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Non-compact Simple Lie Group $F_{4,2}$ of Type F_4

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It is known that there exist three simple Lie groups of type F_4 up to local isomorphism, one of them is compact and the others are non-compact. The compact simple Lie group $F_4 = F_{4(-52)}$ of type F_4 is obtained as the automorphism group $\mathrm{Aut}(\mathfrak{J})$ of the exceptional Jordan algebra $\mathfrak{J} = \mathfrak{J}(3, \mathfrak{C})$ over the Cayley algebra $\mathfrak{C}$ and it is a connected, simply connected, simple (in the sense of the center $z(F_4) = 1$) Lie group. One of the non-compact Lie groups of type F_4 (which is named $F_{4,1} = F_{4(-20)}$) is obtained as the non-Euclidean projective transformation group of the Cayley projective plane $\mathfrak{C}P_2$ ($F_{4,1} = \{\alpha \in E_{6(-26)} \mid \langle \alpha X, \alpha Y\rangle = \langle X, Y\rangle\}$) and it is a connected, simply connected, simple (in the sense of the center $z(F_{4,1}) = 1$) Lie group [5]. In this paper, we investigate the other non-compact simple Lie group $F_{4,2}$ of type F_4. The results are as follows. The connected component group $F_{4,2} = \mathrm{Aut}^0(\mathfrak{J}')$ of the automorphism group $\mathrm{Aut}(\mathfrak{J}')$ of the exceptional Jordan algebra $\mathfrak{J}' = \mathfrak{J}(3, \mathfrak{C}')$ over the split Cayley algebra $\mathfrak{C}'$ is homeomorphic to $(S^3 \times Sp(3))/\mathbf{Z}_2 \times \mathbf{R}^{28}$ and a simple (in the sense of the center $z(F_{4,2}) = 1$) Lie group, and hence the center $z(\widetilde{F}_{4,2})$ of the non-compact simply connected simple Lie group $\widetilde{F}_{4,2} = F_{4(4)}$ of type F_4 is $\mathbf{Z}_2$.

1. Split Cayley algebra $\mathfrak{C}'$

Let $\mathfrak{C}'$ be the split Cayley algebra over the real numbers $\mathbf{R}$ [6]. This algebra $\mathfrak{C}'$ is defined as follows. If in $\mathfrak{C}' = \mathbf{H} \oplus \mathbf{H}e$, where $\mathbf{H}$ is the field of quaternions, we define a multiplication by

$$(a + be)(c + de) = (ac + \overline{d}b) + (b\overline{c} + da)e$$

then $\mathfrak{C}'$ becomes an 8-dim. (non-commutative non-associative) algebra over $\mathbf{R}$ with the conjugation $\overline{a + be} = \overline{a} - be$. And the inner product $(x, y)'$ in $\mathfrak{C}'$ is defined by

$$(a + be, c + de)' = (a, c) - (b, d)$$

2. Jordan algebra $\mathfrak{J}'$ and group $F_{4,2}$

Let $\mathfrak{J}' = \mathfrak{J}(3, \mathfrak{C}')$ be the Jordan algebra consisting of all 3×3 Hermitian matrices X with components in $\mathfrak{C}'$

$$X = X(\xi, x) = \begin{pmatrix} \xi_1 & x_3 & \overline{x}_2 \\ \overline{x}_3 & \xi_2 & x_1 \\ x_2 & \overline{x}_1 & \xi_3 \end{pmatrix}, \qquad \xi_i \in \boldsymbol{R},\ x_i \in \mathfrak{C}'$$

with respect to the composition

$$X \circ Y = \frac{1}{2}(XY + YX)$$

In $\mathfrak{C}'$ we adopt the following notations.

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$F_1(x) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & x \\ 0 & \overline{x} & 0 \end{pmatrix}, \quad F_2(x) = \begin{pmatrix} 0 & 0 & \overline{x} \\ 0 & 0 & 0 \\ x & 0 & 0 \end{pmatrix}, \quad F_3(x) = \begin{pmatrix} 0 & x & 0 \\ \overline{x} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Then these elements generate $\mathfrak{J}'$ additively and the multiplications among them are given as follows.

$$\begin{cases} E_i \circ E_i = E_i, & E_j \circ E_i = 0,\ j \neq i \\ E_i \circ F_i(x) = 0, & 2E_j \circ F_i(x) = F_i(x),\ j \neq i \\ F_i(x) \circ F_i(y) = (x, y)'(E_{i+1} + E_{i+2}), & 2F_i(x) \circ F_{i+1}(y) = F_{i+2}(\overline{xy}) \end{cases}$$

where the indexes are considered as mod 3.

In $\mathfrak{J}'$ we define the inner product $(X, Y)'$ and the trilinear inner product $\mathrm{tr}(X, Y, Z)'$ respectively by

$$(X, Y)' = \mathrm{tr}(X \circ Y) = \sum_{i=1}^{3} (\xi_i \eta_i + 2(x_i, y_i)')$$

$$\mathrm{tr}(X, Y, Z)' = (X \circ Y, Z)' = (X, Y \circ Z)'$$

where $X = X(\xi, x)$, $Y = Y(\eta, y)$.

