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Weighted Trace Functions as Examples of Morse Functions

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Introduction.

In this paper, we give examples of Morse functions on $O(n)$, $U(n)$, $SU(n)$, $Sq(n)$, G_2. $V_{n,m} = O(n)/O(n-m)$ and $G_{n,m} = O(n)/O(m) \times O(n-m)$.

The results are as follows:

We set

$$O(n) = \{(x_{ij}) = Y \in \mathfrak{M}\ (n,\ \boldsymbol{R}) \mid {}^tXX = E\},$$

$$U(n) = \{(x_i + y_{ij}\boldsymbol{i}) = X \in \mathfrak{M}\ (n,\ C) \mid {}^tX\overline{X} = E\},$$

$$S_p(n) = \{(x_{ij} + u_{ij}\boldsymbol{i} + v_{ij}\boldsymbol{j} + w_{ij}\boldsymbol{k}) = X \in \mathfrak{M}\ (n,\ H) \mid {}^t\overline{X}X = E\},$$

$$G_2 = \{X \in \mathfrak{M}\ (8,\ R) \mid X\colon \mathfrak{C} \longrightarrow \mathfrak{C}\ \textit{an automorphism of Cayley numbers}\}.$$

Then the Morse functions of $O(n)$, $U(n)$ and $S_p(n)$ are given by weighted trace functions

$$\varphi(x) = \sum_{i=1}^{n} \alpha_i x_{ii}, \quad 0 < \alpha_1 < \cdots\cdots < \alpha_n.$$

The Morse functions of $V_{n,m}$ are

$$\varphi(x) = \sum_{i=1}^{m} \alpha_i x_{ii}, \quad 0 < \alpha_1 < \cdots\cdots < \alpha_m.$$

The Morse functions of G_2 and $SU(n)$ are given by the same form, but their coefficients α_i need to satisfy some conditions (cf. Lemma 5 of § 4).

The Morse functions of $G_{n,m}$ are given by

$$\varphi(x) = \sum_{i,j} \varepsilon_i \alpha_j x_{ij}, \quad \varepsilon_i = 1, \text{ if } i = 1, \cdots\cdots, k,$$

$$-1, \text{ if } i = k+1, \cdots\cdots, n$$

$$0 < \alpha_1 < \cdots\cdots < \alpha_n.$$

They are different from above functions but calculations are given by the same method.

We note that the Morse indices of the above functions show that they are best possible.

The outline of this paper is as follows: In § 1, we state some general theorems for explicit calculations of critical points. They are proved in § 2. The related results are shown in § 3 using these theorems. § 4 is an appendix but the possibilities of the existence of the Morse functions of G_2 and $SU(n)$ defined in § 3 is shown by Lemma 5 of this section.

In this paper, we refer [1], [2] for the theory of Morse functions and the method of calculations of singularities of mappings.

§ 1. Some Theorems.

We denote by $\boldsymbol{R}^n$ the the n-dimensional Euclidean space.

Let $f=(f^1, \cdots\cdots, f^k)$ be a smooth mapping from $\boldsymbol{R}^n$ to $\boldsymbol{R}^k$ and V the zero set of f, *i.e.*

$$V=\{x\in \boldsymbol{R}^n \mid f(x)=0\}.$$

We assume Df, the Jacobian of f, is not equal to 0 on V. Then, it is easy to see from the implicit function theorem that V is a smooth manifold. For a smooth function φ of $\boldsymbol{R}^n$, we denote $\varphi|_V$ the restriction of φ on V. The gradient vector at p is denoted by $\nabla\varphi(p)$.

Theorem 1. *A point p of V is a critical point of $\varphi|_V$, if and only if $\nabla f(p)$ is a linear combinaton of $\{f^i(p)\}$, i.e. there exist real numbers $\{a_i\}_{i=1,\cdots\cdots,k}$ such that*

$$\nabla\varphi(p)=\sum_{i=1}^{k} a_i \nabla f^i(p).$$

This theorem is proved in the proof of Theorem 2.

