the 2-primary components of $\pi_{15}(R_n)$.

In the sequel, we shall use the abbreviated notation $\pi_i(R_n)$ for $\pi_i(R_n:2)$

The homotopy groups of spinor groups $\boldsymbol{Spin}\ (n)$, $n \leqq 9$, are given in [7] and [9], and we have isomorphisms

(3.1) $$\pi_i(R_5) \approx \pi_i(\boldsymbol{Spin}(5)) \approx \pi_i(\boldsymbol{Sp}(2)),$$

(3.2) $$\pi_i(R_6) \approx \pi_i(\boldsymbol{Spin}(6)) \approx \pi_i(\boldsymbol{SU}(4)).$$

The results for $\pi_{15}(R_n:2)$ are stated as follows:

Proposition 2.1. $\pi_{15}(R_3:2) = \{[\eta_2]\nu'\mu_6\} + \{[\eta_2]\nu'\eta_6\varepsilon_7\} \approx \boldsymbol{Z}_2 + \boldsymbol{Z}_2$

$\pi_{15}(R_4:2) = \{[\eta_2]_4\nu'\mu_6\} + \{[\eta_2]_4\nu'\eta_6\varepsilon_7\} + \{[\iota_3]\nu'\mu_6\} + \{[\iota_3]\nu'\eta_6\varepsilon_7\} \approx \boldsymbol{Z}_2 + \boldsymbol{Z}_2 + \boldsymbol{Z}_2 + \boldsymbol{Z}_2$

$\pi_{15}(R_5:2) = \{[\nu_4\sigma'\eta_{14}]\} \approx \boldsymbol{Z}_2$

$\pi_{15}(R_6:2) = \{[\nu_4\sigma'\eta_{14}]_6\} + \{[\nu_5]\sigma_8\} \approx \boldsymbol{Z}_2 + \boldsymbol{Z}_8$

$\pi_{15}(R_7:2) = \{[\nu_4\sigma'\eta_{14}]_7\} + \{[\nu_5]_7\sigma_8\} + \{[\eta_6]\bar{\nu}_7\} + \{[\eta_6]\varepsilon_7\} \approx \boldsymbol{Z}_2 + \boldsymbol{Z}_2 + \boldsymbol{Z}_2 + \boldsymbol{Z}_2$

$\pi_{15}(R_8:2) = \{[\nu_4\sigma'\eta_{14}]_8 + \{[\nu_5]_8\sigma_8\} + \{[\eta_6]_8\bar{\nu}_7\} + \{[\eta_6]_8\varepsilon_7\} + \{[\iota_7]\sigma'\eta_{14}\} + \{[\iota_7]\bar{\nu}_7\} + \{[\iota_7]\varepsilon_7\}$
$\approx \boldsymbol{Z}_2 + \boldsymbol{Z}_2 + \boldsymbol{Z}_2 + \boldsymbol{Z}_2 + \boldsymbol{Z}_2 + \boldsymbol{Z}_2 + \boldsymbol{Z}_2$

$\pi_{15}(R_9:2) = \{[8\sigma_8]\} + \{[\iota_7]_9\bar{\nu}_7\} + \{[\iota_7]_9\varepsilon_7\} + \{[\nu_5]_9\sigma_8\} \approx \boldsymbol{Z} + \boldsymbol{Z}_2 + \boldsymbol{Z}_2 + \boldsymbol{Z}_2$

$\pi_{15}(R_{10}:2) = \{[8\sigma_8]_{10}\} + \{[\iota_7]_{10}\bar{\nu}_7\} + \{[\nu_5]_{10}\sigma_8\} \approx \boldsymbol{Z} + \boldsymbol{Z}_2 + \boldsymbol{Z}_2$

$\pi_{15}(R_n:2) = \{[8\sigma_8]_n\} + \{[\nu_5]_n\sigma_8\} \approx \boldsymbol{Z} + \boldsymbol{Z}_2$ *for* $n = 11,\ 12$

$\pi_{15}(R_{13}:2) = \{[\eta^3_{12}]\} + \{]\nu_5]_{13}\sigma_8\} \approx \boldsymbol{Z} + \boldsymbol{Z}_2$

$\pi_{15}(R_n:2) = \{[\eta^3_{13}]_n\} \approx \boldsymbol{Z}$ *for* $n = 14,\ 15$

$\pi_{15}(R_{16}:2)=\{[\eta_{12}^3]_{16}\}+[2\iota_{15}]\approx \boldsymbol{Z}+\boldsymbol{Z}$

$\pi_{15}(R_n:2)=\{[\eta_{12}^3]_n\}\approx \boldsymbol{Z}$ *for* $n\geqq 17$.

The following relation hold : $[8\sigma_{10}]_{13}=2[\eta_{12}^3]$.

Proof. The reults for $\pi_{15}(R_3)$ and $\pi_{15}(R_4)$ follow diectly from (2. 3), (2. 4) and Table 1.

R_5 : Since $\pi_7(R_5)\approx \boldsymbol{Z}$ and $\pi_7(R_4)$ is finite,

(3. 3) $i_*:\pi_7(R_4)\longrightarrow\pi_7(R_5)$ *is trivial, i. e.,* $i_*([\eta_2]_4\nu')=i_*([\tau_3]\nu')=0$.

Therefore $i_*:\pi_{15}(R_4)\longrightarrow\pi_{15}(R_5)$ is trivial, too. From this and (2. 6), we have the exact sequence

$$0\longrightarrow\pi_{15}(R_5)\overset{p_*}{\longrightarrow}\pi_{15}^4\overset{\Delta}{\longrightarrow}\pi_{14}(R_4).$$

Using Table 2 and 3, we can prove that the kernel of $\Delta:\pi_{15}^4\longrightarrow\pi_{14}(R_4)$ is generated by $\nu_4\sigma'\eta_{14}$.

In fact, from Table 1, $\pi_{15}^4=\{\nu_4\sigma'\eta_{14}\}+\{\nu_4\bar{\nu}_7\}+\{\nu_4\varepsilon_7\}+\{\varepsilon_4\nu_{12}\}+\{E\nu'\bar{\nu}_7\}+\{E\mu'\}$ $\approx \boldsymbol{Z}_2+\boldsymbol{Z}_2+\boldsymbol{Z}_2+\boldsymbol{Z}_2+\boldsymbol{Z}_2+\boldsymbol{Z}_4$. Then we have

$$\begin{aligned}\Delta(\nu_4\sigma'\eta_{14})&=\Delta(\nu_4\sigma')\eta_{13} &&\text{by (2. 2),}\\&=2([\iota_3]\varepsilon')\eta_{13}&&\text{by Table 3,}\\&=2([\iota_3]\varepsilon'\eta_{13})=0.\end{aligned}$$

$$\Delta(\nu_4\bar{\nu}_7)=\Delta(\nu_4)\bar{\nu}_6=[\iota_3]\nu'\bar{\nu}_6+a[\eta_2]_4\nu'\bar{\nu}_6.$$

$$\Delta(\nu_4\varepsilon_7)=\Delta(\nu_4)\varepsilon_6=[\iota_3]\nu'\varepsilon_6+a[\eta_2]_4\nu'\varepsilon_6.$$

$$\Delta(\varepsilon_4\nu_{12})=\Delta(\iota_4)\varepsilon_3\nu_{11}=2[\iota_3]\varepsilon_3\nu_{11}-[\eta_2]_4\varepsilon_3\nu_{11}=[\eta_2]_4\varepsilon_3\nu_{11}.$$

$$\Delta(E\nu'\varepsilon_7)=\Delta(\iota_4)\varepsilon_6=[\eta_2]_4\varepsilon_6.$$

$$\Delta(E\mu')=\Delta(\tau_4)\eta'=2[\iota_3]\mu'-[\eta_2]_4\mu'.$$

Thus, from the above exact sequence and by definition of $[\nu_4\sigma'\eta_{14}]$, we have

$$\pi_{15}(R_5)=\{[\nu_4\sigma'\eta_{14}]\}\approx \boldsymbol{Z}_2.$$

Let $(2. 6)_n$ denote a part of the exact sequence (2. 6) starting with π_{16}^n and ending in $\pi_{14}(R_n)$, i. e.,

$(2. 6)_n$ $$\pi_{16}^n\overset{\Delta}{\longrightarrow}\pi_{15}(R_n)\overset{i_*}{\longrightarrow}\pi_{15}(R_{n+1})\overset{p_*}{\longrightarrow}\pi_{15}^n\overset{\Delta}{\longrightarrow}\pi_{14}(R_n).$$

