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Notes on the Radical of a Finite Dimensional Algebra

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Throughout this note, A will represent a finite dimensional algebra (with 1) over a field K, N the radical of A, M the left annihilator of N, Λ the center of A, and Π the radical of Λ. Moreover, let G be a finite group, P a p-Sylow subgroup of G, and G' the commutator subgroup of G.

In his recent paper [10], Y. Tsushima proved the following : Let K be an algebraically closed field of characteristic p, and $A=KG$. Then $N=\Pi A$ if and only if $|G'|\not\equiv 0 \pmod p$. In reality, the "if" part is an immeadiate cosequence of K. Morita's theorem [8, Theorem 2]. The "only if" part will be carried over to the case of finite dimensional algebras (Theorem 1).

In [5], M. Hall showed that if A is (two sided) indecomposable then A is either simple or bound (i. e. two sided annihilator of N is contained in N). More recently, S. Asano [1] has proved that if A is an indecomposable quasi-Frobenius algebra and $N^2\neq 0$, then $M\subseteq N^2$. Next, we shall give an alternative proof of this result (see Theorem 2).

Now, let K be an algebraically closed field of characteristic p. In case G is p-solvable, R. J. Clarke [2] characterized G such that Π is an ideal of $A=KG$. Recently, S. Koshitani [6] has proved that if Π is an ideal of $A=KG$, then G is p-solvable. This result will be improved in Theorem 3, whose proof is rather elegant compared with that given in [6].

First we shall prove the following, which has been obtained independently by B. Külshammer [7].

Theorem 1. *Assume that A* is (*two sided*) *indecomposable. If $N=\Pi A$, then A is primary. Moreover, if K is a splitting field for A, then A is a full matrix ring over Λ.*

Proof. Let e and f be primitive idempotents of A. If eAf is contained in N, then $eAf = eNf = e\Pi Af = \Pi eAf$ by assumption. Since Π is nilpotent, $eAf = 0$. Consequently, if $eAf \neq 0$, then the right A-modules eA and fA are isomorphic. Since A is indecomposable, it follows that A contains only one (non-isomorphic) indecomposable right A-submodule and so A is a full matrix ring over a completely primary ring B. Noting that ΠB is the radical of B and Π is nilpotent, we can see that $B = \Lambda$ if K is a splitting field for A.

The proof of the next theorem provides an alternative proof of [1, Theorem 1].

Theorem 2. *Assume that A is a (two sided) indecomposable quasi-Frobenius algebra. Then the following are equivalent.*

(1) $N^2 \neq 0$.

(2) $eN^2 \neq 0$ *for every primitive idempotent* e.

(3) $M \subseteq N^2$.

Proof. Since A is quasi-Frobenius, the implications (2)→(3) and (3)→(1) are trivial. It remains therefore to prove that (1) implies (2). Assume that there exist primitive idempotents e and f of A such that $eN^2 = 0$ and $fN^2 \neq 0$. Then the right A-modules eA and fA are not isomorphic, and so $eAf \subseteq N$, which implies that $0 = eN^2 \supseteq eAf \cdot N$, namely $eAf = eMf$. Since A is quasi-Frobenius, we must obtain $N^2 f \neq 0$, and so $N^2 f \supseteq Mf$. Consequently $0 = eN^2 f \supseteq eMf = eAf$. By symmetry, $fAe = 0$. This contradicts the hypothesis that A is indecomposable. Hence, $eN^2 \neq 0$ for every primitive idempotent e.

In the remainder of this note, we assume that K is an algebraically closed field of characteristic p and $A = KG$. Let $\{B_1, B_2, \cdots, B_s, \cdots, B_t\}$ the set of all blocks of A, where B_1 is a principal block and $\{B_1, B_2, \cdots, B_s\}$ is the set of all blocks containing linear complex characters. Let Δ_i and Γ_i be the sets of all irreducible complex characters and all irreducible Brauer characters contained in B_i, respectively. We set $k_i = |\Delta_i|$ and $l_i = |\Gamma_i|$. Furthermore, k'_i and l'_i will denote the numbers of all linear complex characters and of all linear Brauer characters in B_i, respectively.

The proof of the next lemma is immeadiate, and may be omitted.

Lemma. *The groups of all linear complex characters and of all linear Brauer characters are transitive permutation groups acting by multiplications on $\{\Delta_1, \cdots, \Delta_s\}$ and $\{\Gamma_1, \cdots, \Gamma_s\}$, respectively. In paticular, $|G/G'| = k'_1 s$, $|G/PG'| = l'_1 s$,* and $s \not\equiv 0 \pmod p$.

The above lemma will be used freely in the proof of the following theorem.

Theorem 3. *If Π is an ideal of KG, then G' is either a p-nilpotent group or a p'-group.*

Proof. Assume that $|G'| \equiv 0 \pmod{p}$. Then $k_i > k_i'$ for $i \leq s$ (see [4, Theorem 65.2]) and by [2, Lemma 4] $\Pi = KG\sigma$, where $\sigma = \sum_{x \in G'} x$. Thus, we have

$$\sum_{i=1}^{s} k_i' = |G:G'| = [\Pi:K] = \sum_{i=1}^{t} k_i - t,$$

which implies that $k_i = k_i'+1$ ($i \leq s$) and for $j > s$, B_j contains only one irreducible complex character χ_j such that $\chi_j(1) \equiv 0 \pmod{|P|}$. We may assume that G is not p-nilpotent. Hence, by Thompson's theorem [9, Theorem 1], there exists a non-linear irreducible complex character θ such that $\theta(1) \not\equiv 0 \pmod{p}$. Thus, for $i \leq s$, the degree of any non-linear irreducible complex character contained in B_i is $\theta(1)$. Hence, by $s \not\equiv 0 \pmod{p}$, we obtain

$$|G:G'| = |G| - s\theta(1)^2 - \sum_{j>s}^{t} \chi_j(1)^2 \equiv -s\theta(1)^2 \not\equiv 0 \pmod{p}.$$

It follows then $k_1' = l_1'$. On the other hand, $k_1'+1 = k_1 > l_1$ (see [3, Exercise 86.2]), and so $k_1' = l_1' \leq l_1 \leq k_1'$, which means that G' is p-nilpotent (see [4, Theorem 65.2]).

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