We denote the Hessian of φ at p by $\nabla^2\varphi(p)$, i.e.

$$\nabla^2\varphi(p)=\left(\frac{\partial^2\varphi}{\partial x_i \partial x_j}(p)\right)\ i,\ j=1,\ \cdots\cdots,\ n.$$

The orthogonal projection from $\boldsymbol{R}^n$ to the tangent plane of V at p is denoted by P. We set

$$M_{(p)}=P(V^2\varphi(p)-\sum_{i=1}^{k} a_i \nabla^2 f^1(p))P.$$

Theorem 2. *We assume that p is a critical point of $\varphi|_V$. Then p is nondegenerated if and only if*

$$\text{rank } M_{(p)} = \dim V(=n-k).$$

Moreover, the index of $\varphi|_V$ *at* p *is the number of negative eigenvalues of* $M_{(p)}$.

This theorem is proved in the next section.

Theorem 3. *Let* π: $\tilde{N} \to N$ *be a locally trivial smooth fibre space over a smooth manifold* N, *and* φ: $N \to \boldsymbol{R}$ *a smooth fvnction on* N. *Then* p, *a point of* N *is a critical point of* φ, *if and only if any point of* $\pi^{-1}(p)$ *is a critical point of* $\varphi \circ \pi$.

Moreover, a critical point p *of* φ *is nondegenerated if and only if the rank of* $\nabla^2(f \circ \pi)$ *on* $\pi^{-1}(p)$ *is equal to* dim. N, *and the index of* f *at* p *is the numbers of negative eigenvalues of* $\nabla^2(f \circ \pi)$.

The proof of Theorem 3 is straightforword from the local triviality of π. Therefore, we prove only Theorem 2.

§2. Proof of Theorem 2.

Lemma 4. *In the proof of Theorem 2, we can assume without loss of generality, the followings for some coordinate of* $\boldsymbol{R}^n$.

$$p = 0,$$

$$\nabla f^1(0) = \nabla\varphi(0) = (1, 0, \cdots\cdots, 0),$$

$$\nabla f^i(0) = (0, \cdots\cdots 0, 1, 0, \cdots\cdots, 0) = e_i, \text{ the } i\text{-th canonical base.}$$

Proof. At first, we can choose a coordinate $(x_1, \cdots\cdots, x_n)$ of $\boldsymbol{R}^n$ such that $p = 0$, $\varphi(0) = (1, 0, \cdots\cdots, 0)$ and the tangent space of V at 0 is given by $x_1 = x_2 = x_k = 0$.

Let $g^1 = \sum_i a_i f^i$ and $g^j = \sum_i a_i^j f^i$, where a_i^j is defined by

$$e_j = \sum_i a_i^j \nabla f^i(0).$$

The existence of $\{a_i^j\}$ is verified from rank$(Df) = n - k$. Then V is also defined by the zeros of $\{g^i\}$ and $\{g^i\}$ satisfies the above conditions. We take $\{g^i\}$ the place of $\{f^i\}$, then we have Lemma 4.

Proof of Theorem 2. We can define a local coordinate of V on a neighbourhood of $p = 0$ by $u = (u_1, u_2, \cdots\cdots, u_{n-k})$ such that

$$i(u) = (F_1(u), \cdots\cdots, F_k(u), u_1, \cdots\cdots, u_{n-k})$$

where i is the inclusion of V into $\boldsymbol{R}^n$ and $F_1, \cdots\cdots, F_k$, are some smooth functions.

Then

$$\frac{\partial}{\partial u_i}\varphi(F_1(u),\ \cdots\cdots,\ u_1)=\sum_{\alpha=1}^{k}\frac{\partial\varphi}{\partial x_\alpha}\frac{\partial F_\alpha}{\partial u_i}+\frac{\partial\varphi}{\partial x_{i+k}}$$

$$(\frac{\partial}{\partial u_i}\varphi(F_1(u),\ u_1,\ \cdots\cdots))_0=(\frac{\partial F_1}{\partial u_i})_0$$

On the other hand

$$(*)\qquad 0=(\frac{\partial}{\partial u_i}f^j(F_1(u),\ \cdots\cdots,\ u_1,\ \cdots\cdots))_0$$

$$=(\sum_{\alpha=1}^{k}\frac{\partial f^j}{\partial x_\alpha}\frac{\partial F_\alpha}{\partial u_i}+\frac{\partial f^j}{\partial x_{i+k}})_0=(\frac{\partial F_j}{\partial u_i})_0.$$

