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Explicit Isomorphism between SU(4) and Spin(6)

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It is well known that the special unitary group $SU(4)$ and the spinor group $Spin(6)$ are isomorphic. To prove this it is usually used that their Lie algebras are isomorphic. In this paper, we shall prove it by giving a homomorphism $p: SU(4) \to SO(6)$ explicitly.

1. Preliminaries.

(1) Let $\boldsymbol{C}$ and $\boldsymbol{H}=\boldsymbol{C}\oplus j\boldsymbol{C}$ be the complex and the quaternion fields respectively. $\boldsymbol{H}$ is isomorphic to the space $\mathfrak{H}=\{x\in M(2, \boldsymbol{C})\,|\,xj=j\bar{x}\}$, where $j=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, as algebra, by the correspondence $k: \boldsymbol{H}\to\mathfrak{H}$,

$$k(a+jb)=\begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix}, \qquad a,b\in \boldsymbol{C},$$

and k has the following properties:

$$k(\bar{x})=\boldsymbol{x}^*, \qquad \frac{1}{2}(\boldsymbol{x}\boldsymbol{y}^*+\boldsymbol{y}\boldsymbol{x}^*)=(x,y)E, \qquad \boldsymbol{x}\boldsymbol{x}^*=\boldsymbol{x}^*\boldsymbol{x}=|x|^2E$$

where $\boldsymbol{x}=k(x)$, $\boldsymbol{y}=k(y)$ and E is the unit matrix. This mapping k is naturally extended to the spaces of matrices:

$$k: M(2, \boldsymbol{H})\to M(4, \boldsymbol{C}), \qquad k\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \begin{pmatrix} k(x_{11}) & k(x_{12}) \\ k(x_{21}) & k(x_{22}) \end{pmatrix}.$$

(2) Let $\mathfrak{J}(2, \boldsymbol{H})$ be the vector space of all 2×2 quaternion Hermitian matrices:

$$\mathfrak{J}(2, \boldsymbol{H})=\{X\in M(2, \boldsymbol{H})\,|\,X^*=X\}.$$

In $\mathfrak{J}(2, \boldsymbol{H})$, we define the inner product (X, Y) by

$$(X, Y)=\frac{1}{2}\mathrm{tr}(XY+YX).$$

Let $\mathfrak{J}(2, \boldsymbol{H})^C=\{X=X_1+iX_2\,|\,X_1, X_2\in\mathfrak{J}(2, \boldsymbol{H})\}$ be the complexification of $\mathfrak{J}(2, \boldsymbol{H})$.

In $\mathfrak{J}(2, \boldsymbol{H})^C$ we define the Hermitian inner product $\langle X, Y\rangle$ by

$$\langle X_1+iX_2, Y_1+iY_2\rangle=(X_1, Y_1)+(X_2, Y_2)+i((X_1, Y_2)-(X_2, Y_1)).$$

Furthermore let $\mathfrak{S}(4, \boldsymbol{C})$ be the vector space of all 4×4 complex skew-symmetric matrices:

$$\mathfrak{S}(4, \boldsymbol{C})=\{P\in M(4, \boldsymbol{C})\,|\,{}^tP=-P\}.$$

In $\mathfrak{S}(4, \boldsymbol{C})$ we define the Hermitian inner product $\langle P, Q\rangle$ by

$$\langle P, Q\rangle=-\frac{1}{4}\mathrm{tr}(P\overline{Q}+Q\overline{P}).$$

Then the space $\mathfrak{J}(2, \boldsymbol{H})^C$ is isomorphic to the space $\mathfrak{S}(4, \boldsymbol{C})$ by the correspondence $h: \mathfrak{J}(2, \boldsymbol{H})^C\to\mathfrak{S}(4, \boldsymbol{C})$,

$$h(X_1+iX_2)=(k(X_1)+ik(X_2))J, \qquad J=\begin{pmatrix} j & 0\\ 0 & j\end{pmatrix}.$$

(3) Let $\mathfrak{c}_2$ be the Lie algebra of all 2×2 quaternion skew-Hermitian matrices:

$$\mathfrak{c}_2=\{D\in M(2, \boldsymbol{H})\,|\,D^*=-D\}$$

and $\mathfrak{a}_3$ the Lie algebra of all 4×4 complex skew-Hermitian matrices with zero trace:

$$\mathfrak{a}_3=\{S\in M(4, \boldsymbol{C})\,|\,S^*=-S,\ \mathrm{tr}(S)=0\}.$$

Any element S of $\mathfrak{a}_3$ can be represented by the form

$$\begin{aligned} S&=k(D)+ik(T), \qquad D\in\mathfrak{c}_2,\ T\in\mathfrak{J}(2, \boldsymbol{H}),\ \mathrm{tr}(T)=0\\ &=k(D)+ik(F(a))+itk(E_1-E_2)\end{aligned}$$