The group $F_{4,2}$ is defined by

$$\begin{aligned} F_{4,2} &= \{\alpha \in \mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{J}', \mathfrak{J}') \mid \alpha(X \circ Y) = \alpha X \circ \alpha Y,\ \mathrm{tr}(\alpha X) = \mathrm{tr}(X)\} \\ &= \{\alpha \in \mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{J}', \mathfrak{J}') \mid \alpha(X \circ Y) = \alpha X \circ \alpha Y,\ (\alpha X, \alpha Y)' = (X, Y)'\} \\ &= \{\alpha \in \mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{J}', \mathfrak{J}') \mid (\alpha x, \alpha Y)' = (X, Y)',\ \mathrm{tr}(\alpha X, \alpha Y, \alpha Z)' = \mathrm{tr}(X, Y, Z)'\} \end{aligned}$$

Remark. The auther does not know if the condition $\alpha(X \circ Y) = \alpha X \circ \alpha Y$ implies the condition $\mathrm{tr}(\alpha X) = \mathrm{tr}(X)$ and if $\mathrm{Aut}(\mathfrak{J}') = \{\alpha \in \mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{J}', \mathfrak{J}') \mid \alpha(X \circ Y) = \alpha X \circ \alpha Y\}$

is connected. However it is true that the connected component group $\mathrm{Aut}^0(\mathfrak{J}')$ of $\mathrm{Aut}(\mathfrak{J}')$ is $F_{4,2}$. In the case of the compact group $F_4 = \mathrm{Aut}(\mathfrak{J})$, the condition $\alpha(X \circ Y) = \alpha X \circ \alpha Y$ implies the condition $\mathrm{tr}(\alpha X) = \mathrm{tr}(X)$ [4].

Since the field $\boldsymbol{H}$ of quaternions is a subfield of $\mathfrak{J}'$ regarding $a \in \boldsymbol{H}$ as $a + 0e \in \mathfrak{C}'$, $\mathfrak{J}_{\boldsymbol{H}} = \mathfrak{J}(3, \boldsymbol{H})$ (consisting of all 3×3 Hermitian matrices $X_{\boldsymbol{H}}$ with components in $\boldsymbol{H}$) is a subalgebra of $\mathfrak{J}'$, and any element $X \in \mathfrak{J}'$ can be described as

$$X = \begin{pmatrix} \xi_1 & a_3 & \bar{a}_2 \\ \bar{a}_3 & \xi_2 & a_1 \\ a_2 & \bar{a}_1 & \xi_3 \end{pmatrix} + \begin{pmatrix} 0 & b_3 & -b_2 \\ -b_3 & 0 & b_1 \\ b_2 & -b_1 & 0 \end{pmatrix} e, \quad \xi_i \in \boldsymbol{R},\ a_i,\ b_i \in \boldsymbol{H}$$

We denote this element X by

$$X = X_{\boldsymbol{H}} + F(\boldsymbol{b}e)$$

where $X_{\boldsymbol{H}} \in \mathfrak{J}_{\boldsymbol{H}}$, $\boldsymbol{b} = (b_1, b_2, b_3) \in \boldsymbol{H}^3$.

3. Automorphism group $\mathrm{Aut}(\mathfrak{J}_{\boldsymbol{H}})$

Before we consider the group $F_{4,2}$, we shall investigate the relation between the automorphism group $\mathrm{Aut}(\mathfrak{J}_{\boldsymbol{H}})$ of $\mathfrak{J}_{\boldsymbol{H}}$:

$$\begin{aligned}\mathrm{Aut}(\mathfrak{J}_{\boldsymbol{H}}) &= \{\alpha \in \mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{J}_{\boldsymbol{H}}, \mathfrak{J}_{\boldsymbol{H}}) \mid \alpha(X \circ Y) = \alpha X \circ \alpha Y\} \\ &= \{\alpha \in \mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{J}_{\boldsymbol{H}}, \mathfrak{J}_{\boldsymbol{H}}) \mid \alpha(X \circ Y) = \alpha X \circ \alpha Y,\ \mathrm{tr}(\alpha X) = \mathrm{tr}(X)\}\end{aligned}$$

(cf. Remark of §2) and the symplectic group

$$Sp(3) = \{A \in M(3, \boldsymbol{H}) \mid AA^* = E\}$$

Proposition 1. *The group* $\mathrm{Aut}(\mathfrak{J}_{\boldsymbol{H}})$ *is isomorphic to the group* $Sp(3)/\boldsymbol{Z}_2$, *where* $\boldsymbol{Z}_2 = \{E, -E\}$.

Proof We define a mapping $f : Sp(3) \to \mathrm{Aut}(\mathfrak{J}_{\boldsymbol{H}})$ by

$$f(A)X = AXA^*, \qquad X \in \mathfrak{J}_{\boldsymbol{H}}$$

Obviously f is well-defined and homomorphic. We shall show f is onto. For a given $\alpha \in \mathrm{Aut}(\mathfrak{J}_{\boldsymbol{H}})$, consider αE_i, $i = 1, 2, 3$. Since αE_i satisfies the conditions $(\alpha E_i)^* = \alpha E_i$, $(\alpha E_i)^2 = \alpha E_i$, $\mathrm{tr}(\alpha E_i) = 1$, we can choose a vector $\boldsymbol{b}_i = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \in \boldsymbol{H}^3$, $||\boldsymbol{b}_i|| = 1$ such that $\alpha E_i = \begin{pmatrix} b_1\bar{b}_1 & b_1\bar{b}_2 & b_1\bar{b}_3 \\ b_2\bar{b}_1 & b_2\bar{b}_2 & b_2\bar{b}_3 \\ b_3\bar{b}_1 & b_3\bar{b}_2 & b_3\bar{b}_3 \end{pmatrix}$ (Remember tha αE_i is an element of the quaternionic projective plane $\boldsymbol{H}P_2$). If we construct a matrix $B = (\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3)$, then $B \in Sp(3)$ because $\alpha E_i \circ \alpha E_j = 0$, $i \neq j$, and B satisfies

$$BE_iB^* = \alpha E_i, \quad i = 1, 2, 3$$

Therefore $\beta = f(B)^{-1}\alpha$ satisties

$$\beta E_i = E_i, \quad i = 1, 2, 3$$

If we operate β on $E_i \circ F_i(a) = 0$, $2E_j \circ F_i(a) = F_i(a) (j \neq i)$, then we see that β induces linear transformations β_i of $\boldsymbol{H}$ such that

$$\beta F_i(a) = F_i(\beta_i(a)), \quad i = 1, 2, 3$$

and β_i is orthogonal :

