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Non-compact Simple Lie Group G_2' of Type G_2

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It is known that there exist two simple Lie group of type G_2 up to local isomorphism, one of them is compact and the other is non-compact. The compact simple Lie group $G_2 = G_{2(-14)}$ of type G_2 is obtained as the automorphism group $\mathrm{Aut}(\mathfrak{C})$ of the Cayley algebra $\mathfrak{C}$ and it is a connected, simply connected, simple (in the sense of the center $z(G_2) = 1$) Lie group. It is also known that the Lie algebra $\mathfrak{g}_2'$ of the non-compact simple Lie group $G_{2(2)}$ of type G_2 is obtained as the derivation Lie algebra of the split Cayley algebra $\mathfrak{C}'$ and the maximal compact subgroup of $G_{2(2)}$ has the type of $A_1 \oplus A_1$. In this paper, we investigate some global properties of the automorphism group $G_2' = \mathrm{Aut}(\mathfrak{C}')$ of the split Cayley algebra $\mathfrak{C}'$. The results are as follows. The group G_2' is homeomorphic to $SO(4) \times \boldsymbol{R}^8$ and a simple (in the sense of the center $z(G_2') = 1$) Lie group. And hence the center $z(\widetilde{G}_2')$ of the non-compact simply connected simple Lie group $\widetilde{G}'_2 = G_{2(2)}$ of type G_2 is $\boldsymbol{Z}_2$.

1. Split Cayley algebra $\mathfrak{C}'$

Let $\mathfrak{C}'$ be the split Cayley algebra over the real numbers $\boldsymbol{R}$. This algebra $\mathfrak{C}'$ is defined as follows. Let $\boldsymbol{H} = \{a = a_1 + a_2 i + a_3 j + a_4 k \mid a_i \in \boldsymbol{R}\}$ be the field of quaternions. In $\mathfrak{C}' = \boldsymbol{H} \oplus \boldsymbol{H}e$, if we define a multiplication by

$$(a + be)(c + de) = (ac + \overline{d}b) + (b\overline{c} + da)e$$

then $\mathfrak{C}'$ becomes an 8-dim. (non-commutative non-associative) algebra over $\boldsymbol{R}$ with the conjugation $\overline{a + be} = \overline{a} - be$.

The Q-norm $Q(x)$ and the inner product $(x, y)'$ in $\mathfrak{C}'$ are defined respectively by

$$Q(a + be) = |a|^2 - |b|^2$$

$$(a + be,\ c + de)' = (a,\ c) - (b,\ d)$$

The Q-norm has the following properties.

$$Q(x) = x\overline{x} = \overline{x}x, \quad Q(xy) = Q(x)Q(y), \quad x,\ y \in \mathfrak{C}'$$

Remark. In the usual Cayley algebra $\mathfrak{C} = \boldsymbol{H} + \boldsymbol{H}e$, the multiplication is defined by

$$(a + be)(c + de) = (ac - \bar{d}b) + (b\bar{c} + da)e$$

2. Definition of group G_2'

We denote the automorphism group $\mathrm{Aut}(\mathfrak{C}')$ of the split Cayley algebra $\mathfrak{C}'$ by G_2', i.e.

$$G_2' = \{\alpha \in \mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{C}', \mathfrak{C}') \mid \alpha(xy) = \alpha(x)\alpha(y)\}$$

Obviously $\alpha(1)=1$ holds for $\alpha \in G_2'$. We shall show that the group G_2' is contained in the Lorentz group $O'(\mathfrak{C}')$, where

$$\begin{aligned} O(4, 4) = O'(\mathfrak{C}') &= \{\alpha \in \mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{C}', \mathfrak{C}') \mid Q(\alpha(x)) = Q(x)\} \\ &= \{\alpha \in \mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{C}', \mathfrak{C}') \mid (\alpha(x), \alpha(y))' = (x, y)'\} \end{aligned}$$