R_6 : Consider $(2. 6)_5$. Since $[\iota_3]_5\mu'=4[2\nu_4\sigma']$ in $\pi_{14}(R_5)$, using Table 3 we have

$$\begin{aligned}\Delta(\eta_5\mu_6)&=\Delta(\iota_5)\eta_4\mu_5&&\text{by (2. 2),}\\&=[\iota_3]_5\eta_3^2\mu_5&&\text{by Table 3,}\\&=2[\iota_3]\mu'&&\text{by (7. 7) of [13],}\\&=8[2\nu_4\sigma']\neq 0.\end{aligned}$$

Table 2 : $\pi_i\ (R_n{:}2)$ *for* $3 \leqq i \leqq 14$

$n=$	3	4	5	6	7
$\pi_3\ (R_n{:}2)$	Z	$Z \quad Z$	Z		
generators	$[\eta_2]$	$[\eta_2]_4 \quad [\iota_3]$	$[\iota_3]_5$		
$\pi_4\ (R_n{:}2)$	Z_2	$Z_2 \quad Z_2$	Z_2	0	
generators	$[\eta_2]\eta_3$	$[\eta_2]_4\eta_3 \quad [\iota_3]\eta_3$	$[\iota_3]_5\eta_3$		
$\pi_5\ (R_n{:}2)$	Z_2	$Z_2 \quad Z_2$	Z_2	Z	0
generators	$[\eta_2]\eta_3^2$	$[\eta_2]_4\eta_3^2 \quad [\iota_3]\eta_3^2$	$[\iota_3]_5\eta_3^2$	$[2\iota_5]$	
$\pi_6\ (R_n{:}2)$	Z_4	$Z_4 \quad Z_4$	0	0	0
generators	$[\eta_2]\nu'$	$[\eta_2]_4\nu' \quad [\iota_3]\nu'$			
$\pi_7\ (R_n{:}2)$	Z_2	$Z_2 \quad Z_2$	Z	Z	Z
generators	$[\eta_2]\nu'\eta_6$	$[\eta_2]_4\nu'\eta_6,\ [\iota_3]\nu'\eta_6$	$[4\nu_4]$	$[\eta_5^2]$	$[\eta_6]$
$\pi_8\ (R_n{:}2)$	Z_2	$Z_2 \quad Z_2$	0	Z_8	$Z_2 \quad Z_2$
generators	$\eta_2\nu'\eta_6^2$	$[\eta_2]_4\nu'\eta_6^2, \quad [\iota_3]\nu'\eta_6^2$		$[\nu_5]$	$[\nu_5]_7, \quad [\eta_6]\eta_7$
$\pi_9\ (R_n{:}2)$	0	0	0	Z_2	$Z_2 \quad Z_2$
generators				$[\nu_5]\eta_8$	$[\nu_5]_7\eta_8, \quad [\eta_6]\eta_7^2$
$\pi_{10}\ (R_n{:}2)$	0	0	Z_8	$Z_8 \quad Z_2$	Z_8
generators			$[\nu_4^2]$	$[\nu_4^2]_6, \quad [\nu_5]\eta_8^2$	$[\nu_4^2]_7,$

$n=$	8	9	10	11	12
$\pi_6\ (R_n{:}2)$	0				
generators					
$\pi_7\ (R_n{:}2)$	$Z \quad Z$	Z			
generators	$[\eta_6]_8, \quad [\iota_7]$	$[\iota_7]_9$			
$\pi_8\ (R_n{:}2)$	$Z_2 \quad Z_2 \quad Z_2$	$Z_2 \quad Z_2$	Z_2		
generators	$[\nu_5]_8,\ [\eta_6]_8\eta_7,\ [\iota]_7\eta_7$	$[\nu_5]_9,\ [\iota_7]_9\eta_7$	$[\iota_7]_{10}\eta_7$		
$\pi_9\ (R_n{:}2)$	$Z_2 \quad Z_2 \quad Z_2$	$Z_2 \quad Z_2$	$Z_2 \quad Z$	Z_2	
generators	$[\nu_5]_8\eta_8,\ [\eta_6]_8\eta_7^2,\ [\iota_7]\eta_7^2$	$[\nu_5]_9\eta_8,\ [\iota_7]_9\eta_7^2$	$[\nu_5]_{10}\eta_8, \quad [2\iota_9]$	$[\nu_5]_{11}\eta_8=[\iota_7]_{11}\eta_7^2$	
$\pi_{10}\ (R_n{:}2)$	$Z_8 \quad Z_8$	Z_8	Z_4	Z_2	0
generators	$[\nu_4^2]_8, \quad [\iota_7]\nu_7$	$[\iota_7]_9\nu_7$	$[\iota_7]_{10}\nu_7$	$[\iota_7]_{11}\nu_7$	

$n=$	3	4
$\pi_{11}(R_n:2)$	Z_2	Z_2 Z_2
generators	$[\eta_2]\varepsilon_3$	$[\eta_2]_4\varepsilon_3$, $[\iota_3]\varepsilon_3$
$\pi_{12}(R_n:2)$	Z_2 Z_2	Z_2 Z_2 Z_2 Z_2
generators	$[\eta_2]\eta_3\varepsilon_4$, $[\eta_2]\mu_3$	$[\eta_2]_4\eta_3\varepsilon_4$, $[\eta_2]_4\mu_3$, $[\iota_3]\eta_3\varepsilon_4$, $[\iota_3]\mu_3$
$\pi_{13}(R_n:2)$	Z_4 Z_2	Z_4 Z_2 Z_4 Z_2
generators	$[\eta_2]\varepsilon'$, $[\eta_2]\eta_3\mu_4$	$[\eta_2]_4\varepsilon'$, $[\eta_2]_4\eta_3\mu_4$, $[\iota_3]\varepsilon'$, $[\iota_3]\eta_3\mu_4$
$\pi_{14}(R:2)$	Z_4 Z_2 Z_2	Z_4 Z_2 Z_2 Z_4 Z_2 Z_2
generators	$[\eta_2]\mu'$, $[\eta_2]\nu'\bar{\nu}_6$, $[\eta_2]\nu'\varepsilon_6$	$[\eta_2]_4\mu'$, $[\eta_2]_4\nu'\bar{\nu}_6$, $[\eta_2]_4\nu'\varepsilon_6$, $[\iota_3]\mu'$, $[\iota_3]\varepsilon_3\nu_{11}$, $[\iota_3]\nu'\varepsilon_6$

$n=$	5	6	7	8
$\pi_{11}(R_n:2)$	Z_2	Z_4	Z_2 Z	Z_2 Z
generators	$[\iota_3]_5\varepsilon_3$	$[\nu_5]\sigma_8$	$[\nu_5]_7\nu_8$, $[2\Delta\iota_{13}]$	$[\nu_5]_8\nu_8$, $[2\Delta\iota_{13}]_8$
$\pi_{12}(R_n:2)$	Z_2 Z_2	Z_4	0	0
generators	$[\iota_3]_5\eta_3\varepsilon_4$, $[\iota_3]_5\mu_3$	$\Delta\sigma''$		
$\pi_{13}(R_n:2)$	Z_4 Z_2	Z_4	Z_2	Z_2 Z_2
generators	$[\nu_4^2]\nu_{10}$, $[\iota_3]_5\eta_3\mu_4$	$[\nu_4^2]_6\nu_{10}$	$[\nu_4^2]_7\nu_{10}$	$[\nu_4^2]_8\nu_{10}$, $[\iota_7]\nu_7^2$
$\pi_{14}(R_n:2)$	Z_{16}	Z_{16} Z_2	Z_8 Z_8 Z_2	Z_8 Z_8 Z_2 Z_8
generators	$[2\nu_4\sigma']$	$[\eta_5\varepsilon_6]$, $[\nu_5]\nu_8^2$	$[\eta_5\varepsilon_6]_7$, $[\nu_6+\varepsilon_6]$, $[\nu_5]_7\nu_8^2$	$[\eta_5\varepsilon_6]_8$, $[\nu_6+\varepsilon_6]_8$, $[\nu_5]_8\nu_8^2$, $[\iota_7]\sigma'$

$n=$	9	10	11	12	13
$\pi_{11}(R_n:2)$	Z_2 Z	Z	Z	Z Z	Z
generators	$[\nu_5]_9\nu_8$, $[2\Delta\iota_{13}]_9$	$[2\Delta\iota_{13}]_{10}$	$[2\Delta\iota_{13}]_{11}$	$[2\Delta\iota_{13}]_{12}$, $[2\iota_{11}]$	$[2\Delta\iota_{13}]_{13}$
$\pi_{12}(R_n:2)$	0	Z_4	Z_2	Z_2 Z_2	Z_2
generators		$[2\nu_9]$	$[\eta_{10}^2]$	$[\eta_{10}^2]_{12}$, $[\eta_{11}]$	$[\eta_{11}]_{13}$
$\pi_{13}(R_n:2)$	Z_2	Z_2	Z_2 Z_2	Z_2 Z_2	Z_2
generators	$[\iota_7]_9\nu_7^2$	$[\iota_7]_{10}\nu_7^2$	$[\iota_7]_{11}\nu_7^2$, $[\eta_{10}^2]\eta_{12}$	$[\eta_{10}^2]_{12}\eta_{12}$, $[\eta_{11}]\eta_{12}$	$[\eta_{11}]_{13}\eta_{12}$
$\pi_{14}(R_n:2)$	Z_8 Z_2	Z_8	Z_8	Z_8 Z_4	Z_8
generators	$[\nu_6+\varepsilon_6]_9$, $[\nu_5]_9\nu_8^2$	$[\nu_6+\varepsilon_6]_{10}$	$[\nu_6+\varepsilon_6]_{11}$	$[\nu_6+\varepsilon_6]_{12}$, $\Delta\nu_{12}$	$[\nu_6+\varepsilon_6]_{13}$