$$(\beta_i(a), \beta_i(b)) = (a, b), \quad a, b \in \boldsymbol{H}$$

from $F_i(a) \circ F_i(b) = (a, b)(E_{i+1} + E_{i+2})$. Furthermore $\beta_1, \beta_2, \beta_3$ are combined with the relation

$$\beta_1(a)\beta_2(b) = \overline{\beta_3(\overline{ab})}, \quad a, b \in \boldsymbol{H}$$

from $2F_1(a) \circ F_2(b) = F_3(\overline{ab})$. Put $p = \beta_1(1)$, $q = \beta_2(1)$, then $|p| = 1$, $|q| = 1$ and $\beta_2(a) = \bar{p}\beta_1(a)q$, $\beta_3(a) = \overline{\beta_1(\bar{a})q}$. Furthermore put $\beta_1(a) = p\sigma(a)$, then σ satisfies $\sigma(ab) = \sigma(a)\sigma(b)$, i. e. σ is an automorphism of $\boldsymbol{H}$. Therefore there exists $r \in \boldsymbol{H}$, $|r| = 1$ such that $\sigma(a) = ra\bar{r}$. Hence

$$\beta_1(a) = pra\bar{r}, \quad \beta_2(a) = ra\bar{r}q, \quad \beta_3(a) = \bar{q}ra\bar{r}\bar{p}, \quad a \in \boldsymbol{H}$$

Now, construct a matrix $C = \begin{pmatrix} \bar{q}r & 0 & 0 \\ 0 & pr & 0 \\ 0 & 0 & r \end{pmatrix}$, then $C \in Sp(3)$ and

$$CXC^* = \beta X, \qquad X \in \mathfrak{J}_{\boldsymbol{H}}$$

Therefore $\alpha X = B(CXC^*)B^* = f(BC)X$, hence f is onto. Finally $\mathrm{Ker} f = \{E, -E\}$ is easily obtained. Thus the proof is completed.

4. Compact subgroup $(F_{4,2})_K$ of $F_{4,2}$

We shall consider the following subgroup $(F_{4,2})_K$ of $F_{4,2}$

$$(F_{4,2})_K = \{\alpha \in F_{4,2} | \alpha(\mathfrak{J}_{\boldsymbol{H}}) = \mathfrak{J}_{\boldsymbol{H}}\}$$

Since $\mathfrak{J}_{\boldsymbol{H}}$ and $\mathfrak{J}'_{\boldsymbol{H}e} = \{F(\boldsymbol{a}e) | \boldsymbol{a} \in \boldsymbol{H}^3\}$ are orthogonal with respect to the inner product $(X, Y)'$, $\alpha \in (F_{4,2})_K$ also satisfies $\alpha(\mathfrak{J}'_{\boldsymbol{H}e}) = \mathfrak{J}'_{\boldsymbol{H}e}$.

Proposition 2. *The group* $(F_{4,2})_K$ *is isomorphic to the group* $(S^3 \times Sp(3))/\boldsymbol{Z}_2$, *where* $\boldsymbol{Z}_2 = \{(1, E), (-1, -E)\}$.

Proof Let $S^3 = \{p \in \boldsymbol{H} \mid |p| = 1\}$ and define a mapping $\varphi : S^3 \times Sp(3) \to (F_{4,2})_K$ by

$$\varphi(p, A)(X_{\boldsymbol{H}} + F(ae)) = AX_{\boldsymbol{H}}A^* + F((paA^*)e)$$

In order to show $\alpha = \varphi(p, A) \in (F_{4,2})_K$, since the conditions $\alpha(\mathfrak{J}_{\boldsymbol{H}}) = \mathfrak{J}_{\boldsymbol{H}}$, $\mathrm{tr}(\alpha X) = \mathrm{tr}(X)$ are obviously satisfied, we must prove

$$\alpha(X \circ Y) = \alpha X \circ \alpha Y, \quad X, Y \in \mathfrak{J}'$$

To do this, it is sufficient to show that

(1) $\alpha E_i \circ \alpha F_i(ae) = 0$

(2) $2\alpha E_j \circ \alpha F_i(ae) = \alpha F_i(ae), \ i \neq j$

(3) $\alpha F_i(ae) \circ \alpha F_i(be) = (ae, be)'(\alpha E_{i+1} + \alpha E_{i+2})$

(3)′ $\alpha F_i(ae) \circ \alpha F_i(b) = 0$

(4) $2\alpha F_i(ae) \circ \alpha F_{i+1}(be) = \alpha F_{i+2}(\overline{(ae)(be)})$

(4)′ $2\alpha F_i(ae) \circ \alpha F_{i+1}(b) = \alpha F_{i+2}(\overline{(ae)b})$

where $a, b \in \boldsymbol{H}$. Put $A = (a_{ij})_{i,j=1,2,3}$.

Proof of (1) $\quad 2\alpha E_1 \circ \alpha F_1(ae)$

$$= 2(|a_{11}|^2 E_1 + |a_{21}|^2 E_2 + |a_{31}|^2 E_3 + F_1(a_{21}\overline{a_{31}}) + F_2(a_{31}\overline{a_{11}}) + F_3(a_{11}\overline{a_{21}}))$$

$$\circ (F_1((pa\overline{a_{11}})e) + F_2((pa\overline{a_{21}})e) + F_3((pa\overline{a_{31}})e)$$

$$= F_1((|a_{21}|^2 + |a_{31}|^2)(pa\overline{a_{11}})e + \overline{((pa\overline{a_{21}})e)(a_{11}\overline{a_{21}})} + \overline{(a_{31}\overline{a_{11}})((pa\overline{a_{31}})e)})$$

$$+ F_2(*) + F_3(*)$$

$$= F_1((|a_{21}|^2 + |a_{31}|^2)(pa\overline{a_{11}})e - (pa\overline{a_{21}}a_{21}\overline{a_{11}})e - (pa\overline{a_{31}}a_{31}\overline{a_{11}})e)$$