Lemma 1. *For $\alpha \in G_2'$ and $x \in \mathfrak{C}'$, we have*

(1) *if $xx = -1$ then $Q(\alpha(x)) = 1$ and $\overline{\alpha(x)} = -\alpha(x)$*

(2) *if $xx = 1$, $x \notin \boldsymbol{R}$ then $Q(\alpha(x)) = -1$ and $\overline{\alpha(x)} = -\alpha(x)$*

Proof (1) From $xx = -1$ we have $\alpha(x)\alpha(x) = -1$, so $(Q(\alpha(x)))^2 = 1$, i.e. $Q(\alpha(x)) = \pm 1$. Assume $Q(\alpha(x)) = -1$, then

$$\begin{aligned} \alpha(x) &= -Q(\alpha(x))\alpha(x) = -(\overline{\alpha(x)}\alpha(x))\alpha(x) = -\overline{\alpha(x)}(\alpha(x)\alpha(x)) \\ &= -\overline{\alpha(x)}(-1) = \overline{\alpha(x)} \end{aligned}$$

(Although $\mathfrak{C}'$ is not associative, $(\bar{x}x)y = \bar{x}(xy)$ is always valid). This means $\alpha(x)$ is a real number a, therefore $\alpha(x) = a = \alpha(a)$. Since α is injective, $x = a$ is real and $xx = -1$. This is a contradiction. Hence $Q(\alpha(x)) = 1$ and we get $\overline{\alpha(x)} = -\alpha(x)$ by a calculation similar to the above. (2) is analogously obtained.

Proposition 2. *The group G_2' is a subgroup of the Lorentz group $O'(\mathfrak{C}')$.*

Proof Let $\alpha \in G_2'$. Choose a basis $1, i, j, k, e, ie, je, ke$ in $\mathfrak{C}'$, then these elements except 1 satisfy the conditions of Lemma 1:

$$i^2 = j^2 = k^2 = -1, \quad e^2 = (ie)^2 = (je)^2 = (ke)^2 = 1$$

Therefore, for these elements x we have $\overline{\alpha(x)} = -\alpha(x) = \alpha(\bar{x})$. And since $\overline{\alpha(1)} = 1 = \alpha(\bar{1})$, we get $\overline{\alpha(x)} = \alpha(\bar{x})$ for any element x of $\mathfrak{C}'$. Now

$$Q(\alpha(x)) = \overline{\alpha(x)}\alpha(x) = \alpha(\bar{x})\alpha(x) = \alpha(\bar{x}x) = \alpha(Q(x)) = Q(x), \quad x \in \mathfrak{C}'$$

This means $\alpha \in O'(\mathfrak{C}')$.

3. Compact subgroup $(G_2')_K$ of G_2'

We shall consider the subgroup $(G_2')_K$ of G_2':

$$(G_2')_K = \{\alpha \in G_2' | \alpha(\boldsymbol{H}) = \boldsymbol{H}\}$$

Since $\boldsymbol{H}$ and $\boldsymbol{H}e$ are orthogonal with respect to the inner product $(x, y)'$, $\alpha \in (G_2')_K$ also satisfies $\alpha(\boldsymbol{H}e) = \boldsymbol{H}e$.

To determine this group $(G_2')_K$, we shall give some remarks on the rotation group $SO(n)$ of lower dimensions.

$$SO(4) = \{\sigma \in \mathrm{Iso}_{\boldsymbol{R}}(\boldsymbol{H}, \boldsymbol{H}) || \sigma(x)| = |x|, \ \det\sigma = 1\}$$

$$SO(3) = \{\sigma \in SO(4) | \sigma(1) = 1\}$$

$$= \{\sigma \in \mathrm{Iso}_{\boldsymbol{R}}(\boldsymbol{H}, \boldsymbol{H}) | \sigma(xy) = \sigma(x)\sigma(y)\} = \mathrm{Aut}(\boldsymbol{H})$$

$$S^3 = \{p \in \boldsymbol{H} || p| = 1\}$$

In the topological product of S^3 and $SO(3)$, if we give a multiplication by

$$(q, \tau)(p, \sigma) = (q\tau(p), \tau\sigma)$$

then $S^3 \times SO(3)$ becomes a group. We denote this group by $S^3 \cdot SO(3)$.