Thus we have

$$(\frac{\partial}{\partial u_i}\varphi(F_1(u),\ \cdots\cdots,\ u_1,\ \cdots\cdots))_0=0.$$

We note that Theorem 1 follows from this formula. We also have

$$(\frac{\partial^2\varphi(F_1(u),\ \cdots\cdots,\ u_1,\ \cdots\cdots)}{\partial u_i\partial u_j})_0$$

$$=(\sum_{\alpha,\beta=1}^{k}\frac{\partial^2\varphi}{\partial x_\alpha\partial x_\beta}\frac{\partial F_\alpha}{\partial u_i}\frac{\partial F_\beta}{\partial u_j}+\sum_{\alpha=1}^{k}\frac{\partial^2\varphi}{\partial x_\alpha\partial x_{k+1}}\frac{\partial F_\alpha}{\partial u_i}+$$

$$+\sum_{\alpha,\beta=1}^{k}\frac{\partial\varphi}{\partial x_\beta}\frac{\partial^2F_\beta}{\partial x_\alpha\partial u_j}\frac{\partial F_\alpha}{\partial u_j}+\sum_{\alpha=1}^{k}\frac{\partial\varphi}{\partial x_\alpha}\frac{\partial^2F_\alpha}{\partial u_i\partial u_j}+\sum_{\beta=1}^{k}\frac{\partial^2\varphi}{\partial x_\beta\partial x_{k+i}}\frac{\partial F_\beta}{\partial u_j}+$$

$$+\sum_{\beta=1}^{k}\frac{\partial\varphi}{\partial x_\beta}\frac{\partial^2F_\beta}{\partial x_{k+i}\partial u_j}+\frac{\partial^2\varphi}{\partial x_{k+i}\partial x_{k+j}})_0$$

$$=(\frac{\partial^2\varphi}{\partial x_{k+i}\partial x_{k+j}})_0-(\frac{\partial^2F_1}{\partial u_i\partial u_j})_0.$$

on the other hand, from (*), we have

$$0=(\frac{\partial^2}{\partial u_i\,\partial u_j}f^1(F_1(u),\ \cdots\cdots,\ u_1\cdots\cdots))_0$$

$$=(\frac{\partial^2f^1}{\partial x_{k+i}\partial x_{k+j}})_0+(\frac{\partial^2F_1}{\partial u_i\partial u_j})_0.$$

Thus we have

$$(\frac{\partial^2F_1}{\partial u_i\partial u_j})_0=-(\frac{\partial^2f^1}{\partial x_{k+i}\partial x_{k+j}})_0.$$

Let P be the orthogonal projection to the tangent plane of V at 0, *i.e.*

$$P(x_1, \cdots\cdots, x_n) = (0, \cdots\cdots, 0, x_{k+1}, \cdots\cdots, x_n).$$

Then

$$M_{(0)} = P(\frac{\partial^2\varphi}{\partial x_i \partial x_j} - \frac{\partial^2 f^1}{\partial x_i \partial x_j})_0 P$$

$$= 0_k \oplus (\frac{\partial^2 \varphi | \mathrm{V}}{\partial u_i \partial u_j})_0.$$

Hence $P(\frac{\partial^2\varphi}{\partial x_i \partial x_j} - \sum_{h=1}^{k} a_h \frac{\partial^2 f^h}{\partial x_i \partial x_j})_0 P$ is similar to $0_k \oplus (\frac{\partial^2\varphi | \mathrm{V}}{\partial u_i \partial u_j})_0$ for a general coordinate without conditions of Lemma 4. Thus we have Theorem 2.

§ 3. Some examples.

In this section we give Morse functions of some spaces as examples of Theorems 1, 2 and 3.

1. $O(n)$, $U(n)$ and $Sp(n)$.

They are represented in $\boldsymbol{R}^{n^2}$, $\boldsymbol{R}^{2n^2}$, R^{4n^2}, as in the introduction. And we set $\varphi(x) = \sum_{i=1}^{n} \alpha_i x_{ii}$, $0 < \alpha_1 < \cdots\cdots < \alpha_n$,

Then nondegenerated critical points of each case are commonly

$$\left\{ \begin{pmatrix} \varepsilon_1 & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \varepsilon_n \end{pmatrix} = p \;\middle|\; \varepsilon_i = \pm 1 \right\}.$$

and the the index at p is $\sum_{i=1}^{n} (\frac{\varepsilon_i + 1}{2})(ai - 1)$, where a is 1, 2 and 4 respectively $O(n)$, $U(n)$ and $Sp(n)$. We give a proof of the case of $U(n)$. The others are obtained similarly.