$n=$	14	15	16
$\pi_{12}(R_n:2)$	0	0	
generators			
$\pi_{13}(R_n:2)$	Z	0	0
generators	$[2\iota_{13}]$		
$\pi_{14}(R_n:2)$	Z_4	Z_2	0
generators	$[\nu_6+\varepsilon_6]_{14}$	$[\nu_6+\varepsilon_6]_{15}$	

Table 3 :

$\alpha \in \pi_i^n$	ι_4	ι_5	ι_8	ι_9
$\Delta\alpha$	$2[\iota_3] - [\eta_2]_4$	$[\iota_3]_5\eta_3$	$2[\iota_7] - [\eta_6]_8$	$[\nu_5]_9 + [\iota_7]_9\eta_7$

$\alpha \in \pi_i^n$	ι_{11}	ι_{13}	ι_{15}	ν_5	η_6	ν_4^2	ν_8
$\Delta\alpha$	$[\iota_7]_{11}\nu_7$	$[\eta_{11}]_{13}$	$[\nu_6+\varepsilon_6]_5$	0	0	0	$2[\iota_7]\nu_7 - [\nu_4^2]$

$\alpha \in \pi_i^n$	$\nu_4\sigma'$	σ_8	$E\sigma'$	ν_6
$\Delta\alpha$	$2[\iota_3]\varepsilon'$	$[\iota_7]\sigma' - [\nu_6 + \varepsilon_6]$	$2[\iota_7]\sigma' - [\eta_5\varepsilon_6]$	$2[\nu_5]$

$\alpha \in \pi_i^n$	ν_4
$\Delta\alpha$	$[\iota_3]\nu' + a[\eta_2]_4\nu'$ *for* $0 \leqq a \leqq 3$

From Table 2, there exists an element $[\nu_5]\sigma_8 \in \pi_{15}(R_6)$ such that $p_*([\nu_5]\sigma_8) = \nu_5\sigma_8$. Since $[\nu_5]$ is of order 8, $[\nu_5]\ \sigma_8$ is of order 8. On the other hand, by (3.2) and Theorem 6.1 of [7], $\pi_{15}(R_6) \approx \boldsymbol{Z}_8 + \boldsymbol{Z}_2$. Thus, from $(2.6)_5$, we have

$$\pi_{15}(R_6) = \{[\nu_5]\sigma_8\} + \{[\nu_4\sigma'\eta_{14}]_6\} \approx \boldsymbol{Z}_8 + \boldsymbol{Z}_2$$

and

(3.4) $\qquad i_* : \pi_{15}(R_5) \longrightarrow \pi_{15}(R_6)$ *is a monomorphism.*

R_7 : Consider $(2.6)_6$. From Table 1 and 3, we have relations;

$$\Delta(\eta_6\varepsilon_7) = \Delta(\nu^3{}_6 = \eta_6\bar{\nu}_7) = 0 \qquad \text{by } \Delta(\eta_6) = 0,$$
$$\Delta(\mu_6) \neq 0 \qquad \text{by Table 2.}$$

(3.5)
$$\Delta(\nu_6\sigma_9) = \Delta(\nu_6)\sigma_8 = 2[\nu_5]\sigma_8 \qquad \text{by Table 3 and (2.2),}$$
$$\Delta(\eta_6\mu_7) = 0 \qquad \text{by Table 3.}$$

Therefore, from $(2.6)_6$, we have the following exact sequence

$$0 \longrightarrow \{[\nu_5]_6\sigma_8\} + \{[\nu\ \sigma'\eta_{14}]_6\} \xrightarrow{i_*} \pi_{15}(R_7) \xrightarrow{p_*} \{\nu_6^3\} + \{\eta_6\varepsilon_7\} \longrightarrow 0.$$

From Table 2, there exist a element $[\eta_6]\varepsilon_7$ and $[\eta_6]\bar{\nu}_7$ in $\pi_{15}(R_7)$ such that $p_*([\eta_6]\varepsilon_7) = \eta_6\ \varepsilon_7$ and $p_*([\eta_6]\nu_7) = \eta_6\bar{\nu}_7 = \nu_6^3$ (by Lemma 6.3 of [13]).
Since ν_7 and ε_7 are of order 2, $[\eta_6]\nu_7$ and $[\eta_6]\varepsilon_7$ are of order 2. Thus we have

$$\pi_{15}(R_7) = \{[\eta_6]\bar{\nu}_7\} + \{[\eta_6]\varepsilon_7\} + \{[\nu_5]_7\sigma_8\} + \{[\nu_4\sigma'\eta_{14}]_7\} \approx \boldsymbol{Z}_2 + \boldsymbol{Z}_2 + \boldsymbol{Z}_2 + \boldsymbol{Z}_2.$$

R_8 : The result for $\pi_{15}(R_8)$ follows from the result for $\pi_{15}(R_7)$, (2.4) and Table 1.

R_9 : From Table 3, we have following relations

$$\begin{aligned}(3.6)\qquad &\Delta(\bar{\nu}_8)=\Delta(\iota_8)\nu_7=[\eta_6]_8\nu_7,\\ &\Delta(\varepsilon_8)=\Delta(\iota_8)\varepsilon_7=[\eta_6]_8\varepsilon_7,\\ &\Delta(\sigma_8\eta_{15})=[\iota_7]\sigma'\eta_{14},\\ &\Delta(E\sigma'\eta_{15})=[\nu_4\sigma'\eta_{14}]_8.\end{aligned}$$

From $(2.6)_8$ and Table 3, we have the exact sequence

$$0\longrightarrow\{[\iota_7]_9\bar{\nu}_7\}+\{[\iota_7]_9\varepsilon_7\}+\{[\nu_5]_9\sigma_8\}\longrightarrow\pi_{15}(R_9)\xrightarrow{p_*}\{8\sigma_8\}\longrightarrow 0.$$

Thus, from the fact that $8\sigma_8$ is of order infinite, we conclude

$$\pi_{15}(R_9)=\{[8\sigma_8]\}+\{[\iota_7]_9\bar{\nu}_7\}+\{[\iota_7]_9\varepsilon_7\}+\{[\nu_5]_9\sigma_8\}\approx \boldsymbol{Z}+\boldsymbol{Z}_2+\boldsymbol{Z}_2+\boldsymbol{Z}_2.$$

Moreover, from the exactness of the sequence (2.6), we have

(3.7) $i_*\ :\ \pi_{16}(R_8)\longrightarrow\pi_{16}(R_9)$ *is an epimorphism.*

R_{10} : Consider $(2.6)_9$. Using Table 3, we have

$$\begin{aligned}\Delta(\nu_9^2)&=\Delta(\iota_9)\nu_9^2 &&\text{by (2.2)}\\ &=[\nu_5]_9\nu_8^2+[\iota_7]_9\eta_7\nu_8^2 &&\text{by Table 3}\\ &=[\nu_5]_9\nu_8^2 &&\text{by } \pi_{11}^7=0.\end{aligned}$$

Thus, from $(2.6)_9$, we have the exact sequence

$$\pi_{16}^6=\{\sigma_9\}\xrightarrow{\Delta}\pi_{15}(R_9)\xrightarrow{i_*}\pi_{15}(R_{10})\longrightarrow 0.$$

For the homomorphism $\Delta\ :\ \pi_{16}^9\longrightarrow\pi_{15}(R_9)$, we have

$$\begin{aligned}(3.8)\qquad \Delta(\sigma_9)&=\Delta(\iota_9)\sigma_8 &&\text{by (2.2)}\\ &=[\iota_7]_9\eta_7\sigma_8+[\nu_5]_9\sigma_8 &&\text{by Table 3}\\ &=[\iota_7]_9\bar{\nu}_7+[\iota_7]_9\varepsilon_7+[\nu_5]_9\sigma_8 &&\text{by (7.4) of [13].}\end{aligned}$$