Lemma 3. *The rotation group $SO(4)$ is isomorphic to the group $S^3 \cdot SO(3)$.*

Proof Define a mapping $f : S^3 \cdot SO(3) \to SO(4)$ by

$$f(p, \sigma) = L_p\sigma$$

where $L_p \in SO(4)$ is defined by $L_p(x) = px$, $x \in \boldsymbol{H}$. Then it is easy to verify that f is an isomorphism.

Proposition 4. *The group $(G_2')_K$ is isomorphic to the rotation group $SO(4)$.*

Proof Define a mapping $\varphi : S^3 \cdot SO(3) \to (G_2')_K$ by

$$\varphi(p, \sigma)(a + be) = \sigma(a) + (p\sigma(b))e, \quad a, b \in \boldsymbol{H}$$

First we must show $\varphi(p, \sigma) \in (G_2')_K$. However it is easy ; in fact, $\varphi(p, \sigma)(\boldsymbol{H}) = \boldsymbol{H}$ is trivial and

$$(\varphi(p, \sigma)(a + be))(\varphi(p, \sigma)(c + de))$$

$$= (\sigma(a) + (p\sigma(b))e)(\sigma(c) + (p\sigma(d))e)$$
$$= (\sigma(a)\sigma(c) + \overline{p\sigma(d)}p\sigma(b)) + (p\sigma(b)\overline{\sigma(c)} + p\sigma(d)\sigma(a))e$$
$$= \sigma(ac + \bar{d}b) + (p\sigma(b\bar{c} + da))e$$
$$= \varphi(p, \sigma)((ac + \bar{d}b) + (b\bar{c} + da)e) = \varphi(p, \sigma)((a + bc)(c + de))$$

φ is a homomorphism because

$$\varphi(q, \tau)(\varphi(p, \sigma)(a + be)) = \varphi(q, \tau)(\sigma(a) + (p\sigma(b))e)$$
$$= \tau\sigma(a) + (q\tau(p\sigma(b))e = \tau\sigma(a) + (q\tau(p)\tau\sigma(b))e$$
$$= \varphi(q\tau(p), \tau\sigma)(a + be) = \varphi((p, \tau)(p, \sigma))(a + be)$$

Next we shall show that φ is onto. For a given $\alpha \in (G_2')_K$, consider the restriction $\sigma = \alpha|\boldsymbol{H}$ of α to $\boldsymbol{H}$, then $\alpha \in \mathrm{Aut}(\boldsymbol{H}) = SO(3)$. Put $\beta = \alpha\varphi(1, \sigma)^{-1}$, then $\beta|\boldsymbol{H} = 1$. Set $\beta(e) = pe$, $p \in \boldsymbol{H}$, then $|p| = 1$ and

$$\beta(a + be) = a + b\beta(e) = a + b(pe) = a + (pb)e = \varphi(p, 1)(a + be)$$

i.e. $\beta = \varphi(p, 1)$. Hence $\alpha = \beta\varphi(1, \sigma) = \varphi(p, 1)\varphi(1, \sigma) = \varphi(p, \sigma)$, so that φ is onto. Obviously φ is one-to-one. Thus the proof is completed.

Remark. The compact Lie group $G_2 = \mathrm{Aut}(\mathfrak{C})$ also contains a subgroup $(G_2)_K$ which is isomorphic to $SO(4)$ by a mapping $\varphi : S^3 \cdot SO(3) \to G_2$,

$$\varphi(p, \sigma)(a + be) = \sigma(a) + (p\sigma(b))e$$

4. Polar decompsition of G_2'

To give a polar decomposition of G_2' we use the following

Lemma 5 ([1] *p.* 345). *Let G be a real algebraic subgroup of the general linear group $GL(n, \boldsymbol{R})$ such that the condition $A \in G$ implies ${}^tA \in G$. Then G is homeomorphic to the topological product of $G \cap O(n)$ (which is a maximal compact subgroup of G) and a Euclidean space $\boldsymbol{R}^d$:*

