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Non-compact simple Lie group $E_{7(-25)}$ of type E_7

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It is known that there exist four simple Lie groups of type E_7 up to local isomorphism, one of them is compact and the others are non-compact. As for the compact case, it is known that the following group

$$E_7 = \left\{ \alpha \in \mathrm{Iso}_C(\mathfrak{P}^C, \mathfrak{P}^C) \;\middle|\; \begin{array}{l} \alpha\mathfrak{M}^C = \mathfrak{M}^C, \ \{\alpha\dot{1}, \alpha\dot{\mathrm{i}}\} = 1 \\ \langle\alpha P, \alpha Q\rangle = \langle P, Q\rangle \end{array} \right\}$$

is a simply connected compact simple Lie group of type E_7 [4]. As for one of non-compact cases, H. Freudenthal showed in [2] that the Lie algebra of the group

$$E_{7,1} = \{ \alpha \in \mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{P}, \mathfrak{P}) \mid \alpha\mathfrak{M} = \mathfrak{M}, \ \{\alpha P, \alpha Q\} = \{P, Q\}\}$$

is a simple Lie algebra of type E_7, where $\mathfrak{M}$ is the Freudenthal's manifold in $\mathfrak{P} = \mathfrak{J} \oplus \mathfrak{J} \oplus \boldsymbol{R} \oplus \boldsymbol{R}$, $\mathfrak{M}^C$, $\mathfrak{P}^C$ the complexification of $\mathfrak{M}$, $\mathfrak{P}$ respectively and $\{P, Q\}$, $\langle P, Q\rangle$ inner products in $\mathfrak{P}$ or $\mathfrak{P}^C$. In this paper, we shall investigate the structures of this group $E_{7,1}$. Our results are as follows. The group $E_{7,1}$ is a connected non-compact simple Lie group of type E_7 and its center is the cyclic group of order 2 :

$$z(E_{7,1}) = \{1, -1\}.$$

The polar decomposition of the group $E_{7,1}$ is given by

$$E_{7,1} \simeq (U(1) \times E_6)/\boldsymbol{Z}_3 \times \boldsymbol{R}^{54}.$$

In order to give the above decomposition, we construct another group

$$E_{7,\iota} = \left\{ \alpha \in \mathrm{Iso}_C(\mathfrak{P}^C, \mathfrak{P}^C) \;\middle|\; \begin{array}{l} \alpha\mathfrak{M}^C = \mathfrak{M}^C, \ \{\alpha\dot{1}, \alpha\dot{\mathrm{i}}\} = 1 \\ \langle\alpha P, \alpha Q\rangle_\iota = \langle P, Q\rangle_\iota \end{array} \right\}$$

(where $\langle P, Q\rangle_\iota$ is another inner product in $\mathfrak{P}^C$) which is isomorphic to $E_{7,1}$ and find the subgroup $(U(1) \times E_6)/\boldsymbol{Z}_3$ explicitly in this group $E_{7,\iota}$.

1. Preliminaries.

Let $\mathfrak{C}$ denote the Cayley algebra over the field of real numbers $\boldsymbol{R}$ and $\mathfrak{J}$ the Jordan algebra consisting of all 3×3 Hermitian matrices in $\mathfrak{C}$ with respect to the multiplication $X\circ Y=\frac{1}{2}(XY+YX)$. In $\mathfrak{J}$, the positive definite symmetric inner product $(X,\ Y)$, the crossed product $X\times Y$, the cubic form $(X,\ Y,\ Z)$ and the determinant $\det X$ are defined respectively by

$$
\begin{aligned}
&(X,\ Y)=\mathrm{tr}(X\circ Y),\\
&X\times Y=\frac{1}{2}(2X\circ Y-\mathrm{tr}(X)Y-\mathrm{tr}(Y)X+(\mathrm{tr}(X)\mathrm{tr}(Y)-(X,\ Y))E),\\
&(X,\ Y,\ Z)=(X\times Y,\ Z)=(X,\ Y\times Z),\\
&\det X=\frac{1}{3}(X,\ X,\ X)
\end{aligned}
$$

where E is the 3×3 unit matrix.