Proof of the case of $U(n)$.

we set

$$U(n) = \{(x_{ij} + y_{ij}\boldsymbol{i}) = x \in \mathfrak{M}(n, \boldsymbol{C}) \mid f_{kl} = \sum_{i=1}^{n} (x_{ki}x_{li} + y_{ki}y_{li}) - \delta_{kl} = 0 \quad \text{for}$$

$$1 \leq k \leq l \leq n, \quad g_{kl} = \sum_{i=1}^{n} (x_{ki}x_{li} - x_{lj}y_{ki}) = 0 \;\; for \;\; 1 \leq k < l \leq n\}$$

It is easily verefied that $\{\nabla f_{kk}, \nabla f_{kl}, \nabla g_{kl}\}$ is linearly independent. The Grammian of $\{\nabla f_{kk}, \nabla f_{kl}, \nabla g_{kl}, \nabla \varphi\}$ is

$$G=\left|\begin{array}{cccccccc} 4 & & & & & & \vdots & \\ & \ddots & & & & 0 & \cdots 2\alpha_l x_{ll}, \cdots\cdots & :ll\\ & & 4 & & & & \vdots & \\ & & & 2 & & & \vdots & \\ & & & & \ddots & & \cdots\alpha_l x_{kl}+\alpha_k x_{lk}, \cdots & :lk\\ & 0 & & & & \ddots & \vdots & \\ & & & & & & -\alpha_l y_{kl}+\alpha_k y_{lk}\cdots & :lk\\ & & & & & 2 & \vdots & \\ \cdots, & 2\alpha_l x_{ll}, & \cdots, & \alpha_l x_{kl}+\alpha_k x_{lk}, & \cdots, & -\alpha_l y_{kl}+\alpha_k y_{lk} \cdots, & \sum_{i=1}^{n}\alpha_i^2, & \end{array}\right|,$$

Now, we assume $G=0$ on $U(n)$. Then we have

$$\alpha_l x_{kl}=\alpha_k x_{lk},\quad \alpha_l y_{kl}=-\alpha_k y_{lk},$$

From $f_{11}(X)=f_{11}({}^tX)=0$, we have

$$1-x_{11}^2-y_{11}^2=x_{12}^2+y_{12}^2+\cdots+x_{ln}^2+y_{ln}^2$$
$$=x_{21}^2+y_{21}^2+\cdots+x_{nl}^2+y_{nl}^2.$$
$$\left(\frac{\alpha_2^2}{\alpha_1^2}-1\right)x_{12}^2+\cdots+\left(\frac{\alpha_n^2}{\alpha_1^2}-1\right)x_{ln}^2=0.$$

Thus we have

$$x_{12}=y_{12}=x_{13}=y_{13}=\cdots=x_{ln}=y_{ln}=0,$$
$$x_{21}=y_{21}=x_{31}=y_{31}=\cdots=x_{nl}=y_{nl}=0.$$

Inductively, from $f_{ii}(X)=f_{ii}({}^tX)=0$, we obtain

$$x_{ij}=y_{ij}=0,\ \text{for } i\neq j.$$

And from $\alpha_i y_{ii}=-\alpha_i y_{ii}$, we also have $y_{ii}=0$.

Thus, $x_{ii}=\pm 1$. Hence from Theorem 2, we have that the critical points

$$\text{are}\left\{\left(\begin{array}{ccc}\varepsilon_1 & & 0\\ & \ddots & \\ 0 & & \varepsilon_n\end{array}\right)\middle|\ \varepsilon_i=\pm 1\right\}$$

Now, we shall calculate the index at $p=\begin{pmatrix} \varepsilon_1 & & & 0 \\ & \cdot & & \\ & & \cdot & \\ 0 & & & \varepsilon_n \end{pmatrix}$.