where $F(a)=\begin{pmatrix} 0 & a\\ \bar{a} & 0\end{pmatrix}$, $a\in\boldsymbol{H}$, $E_1-E_2=\begin{pmatrix} 1 & 0\\ 0 & -1\end{pmatrix}$ and $t\in\boldsymbol{R}$ ($\boldsymbol{R}$ is the field of real numbers). In fact, for $S\in\mathfrak{a}_3$, put $D_1=\frac{1}{2}(S-J\overline{S}J)$ and $T_1=-\frac{i}{2}(S+J\overline{S}J)$, then we have $S=D_1+iT_1$, $D_1{}^*=-D_1$, $D_1J=J\overline{D}_1$ and $T_1{}^*=T_1$, $T_1J=J\overline{T}_1$, $\mathrm{tr}(T_1)=0$. So $D=k^{-1}(D_1)$ and $T=k^{-1}(T_1)$ satisfy the required conditions.

2. Low dimensional spinor groups.

We define the low dimensional symplectic groups, the special unitary group and the orthogonal groups by

$$Sp(1)=\{a\in\boldsymbol{H}\ |\ |a|=1\},$$

$$Sp(2)=\{A\in M(2, \boldsymbol{H})\,|\,A^*A=E\},$$

$$SU(4)=\{A\in M(4, \boldsymbol{C})\,|\,A^*A=E,\ \det A=1\},$$

$$SO(3)=SO(\boldsymbol{H}_0)=\{\alpha\in\mathrm{Iso}_{\boldsymbol{R}}(\boldsymbol{H}_0, \boldsymbol{H}_0)\,|\,(\alpha x, \alpha y)=(x, y),\ \det\alpha=1\}$$

where $\boldsymbol{H}_0=\{x\in\boldsymbol{H}\,|\,\bar{x}=-x\}$,

$$SO(4)=SO(\boldsymbol{H})=\{\alpha\in\mathrm{Iso}_{\boldsymbol{R}}(\boldsymbol{H},\ \boldsymbol{H})\,|\,(\alpha x,\ \alpha y)=(x,\ y),\ \det\alpha=1\},$$

$$SO(5)=SO(\mathfrak{J}_0)=\{\alpha\in\mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{J}_0,\ \mathfrak{J}_0)\,|\,(\alpha X,\ \alpha Y)=(X,\ Y),\ \det\alpha=1\}$$

where $\mathfrak{J}_0=\mathfrak{J}(2,\ \boldsymbol{H})_0=\{X\in\mathfrak{J}(2,\ \boldsymbol{H})\,|\,\mathrm{tr}(X)=0\}$ and

$$SO(6)=SO(V)=\{\alpha\in\mathrm{Iso}_{\boldsymbol{R}}(V,\ V)\,|\,\langle\alpha X,\ \alpha Y\rangle=\langle X,\ Y\rangle,\ \det\alpha=1\}$$

where $V=\left\{\begin{pmatrix}\xi & x\\ \bar{x} & -\bar{\xi}\end{pmatrix}\Big|\,\xi\in\boldsymbol{C},\ x\in\boldsymbol{H}\right\}\subset\mathfrak{J}(2,\ \boldsymbol{H})^{C}$.

We note that the restriction of the mepping h of the section 1 on V is an isometry:

$$\langle h(X),\ h(Y)\rangle=\langle X,\ Y\rangle,\qquad X,\ Y\in V$$

and the group $Sp(2)$ acts on the space $\mathfrak{J}(2,\ \boldsymbol{H})$ by $\mu: Sp(2)\times\mathfrak{J}(2,\ \boldsymbol{H})\to\mathfrak{J}(2,\ \boldsymbol{H})$, $\mu(A,\ X)=AXA^*$ and it holds that

$$(AXA^*,\ AYA^*)=(X,\ Y),\qquad \mathrm{tr}(AXA^*)=\mathrm{tr}(X).$$

On the other hand, the group $SU(4)$ acts on the space $\mathfrak{S}(4,\ \boldsymbol{C})$ by $\mu: SU(4)\times\mathfrak{S}(4,\ \boldsymbol{C})\to\mathfrak{S}(4,\ \boldsymbol{C})$, $\mu(A,\ P)=AP{}^tA$ and it holds that

$$\langle AP{}^tA,\ AQ{}^tA\rangle=\langle P,\ Q\rangle.$$

Now we define the following homomorphisms.