$$+ F_2(*) + F_3(*)$$

$$= F_1((|a_{21}|^2 + |a_{31}|^2 - |a_{21}|^2 - |a_{31}|^2)(pa\overline{a_{11}})e) + F_2(*) + F_3(*)$$

$$= F_1(0) + F_2(0) + F_3(0) = 0$$

Proof of (4) $\quad 2\alpha F_1(ae) \circ \alpha F_2(be)$

$$= 2(F_1((pa\overline{a_{11}})e) + F_2((pa\overline{a_{21}})e) + F_3((pa\overline{a_{31}})e))$$

$$\circ (F_1((pb\overline{a_{12}})e) + F_2((pb\overline{a_{22}})e) + F_3((pb\overline{a_{32}})e))$$

$$= 2((pa\overline{a_{11}})e,\ (pb\overline{a_{12}})e)'(E_2+E_3) + 2((pa\overline{a_{21}})e,\ (pb\overline{a_{22}})e)'(E_3+E_1)$$
$$+ 2((pa\overline{a_{31}})e,\ (pb\overline{a_{32}})e)'(E_1+E_2)$$
$$+ F_1(\overline{((pa\overline{a_{21}})e)((pb\overline{a_{32}})e)} + \overline{((pb\overline{a_{22}})e)((pa\overline{a_{31}})e)}) + F_2(*) + F_3(*)$$
$$= (-2(a\overline{a_{21}},\ b\overline{a_{22}}) - 2(a\overline{a_{31}},\ b\overline{a_{32}}))E_1 + *E_2 + *E_3$$
$$+ F_1((a_{21}\overline{a}\overline{p})(pb\overline{a_{32}} + (a_{22}\overline{b}\overline{p})(pa\overline{a_{31}})) + F_2(*) + F_3(*)$$
$$= (-2(\overline{b}a,\ \overline{a_{22}}a_{21} + \overline{a_{32}}a_{31}))E_1 + *E_2 + *E_3$$
$$+ F_1(a_{21}\overline{a}b\overline{a_{32}} + a_{22}\overline{b}a\overline{a_{31}}) + F_2(*) + F_3(*)$$
$$= 2(\overline{b}a,\ \overline{a_{12}}a_{11})E_1 + *E_2 + *E_3$$
$$+ F_1(a_{21}\overline{a}b\overline{a_{32}} + a_{22}\overline{b}a\overline{a_{31}}) + F_2(*) + F_3(*)$$
$$= 2(\overline{(ae)}(be),\ \overline{a_{11}}a_{12})E_1 + *E_2 + *E_3$$
$$+ F_1(a_{21}\overline{(ae)}(be)\overline{a_{32}} + a_{22}(ae)(be)\overline{a_{31}}) + F_2(*) + F_3(*)$$
$$= \alpha F_3(\overline{(ae)}(be))$$

The other formulae are also proved by calculations similar to the above. Obviously φ is a homorphism. Next we shall prove φ is onto.

For a given $\alpha \in (F_{4,2})_K$, consider the restiction $\alpha|\mathfrak{J}_H$ of α to $\mathfrak{J}_H$. Since $\alpha|\mathfrak{J}_H$ is an automorphism of $\mathfrak{J}_H$, there exists an element $A \in Sp(3)$ such that

$$\alpha X_H = AX_HA^*, \quad X_H \in \mathfrak{J}_H$$

by Proposition 1. Put $\beta = \varphi(1, A)^{-1}\alpha$, then $\beta|\mathfrak{J}_H = 1$. In particular β satisfies $\beta E_i = E_i$, $i = 1, 2, 3$, hence β induces linear transformations β_i of $\mathfrak{J}'$ such that

$$\beta F_i(x) = F_i(\beta_i(x)), \quad i = 1, 2, 3$$

(the proof is the same as Proposition 1). Furthermore β_i satisfies

$$(\beta_i(u),\ a)' = 0, \quad (\beta_i(u),\ \beta_i(v))' = (u,\ v)', \quad a \in \boldsymbol{H},\ u, v \in \boldsymbol{H}e$$

from $F_i(u)\circ F_i(a) = 0$, $F_i(u)\circ F_i(v) = (u, v)'(E_{i+1} + E_{i+2})$ respectively. Hence β_i induces an orthogonal transformation of $\boldsymbol{H}e$. And from $2F_3(a)\circ F_1(u) = -F_2(au)$ we get

$$a\beta_1(u) = \beta_2(au), \quad a \in \boldsymbol{H},\ u \in \boldsymbol{H}e$$

Put $a = 1$, then we have $\beta_1 = \beta_2$, similarly $\beta_1 = \beta_3$ $(= \beta')$. Therefore β' satisfies

$$\beta'(au) = a\beta'(u), \quad a \in \boldsymbol{H},\ u \in \boldsymbol{H}e$$

Set $\beta'(e) = pe,\ p \in \boldsymbol{H}$, *then* $|p| = 1$ and

$$\beta'(ae) = a\beta'(e) = a(pe) = (pa)e, \quad a \in \boldsymbol{H}$$

Therefore

$$\beta X = \beta(X_{\boldsymbol{H}} + F(ae)) = X_{\boldsymbol{H}} + F((pa)e) = \varphi(p, E)X, \quad X \in \mathfrak{J}'$$

Hence $\alpha = \varphi(1, A)\varphi(p, E) = \varphi(p, A)$, i.e. φ is onto. Finally $\operatorname{Ker}\varphi = \{(1, E), (-1, -E)\}$ is easily obtained. Thus the proof of Proposition 2 is completed.