By simple calculation, we have

$$\nabla\varphi(p) = \sum_{i=1}^{n} \frac{\alpha_i\varepsilon_i}{2} \nabla f_{ii}(p).$$

Let the coordinate (x_{ij}) be ordered as

$$(x_{11},\ x_{22},\ \cdots\cdots,\ x_{nn},\ x_{12},\ x_{21},\ \cdots\cdots,\ x_{n,\,n-1},\ y_{11},\ \cdots\cdots,\ y_{n,\,n-1})$$

Then P, the projection of Theorem 2, is represented as

$$P = \begin{pmatrix} 0 & & & & & & & & & & & \\ & \cdot & & & & & & & & & 0 & \\ & & \cdot & & & & & & & & & \\ & & & 0 & & & & & & & & \\ & & & & A_{12} & & & & & & & \\ & & & & & A_{13} & & & & & & \\ & & & & & & \cdot & & & & & \\ & & & & & & & \cdot & & & & \\ & & & & & & & & A_{n-1,\,n} & & & \\ & & & & & & & & & 1 & & \\ & & & & & & & & & & \cdot & \\ & & & & & & & & & & & \cdot \\ & & 0 & & & & & & & & & 1 \\ & & & & & & & & & & & A_{12} \\ & & & & & & & & & & & \cdot \\ & & & & & & & & & & & \cdot \\ & & & & & & & & & & & A_{n-1,\,n} \end{pmatrix}$$

where

$$A_{ij} = \begin{pmatrix} 1, & \varepsilon_i\varepsilon_j \\ \varepsilon_i\varepsilon_j, & 1 \end{pmatrix}$$

Hence we get

$$M_{(p)} = P\,(-\sum_{i=1}^{n} \frac{\alpha_1\varepsilon_1}{2} \nabla f_{ii(p)})\,P.$$

$$= \begin{pmatrix} 0 & & & & & & & & & & & \\ & \cdot & & & & & & & & & 0 & \\ & & \cdot & & & & & & & & & \\ & & & 0 & & & & & & & & \\ & & & & B_{12} & & & & & & & \\ & & & & & \cdot & & & & & & \\ & & & & & & \cdot & & & & & \\ & & & & & & & B_{n-1,\,n} & & & & \\ & & & & & & & & -\varepsilon_1\alpha_1 & & & \\ & & & & & & & & & \cdot & & \\ & & & & & & & & & & \cdot & \\ & & 0 & & & & & & & & & -\varepsilon_n\alpha_n \\ & & & & & & & & & & & B_{12} \\ & & & & & & & & & & & \cdot \\ & & & & & & & & & & & \cdot \\ & & & & & & & & & & & B_{n-1,\,n} \end{pmatrix}$$

where $B_{ij}=\begin{pmatrix} -\varepsilon_i\alpha_i-\varepsilon_j\alpha_j, & -\varepsilon_i\alpha_j-\varepsilon_j\alpha_i \\ -\varepsilon_i\alpha_j-\varepsilon_j\alpha_i, & -\varepsilon_i\alpha_i-\varepsilon_j\alpha_j \end{pmatrix}$

Therefore the eigen values of $M_{(p)}$ are

$$\begin{cases} 0, \quad \text{multiplicity } n, \\ -(\alpha_i-\alpha_j)(\varepsilon_i-\varepsilon_j), \quad i\neq j, \\ -(\alpha_i+\alpha_j)(\varepsilon_i+\varepsilon_j), \quad i\neq j, \\ -\alpha_i\varepsilon_i, \quad i=1, \cdots\cdots, n. \end{cases}$$

Hence we have rank $M_{(p)}=n^2$.

Thus, each critical point is nondegenerated and the index is $\sum_{i=1}^{n}((\varepsilon_i+1)/2)(2(n-i)+1)$ by Theorem 2. Therefore, $\varphi(x)$ is a Morse function of $U(n)$ and it is best possible.

2. $V_{n,m}$, $\boldsymbol{C}V_{n,m}$ and $\boldsymbol{H}V_{n,m}$.