Therefore, from the above exact sequence, we have

$$\pi_{15}(R_{10})=\{[8\sigma_8]_{10}\}+\{[\iota_7]_{10}\bar{\nu}_7\}+\{[\nu_5]_{10}\sigma_8\}\approx\boldsymbol{Z}+\boldsymbol{Z}_2+\boldsymbol{Z}_2.$$

R_{11} : Consider the diagram (2.7)

$$\begin{array}{ccccccc}\pi_{18}^{10}=\{\nu_{10}^2\} & \xrightarrow{\Delta} & \pi_{15}(R_{10}) & \xrightarrow{i_*} & \pi_{15}(R_{11}) & \longrightarrow & \pi_{15}^{10}=0\\ \downarrow E^{11} & & \downarrow J & & \downarrow J & & \\ \pi_{27}^{21}=\{\nu_{21}^2\} & \xrightarrow{\Delta} & \pi_{25}^{10} & \xrightarrow{E} & \pi_{26}^{11} & \longrightarrow & \pi_{26}^{21}=0\end{array}$$

Then we have

$$\Delta E^{11}\nu_{10}^2 = \Delta(\nu_{21}^2) = \sigma_{10}\bar{\nu}_{17} \neq 0 \qquad \text{by (10. 20) of [13].}$$

Therefore, from the above diagram and $\pi_{16}^{10} \approx \boldsymbol{Z}_2$, we have that $\Delta : \pi_{16}^{10} \longrightarrow \pi_{15}(R_{10})$ must be non-trivial. On the other hand, we have

$$\begin{aligned} \mathbf{J}([\iota_7]_{10}\bar{\nu}_7) &= \mathbf{J}([\iota_7]_{10})E^{10}\bar{\nu}_7 && \text{by (2. 8)} \\ &= \sigma_{10}\bar{\nu}_{17} && \text{by (2. 9).} \\ \mathbf{J}([\nu_5]_{10}\sigma_8) &= \mathbf{J}([\nu_5]_{10})E^{10}\sigma_8 && \text{by (2. 8)} \\ &= \nu_{10}\sigma_{18} && \text{by (2. 9)} \\ &= 0 && \text{by Lemma 10. 7 of [13].} \end{aligned}$$

From the above diagram we have

$$\Delta(\nu_{10}{}^2) = [\iota_7]_{10}\bar{\nu}_7 + x[\nu_5]_{10}\sigma_8$$

where x = 1 or 0. Therefore, the exactness of the upper row sequence of the above diagram, we have

$$\pi_{15}(R_{11}) = \{[8\sigma_8]_{11}\} + \{[\nu_5]_{11}\sigma_8\} \approx \boldsymbol{Z} + \boldsymbol{Z}_2,$$

and from the exact sequence (2. 6) we have that

(3. 9) $\quad i_* : \pi_{16}(R_{10}) \longrightarrow \pi_{16}(R_{11})$ *is an epimorphism.*

R_{12} : Since $\pi_{16}^{11} = \pi_{15}^{10} = 0$, we have the result for $\pi_{15}(R_{12})$ from $(2.6)_{10}$.

To show the result for R_{13}, we shall need the following

Lemma 3. 10. (SUGAWARA [11]). *Let α be an element of $\pi_{r+1}(S^n)$. Then We have*

$$E^{n+3}p_*\Delta(\alpha) = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ 2\ E^{n+2}\alpha & \text{if } n \text{ is even,} \end{cases}$$

where $\Delta : \pi_{r+1}(S^n) \longrightarrow \pi_r(R_n)$ is boundary homomor phlsm and $p_* : \pi_r(R_n) \longrightarrow \pi_r(S^{n-1})$ is a homomor phism induced by the bundle projection $p : R_n \longrightarrow S^{n-1}$.

R_{13} : From (3. 10), $E^{15}p_*\Delta(\nu_{12}) = E^{15}(2\nu_{11})$. Since $E^{15} : \pi_{14}^{11} \longrightarrow \pi_{29}^{26}$ is an isomorphism, we have $p_*\Delta(\nu_{12}) = 2\nu_{11}$. By definition of $[2\nu_{11}]$, $\Delta(\nu_{11}) = [2\nu_{11}]$. From the fact that order of $[2\nu_{11}]$ is equal to 4, we have that the kernel of $\Delta : \pi_{15}^{12} \longrightarrow \pi_{14}(R_{12})$ is generated by $4\nu_{12} = \eta_{12}^3$. Thus there exists an element $[\eta_{12}^3] \in \pi_{15}(R_{13})$ such that $p_*([\eta_{12}^3]) = 4\nu_{12}$.

Consider the following diagram

(3. 11)
$$\begin{array}{ccccccccc} 0 = \pi_{16}^{12} & \longrightarrow & \pi_{15}(R_{12}) & \xrightarrow{i_*} & \pi_{15}(R_{13}) & \xrightarrow{p_*} & \{\eta_{12}^3\} & \longrightarrow & 0 \\ & & \downarrow \mathrm{J} & E & \downarrow \mathrm{J} & H & \downarrow E^{13} & & \\ 0 = \pi_{29}^{25} & \longrightarrow & \pi_{27}^{12} & \longrightarrow & \pi_{28}^{13} & \longrightarrow & \{\eta_{25}^3\} & \longrightarrow & 0 \end{array}$$

of (2.7). The upper row sequence is exact by the above fact and the lower sequence is exact by (10.11) of [13]. We have

$$\pi^{12}_{27} = \{E^3\rho'\} + \{\varepsilon_{12}\} \approx Z_{16} + Z_2 \qquad \text{by Theorem 10.5 of [13],}$$
$$\pi^{13}_{27} = \{\rho_{13}\} + \{\varepsilon_{13}\} \approx Z_{32} + Z_2 \qquad \text{by Theorem 10.10 of [13],}$$
$$E^4\rho' = 2\rho_{13} \qquad \text{by Lemma 10.9 of [13],}$$
and
$$H(\rho_{13}) = 4\nu_{25} = \eta^3_{25} \qquad \text{by (10.11) of [13].}$$

Next we prove

Lemma 3.12 $\mathbf{J}([8\sigma_8]_{12}) = E^3\rho' + x\varepsilon_{12}$ *for* $x = 0$ *or* 1.

Proof. Consider the diagram

$$\begin{array}{ccccc} \pi_{15}(R_8) & \xrightarrow{i_*} & \pi_{15}(R_9) & \xrightarrow{p_*} & \pi^8_{15} \\ \downarrow \mathrm{J} & & \downarrow \mathrm{J} & & \downarrow E^9 \\ \pi_{23}{}^8 & \xrightarrow{E} & \pi^9_{24} & \xrightarrow{H} & \pi^{17}_{24} \end{array}$$

of (2.7). Then we have

$$\begin{aligned} H\, J([8\sigma_8]) &= \pm E^9(8\sigma_8) \\ &= 8\sigma_{17} \\ &= H(\rho') \qquad \text{by (10.2) of [13].} \end{aligned}$$

Thus, $\mathbf{J}([8\sigma_8] \equiv \rho'$ mod $E\pi^8_{23}$.
By definition of $[8\sigma_8]_{12}$ and the diagram (2.7),

$$\begin{aligned} \mathbf{J}([8\sigma_8]_{12}) &= \mathbf{J}(j_*([8\sigma_8])) \\ &= E^3\mathbf{J}([8\sigma_8]) \\ &\equiv E^3\rho' \mod E^4\pi^8_{23} \\ &= E^3\rho' + x\varepsilon_{12}, \end{aligned}$$

where $x = 0$ or 1 and $j_* : \pi_{15}(R_9) \longrightarrow \pi_{15}(R_{12})$ is a homomorphism induced by the inclusion map $j : R_9 \longrightarrow R_{12}$. q.e.d.