$$G \simeq (G \cap O(n)) \times \boldsymbol{R}^d, \quad d = \dim(\mathfrak{g} \cap \mathfrak{h}(n))$$

where $O(n)$ is the orthogonal subgroup of $GL(n, \boldsymbol{R})$, $\mathfrak{g}$ the Lie algebra of G and $\mathfrak{h}(n)$ the vector space of all real symmetric matrices of degree n.

To use the above lemma, we define a positive definite inner product (x, y) in $\mathfrak{C}'$ by

$$(a + be, c + de) = (a, c) + (b, d)$$

Two inner products (x, y), $(x, y)'$ in $\mathfrak{C}'$ are combined with the following relations

$$(x, y) = (x, \gamma(y))', \quad (x, y)' = (x, \gamma(y))$$

where $\gamma = \varphi(-1, 1)$. We denote by ${}^t\alpha$ the transpose of $\alpha \in \mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{C}', \mathfrak{C}')$ with respect to (x, y) : $(\alpha(x), y) = (x, {}^t\alpha(y))$.

Lemma 6. *G_2' is a real algebraic subgroup of $GL(8, \boldsymbol{R}) = \mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{C}', \mathfrak{C}')$ and satisfies the condition $\alpha \in G_2'$ implies ${}^t\alpha \in G_2'$.*

Proof Since $(x, \gamma(y)) = (x, y)' = (\alpha(x), \alpha(y))' = (\alpha(x), \gamma\alpha(y)) = (x, {}^t\alpha\gamma\alpha(y))$ for $\alpha \in G_2'$, we have $\gamma = {}^t\alpha\gamma\alpha$. Hence ${}^t\alpha = \gamma\alpha^{-1}\gamma \in G_2'$. It is trivial that G_2' is real algebraic because it is defined by the algebraic relation $\alpha(xy) = \alpha(x)\alpha(y)$.

Let $O(\mathfrak{C}')$ be the orthogonal subgroup of $\mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{C}', \mathfrak{C}')$:

$$O(8) = O(\mathfrak{C}') = \{\alpha \in \mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{C}', \mathfrak{C}') \mid (\alpha(x), \alpha(y)) = (x, y)\}$$

Then $\alpha \in G_2' \cap O(\mathfrak{C}')$ induces a linear transformation of $\boldsymbol{H}$. In fact, $(\alpha(a), u) = -(\alpha(a), u)' = -(a, \alpha^{-1}(u))' = -(a, \alpha^{-1}(u)) = -(\alpha(a), u)$, for $a \in \boldsymbol{H}$, $u \in \boldsymbol{H}e$, therefore $(\alpha(a), u) = 0$ and $\alpha(a) \in \boldsymbol{H}$. Hence we have

$$G_2' \cap O(\mathfrak{C}') = (G_2')_K \cong SO(4)$$

by Proposition 4. Next we shall determine the Euclidian part $\mathfrak{g}_2' \cap \mathfrak{h}(\mathfrak{C}')$ of G_2', where

$$\mathfrak{g}_2' = \{\varsigma \in \mathrm{Hom}_{\boldsymbol{R}}(\mathfrak{C}', \mathfrak{C}') \mid \varsigma(xy) = \varsigma(x)y + x\varsigma(y)\}$$

$$\mathfrak{h}(8) = \mathfrak{h}(\mathfrak{C}') = \{\varsigma \in \mathrm{Hom}_{\boldsymbol{R}}(\mathfrak{C}', \mathfrak{C}') \mid (\varsigma(x, y) = (x, \varsigma(y))\}$$