We set

$$V_{n,m}=O(n)/O(n-m); \text{ Real Stiefel manifold.}$$
$$\boldsymbol{C}V_{n,m}=U(n)/U(n-m): \text{ Complex Stiefel manifold.}$$
$$\boldsymbol{H}V_{n,m}=Sp(n)/Sp(n-m); \text{ Quarternionic Stiefel manifold.}$$

We use the same coordinate as in **1.** and consider that the groups $o(n-m)$, $U(n-m)$ and $Sp(n-m)$ act on the last $(n-m)$-coordinates. We set

$$\varphi(x)=\sum_{i=1}^{m}\alpha_i x_{ii}, \quad 0<\alpha_1<\cdots\cdots<\alpha_m.$$

Then $\varphi(x)$ is invariant under the actions of $O(n-m)$, etc. By the same calculations as above and by Theorem 3, we have the following results.

The critical points of $\varphi|_V$ are $\left\{ p=\begin{pmatrix} \varepsilon_1 & & 0 \\ & \ddots & \\ 0 & & \varepsilon_m \\ \hline & * & \end{pmatrix} \middle| \varepsilon_i=\pm 1 \right\}$.

In $O(n)$-case, the eigenvalues at p are

$$\begin{cases} 0, \quad \text{multiplicity } (n+(n-m)(n-m-1)), \\ -(\alpha_i-\alpha_j)(\varepsilon_i-\varepsilon_j), \quad i<j\leqq m, \\ -(\alpha_i+\alpha_j)(\varepsilon_i+\varepsilon_j), \quad i<j\leqq m, \\ -\alpha_i(\varepsilon_i-\varepsilon_j), \quad i\leqq m<j, \\ -\alpha_i(\varepsilon_i+\varepsilon_j), \quad i\leqq m<j. \end{cases}$$

Thus critical points are nondegenerated and the index at p is $\sum_{i=1}^{m}((\varepsilon_i+1)/2)(n+i-m-1)$.

In $CV_{n,m}$ (resp. $HV_{n,m}$)-case, similar calculations show that the index at p is $\sum_{i=1}^{m}((\varepsilon_i+1)/2)(2(n-m+i)-1)$ (resp. $\sum_{i=1}^{m}((\varepsilon_i+1)/2)(4(n-m+i)-1)$. Therefore these $\varphi(x)$ are Morse functions of $V_{n,m}$, $CV_{n,m}$ and $HV_{n,m}$.

3. G_2.

We give an order of the cannonical base of $\mathfrak{C}$, the Cayley number field, by

$$(1,\ e_1,\ e_2,\ e_3,\ e_2e_3,\ e_3e_1,\ e_1e_2,\ e_1(e_2e_3)).$$

G_2 is represented by

$$G_2=\{X\in \mathfrak{M}(8,\ \boldsymbol{R})|\ X\colon\ \mathfrak{C}\to\mathfrak{C};\ \textit{an automorphism of Cayley numbers}\}.$$

Simple calculations show that G_2 is given simply, by

$$\begin{aligned} G_2=\{X|\ X={}^t(x_{11},\ \cdots\cdots,\ x_{17},\ x_{21},\ \cdots\cdots,\ x_{27},\ x_{31},\ \cdots\cdots,\ x_{37}),\\ f_{ij}=\sum_{l=1}^{7}x_{il}x_{jl}-\delta_{ij}=0,\ \ 1\leqq i\leqq j\leqq 3,\\ g=\mathrm{Re.}\,(x_1(x_2x_3)+(x_1x_2)x_3)=0\},\end{aligned}$$

where $x_i=(0,\ x_{i1},\ \cdots\cdots,\ x_{i7})$ and the product is of Cayley numbers.

We set

$$\varphi(x)=\alpha x_{11}+\beta x_{22}+\gamma x_{33},$$

where $(\alpha,\ \beta,\ \gamma)$ satisfies the conditions of Lemma 5 (given in the last section of this paper).

The Grammian of $(\nabla f_{ij},\ \nabla g, \nabla\varphi)$ is

$$G=\begin{vmatrix} 4 & & & & & & & 2\alpha x_{11}\\ & 4 & & & 0 & & & \vdots\\ & & 4 & & & & & \beta x_{32}+\gamma x_{23}\\ & & & 2 & & & & \vdots\\ & & & & 2 & & & \\ & 0 & & & & 2 & & \vdots\\ & & & & & & 3 & (x_2x_3)_1+(x_3x_1)_2+(x_1x_2)_3\\ 2x_{11}, & \cdots, & x_{32}+x_{23}, & \cdots, & (x_2x_3)_1+(x_3x_1)_2+(x_1x_2)_3, & & & \alpha^2+\beta^2+\gamma^2\end{vmatrix}.$$