From the above diagram, we have

$$\mathbf{J}([\eta^3_{12}]) \equiv \rho_{13} \mod E\pi^{12}_{27}.$$

Therefore, from (3.11),

$$[8\sigma_8]_{13} = 2[\eta^3_{12}].$$

Thus we have, from the exactness of the upper sequence of the diagram (3.11),

$$\pi_{15}(R_{13}) = \{[\eta^3_{12}]\} + \{[\nu_5]_{13}\sigma_8\} \approx Z + Z_2.$$

$\boldsymbol{R}_{14}$ and $\boldsymbol{R}_{15}$: Consider $(2.6)_{13}$ and $(2.6)_{14}$. According to Theorem 3 of [5], we have

(3.13) $\Delta(\nu_{13}) \neq 0,\ \ 2\Delta(\nu_{13}) = 0\ ;\ \ \Delta(\eta_{13}^2) \neq 0,\ \ \Delta(\eta_{14}) \neq 0.$

Therefore we conclude

$$\pi_{15}(R_{n+1}) = \{[\eta_{13}^2]_{n+1}\} \approx Z \qquad \text{for } n = 13 \text{ and } 14.$$

R_{16} : Consider $(2.6)_{15}$. From Theorem 23.4 of [12], we have

(3.14) $p_*\Delta(\iota_{16}) = 2\iota_{15}.$

On the other hand $\Delta : \pi_{16}^{15} \approx Z_2 \longrightarrow \pi_{15}(R_{15}) \approx Z$ is trivial. Thus we have the exact sequence

$$0 \longrightarrow \pi_{15}(\mathrm{R}_{15}) \longrightarrow \pi_{15}(R_{16}) \longrightarrow \{2\iota_{15}\} \approx Z \longrightarrow 0.$$

Therefore we obtain that

$$\pi_{15}(R_{16}) = \{[\eta_{12}^3]\} + \{[2\iota_{15}]\} \approx Z + Z.$$

R_n for $n \geqq 17$: From (3.12), $(2.6)_{16}$ and the stability of $\pi_{15}(R_n)$, we have

$$\pi_{15}(R_n) = \{[\eta_{12}^3]_n\} \approx Z \qquad \text{for } n \geqq 17.$$

4. Groups π_{16} (R_n) and their generators

The results for $\pi_{16}(R_n:2)$ are stated as follows:

Proposition 4.1. $\pi_{16}(R_3:2) = \{[\eta_2]\nu'\eta_6\mu_7\} \approx Z_2$

$\pi_{16}(R_4:2) = \{[\iota_3]\nu'\eta_6\mu_7]\} + \{[\eta_2]_4\nu'\eta_6\mu_7\} \approx Z_2 + Z_2$

$\pi_{16}(R_5:2) = \{[\nu_4\sigma'\eta_{14}]\eta_{15}\} + \{[\nu_4^2]\}\nu_{10}^2\} \approx Z_2 + Z_2$

$\pi_{16}(R_6:2) = \{[\nu_4\sigma'\eta_{14}]_6\eta_{15}\} + \{[\nu_4^2]_6\nu_{10}^2\} + \{[\zeta_5]\} + \{[\nu_5]\bar{\nu}_8\} + \{[\nu_5]\varepsilon_8\}$
$\approx Z_2 + Z_2 + Z_8 + Z_2 + Z_2$

$\pi_{16}(R_7:2) = \{[\nu_4\sigma'\eta_{14}]_7\eta_{15}\} + \{[\nu_4^2]_7\nu_{10}^2\} + \{[\zeta_5]_7\} + \{[\nu_5]_7\bar{\nu}_8\} + \{[\nu_5]_7\varepsilon_8\} + \{[\eta_6]\eta_7\varepsilon_8\} + [\eta_6]\mu_7\}$
$\approx Z_2 + Z_2 + Z_2 + Z_2 + Z_2 + Z_2 + Z_2$

$\pi_{16}(R_8:2) = \{[\nu_4\sigma'\eta_{14}]_8\eta_{15}\} + \{[\nu_4^2]_8\nu_{10}^2\} + \{[\zeta_5]_8\} + \{[\nu_5]_8\bar{\nu}_8\} + \{[\nu_5]_8\varepsilon_8\} + \{[\eta_6]_8\mu_7\} +$
$\{[\eta_6]_8\eta_7\varepsilon_8\} + \{[\iota_7]\sigma'\eta_{14}^2\} + \{[\iota_7]\nu_7^3\} + \{[\iota_7]\mu_7\} + \{[\iota_7]\eta_7\varepsilon_8\}$
$\approx Z_2 + Z_2 + Z_2 + Z_2 + Z_2 + Z_2 + Z_2 + Z_2 + Z_2 + Z_2 + Z_2$

$\pi_{16}(R_9:2) = \{[\zeta_5]_9\} + \{[\nu_5]_9\bar{\nu}_8\} + \{[\nu_5]_9\varepsilon_8\} + \{[\iota_7]_9\nu_7^3\} + \{[\iota_7]_9\mu_7\} + \{[\iota_7]_9\eta_7\varepsilon_8\}$
$\approx Z_2 + Z_2 + Z_2 + Z_2 + Z_2 + Z_2$

$\pi_{16}(R_{10}:2) = \{[2\sigma_9]\} + \{[\iota_7]_{10}\nu_7^3\} + \{[\iota_7]_{10}\mu_7\} \approx Z_{16} + Z_2 + Z_2$

$\pi_{16}(R_{11}:2) = \{[\iota_7]_{11}\nu_7^3\} + \{[\iota_7]_{11}\mu_7\} \approx Z_2 + Z_2$

$\pi_{16}(R_n:2) = \{[\iota_7]_n\mu_7\} \approx Z_2$ *for* $n = 12,\ 13$

$\pi_{16}(R_{14}:2) = \{[\iota_7]_{14}\mu_7\} + \{[2\nu_{13}]\} \approx Z_2 + Z_4$

$\pi_{16}(R_{15}:2)=\{[\iota_7]_{15}\mu_7\}+\{[\eta_{14}^2]\}\approx Z_2+Z_2$
$\pi_{16}(R_{16}:2)=\{[\eta_{14}^2]_{16}\}+\{[\iota_7]_{16}\mu_7\}+\{[\eta_{15}]\}\approx Z_2+Z_2+Z_2$
$\pi_{16}(R_{17}:2)=\{[\iota_7]_{17}\mu_7\}+\{[\eta_{15}]_{17}\}\approx Z_2+Z_2$
$\pi_{16}(R_n:2)=\{[\iota_7]_n\mu_7\}\approx Z_2$ *for* $n>17$.

Proof. From (2.3), (2.4) and Table 1, we have the results for $\pi_{16}(R_3)$ and $\pi_{16}(R_4)$.

R_5 : Consider the exact sequence

$$\pi_{16}(R_4)\xrightarrow{i_*}\pi_{16}(R_5)\xrightarrow{p_*}\pi_{16}^4\xrightarrow{\Delta}\pi_{15}(R_4).$$

By the same argument as in the case of $i_*:\pi_{15}(R_4)\longrightarrow\pi_{15}(R_5)$, we have that $i_*:\pi_{16}(R_4)\longrightarrow\pi_{16}(R_5)$ is trivial. On the other hand from Table 3 we can prove that the kernel of $\Delta:\pi_{16}^4\longrightarrow\pi_{15}(R_4)$ is generated by $\nu_4\sigma'\eta_{14}^2$ and ν_4^4. Then, the exactness of the above sequence, we have an isomorphism

$$p_*:\pi_{16}(R_5)\longrightarrow\{\nu_4\sigma'\eta_{14}^2\}+\{\nu_4^4\}.$$

From Table 3 and the result for $\pi_{15}(R_5)$, there exist elements $[\nu_4\sigma'\eta_{14}]\in\pi_{15}(R_5)$ and $[\nu_4^2]\in\pi_{10}(R_5)$ such that $p_*([\nu_4\sigma'\eta_{14}])=\nu_4\sigma'\eta_{14}$ and $p_*([\nu_4^2])=\nu_4^2$. Therefore we obtain from (2.2) that

$$\pi_{16}(R_5)=\{[\nu_4\sigma'\eta_{14}]\eta_{15}\}+\{[\nu_4^2]\nu_{10}^2\}\approx Z_2+Z_2.$$

R_6 : From Table 3 and (3.4), it follows that the sequence

$$0\longrightarrow\pi_{16}(R_5)\xrightarrow{i_*}\pi_{16}(R_6)\xrightarrow{p_*}\pi_{16}^5\longrightarrow 0$$

is exact. From (3.2) and Theorem 6.1 of [7], the above sequence splits. From Table 3, there exists an element $[\nu_5]\in\pi_8(R_6)$ such that $p_*([\nu_5])=\nu_5$.
Thus we have from (2.2)

$$\pi_{16}(R_6)=\{[\zeta_5]\}+\{[\bar{\nu}_5]\nu_8\}+\{[\nu_5]\varepsilon_8\}+\{[\nu_4\sigma'\eta_{14}]_6\eta_{15}\}+\{[\nu_4^2]_6\nu_{10}^2\}$$
$$\approx Z_8+Z_2+Z_2+Z_2+Z_2.$$

R_7 : From (3.5), the exact sequence (2.6) yields the following exact sequence

$$\pi_{17}^6\xrightarrow{\Delta}\pi_{16}(R_6)\xrightarrow{i_*}\pi_{16}(R_7)\xrightarrow{p_*}\{4\nu_6\sigma_9\}+\{\eta_6\mu_7\}\longrightarrow 0.$$

Now we have following relations :

(4.1)
$$\Delta(\zeta_6)=2[\zeta_5]+a[\bar{\nu}_5]\bar{\nu}_8+b[\nu_5]\varepsilon_8 \qquad a,\ b=0\text{ or }1,$$
$$\Delta(\nu_6\bar{\nu}_{14})=0.$$