Remark. The compact Lie group $F_4 = \operatorname{Aut}(\mathfrak{J})$ also contains a subgroup $(F_4)_K$ which is isomorphic to $(S^3 \times Sp(3))/\boldsymbol{Z}_2$ by a mapping $\varphi : S^3 \times Sp(3) \to F_4$,

$$\varphi(p, A)(X_{\boldsymbol{H}} + F(ae)) = AX_{\boldsymbol{H}}A^* + F((paA^*)e)$$

5. Lie algebra $\mathfrak{f}_{4,2}$ of $F_{4,2}$

We consider the Lie algebra $\mathfrak{f}_{4,2}$ of $F_{4,2}$:

$$\begin{aligned}
\mathfrak{f}_{4,2} &= \{\varsigma \in \operatorname{Hom}_{\boldsymbol{R}}(\mathfrak{J}', \mathfrak{J}') \mid \varsigma(X \circ Y) = \varsigma X \circ Y + X \circ \varsigma Y,\ \operatorname{tr}(\varsigma X) = 0\} \\
&= \left\{\varsigma \in \operatorname{Hom}_{\boldsymbol{R}}(\mathfrak{J}', \mathfrak{J}') \,\middle|\, \begin{array}{l} (\varsigma X, Y)' + (X, \varsigma Y)' = 0 \\ \operatorname{tr}(\varsigma X, Y, Z)' + \operatorname{tr}(X, \varsigma Y, Z)' + \operatorname{tr}(X, Y, \varsigma Z)' = 0 \end{array}\right\} \\
&= \{\varsigma \in \operatorname{Hom}_{\boldsymbol{R}}(\mathfrak{J}', \mathfrak{J}') \mid \varsigma(X \circ Y) = \varsigma X \circ Y + X \circ \varsigma Y\}
\end{aligned}$$

(the last equality is proved in Remark of Proposition 6). The structure of the Lie algebra $\mathfrak{f}_{4,2}$ is analogous to the Lie algebra $\mathfrak{f}_4$ of the compact Lie group $F_4 = \operatorname{Aut}(\mathfrak{J})$ [1], [4]. However we give an outline of the proof of Proposition 6.

Let M'^- be the vector space over $\boldsymbol{R}$ consisting of all 3×3 skew-Hermitian matrices A ; $A^* = -A$ with components in $\mathfrak{J}_{\prime}$. Any element $A \in M'^-$ induces a linear transformation $\tilde{A}$ of $\mathfrak{J}'$ by

$$\tilde{A}X = AX - XA, \quad X \in \mathfrak{J}'$$

Lemma 3 ([1]). *If* $A \in M'^-$, $\operatorname{tr}(A) = 0$, *then* $\tilde{A} \in \mathfrak{f}_{4,2}$, *i.e.* $\tilde{A}$ *satisfies*

(1) $(\tilde{A}X, Y)' + (X, \tilde{A}Y) = 0$

(2) $\operatorname{tr}(\tilde{A}X, Y, Z)' + \operatorname{tr}(X, \tilde{A}Y, Z)' + \operatorname{tr}(X, Y, \tilde{A}Z)' = 0$

We adopt the following notations in M'^-.

$$A_1(r) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & r \\ 0 & -\bar{r} & 0 \end{pmatrix}, \quad A_2(r) = \begin{pmatrix} 0 & 0 & -\bar{r} \\ 0 & 0 & 0 \\ r & 0 & 0 \end{pmatrix}, \quad A_3(r) = \begin{pmatrix} 0 & r & 0 \\ -\bar{r} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Then $\widetilde{A}_i(r) \in \mathfrak{f}_{4,2}$ and the following formulae are hold.

$$\begin{cases} \widetilde{A}_i(r)E_i = 0, & \widetilde{A}_i(r)F_i(x) = 2(r, x)'(E_{i+1} - E_{i+2}), \\ \widetilde{A}_i(r)E_{i+1} = -F_i(r), & \widetilde{A}_i(r)F_{i+1}(x) = F_{i+2}(\overline{rx}), \\ \widetilde{A}_i(r)E_{i+2} = F_i(r), & \widetilde{A}_i(r)F_{i+2}(x) = -F_{i+1}(\overline{xr}) \end{cases}$$

where the indexes are considered mod 3.

Let $\mathfrak{o}'(\mathfrak{C}')$ be the Lie algebra

$$\mathfrak{o}(4, 4) = \mathfrak{o}'(\mathfrak{C}') = \{D \in \mathrm{Hom}_{\boldsymbol{R}}(\mathfrak{C}', \mathfrak{C}') \mid (Dx, y)' + (x, Dy)' = 0\}$$

of the Lorentz group $O(4, 4) = O'(\mathfrak{C}') = \{\sigma \in \mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{C}', \mathfrak{C}') \mid (\sigma x, \sigma y)' = (x, y)'\}$.

Lemma 4 (Principle of the infinitesimal triality in $\mathfrak{o}'(\mathfrak{C}')$ [1], [3]). *For any element $D_1 \in \mathfrak{o}'(\mathfrak{C}')$, there exist $D_2, D_3 \in \mathfrak{o}'(\mathfrak{C}')$ uniquely such that*

$$D_1(x)y + xD_2(y) = \overline{D_3(\overline{xy})}, \quad x, y \in \mathfrak{C}'$$

Lemma 5 ([1]). *The Lie algebra $\mathfrak{o}'_{\mathfrak{C}'} = \{\delta \in \mathfrak{f}_{4,2} \mid \delta E_i = 0, \ i = 1, 2, 3\}$ is isomorphic to the Lie algebra $\mathfrak{o}'(\mathfrak{C}')$ by the correspondence*

$$D_1 \in \mathfrak{o}'(\mathfrak{C}') \to \delta = \delta(D_1, D_2, D_3) \in \mathfrak{o}'_{\mathfrak{C}'}$$

$$\delta \begin{pmatrix} \xi_1 & x_3 & \overline{x}_2 \\ \overline{x}_3 & \xi_2 & x_1 \\ x_2 & \overline{x}_1 & \xi_3 \end{pmatrix} = \begin{pmatrix} 0 & D_3x_3 & \overline{D_2x_2} \\ \overline{D_3x_3} & 0 & D_1x_1 \\ D_2x_2 & \overline{D_1x_1} & 0 \end{pmatrix}$$

where D_2, D_3 are elements of $\mathfrak{o}'(\mathfrak{C}')$ determined by Principle of the infinitesimal triality in $\mathfrak{o}'(\mathfrak{C}')$.