Now, we assume $G=0$, then, similarly in $U(n)$-case, the critical points are of the following forms.

$$p=\begin{pmatrix}\varepsilon_1, & 0, & 0, & \beta\gamma\mu, & 0, & 0, & 0\\ 0\,, & \varepsilon_2, & 0, & 0, & \gamma\alpha\mu, & 0, & 0\\ 0\,, & 0, & \varepsilon_3, & 0, & 0, & \alpha\beta\mu, & 0\end{pmatrix}.$$

We put $\beta\gamma\mu=\sin\theta_1$,

$\gamma\alpha\mu=\sin\theta_2$,

$\alpha\beta\mu=\sin\theta_3$.

Then $\frac{\sin\theta_1}{\alpha}=\frac{\sin\theta_2}{\beta}=\frac{\sin\theta_3}{\gamma}$. Since $g=0$ on G_2, we have

$$\theta_1\pm\theta_2\pm\theta_3\equiv 0 \pmod{2\pi}.$$

From Lemma 5 we have $\theta_1\equiv\theta_2\equiv\theta_3\equiv 0 \pmod{\pi}$. That is,

$$\mu=0,\ \varepsilon_1,\ \varepsilon_2,\ \varepsilon_3=\pm 1.$$

Hence the nondegenerated critical points are

$$\left\{P(\varepsilon_1,\ \varepsilon_2,\ \varepsilon_3)=\begin{pmatrix}\varepsilon_1 & 0 & 0 & \\ 0 & \varepsilon_2 & 0 & 0\\ 0 & 0 & \varepsilon_3 & \end{pmatrix}\,\middle|\,\varepsilon_1,\ \varepsilon_2,\ \varepsilon_3\pm 1\right\},$$

And straightforward calucuiations show that $I(\varepsilon_1,\ \varepsilon_2,\ \varepsilon_3)$, the index at $P(\varepsilon_1,\ \varepsilon_2,\ \varepsilon_3)$, is

$I(-1,\ -1,\ -1)=0$ $\quad I(-1,\ -1,\ 1)=3$

$I(-1,\ 1,\ -1)=4$ $\quad I(1,\ -1,\ -1,)=5$

$I(-1,\ 1,\ 1)=9$ $\quad I(1,\ -1,\ 1,)=10$

$I(1,\ 1,\ -1)=11$ $\quad I(1,\ 1,\ 1)=14$,

Therefore $\varphi(X)$ is a Morse function of G_2 and it is best possible.

4, $SU(n)$.

We give the special unitary group $SU(n)$ by

$$SU(n)=\{(x_{ij}+y_{ij}\boldsymbol{i})=X\,|\,{}^t\overline{X}X=E, \det X=1\}$$

Let $\varphi(X)=\sum_{i=1}^{n}\alpha_i x_{ii}$, and α_i satisfy the conditions of Lemma 5.

Straightforward caluculations show that critical points are of following forms.

$$p=\begin{pmatrix}\varepsilon_1+\alpha_1\mu\boldsymbol{i} & & & 0\\ & \cdot & & \\ & & \cdot & \\ 0 & & \cdot & \varepsilon_n+\alpha_n\mu\boldsymbol{i}\end{pmatrix}$$

Now we put $\varepsilon_i + \alpha_i \mu \boldsymbol{i} = e^{\theta \boldsymbol{i}}$. Since det $P = 0$,

$$\sum_{i=1}^{n} \theta_i \equiv 0 \pmod{2\pi}.$$

And $\frac{\sin \theta_1}{\alpha_1} = \cdots\cdots = \frac{\sin \theta_n}{\alpha_n}$ $(=\mu)$. By Lemma 5, the critical points of φ are

$$\left\{ \begin{pmatrix} \varepsilon_1 & & & 0 \\ & \cdot & & \\ & & \cdot & \\ 0 & & & \varepsilon_n \end{pmatrix} = p \mid \prod_{i=1}^{n} \varepsilon_i = 1 \right\}.$$

Simmilarly as in **1.**, we have that the index at p is equal to

$$\sum_{i=1}^{n} ((\varepsilon_i + 1)/2)(2i - 1).$$

Therefore $\varphi(x)$ is a Morse function of $SU(n)$ and it is best possible.