$$p_1: Sp(1)\to SO(3),\qquad p_1(a)x=ax\bar{a},\ x\in\boldsymbol{H}_0,$$

$$p_2: Sp(1)\times Sp(1)\to SO(4),\qquad p_2(a,\ b)x=ax\bar{b},\ x\in\boldsymbol{H},$$

$$p_3: Sp(2)\to SO(5),\qquad p_3(A)X=AXA^*,\ X\in\mathfrak{J}_0,$$

$$p=p_4: SU(4)\to SO(6),\qquad p(A)X=h^{-1}(Ah(X){}^tA),\ X\in V.$$

Then we have

Theorem 1. *The following diagram is commutative*

$$\begin{array}{ccccccc}
Sp(1) & \xrightarrow{k_1} & Sp(1)\times Sp(1) & \xrightarrow{k_2} & Sp(2) & \xrightarrow{k} & SU(4)\\
\downarrow p_1 & & \downarrow p_2 & & \downarrow p_3 & & \downarrow p\\
SO(3) & \xrightarrow{j_1} & SO(4) & \xrightarrow{j_2} & SO(5) & \xrightarrow{j} & SO(6)
\end{array}$$

where k_1 is the diagonal mapping and k_2, j_1, j_2, j are natural inclusions. And each mapping p_i is the universal covering homomorphism. In particular, we have the following isomorphisms.

$$Sp(1)\cong Spin(3),\qquad Sp(1)\times Sp(1)\cong Spin(4),$$

$$Sp(2)\cong Spin(5),\qquad SU(4)\cong Spin(6).$$

Proof. As for the mapping p_1, p_2, they are well known (Chap. I [1]). The mapping p_3 is also well known, however we will give a proof that p_3 is onto by using the following

Lemma 2. *Let G, G' be groups, H, H' subgroups of G, G' respectively and $p: G\to G'$ a homomorphism satisfying $p(H)\subset H'$. If $p'=p|H: H\to H'$ and $\bar{p}: G/H\to G'/H'$ (the induced mapping of p) are both onto, then $p: G\to G'$ is also onto.*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & H & \longrightarrow & G & \longrightarrow & G/H & \longrightarrow & * \\ & & \downarrow p' & & \downarrow p & & \downarrow \bar{p} & & \\ 1 & \longrightarrow & H' & \longrightarrow & G' & \longrightarrow & G'/H' & \longrightarrow & * \end{array}$$

Proof of Lemma 2 is easy (Lemma 1.50 [2]).

Let S^4 be the unit sphere in $\mathfrak{J}(2, \boldsymbol{H})_0$:

$$S^4=\{X\in\mathfrak{J}(2, \boldsymbol{H})_0\,|\,\langle X, X\rangle=2\}.$$

By using that any element of $\mathfrak{J}(2, \boldsymbol{H})$ can be transformed in a diagonal form by the action μ of $Sp(2)$, we see that any element X of S^4 can be transformed to $E_1-E_2=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ by $Sp(2)$. This shows that the group $Sp(2)$ acts transitively on S^4. Since the isotropy subgroup of $Sp(2)$ at E_1-E_2 is $k_2(Sp(1)\times Sp(1))$, we have the following homeomorphism

$$Sp(2)/k_2(Sp(1)\times Sp(1))\simeq S^4.$$

Thus we have the following diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & Sp(1)\times Sp(1) & \xrightarrow{k_2} & Sp(2) & \longrightarrow & S^4 & \longrightarrow & * \\ & & \downarrow p_2 & & \downarrow p_3 & & \| & & \\ 1 & \longrightarrow & SO(4) & \xrightarrow{j_2} & SO(5) & \longrightarrow & S^4 & \longrightarrow & * \end{array}$$

Therefore, from Lemma 2, we see that p_3 is onto. $\mathrm{Ker}\, p_3=\boldsymbol{Z}_2=\{E, -E\}$ is easily obtained.

Now, we consider the mapping $p: SU(4)\to SO(6)$. In order to prove that the mapping p is well-defined, first we have to show that, for $A\in SU(4)$ and $X\in V$, we have

$$p(A)X=h^{-1}(Ah(X)^tA)\in V.$$

Since any element S of the Lie algebra $\mathfrak{a}_3$ of $SU(4)$ is represented by the form

$$S=k(D)+ik(F(a))+itk(E_1-E_2)$$

as §1 (3), the group $SU(4)$ is generated by the elements such as $\exp k(D)$, $\exp ik(F(a))$ and $\exp itk(E_1-E_2)$. For $A=k(A_1)$ where $A_1=\exp D\in Sp(2)$, $X\in V$, we have

$$h^{-1}(Ah(X)^tA)=h^{-1}(k(A_1)k(X)J^tk(A_1))=h^{-1}(k(A_1)k(X)k(A_1)^*J)$$
$$=h^{-1}(k(A_1XA_1^*)J)=A_1XA_1^*\in V.$$