Since $\mathfrak{g}_2'$ is a Lie subalgebra of the Lie algebra $\mathfrak{o}(4, 4) = \mathfrak{o}'(\mathfrak{C}') = \{\varsigma \in \mathrm{Hom}_{\boldsymbol{R}}(\mathfrak{C}', \mathfrak{C}') \mid (\varsigma(x), y)' + (x, \varsigma(y))' = 0\}$, $\varsigma \in \mathfrak{g}_2' \cap \mathfrak{h}(\mathfrak{C}')$ induces a linear transformation $\varsigma|\boldsymbol{H} : \boldsymbol{H} \to \boldsymbol{H}e$ and $\varsigma|\boldsymbol{H}e = {}^t(\varsigma|\boldsymbol{H})$. Put

$$\varsigma(i) = pe, \quad \varsigma(j) = qe, \quad p, q \in \boldsymbol{H}$$

then $\varsigma(k)$ is determined by $\varsigma(k) = \varsigma(ij) = \varsigma(i)j + i\varsigma(j) = (pe)j + i(qe) = (-pj + qi)e$ and obviously $\varsigma(1) = 0$. Conversely two elements $p, q \in \boldsymbol{H}$ determine ς of $\mathfrak{g}_2' \cap \mathfrak{h}(\mathfrak{C}')$. Hence

$$\dim(\mathfrak{g}_2' \cap \mathfrak{h}(\mathfrak{C}')) = 4 + 4 = 8$$

Thus we have the following

Theorem 7. *The group G_2' is homeomorphic to the topological product of the rotation group $SO(4)$ and an 8-dim. Euclidean space $\boldsymbol{R}^8$:*

$$G_2' \simeq SO(4) \times \boldsymbol{R}^8$$

In particular, G_2' *is a connected (but not simply connected) Lie group.*

5. Simplicity of G_2'

Lemma 8. *The Lie algebra* $\mathfrak{g}_2'$ *of* G_2' *is simple.*

Proof The complexification $\mathfrak{g}_2'^C$ of the Lie algebra $\mathfrak{g}_2'$ is isomorphic to the complexification $\mathfrak{g}_2^C$ of the derivation Lie algebra $\mathfrak{g}_2$ of the Cayley algebra $\mathfrak{C}$, because the complexification $\mathfrak{C}'^C$ of $\mathfrak{C}'$ is isomorphic to the one $\mathfrak{C}^C$ of $\mathfrak{C}$. As is well known $\mathfrak{g}_2^C$ is simple, so that $\mathfrak{g}_2'^C$ is so, hence $\mathfrak{G}_2'$ is also simple.

Since G_2' is a connected group from Theorem 7 and a simple group as Lie group from Lemma 8, any normal subgroup of G_2' is contained in the center $z(G_2')$ of G_2'. We shall show $z(G_2') = 1$.

Let $\alpha \in z(G_2')$. First we show that α induces a linear transformation of $\boldsymbol{H}$: $\alpha \in (G_2')_K$. In fact, put $\alpha(a) = c + de$ for $a \in \boldsymbol{H}$, then from the commutativity condition $\varphi(-1, 1)\alpha = \alpha\varphi(-1, 1)$, we have

$$c - de = \varphi(-1, 1)(c + de) = \varphi(-1, 1)\alpha(a) = \alpha\varphi(-1, 1)(a) = \alpha(a) = c + de$$

Therefore $de = 0$ and $\alpha(a) = c \in \boldsymbol{H}$. Hence there exists an element $(p, \sigma) \in S^3 \cdot SO(3)$ such that $\alpha = \varphi(p, \sigma)$ by Proposition 4. Furthermore from the commutativity condition $\alpha\varphi(q, 1) = \varphi(q, 1)\alpha$, $\alpha\varphi(1, \sigma) = \varphi(1, \tau)\alpha$, we have

$$pq = qp \qquad \text{for all } q \in S^3$$
$$\sigma\tau = \tau\sigma \qquad \text{for all } \tau \in SO(3)$$

so that $p = \pm 1$, $\sigma = 1$. Hence $\alpha = \varphi(1, 1)$ or $\alpha = \varphi(-1, 1)$. We shall show that $\varphi(-1, 1)$ is not an element of the center $z(G_2')$ using the following β. Let β be a linear transformation of $\mathfrak{C}'$ satisfying