In fact, since $E^9 p_* \Delta(\zeta_6) = 2E^9\zeta_5$ by (3.10), we have $\Delta(\zeta_6) = 2[\zeta_5] + a[\nu_5]\bar{\nu}_8 + b[\nu_5]\varepsilon_8$ for some integers a, b. And

$$\begin{aligned}\Delta(\nu_6\nu_{14}) &= \Delta((\nu_6+\varepsilon_6)\nu_{14}) && \text{by } \varepsilon_6\nu_{14}=0,\\ &= \Delta(\nu_6+\varepsilon_6)\nu_{13} && \text{by (2.2)}\\ &= 0 && \text{by Table 2.}\end{aligned}$$

By (7.10) of [13], $4\nu_6\sigma_9 = \eta_6^2\varepsilon_8$. From Table 2, there exist elements $[\eta_6]\eta_7\mu_8$ and $[\eta_6]\mu_7$ such that $p_*([\eta_6]\eta_7\mu_8) = \eta_6^2\mu_8 = 4\nu_6\sigma_9$ and $p_*([\eta_6]\mu_7) = \eta_6\mu_7$. Since $\eta_7\varepsilon_8$ and μ_7 are of order 2, it follows that $[\eta_6]\eta_7\varepsilon_8$ and $[\eta_6]\mu_7$ are of order 2. Thus, from the above exact sequence, we have

$$\begin{aligned}\pi_{16}(R_7) = &\{[[\zeta_5]_7\}+\{[\nu_5]_7\bar{\nu}_8\}+\{[\nu_5]_7\varepsilon_8\}+\{[\nu_4\sigma'\eta_{14}]_7\eta_{15}\}+\{[\nu_4^2]_7\nu_{10}^2\}\\ &+\{[\eta_6]\eta_7\varepsilon_8\}+\{[\eta_6]\mu_7\} \approx \boldsymbol{Z}_2+\boldsymbol{Z}_2+\boldsymbol{Z}_2+\boldsymbol{Z}_2+\boldsymbol{Z}_2+\boldsymbol{Z}_2+\boldsymbol{Z}_2.\end{aligned}$$

Moreover, by Lemma 6.7 of [13], we obtain that the kernel of

$$(4.2)\qquad \Delta\ :\ \pi_{17}^6 \longrightarrow \pi_{16}(R_6)$$

is generated by $4\zeta_6 = \eta_6^2\mu_8$ and $\bar{\nu}_6\nu_{16}$.

$\boldsymbol{R}_8$: By (2.4) and Table 1, the results for $\pi_{16}(R_8)$ are given.

$\boldsymbol{R}_9$: By (3.7), (2.6) yields the following exact sequence

$$\pi_{17}^8 \xrightarrow{\Delta} \pi_{16}(R_8) \xrightarrow{i_*} \pi_{16}(R_9) \longrightarrow 0.$$

For the homomorphism $\Delta : \pi_{17}^8 \longrightarrow \pi_{16}(R_8)$, making use of the Table 1 and 3 and the formula (2.2), we have that

$$(4.3)\qquad \begin{aligned}&\Delta(\nu_8^3) = [\nu_4^2]_8\nu_{10}^2,\\ &\Delta(\mu_8) = [\eta_6]_8\mu_7,\\ &\Delta(\sigma_8\eta_{15}^2) = [\iota_7]\sigma'\eta_{14}^2,\\ &\Delta(E\sigma'\eta_{15}^2) = [\nu_4\sigma'\eta_{14}]_8\eta_{15} && \text{by (3.6)},\\ &\Delta(\eta_8\varepsilon_9) = [\eta_6]_8\eta_7\varepsilon_8.\end{aligned}$$

Thus, from the exactness of the above sequence, we have

$$\begin{aligned}\pi_{16}(R_9) = &\{[[\zeta_5]_9\}+\{[\nu_5]_9\bar{\nu}_8\}+\}[\nu_5]_9\varepsilon_8\}+\{[\iota_7]_9\nu_3^7\}+\{[\iota_7]_9\mu_7\}+\{[\iota_7]_9\eta_7\varepsilon_8\}\\ &\approx \boldsymbol{Z}_2+\boldsymbol{Z}_2+\boldsymbol{Z}_2+\boldsymbol{Z}_2+\boldsymbol{Z}_2+\boldsymbol{Z}_2.\end{aligned}$$

From (4.3) and the exact sequence (2.6),

(4.4) $\qquad i_*\ :\ \pi_{17}(R_8) \longrightarrow \pi_{17}(R_9)$ *is an epimorphism.*

$\boldsymbol{R}_{10}$: From (3.8) and (2.6), we obtain that the sequence

$$\pi^6_{17} \xrightarrow{\Delta} \pi_{16}(R_9) \xrightarrow{i_*} \pi_{16}(R_{10}) \xrightarrow{p_*} \{2\sigma_9\} \longrightarrow 0$$

is exact. By the use of (2.2), (3.8) and Tables 1 and 3, we have the following relations :

$$\begin{aligned} (4.5)\qquad \Delta(\varepsilon_9) &= \Delta(\iota_9)\varepsilon_8 = [\nu_5]_9\varepsilon_8 + [\iota_7]_9\eta_7\varepsilon_8, \\ \Delta(\nu_9) &= \Delta(\iota_9)\nu_8 = [\nu_5]_9\nu_8 + [\iota_7]_9\eta_7\nu_8 \\ &= [\nu_5]_9\nu_8 + [\iota_7]_9\nu_7^3 \qquad \text{by Lemma 6.3 of [13]}, \\ \Delta(\sigma_9\eta_{16}) &= \Delta(\sigma_9)\eta_{15} = [\iota_7]_9\bar{\nu}_7\eta_{15} + [\iota_7]_9\varepsilon_7\eta_{15} + [\nu_5]_9\sigma_8\eta_{15} \\ &= [\iota_7]_9\nu_7^3 + [\iota_7]_9\eta_7\varepsilon_8 + [\nu_5]_9\sigma_8\eta_{15} \text{ by Lemma 6.3 and (7.5) of [13.]}, \end{aligned}$$

where $[\nu_5]_9\sigma_8\eta_{15}$ is a linear combination of $[\zeta_5]_9$, $[\nu_5]_9\bar{\nu}_8$ and $[\nu_5]_9\varepsilon_8$.
Thus, from the above exact sequence, it follows that the sequence

$$(4.6)\qquad 0 \longrightarrow \{[\iota_7]_9\mu_7\} + \{[\zeta_5]_9\} + \{[\iota_7]_9\nu_7^3\} \xrightarrow{i_*} \pi_{16}(R_{10}) \xrightarrow{p_*} \{2\sigma_9\} \longrightarrow 0$$

is exact. Consider the diagram

$$\begin{array}{ccccc} \pi^{10}_{17} = \{\sigma_{10}\} & \xrightarrow{\Delta} & \pi_{16}(R_{10}) & \xrightarrow{p_*} & \pi^9_{16} = \{\sigma_9\} \\ \approx\downarrow E^{11} & & \downarrow J & & \approx\downarrow E^{10} \\ \pi^{21}_{28} = \{\sigma_{21}\} & \xrightarrow{\Delta} & \pi^{10}_{26} & \xrightarrow{H} & \pi^{19}_{26} = \{\sigma_{19}\} \end{array}$$

of (2.7). Then we have

$$E^{13}p_*\Delta(\sigma_{10}) = 2E^{13}\sigma_9 \quad \text{by (3.10)}.$$

Since E^{13} : $\pi^9_{16} \longrightarrow \pi^{22}_{29}$ is an isomorphism, $p_*\Delta(\sigma_{10}) = 2\sigma_6$.
Therefore, by definition of $[2\sigma_9]$, we have

$$(4.7)\qquad \Delta(\sigma_{10}) = [2\sigma_9].$$

By (12.19) of [13], the homomorphiom ΔE^{11} : $\pi^{10}_{17} \longrightarrow \pi^{10}_{62}$ is a monomorphism. On the other hand

$$\begin{aligned} H\Delta E^{11}(\sigma_{10}) &= \pm 2\sigma_{19} \qquad \text{by Proposition 2.5 and 2.7 of [13]}, \\ &= \pm HJ\Delta(\sigma_{10}) \end{aligned}$$

Thus, from the exact sequence $\pi^9_{55} \xrightarrow{E} \pi^{10}_{26} \xrightarrow{H} \pi^{19}_{26}$ of [13] and (4.6), we have

$$(4.8)\qquad J([2\sigma_9]) \equiv \pm\Delta(\sigma_{21}) \quad \text{mod } E\pi^9_{25}.$$