For $A=\exp ik(F(a))$, $X\in V$, we have

$$h^{-1}(Ah(X)^tA)=h^{-1}((\exp ik(F(a)))k(X)J^t(\exp ik(F(a))))$$
$$=h^{-1}((\exp ik(F(a)))k(X)(\exp ik(F(a)))J)$$
$$=h^{-1}(k((\exp iF(a))X(\exp iF(a)))J)$$
$$=(\exp iF(a))X(\exp iF(a))$$
$$=\begin{pmatrix} \cos|a| & i\frac{a}{|a|}\sin|a| \\ i\frac{\bar{a}}{|a|}\sin|a| & \cos|a| \end{pmatrix}\begin{pmatrix} \xi & x \\ \bar{x} & -\bar{\xi} \end{pmatrix}\begin{pmatrix} \cos|a| & i\frac{a}{|a|}\sin|a| \\ i\frac{\bar{a}}{|a|}\sin|a| & \cos|a| \end{pmatrix}$$
$$=\begin{pmatrix} \eta & y \\ \bar{y} & -\bar{\eta} \end{pmatrix}\in V,$$

where

$$\eta=\xi\cos^2|a|+\bar{\xi}\sin^2|a|+i\frac{2(a,\ x)}{|a|}\sin|a|\cos|a|,$$
$$y=x-\frac{2(a,\ x)}{|a|^2}\sin^2|a|+i\frac{(\xi-\bar{\xi})a}{|a|}\sin|a|\cos|a|.$$

For $A(t)=\exp\frac{it}{2}k(E_1-E_2)$, $X=\begin{pmatrix} \xi & x \\ \bar{x} & -\bar{\xi} \end{pmatrix}\in V$, it is easy to verify that

$$h^{-1}(A(t)h(X)^tA(t))=\begin{pmatrix} e^{it}\xi & x \\ \bar{x} & -e^{-it}\bar{\xi} \end{pmatrix}\in V.$$

Thus $p(A)X\in V$ is proved. For $A\in SU(4)$, we see that $p(A)\in O(8)=O(V)=\{\alpha\in \mathrm{Iso}_{\boldsymbol{R}}(V,\ V)\,|\,\langle\alpha X,\ \alpha Y\rangle=\langle X,\ Y\rangle\}$, because

$$\langle p(A)X,\ p(A)Y\rangle=\langle h(p(A)X),\ h(p(A)Y)\rangle$$
$$=\langle Ah(X)^tA,\ Ah(Y)^tA\rangle=\langle h(X),\ h(Y)\rangle=\langle X,\ Y\rangle.$$

Since $SU(4)$ is connected, $p(SU(4))$ is contained in the connected component $SO(6)$ of identity E in $O(V)$, i.e. $p(SU(4))\subset SO(6)$. Thus we see that the mapping p is well-defined.

Let S^5 be the unit sphere in V:

$$S^5=\{X\in V\,|\,\langle X,\ X\rangle=2\}$$

We shall prove that the group $SU(4)$ acts transitively on S^5. To prove this, it is sufficient to show that any element X of S^5 can be transformed to $i(E_1+E_2)=\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$. For a given $X\in S^5$, operate some element $A(t_0)=\exp\frac{it}{2}(E_1-E_2)$, then we have

$$p(A(t_0))X\in S^4.$$

Since $Sp(2)$ acts transitively on S^4, there exists $A\in Sp(2)$ such that

$$p(k(A)A(t_0))X=E_1-E_2,$$

and then operate $A\left(\frac{\pi}{2}\right)=\exp\frac{i\pi}{4}(E_1-E_2)$ on it, then we have

$$p(A\left(\frac{\pi}{2}\right)k(A)A(t_0))X=i(E_1+E_2).$$

This implies the transitivity of $SU(4)$. Since the isotropy subgroup of $SU(4)$ at $i(E_1+E_2)$ is $k(Sp(2))$, we have the following homeomorphism

$$SU(4)/k(Sp(2))\simeq S^5.$$

Thus we have the following commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & Sp(2) & \overset{k}{\longrightarrow} & SU(4) & \longrightarrow & S^5 & \longrightarrow & * \\ & & \downarrow p_3 & & \downarrow p & & \| & & \\ 1 & \longrightarrow & SO(5) & \underset{j}{\longrightarrow} & SO(6) & \longrightarrow & S^5 & \longrightarrow & * \end{array}$$

Therefore, from Lemma 2, we see that p is onto. $\mathrm{Ker}\,p=\{E, -E\}$ is easily obtained. Thus the proof of Theorem 1 is completed.

References

[1] C. CHEVALLY ; *Theory of Lie Groups I*, Princeton Univ. Press, 1946.
[2] I. YOKOTA ; *Groups and Representations* (in Japanese), Shokabo, 1973.