Proposition 6. *Any element ς of the Lie algebra $\mathfrak{f}_{4,2} = \{\varsigma \in \mathrm{Hom}_{\boldsymbol{R}}(\mathfrak{C}', \mathfrak{C}') \mid \varsigma(X \circ Y) = \varsigma X \circ Y + X \circ \varsigma Y\}$ is uniquely reqresented by the form*

$$\varsigma = \delta + \widetilde{A}, \quad \delta \in \mathfrak{o}'_{\mathfrak{C}'}, \quad A \in M'_-, \quad \mathrm{diag}\, A = 0$$

where $\mathrm{diag}\, A = 0$ *means that all diagonal elements of A are* 0.

Proof From $E_i \circ E_i = E_i$, $E_i \circ E_j = 0$, $j \neq i$, we have $2E_i \circ \varsigma E_i = \varsigma E_i$, $\varsigma E_i \circ E_j + \circ E_i \circ \varsigma E_j = 0$. Hence ςE_i, $i = 1, 2, 3$ have the following form

$$\varsigma E_1 = \begin{pmatrix} 0 & -r_3 & \overline{r}_2 \\ -\overline{r}_3 & 0 & 0 \\ r_2 & 0 & 0 \end{pmatrix}, \quad \varsigma E_2 = \begin{pmatrix} 0 & r_3 & 0 \\ \overline{r_3} & 0 & -r_1 \\ 0 & -\overline{r_1} & 0 \end{pmatrix}, \quad \varsigma E_3 = \begin{pmatrix} 0 & 0 & -\overline{r_2} \\ 0 & 0 & r_1 \\ -r_2 & \overline{r_1} & 0 \end{pmatrix}$$

Construct a matrix $A=\begin{pmatrix} 0 & r_3 & -\bar{r}_2 \\ -\bar{r}_3 & 0 & r_1 \\ r_2 & -\bar{r}_1 & 0 \end{pmatrix}$, then $A \in M'$-, diag $A=0$ and $\widetilde{A}$ satisfies

$$\widetilde{A}E_i = \varsigma E_i, \quad i=1, 2, 3$$

so that $\delta=\varsigma-\widetilde{A} \in \mathfrak{v}'_{\mathfrak{C}'}$. Hence $\varsigma=\delta+\widetilde{A}$, $\delta \in \mathfrak{v}'_{\mathfrak{C}'}$, $A \in M'$-, diag $A=0$. To prove the uniqueness, it sufficient to show

$$\delta+\widetilde{A}=0, \quad \delta \in \mathfrak{v}'_{\mathfrak{C}'}, \quad A \in M'\text{-}, \quad \operatorname{diag} A=0 \Rightarrow \delta=0, \quad A=0$$

However it is easily obtained if we operate $\delta+\widetilde{A}$ on E_i, $i=1, 2, 3$.

Remark. Any element ς of the Lie algebra $\mathfrak{f}_{4,2}=\{\varsigma \in \operatorname{Hom}_{\boldsymbol{R}}(\mathfrak{J}', \mathfrak{J}') | \varsigma(X \circ Y)=\varsigma X \circ Y+X \circ \varsigma Y\}$ satisfies the condition $\operatorname{tr}(\varsigma X)=0$. In fact, for $\varsigma=\delta+\widetilde{A} \in \mathfrak{f}_{4,2}$, $\operatorname{tr}(\delta X)=0$ is satisfied trivially and $\operatorname{tr}(\widetilde{A}X)=\operatorname{Re}(\operatorname{tr}(\widetilde{A}X))=\operatorname{Re}(\operatorname{tr}(AX-XA))=\operatorname{Re}(\sum_{i,j} a_{ij}x_{ji}-\sum_{i,j} x_{ij}a_{ji})=\sum_{i,j}\operatorname{Re}(a_{ij}x_{ji}-x_{ji}a_{ij})=0$ for $A=(a_{ij})_{i,j=1,2,3}$, $X=(x_{ij})_{i,j=1,2,3}$.

6. Polar decomposition of $F_{4,2}$

To give a polar decomposition of $F_{4,2}$ we use the following

Lemma 7 ([2] *p.* 345). *Let G be a real algebraic subgroup of the general linear group $GL(n, \boldsymbol{R})$ such that the condition $A \in G$ implies ${}^tA \in G$. Then G is homeomorphic to the topological product of $G \cap O(n)$ (which is a maximal compact subgroup of G) and a Euclidean space $\boldsymbol{R}^d$:*

$$G \simeq (G \cap O(n)) \times \boldsymbol{R}^d, \quad d=\dim(\mathfrak{g} \cap \mathfrak{h}(n))$$

where $O(n)$ is the orthogonal subgroup of $GL(n, \boldsymbol{R})$, $\mathfrak{g}$ the Lie algebra of G and $\mathfrak{h}(n)$ the vector space of all real symmetric matrices of degree n.

To use the above lemma, we define the positive definite inner products (x, y) in $\mathfrak{C}'$ and (X, Y) in $\mathfrak{J}'$ respectively by

$$(x, y)=(a, c)+(b, d)$$

$$(X, Y)=\sum_{i=1}^{3}(\xi_i\eta_i+2(x_i, y_i))$$

for $x=a+be$, $y=c+de$ and $X=X(\xi, x)$, $Y=Y(\eta, y)$. Two inner products (X, Y), $(X, Y)'$ in $\mathfrak{J}'$ are combined with the following relations

$$(X, Y)=(X, \gamma Y)', \quad (X, Y)'=(X, \gamma Y)$$

where $\gamma=\varphi(-1, E)$. We denote by ${}^t\alpha$ the transpose of $\alpha \in \operatorname{Iso}_{\boldsymbol{R}}(\mathfrak{J}', \mathfrak{J}')$ with respect to (X, Y) : $(\alpha X, Y)=(X, {}^t\alpha Y)$.