5. $G_{n,m} = 0(n)/0(m) \times 0(n-m)$.

We use the same coordinates as in **1**, for $0(n)$.

We set

$$\varphi(x) = \sum_{i,j}^{n} \varepsilon_i \alpha_j x_{ij}, \quad \varepsilon_i = \begin{cases} 1, & i = 1, \cdots\cdots, m \\ -1, & i = m+1, \cdots\cdots, n, \end{cases}$$
$$0 < \alpha_1 < \cdots\cdots < \alpha_n.$$

Since φ is $0(n) \times 0(n-m)$-invariant, φ is a smooth function on $G_{n,m}$. This φ has a different form of φ in $1 \sim 4$, but by Theorems 2 and 3, straightforword calculations give us the following results.

Let τ be a combination of m-elements in the set of n-elements, then τ is represented in $O(n)$ as $\tau = (\tau_{ij})$, $\tau_{ij} = \delta_{i,\tau j}$ for $j \leqq m$ and $\tau_{ij} = \delta_{i,\bar{\tau}(j-m)}$ for $j > m$, where $\bar{\tau}$ is the complementary combination of τ. Then the critical points of $\varphi(x)$ are $\{\tau \mid \tau$ is a combination of m-elements in the set of n-elements$\}$, and the index at τ is the number of positive $(\tau_i - \bar{\tau}_j)$'s, where $i = 1, \cdots\cdots, m$ and $j = 1, \cdots\cdots, n-m$.

Therefore $\varphi(x)$ is a Morse function of $G_{n,m}$ and it is best possible.

§ 4. Appendix.

Lemma 5. *There exist positive numbers $\alpha_1, \cdots\cdots, \alpha_n$ such that*

$$0 < \alpha_1 < \alpha_2 < \cdots\cdots < \alpha_n,$$

and the equations

$$(*) \qquad \theta_1 \pm \cdots\cdots \pm \theta_n \equiv 0 \pmod{\pi}, \quad \frac{\sin \theta_1}{\alpha_1} = \cdots\cdots = \frac{\sin \theta_n}{\alpha_n},$$

have only the trivial solution $\theta_1 \equiv \cdots\cdots \equiv \theta_n \equiv 0 \pmod{\pi}$.

Proof. We put $\alpha_n = 1$, $\alpha_{n-1} = (n-1)\varepsilon$, $\cdots\cdots$, $\alpha_1 = \varepsilon$, where $0 < \varepsilon < 1/(n^2)$. If (*) has nontrivial solutions $(\theta_1, \cdots\cdots, \theta_n)$, then we can assume $\theta_i \in (0, \pi)$, (if not, we take $-\theta_i$). Moreover, we can assume $\theta_i \in (0, \pi/2]$, because if $\theta_i \in (\pi/2, \pi)$, then we may take $\theta_i' = \pi - \theta_i$ in the place of θ_i, then we have

$$\theta_1 \pm \cdots\cdots \pm \theta_i' \pm \cdots\cdots \equiv 0 \pmod{\pi}, \quad \sin \theta_i = \sin \theta_i'.$$

For $\varepsilon < 1/(n^2)$, we have

$$1 \geqq \sin \theta_n = \frac{1}{i\varepsilon} \sin \theta_i > n \sin \theta_i.$$

Then $\theta_1, \cdots\cdots, \theta_{n-1} \in (0, \pi/2n)$ and $\theta_n \in (0, \pi/2]$. Thus

$$\begin{aligned} \sin \theta_n &= \sin(\pm \theta_1 \pm \cdots\cdots \pm \theta_n) \\ &< \sin(\theta_1 + \cdots\cdots + \theta_n) \\ &< \sin \theta_1 + \cdots\cdots + \sin \theta_n = \frac{n(n-1)}{2} \varepsilon \sin \theta_n \\ &< \sin \theta_n. \end{aligned}$$

This is a cotradiction. Therefore, we have the Lemma.

Referenes.

[1]. MILNOR, J.: *Morse Theory*, Ann. of Math. Studies, 51. Princeton, 1963.

[2]. MILNOR, J.: *Singular Points of Complex Hypersurfaces*, Ann. of Math. Studies, 61. Priceton, 1968.