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On Sexauer–Warnock Theorem

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In this paper, we shall give an alternative proof of "sufficiency" part of Sexauer-Warnock theorem [4, Th.]. It is an easy consequence of the following theorem, which is an improvement of Theorem 1 in the previous paper [3]. As to notations, we shall follow [3].

Theorem. *Let* $\{A_\lambda\}_{\lambda\in\Lambda}$ *be a family of left ideals of* R *in* J *and* $\mathfrak{A}=\{(a_{\lambda\nu})\in(R)_\Lambda;\ a_{\lambda\nu}\in A_\nu$ *for every* $\lambda\in\Lambda\}$. *Then the following conditions are equivalent.*

(1) $\{A_\lambda\}_{\lambda\in\Lambda}$ *is a left* T*-nilpotent family.*[1)]

(2) *Let* M *be an* R*-module and* N *an* R*-submodule of* M. *Suppose that for every finite subset* F *of* Λ, $M=N+(\sum_{\lambda\in\Lambda-F}A_\lambda)M$. *Then* $M=N$.

(3) $\mathfrak{A}$ *is contained in the radical of* $(R)_\Lambda$.

Proof. (1) ⇨ (2) is by the method due to Bass [1, pp. 473–474]. (3) ⇨ (1); (Cf. [3, Th. 1]). (2) ⇨ (3); Let X be an element of $\mathfrak{A}$, A_λ^* the left ideal of R generated by the elements of the λ-column of X, and for an arbitary finite subset F of Λ, X_F a matrix such that for $\lambda\in F$ the λ-column of X is that of X and O's elsewhere. Since X_F is contained in $(J)_\Lambda$, by the same method as in [2, Th. 1. 7. 3], we can see that X_F is contained in the radical of $(R)_\Lambda$. Let X_F' be the quasi-inverse of X_F, and $Y=(I-X_F')(X-X_F)$. Since for $\lambda\in F$ the λ-column of $X-X_F$ is 0,

$$R^{(\Lambda)}=R^{(\Lambda)}(I-Y)+R^{(\Lambda)}\cdot Y=R^{(\Lambda)}(I-Y)+(\sum_{\lambda\in\Lambda-F}A_\lambda^*)R^{(\Lambda)}=R^{(\Lambda)}(I-X)+(\sum_{\lambda\in\Lambda-F}A_\lambda)R^{(\Lambda)}.$$

Hence $R^{(\Lambda)}=R^{(\Lambda)}(I-X)$ and X is quasi-regular.

References

[1] H. BASS; Finistic dimension and a homological generalization of semi-primary rings, Trans. Amer. Math. Soc., 95 (1960), 466–488.

[2] N. JACOBSON; *Structure of rings*, Providence, 1956.

[3] K. MOTOSE; A note on perfect rings and semi-perfect rings, J. Fac. Sci. Shinshu

1) In this paper, the term "a left T-nilpotent family" will mean "a right vanishing family" (Cf. [4, pp. 288]).

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[4] N. E. SEXAUER and J. E. WARNOCK; The radical of the row-finite matrices over an arbitary ring, Trans. Amer. Math. Soc., 139 (1969), 287–295.