Therefore it follows that $[2\sigma_9]$ is of order 16. We have a relation :

$$\begin{aligned} J(8[2\sigma_9]) &= J\Delta(8\sigma_{10}) \qquad \text{by (4.6)}, \\ &= \pm\Delta E^{11}(8\sigma_{10}), \end{aligned}$$

$$
\begin{aligned}
&= \pm\Delta(8\sigma_{21}), \\
&= \sigma_{10}\mu_{17} + \mu_{10}\sigma_{19} && \text{(cf. the page 156 of [13]),} \\
&= \mathbf{J}([\iota_7]_{10}\mu_7) + \mu_{10}\sigma_{19} && \text{by (2.9).}
\end{aligned}
$$

Thus we have $\mathbf{J}(8[2\sigma_9] - [\iota_7]_{10}\mu_7) \neq 0$. From the exact sequence (4.6),

$$8[2\sigma_9] - [\iota_7]_{10}\mu_7 = x[\zeta_5]_{10} + y[\iota_7]_{10}\nu_7^3.$$

for some integers x, y (x, $y = 0$ or 1). On the other hand,

$$
\begin{aligned}
\mathbf{J}([\iota_7]_{10}\nu_7{}^3) &= \mathbf{J}([\iota_7]_{10})E^{10}\nu_7{}^3 && \text{by (2.8),} \\
&= \sigma_{10}\nu_{17}^3 = 0 && \text{by (7.1) of [13] and (2.9).}
\end{aligned}
$$

Therefore we have $x = 1$ and $\mathbf{J}([\zeta_5]_{10}) = \mu_{10}\sigma_{19}$. Thus we have obtaind a relation

$$8[2\sigma_9] = [\iota_7]_{10}\mu_7 + [\zeta_5]_{10} + y[\iota_7]_{10}\nu_7^3,$$

where $y = 0$ or 1. It follows from the exactness of the above sequence that

$$\pi_{16}(R_{10}) = \{[2\sigma_9]\} + \{[\iota_7]_{10}\mu_7\} + \{[\iota_7]_{10}\nu_7{}^3\} \approx \boldsymbol{Z}_{16} + \boldsymbol{Z}_2 + \boldsymbol{Z}_2.$$

$\boldsymbol{R}_{11}$: By (3.9), we have an exact sequence

$$\pi_{17}^{10} \xrightarrow{\Delta} \pi_{16}(R_{10}) \xrightarrow{i_*} \pi_{16}(R_{11}) \longrightarrow 0.$$

Then it follows from (4.7) that

(4.9) $\quad \Delta : \pi_{17}^{10} \longrightarrow \pi_6^1(R_{10})$ *is a monomorphism*

and

$$\pi_{16}(R_{11}) = \{[\iota_7]_{11}\mu_7\} + \{[\iota_7]_{11}\nu_7^3\} \approx \boldsymbol{Z}_2 + \boldsymbol{Z}_2.$$

$\boldsymbol{R}'_{12}$: Consider the exact sequence

$$\pi_{17}^{11} = \{\nu_{11}^2\} \xrightarrow{\Delta} \pi_{16}(R_{11}) \xrightarrow{i_*} \pi_{16}(R_{12})p^* \longrightarrow \pi_{16}^{11} = 0$$

of (2.6). We have the relation;

$$
\begin{aligned}
\Delta(\nu_{11}^2) &= \Delta(\iota_{11})\nu_{10}^2 && \text{by(2.2),} \\
&= [\iota_7]_{11}\nu_7^3 && \text{by Table 3.}
\end{aligned}
$$

Then it follows from the exactness of the above sequence that

(4.10) $\quad \Delta : \pi_{17}^{11} \longrightarrow \pi_{16}(R_{11}) \quad$ *is a monomorphism*

and

$$\pi_{16}(R_{12})=\{[\iota_7]_{11}\mu_7\}\approx \boldsymbol{Z}_2.$$

$\boldsymbol{R}_{13}$: $\pi^{12}_{17}=\pi^{12}_{16}=0$ by Table 1. Then, from the exactness of (2. 6), it follows that

$$\pi_{16}(R_{13})=\{[\iota_7]_{13}\mu_7\}\approx \boldsymbol{Z}_2.$$

$\boldsymbol{R}_{14}$: From (3. 13) and (2. 6), the following sequence

$$0=\pi^{13}_{17}\longrightarrow \pi_{16}(R_{13})\xrightarrow{i_*}\pi_{16}(R_{14})\xrightarrow{p_*}\{2\nu_{13}\}\longrightarrow 0$$

is exact. Consider the diagram

$$\begin{array}{ccccc}
\pi^{14}_{17}=\{\nu_{14}\} & \xrightarrow{\Delta} & \pi_{16}(R_{14}) & \xrightarrow{p_*} & \pi^{13}_{16}=\{\nu_{13}\} \\
\approx\big\downarrow E^{15} & & \big\downarrow \mathbf{J} & & H\approx\big\downarrow E^{14} \\
\pi^{29}_{32}=\{\nu_{29}\} & \xrightarrow{\Delta} & \pi^{14}_{30} & \longrightarrow & \pi^{27}_{30}=\{\nu_{27}\}
\end{array}$$

of (2. 7). Then we have

$$E^{17}p_*\Delta\nu_{14}=2E^{17}\nu_{13} \qquad \text{by (3. 10).}$$

Since E^{17} : $\pi^{13}_{16}\longrightarrow\pi^{30}_{33}$ is an isomorphism, we have $p_*\Delta\nu_{14}=2\nu_{13}$. Then, by definition of $[2\nu_{13}]$, we have

(4. 11) $$\Delta\nu_{14}=[2\nu_{13}].$$

On the other hand,

$$\pi^{14}_{30}=\{\omega_{14}\}+\{\sigma_{14}\mu_{21}\}\approx \boldsymbol{Z}_8+\boldsymbol{Z}_2 \qquad \text{by Theorem 12. 16 of [13],}$$

$$H(\omega_{14})=\nu_{27} \qquad \text{by Lemma 12. 15 of [13],}$$

and $$\Delta\nu_{29}=\pm 2\omega_{14} \qquad \text{(cf. page 159 or [13]).}$$

Thus, from the above diagram

(4. 12) $$\begin{aligned}\mathbf{J}([2\nu_{13}])&=\mathbf{J}\Delta\nu_{14}\\ &=\pm\Delta E^{15}\nu_{14}\\ &=\pm\Delta\nu_{29}\\ &=\pm 2\omega_{14}.\end{aligned}$$

If $[2\nu_{13}]$ is of order 8, then, from the above exact sequence, we have $i_*([\iota_7]_{13}\mu_7)=[\iota_7]_{14}\mu_7=4[2\nu_{13}]$, and

$$\begin{aligned}0\neq\sigma_{14}\mu_{21}&=\mathbf{J}([\iota_7]_{14}\mu_7) && \text{by (2. 9)}\\ &=\mathbf{J}4(2\nu_{13})\\ &=\pm 8\omega_{14} && \text{by (4. 12)}\\ &=0\end{aligned}$$

This is a contradiction, and hence $[2\nu_{13}]$ must be of order 4. From the exactness of the above sequence, we have

$$\pi_{16}(R_{14}) = \{[2\nu_{13}]\} + \{[\iota_7]_{14}\mu_7\} \approx \boldsymbol{Z}_4 + \boldsymbol{Z}_2.$$

Moreover,

(4.13) *The kernel of* $\Delta : \pi_{17}{}^{14} \longrightarrow \pi_{16}(R_{14})$ *is* $\{4\nu_{14}\} = \{\eta_{16}^3\}$.

$\boldsymbol{R}_{15}$: From (2.6), (4.10) and $\pi_{15}(R_{14}) \approx \boldsymbol{Z}$, it follows that the sequence

$$0 \longrightarrow [\iota_7]_{15}\mu_7\} \longrightarrow \pi_{16}(R_{15}) \xrightarrow{p_*} \pi_{16}^{14} = \{\eta_{14}^2\}_4 \longrightarrow 0$$

is exact. Consider the diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & \{[\iota_7]_{15}\mu_7\} & \longrightarrow & \pi_{16}(R_{15}) & \xrightarrow{p_*} & \{\eta_{14}^2\} & \longrightarrow 0 \\ & \downarrow \mathrm{J} & & \downarrow \mathrm{J} & H & \approx \downarrow E^{15} & \\ 0 \longrightarrow & \{\sigma_{15}\mu_{22}\}+\{\omega_{15}\} & \longrightarrow & \pi_{31}^{15} & \longrightarrow & \{\eta_{29}^2\} & \longrightarrow 0 \end{array}$$

of (2.7), where the lower sequence is exact by Lemma 12.14 and (12.20) of [13]. We have

$$\mathrm{J}([\iota_7]_{15}\mu_7) = \sigma_{15}\mu_{22} \qquad \text{by (2.8) and (2.9).}$$

Thus, from the above diagram, we have

(4.14) $\mathrm{J} : \pi_{16}(R_{15}) \longrightarrow \pi_{31}^{15}$ *is a monomorphism.*

On the other hand,

$$\pi_{31}^{15} = \{\omega_{15}\}+\{\sigma_{15}\mu_{22}\}+\{\mu^{*\prime}\} \approx \boldsymbol{Z}_2+\boldsymbol{Z}_2+\boldsymbol{Z}_2 \quad \text{by Theorem 12.16 of [13]}$$

and

$$H(\eta^{*\prime}) = \eta_{29}^2 \qquad \text{by Lemma 12.14 of [13].}$$

Therefore, there exists an element $[\eta_{14}^2] \in \pi_{16}(R_{15})$ of order 2 such that $p_*([\eta_{14}^2]) = \eta_{14}^2$. Thus, from the exactness of the above sequence, we have

$$\pi_{16}(R_{15}) = \{[\iota_7]_{15}\mu_7\} + \{[\eta_{14}^2]\} \approx \boldsymbol{Z}_2 + \boldsymbol{Z}_2$$

and

(4.15) $\mathrm{J}([\eta_{14}^2]) = \eta^{*\prime}$.