$$\beta(1) = 1, \quad \beta(i) = ie, \quad \beta(j) = je, \quad \beta(k) = k,$$
$$\beta(e) = e, \quad \beta(ie) = i, \quad \beta(je) = j, \quad \beta(ke) = ke$$

then $\beta \in G_2'$ *and* $\varphi(-1, 1)$ does not commute with β because $\varphi(-1, 1)\beta(i) = \varphi(-1, 1)(ie) = -ie$ and $\beta\varphi(-1, 1)(i) = \beta(i) = ie$. Thus we have the following

Theorem 9. *The group* G_2' *is simple (in the algebraic sense) Lie group.*

Since the fundamental group of G_2' is $\boldsymbol{Z}_2$ from Theorem 7 and G_2' is a simple group, we have the following

Theorem 10. *The center* $z(\widetilde{G}_2')$ *of the non-compact simply connected Lie group*

$\widetilde{G}_2' = G_{2(2)}$ of type G_2 is $\boldsymbol{Z}_2$.

6. Some subgroups of G_2'

I. We shall investigate the structures of subgroups of G_2' under which some elememts of the basis $1, i, j, k, e, ie, je, ke$ of $\mathfrak{C}'$ are invariant.

(1) The group $G_2'(i) = \{\alpha \in G_2' | \alpha(i) = i\}$ is isomorphic to the group

$$SU(1, 2) = \{A \in M(3, \boldsymbol{C}) | AJA^* = J, \ \det A = 1\}$$

where $\boldsymbol{C}$ is the complex numbers, A^* the conjugate transposed matrix of A and $J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$. And $G_2'(i)$ is homeomorphic to

$$G_2'(i) \simeq U(2) \times \boldsymbol{R}^4$$

where $U(2)$ is the unitary group.

(2) The group $G_2'(e) = \{\alpha \in G_2' | \alpha(e) = e\}$ is isomorphic to the group

$$SU(3, \mathfrak{a}) = \{A \in M(3, \mathfrak{a}) | AA^* = E, \ \det A = 1\}$$

where $\mathfrak{a}$ is the algebra $\mathfrak{a} = \{a + b\iota | a, b \in \boldsymbol{R}, \iota^2 = 1\}$ with the conjugation $\overline{a + b\iota} = a - b\iota$ and A^* is the conjugate transposed matrix of A. And $G_2'(e)$ is homeomorphic to

$$G_2'(e) \simeq SO(3) \times \boldsymbol{R}^5$$

(3) The group $G_2'(i, j, k) = \{\alpha \in G_2' | \alpha(i) = i, \ \alpha(j) = j, \ \alpha(k) = k\}$ is isomorphic to the group S^3.

(4) The group $G_2'(i, e, ie) = \{\alpha \in G_2' | \alpha(i) = i, \ \alpha(e) = e, \ \alpha(ie) = ie\}$ is isomorphic to the general linear group $GL(2, \boldsymbol{R})$, hence $G_2'(i, e, ie)$ is homeomorphic to $S^1 \times \boldsymbol{R}^2$.

II. Let $M_{m,n}$ be the manifold defined by

$$M_{m,n} = \{(x_1, \cdots, x_m, x_{m+1}, \cdots, x_{m+n}) \in \boldsymbol{R}^{m+n} | \sum_{i=1}^{m} x_i^2 - \sum_{j=m+1}^{m+n} x_j^2 = 1\}$$

Then we have the following homogeneous spaces.

$$G_2'/G_2'(i) \simeq M_{3,4}, \qquad G_2'(i)/G_2'(i, j, k) \simeq M_{2,4},$$

$$G_2'/G_2'(e) \simeq M_{4,3}, \qquad G_2'(e)/G_2'(i, e, ie) \simeq M_{3,3}$$

References

[1] S. HELGASON ; *Differential Geometry and Symmetric Spaces*, Pure and Applied Math., 1962.