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Note on the Endomorphism Ring

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Let R be a ring with 1, ${}_RM$ a unital left R-module, $J(M)$ the radical of M and S the R-endomorphism ring of M acting on the right. As to other notations and terminologies used here, we shall follow [1] and [5]. In this note, we shall give a sufficient condition for $J(S)$ $(=J({}_SS))$ to be T-nilpotent.

Theorem 1. *If* ${}_RM$ *is quasi-projective and satisfies the descending chain condition for small submodules,*[1)] *then* $J(S)$ *is right* T*-nilpotent.*

Proof. We note first that for every element s of $J(S)$, *Ms is small in* M ([7, Lemma 1]). Let $\{s_i\}_{i=1,2,\cdots}$ be an arbitrary sequence of elements of $J(S)$. Then we have a descending chain of small submodules $Ms_1 \supseteq Ms_2s_1 \supseteq \cdots \supseteq Ms_ns_{n-1}\cdots s_1 \supseteq \cdots$ and hence there exists a natural number n such that $Ms_ns_{n-1}\cdots s_1 = Ms_{n+1}s_ns_{n-1}\cdots s_1$. Therefore, $M = Ms_{n+1} + Ker\ s_ns_{n-1}\cdots s_1$. Since Ms_{n+1} is small in M, $M = Ker\ s_ns_{n-1}\cdots s_1$ and hence $s_ns_{n-1}\cdots s_1 = 0$.

For a ring with 1, the following contains [3, Th. 3.1.1].

Corollary 1. *If* ${}_RM$ *is quasi-projective and* $J(M)$ *is Artinian, then* $J(S)$ *is nilpotent.*

Proof. Since $MJ(S)$ is a sum of small submodules, we obtain the descending chain $J(M) \supseteq MJ(S) \supseteq MJ(S)^2 \supseteq \cdots \supseteq MJ(S)^k \supseteq \cdots$, and so $MJ(S)^k = MJ(S)^{k+1}$ for some integer k. Since $J(S)$ is right T-nilpotent (Th. 1), the argument used in [1, pp. 473–474] implies $MJ(S)^k = 0$ and hence $J(S)^k = 0$.

Combining [5, Th. 3. 10] with Cor. 1, we can see the following

Corollary 2. *If* ${}_RM$ *is finitely generated projective and Artinian, then* S *is a semi-primary ring with nilpotent radical.*

The next is a dual of Th. 1

Theorem 2. *If* ${}_RM$ *is quasi-injective and satisfies the ascending chain condition for essential submodules, then* $J(S)$ *is left* T*-nilpotent.*

Proof. One may remark first that for every element s of $J(S)$, $Ker\ s$ is essential in M ([4, Ex. 4.4.8]). Let $\{s_i\}_{i=1,2,\cdots}$ be an arbitrary sequence of elements of $J(S)$. Then we have the ascending chain of essential submodules $Ker\ s_1$

1) "small" means "d-dense" in the sense of [5].

$\subseteq Ker\ s_1s_2 \subseteq \cdots \subseteq Ker\ s_1s_2\cdots s_n \subseteq \cdots$ and hence there exists a natural number n such that $Ker\ s_1s_2\cdots s_n = Ker\ s_1s_2\cdots s_ns_{n+1}$. Therefore, $Ker\ s_{n+1} \cap Ms_1s_2\cdots s_n = 0$. Since $Ker\ s_{n+1}$ is essential in M, $Ms_1s_2\cdots s_n = 0$ and hence $s_1s_2\cdots s_n = 0$.

Corollary 3. *If $_RM$ is quasi-injective and Noetherian, then $J(S)$ is nilpotent.*

Proof. Since M is Noetherian, S satisfies the ascending chain condition for left annihilators ([6, Lemma]). On the other hand, $J(S)$ is left T-nilpotent by Th. 2 and hence $J(S)$ is nilpotent by [2, Prop. 1.5].

Combining [4, Prop. 4.4.2] with Cor. 3, we can see the following.

Corollary 4. *If $_RM$ is injective and Noetherian, then S is a semi-primary ring with nilpotent radical.*

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