Lemma 8. *$F_{4,2}$ is a real algebraic subgroup of the general linear group $GL(27, \boldsymbol{R})=\mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{J}', \mathfrak{J}')$ and satisfies the condition $\alpha\in F_{4,2}$ implies ${}^t\alpha\in F_{4,2}$.*

Proof Since $(X, \gamma Y)=(X, Y)'=(\alpha X, \alpha Y)'=(\alpha X, \gamma\alpha Y)=(X, {}^t\alpha\gamma\alpha Y)$ for $\alpha\in F_{4,2}$, we have $\gamma={}^t\alpha\gamma\alpha$. Hence ${}^t\alpha=\gamma\alpha^{-1}\gamma\in F_{4,2}$. It is trivial that $F_{4,2}$ is real algebraic, because $F_{4,2}$ is defined by the algebraic relations $\alpha(X\circ Y)=\alpha X\circ\alpha Y$, $\mathrm{tr}(\alpha X)=\mathrm{tr}(X)$.

Let $O(\mathfrak{J}')$ *be the orthogonal subgroup of* $\mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{J}', \mathfrak{J}')$:

$$O(27)=O(\mathfrak{J}')=\{\alpha\in\mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{J}', \mathfrak{J}')\,|\,(\alpha X, \alpha Y)=(X, Y)\}$$

Then $\alpha\in F_{4,2}\cap O(\mathfrak{J}')$ induces a linear transformation of $\mathfrak{J}_H$. In fact, $(\alpha X_H, Y)=-(\alpha X_H, Y)'=-(X_H, \alpha^{-1}Y)'=-(X_H, \alpha^{-1}Y)=-(\alpha X_H, Y)$ for $X_H\in\mathfrak{J}_H$, $Y\in\mathfrak{J}'_{He}$, therefore $(\alpha X_H, Y)=0$ *and* $\alpha X_H\in\mathfrak{J}_H$. Hence we have

$$F_{4,2}\cap O(\mathfrak{J}')=(F_{4,2})_K\cong(S^3\times Sp(3))/\boldsymbol{Z}_2$$

by Proposition 2. Next we shall determine the Euclidean part $\mathfrak{f}_{4,2}\cap\mathfrak{h}(\mathfrak{J}')$ of $F_{4,2}$, where

$$\mathfrak{h}(27)=\mathfrak{h}(\mathfrak{J}')=\{\varsigma\in\mathrm{Hom}_{\boldsymbol{R}}(\mathfrak{J}', \mathfrak{J}')\,|\,(\varsigma X, Y)=(X, \varsigma Y)\}$$

Let $\varsigma\in\mathfrak{f}_{4,2}\cap\mathfrak{h}(\mathfrak{J}')$. Represent ς in the form

$$\varsigma=\delta+\tilde{A}_1(r_1)+\tilde{A}_2(r_2)+\tilde{A}_3(r_3)$$

where $\delta=\delta(D_1, D_2, D_3)\in\mathfrak{o}'_{\mathfrak{C}'}$, $r_i\in\mathfrak{C}'$. Since ς satisfies $(\varsigma E_{i+1}, F_i(x))=(E_{i+1}, \varsigma F_i(x))$, we have

$$((\delta+\sum_{j=1}^{3}\tilde{A}_j(r_j))E_{i+1}, F_i(x))=(E_{i+1}, (\delta+\sum_{j=1}^{3}\tilde{A}_j(r_j))F_i(x))$$

$$(-F_i(r_i)+F_{i+2}(r_{i+2}), F_i(x))$$
$$=(E_{i+1}, F_i(D_ix)+2(r_i, x)'(E_{i+1}-E_{i+2})-F_{i+2}(\overline{xr_{i+1}})+F_{i+1}(\overline{r_{i+2}x}))$$

Hence we have

$$-(r_i, x)=(r_i, x)', \quad x\in\mathfrak{C}'$$

From this we get $r_i\in\boldsymbol{H}e$, $i=1, 2, 3$. Next, from the condition $(\varsigma F_1(x), F_1(y))=(F_1(x), \varsigma F_1(y))$, we have

$$(F_1(D_1x)+2(r_1, x)'(E_2-E_3)-F_3(\overline{xr_2})+F_2(\overline{r_3x}), F_1(y))$$
$$=(F_1(x), F_1(D_1y)+2(r_1, y)'(E_2-E_3)-F_3(\overline{yr_2})+F_2(\overline{r_3y}))$$

Hence we have

$$(D_1x, y) = (x, D_1y), \quad x, y \in \mathfrak{C}'$$

Since D_1 also satisfies $(D_1x, y)' + (x, D_1y)' = 0$, D_1 induces a linear transformation $D_1|\boldsymbol{H} : \boldsymbol{H} \to \boldsymbol{H}e$. Conversely

$$\delta(D_1, D_2, D_3) + \sum_{i=1}^{3} \widetilde{A}_i(r_i), \quad D_1|\boldsymbol{H} \in \mathrm{Hom}_{\boldsymbol{R}}(\boldsymbol{H}, \boldsymbol{H}e), \ r_i \in \boldsymbol{H}e$$

is an element of $\mathfrak{f}_{4,2} \cap \mathfrak{h}(\mathfrak{J}')$. Hence

$$\dim(\mathfrak{f}_{4,2} \cap \mathfrak{h}(\mathfrak{J}')) = 4 \times 4 + 4 \times 3 = 28$$

Thus we have the following

Theorem 9. *The group $F_{4,2}$ is homeomorphic to the topological product of the group $(S^3 \times Sp(3))/\boldsymbol{Z}_2$ and a 28-dim. Euclidean space $\boldsymbol{R}^{28}$:*

$$F_{4,2} \simeq (S^3 \times Sp(3))/\boldsymbol{Z}_2 \times \boldsymbol{R}^{28}$$

In particular, $F_{4,2}$ is a connected (but not simply connected) Lie group.