$\boldsymbol{R}_{16}$: Consider the diagram

$$\begin{array}{ccccccc}
\pi^{15}_{17} = \{\eta^2_{15}\} & \xrightarrow{\Delta} & \pi_{16}(R_{15}) & \xrightarrow{i_*} & \pi_{16}(R_{16}) & \xrightarrow{p_*} & \pi^{15}_{61} = \{\eta_{15}\} \\
\approx \downarrow E^{16} & & \downarrow \mathbf{J} & & \downarrow \mathbf{J} & & \approx \downarrow E^{16} \\
\pi^{31}_{33} = \{\eta^2_{31}\} & \xrightarrow{\Delta} & \pi^{15}_{31} & \xrightarrow{E} & \pi^{16}_{32} & \xrightarrow{H} & \pi^{18}_{32} = \{\eta_{31}\}
\end{array}$$

fo (2. 7). From (4. 14) and by five lemma, we have

(4. 16) $\quad \mathbf{J} : \pi_{16}(R_{16}) \longrightarrow \pi^{16}_{32}$ *is a monomorphism.*

From Lemma 12. 14 of [13], $\Delta : \pi^{31}_{33} \longrightarrow \pi^{15}_{31}$ is trivial. Thus, from (4. 14) and the above diagram, it follows that

(4. 17) $\quad \Delta : {\pi_{17}}^{15} \longrightarrow \pi_{16}(R_{15})$ *is trivial.*

On the other hand, from Lemma 12. 14 of [13], there exists an element $\eta^*_{16} \in \pi^{16}_{32}$ such that $H(\eta^*_{16}) = \eta_{31}$ and $2\eta_{16} = 0$. Moreover, from the above diagram, there exists an element $[\eta_{15}] \in \pi_{16}(R_{16})$ such that

$$p_*([\eta_{15}]) = \eta_{15},$$
$$2[\eta_{15}] = 0,$$

and

(4. 17) $\quad \mathbf{J}([\eta_{15}]) = \eta^*_{16}.$

Thus, from theexact sequence

$$0 \longrightarrow \pi_{16}(R_{15}) \xrightarrow{i_*} \pi_{16}(R_{16}) \xrightarrow{p_*} \pi^{15}_{16} \longrightarrow 0,$$

we have

$$\pi_{16}(R_{16}) = \{[\eta_{15}]\} + \{[\eta^2_{14}]_{16}\} + \{[\iota_7]_{16}\mu_7\}$$
$$\approx \mathbf{Z}_2 + \mathbf{Z}_2 + \mathbf{Z}_2.$$

From the above diagram, we have also

(4. 18)
$$\begin{aligned} \mathbf{J}([\eta^2_{14}]_{16}) &= \mathbf{J}(i_*[\eta^2_{14}]) \\ &= EJ([\eta^2_{14}]) \qquad \text{by (4. 15)} \\ &= E\eta^{*\prime}. \end{aligned}$$

$\boldsymbol{R}_{17}$: Consdier the diagram

$$\begin{array}{ccccccc}
\pi^{16}_{17} = \{\eta_{16}\} & \xrightarrow{\Delta} & \pi_{16}(R_{16}) & \xrightarrow{i_*} & \pi_{16}(R_{17}) & \longrightarrow & 0 \\
\approx \downarrow E^{17} & & \downarrow \mathbf{J} & & \downarrow \mathbf{J} & & \\
\pi^{33}_{34} = \{\eta_{33}\} & \xrightarrow{\Delta} & \pi^{16}_{33} & \xrightarrow{E} & \pi^{17}_{32} & \xrightarrow{H} & \pi^{33}_{33} = \{\iota_{33}\}
\end{array}$$

of (2. 7), where the upper sequence is exact by (3. 14).

We have a relation

$$\Delta(\eta_{33}) \equiv E\eta^{*\prime} \qquad \text{mod } E^2\pi^{14}_{30} \qquad \text{(cf. page 160 of [13]).}$$

Thus, from (4. 18) and (4. 16), we have

(4. 19) $$\Delta(\eta_{16}) = [\eta^2_{14}]_{16},$$

and

$$\pi_{16}(R_{17}) = [\{\eta_{15}]_{17}\} + \{[\iota_7]_{17}\mu_7\} \approx Z_2 + Z_2.$$

Also, we obtain

(4. 20) $\mathbf{J} : \pi_{16}(R_{17}) \longrightarrow \pi^{17}_{33}$ *is a monomorphism.*

R_n for *$n \geqq 18$* : Consider the diagram

$$\begin{array}{ccccccc}
\pi^{17}_{17} & \xrightarrow{\Delta} & \pi_{16}(R_{17}) & \xrightarrow{i_*} & \pi_{16}(R_{18}) & \longrightarrow & \pi^{16}_{17} = 0 \\
\approx\downarrow E^{18} & & \downarrow \mathbf{J} & & \downarrow \mathbf{J} & & \\
\pi^{35}_{36} & \xrightarrow{\Delta} & \pi^{17}_{33} & \xrightarrow{E} & \pi^{18}_{34} & \longrightarrow & \pi^{35}_{34} = 0
\end{array}$$

of (2. 7). Then

$$\begin{aligned} \mathbf{J}\Delta(\iota_{17}) &= \Delta\iota_{35} = \eta^*_{17} && \text{(cf. page 160 of [13])} \\ &= \mathbf{J}([\eta_{15}]_{17}) && \text{by (4. 20).} \end{aligned}$$

Thus, from (4. 20) and the stability of $\pi_i(R_n)$, it follows that

$$\pi_{16}(R_n) = \{[\iota_7]_n\mu_7\} \approx Z_2 \qquad \text{for } n \geqq 18.$$

Reference

[1] BOTT, R. : The stable homotopy of the classical groups, Proc. Nat. Acad. Sci. U. S. A., 43 (1957), 933–935.

[2] HILTON, P. J. : A note on the P-homomorphism in homotopy groups of spheres, Proc. Cambridge Philos. Soc., 51 (1955), 230–233.

[3] HILTON. P. J. -WHITEHEAD, J. H. C. : Note on the Whitehead product, Ann. of Math. 58 (1953), 429–442.

[4] HARRIS, B. : On the homotopy groups of the classical groups, Ann. of Math. 74 (1961), 407–413.

[5] KERVAIRE, M. : Some non-stable homotopy groups of Lie groups, Illi. Jour. Math. 4 (1960), 161–169.

[6] MILNOR, J. -KERVAIRE, M. : Bernoulli number, homotopy groups and a theorem of Rohlin, Proc. International Congress of Math. Edinburgh, 1958.

[7] MIMURA, M. -TODA, H. : The homotopy groups of $Sp(2)$, $SU(3)$ and $SU(4)$, J. of Math. of Kyoto Univ. 3 (1964), 217–250.

[8] MIMURA, M. -TODA, H. : Homotopy groups of symplectic groups, J. of Math. of Kyoto Univ. 3 (1964), 251–273.

[9] MIMURA, M. : The homotopy groups of Lie groups of low lank, J. of Math. of Kyoto Univ. 6 (1967), 113–176.

[10] SUGAWARA, M. : On the homotopy groups of rotation groups, Math. Jour. of Okayama Univ., 1 (1953), 11–21.

[11] SUGAWARA, M. : Some remarks on homotopy groups of rotation groups, Math. Jour. of Okayama Univ., 2 (1954).

[12] STEENROD, N. E. : *The Topology of Fibre Bundles* Princiton, (1951)

[13] TODA, H. : *Composition methods in homotopy groups of spheres,* Ann. of Math. studies No. 49, Rrinciton, (1962).