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A Covering Theorem for Simple Rings

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Let U be a simple Artinian ring with 1 and S, T_1, $\cdots$, T_n simple Artinian subrings of U containing 1. The purpose of this paper is to prove the following theorem which contains Theorem 2 of [4].

Theorem. *If* $S \subseteqq T_1 \cup T_2 \cup \cdots \cup T_n$ *and* $S \cap T_i$ *is a simple Artinian ring for each* i, *then* S *is contained in one of the* T_i.

Proof. Since $S=(T_1 \cap S) \cup (T_2 \cap S) \cup \cdots \cup (T_n \cap S)$, it suffices to prove the case $S = T_1 \cup T_2 \cup \cdots \cup T_n$.

Case I : Let S be an infinite simple Artinian ring.

Suppose $n>1$ and no T_i can be dropped out from $S=T_1 \cup T_2 \cup \cdots \cup T_n$. Namely, every T_i contains an element t_i with $t_i \notin T_j$ for $j \neq i$. We set $S_0 = T_1 \cap T_2 \cap \cdots \cap T_n$ and $D_0 = D \cap T_1 \cap T_2 \cap \cdots \cap T_n$ where D is a division ring component of S respectively. Then, by B. H. Neuman's theorem [2, (4.4) or 3, Lemma 1], the additive group S_0 is of finite index in S. Therefore, $D_0 = S_0 \cap D$ is of finite index in D and hence D_0 must be an infinite subring of S. Further, noting that an arbitrary element of a simple Artinian ring is either a unit or a left zero divisor, we can easily see that D_0 is a division ring. Accordingly, $\{t_1 + dt_2 | d \in D_0\}$ is an infinite subset of $S = T_1 \cup T_2 \cup \cdots \cup T_n$, and hence, there exists one T_k containing $t_1 + dt_2$, $t_1 + d't_2$ with $d \neq d'$ ($d, d' \in D_0$). Hence, we have a contradiction t_1, $t_2 \in T_k$.

Case II : Let S be a finite simple ring.

Let $C = GF(p^g)$ be the center of S where p is a prime, c a generator of cyclic group $C^* = C - \{0\}$ and e a primitive idempotent of S. Then ce is contained in $T = T_k$ for some k, and so $e = (ce)^{p^g-1}$ is in T. Noting that $[T|T]$ (the capacity of T) is a divisor of $[S|S]$ [1, Prop. 6.6.2] and e is primitive in S (and so in T), we can easily see that $[S|S] = [T|T]$. If $T = \sum_{i,j} C_0 f_{ij}$, where $\{f_{ij}'s\}$ is a system of matrix units of T such that $C_0 = V_T(\{f_{ij}'s\})$ is the center of T then $S = \sum_{i,j} C f_{ij}$. If $e = \sum_{i,j} a_{ij} f_{ij}$ ($a_{ij} \in C_0$) and $a_{kl} \neq 0$ for some k, l, then $\sum_i f_{ik} c e f_{li} = c a_{kl}$ is contained in T and c is so in C_0. Therefore, $C \subseteqq C_0$ and $S = T$.

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