7. Simplicity of $F_{4,2}$

Lemma 10. *The Lie algebra $\mathfrak{f}_{4,2}$ of $F_{4,2}$ is simple.*

Proof The complexification $\mathfrak{f}_{4,2}{}^C$ of the Lie algebra $\mathfrak{f}_{4,2}$ is isomorphic to the complexification $\mathfrak{f}_4{}^C$ of the Lie algebra $\mathfrak{f}_4 = \{\varsigma \in \mathrm{Hom}_{\boldsymbol{R}}(\mathfrak{J}, \mathfrak{J}) | \varsigma(X \circ Y) = \varsigma X \circ Y + X \circ \varsigma Y\}$ of the compact Lie group $F_4 = \mathrm{Aut}(\mathfrak{J})$, because the complexification $\mathfrak{C}'^C$ of $\mathfrak{C}'$ is isomorphic to the one $\mathfrak{C}^C$ of the Cayley algebra $\mathfrak{C}$. As is well known $\mathfrak{f}_4{}^C$ is simple, so that $\mathfrak{f}_{4,2}{}^C$ is so, hence $\mathfrak{f}_{4,2}$ is also simple.

Since $F_{4,2}$ is a connected group from Theorem 9 and a simple group as Lie group from Lemma 10, any normal subgroup of $F_{4,2}$ is contained in the center $z(F_{4,2})$ of $F_{4,2}$. We shall show $z(F_{4,2}) = 1$.

Let $\alpha \in z(F_{4,2})$. First we show that α induces a linear transformation of $\mathfrak{J}_{\boldsymbol{H}}$: $\alpha \in (F_{4,2})_K$. In fact, put $\alpha X_{\boldsymbol{H}} = Y_{\boldsymbol{H}} + F(\boldsymbol{a}e)$ for $X_{\boldsymbol{H}} \in \mathfrak{J}_{\boldsymbol{H}}$, then the commutativity condition $\varphi(-1, E)\alpha = \alpha\varphi(-1, E)$, we have

$$Y_{\boldsymbol{H}} + F(-\boldsymbol{a}e) = \varphi(-1, E)(Y_{\boldsymbol{H}} + F(\boldsymbol{a}e)) = \varphi(-1, E)\alpha X_{\boldsymbol{H}}$$
$$= \alpha\varphi(-1, E)X_{\boldsymbol{H}} = \alpha X_{\boldsymbol{H}} = Y_{\boldsymbol{H}} + F(\boldsymbol{a}e)$$

Therefore $F(\boldsymbol{a}e) = 0$ and $\alpha X_{\boldsymbol{H}} = Y_{\boldsymbol{H}} \in \mathfrak{J}_{\boldsymbol{H}}$. Hence there exists an element $(p, A) \in S^3 \times Sp(3)$ such that $\alpha = \varphi(p, A)$ by Proposition 2. Furthermore from the commutativity condition $\alpha\varphi(q, E) = \varphi(q, E)\alpha$, $\alpha\varphi(1, B) = \varphi(1, B)\alpha$, we have

$$pq = qp \qquad \text{for all} \quad q \in S^3$$
$$AB = BA \qquad \text{for all} \quad B \in Sp(3)$$

so that $p = \pm 1$, $A = \pm E$. Hence $\alpha = \varphi(1, E)$ or $\alpha = \varphi(-1, E)$. We shall show that $\varphi(-1, E)$ is not an element of the center $z(F_{4,2})$ using the following

Lemma 11. *The following mapping* $\beta : \mathfrak{J}' \to \mathfrak{J}'$, $\beta X = Y$, $X = X(\xi, x)$, $Y = Y(\eta, y)$:

$$\begin{cases} \eta_1 = \xi_1 \\ \eta_2 = (e, x_1)' \sinh 2 + \dfrac{\xi_2 - \xi_3}{2} \cosh 2 + \dfrac{\xi_2 + \xi_3}{2} \\ \eta_3 = -(e, x_1)' \sinh 2 - \dfrac{\xi_2 - \xi_3}{2} \cosh 2 + \dfrac{\xi_2 + \xi_3}{2} \end{cases}$$

$$\begin{cases} y_1 = x_1 - 2e(e, x_1)' \sinh^2 1 - \dfrac{e(\xi_2 - \xi_3)}{2} \sinh 2 \\ y_2 = x_2 \cosh 1 - \overline{x_3 e} \sinh 1 \\ y_3 = x_3 \cosh 1 + \overline{e x_2} \sinh 1 \end{cases}$$

is an element of $F_{4,2}$.

Proof This mapping β is $\exp \widetilde{A}_1(e)$, $\widetilde{A}_1(e) \in \mathfrak{f}_{4,2}$, hence $\beta \in F_{4,2}$.

$\varphi(-1, E)$ does not commute with β, because $\varphi(-1, E)\beta F_2(e) = \varphi(-1, E)(F_3(\sinh 1) + F_2(e\cosh 1)) = F_3(\sinh 1) - F_2(e \cosh 1)$ and $\beta\varphi(-1, E)F_2(e) = \beta(F_2(e)) = -F_3(\sinh 1) - F_2(e \cosh 1)$. Thus we have the following

Theorem 12. *The group* $F_{4,2}$ *is a simple (in the algebraic sense) Lie group.*

Since the fundamental group of $F_{4,2}$ is Z_2 from Theorem 9 and $F_{4,2}$ is a simple group, we have the following

Theorem 13. *The center* $z(\widetilde{F}_{4,2})$ *of the non-compact simply connected Lie group* $\widetilde{F}_{4,2} = F_{4(4)}$ *of type* F_4 *is* Z_2.

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