Now we define a 56 dimensional vector space $\mathfrak{P}$ by

$$\mathfrak{P}=\mathfrak{J}\oplus\mathfrak{J}\oplus\boldsymbol{R}\oplus\boldsymbol{R}.$$

An element $P=\begin{pmatrix}X\\ Y\\ \xi\\ \eta\end{pmatrix}$ of $\mathfrak{P}$ is often denoted by $P=X+\dot{Y}+\xi+\dot{\eta}$ briefly. We define a bilinear mapping $\times:\ \mathfrak{P}\times\mathfrak{P}\longrightarrow\mathfrak{J}\oplus\mathfrak{J}\oplus\boldsymbol{R}$ by

$$P\times Q=\begin{pmatrix}X\\ Y\\ \xi\\ \eta\end{pmatrix}\times\begin{pmatrix}Z\\ W\\ \zeta\\ \omega\end{pmatrix}=\begin{pmatrix}2X\times Z-\eta W-\omega Y\\ 2Y\times W-\xi Z-\zeta X\\ (X,\ W)+(Y,\ Z)-3(\xi\omega+\eta\zeta)\end{pmatrix}$$

and a space $\mathfrak{M}$ by

$$
\begin{aligned}
\mathfrak{M}&=\{\,L\in\mathfrak{P}\mid L\times L=0,\ L\neq 0\}\\
&=\left\{L=\begin{pmatrix}M\\ N\\ \mu\\ \nu\end{pmatrix}\in\mathfrak{P}\ \middle|\ \begin{array}{l}M\times M=\nu N\\ N\times N=\mu M\\ (M,\ N)=3\mu\nu\\ L\neq 0\end{array}\right\}.
\end{aligned}
$$

For example, the following elements of $\mathfrak{P}$

$$\begin{pmatrix} X \\ \frac{1}{\eta}(X\times X) \\ \frac{1}{\eta^2}\det X \\ \eta \end{pmatrix}, \quad \begin{pmatrix} \frac{1}{\xi}(Y\times Y) \\ Y \\ \xi \\ \frac{1}{\xi^2}\det Y \end{pmatrix}, \quad 1 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \dot{1} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

where $\eta \neq 0$, $\xi \neq 0$, belong to $\mathfrak{M}$. Finally in $\mathfrak{P}$ we define the skew-symmetric inner product $\{P, Q\}$ by

$$\{P, Q\} = (X, W) - (Z, Y) + \xi\omega - \zeta\eta$$

for $P = X + \dot{Y} + \xi + \dot{\eta}$, $Q = Z + \dot{W} + \zeta + \dot{\omega} \in \mathfrak{P}$.

2. Group $E_{7,1}$ and its Lie algebra $\mathfrak{e}_{7,1}$.

The group $E_{7,1}$ is defined to be the group of linear isomorphisms of $\mathfrak{P}$ leaving the space $\mathfrak{M}$ and the skew-symmetric inner product $\{P, Q\}$ invariant :

$$E_{7,1} = \{\alpha \in \mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{P}, \mathfrak{P}) \mid \alpha\mathfrak{M} = \mathfrak{M},\ \{\alpha P, \alpha Q\} = \{P, Q\}\}.$$

We define a subgroup $E_{6,1}$ of $E_{7,1}$ by

$$E_{6,1} = \{\alpha \in E_{7,1} \mid \alpha 1 = 1,\ \alpha\dot{1} = \dot{1}\}.$$

Proposition 1. *The group $E_{6,1}$ is a simply connected non-compact simple Lie group of type $E_{6(-26)}$.*

Proof. We define a group $E_{6(-26)}$ by

$$\begin{aligned} E_{6(-26)} &= \{\beta \in \mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{J}, \mathfrak{J}) \mid \det\beta X = \det X\} \\ &= \{\beta \in \mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{J}, \mathfrak{J}) \mid \beta X \times \beta Y = {}^t\beta^{-1}(X \times Y)\} \end{aligned}$$

where ${}^t\beta$ is the transpose of β with respect to the inner product (X, Y) : $(\beta X, Y) = (X, {}^t\beta Y)$. Then $E_{6(-26)}$ is a simply connected simple Lie group of type E_6 [1] and moreover of type $E_{6(-26)}$, since its polar decomposition is given by

$$E_{6(-26)} \simeq F_4 \times \boldsymbol{R}^{26}$$

where F_4 is a simply connected compact simple Lie group of type F_4 [1]. We shall show that the group $E_{6,1}$ is isomorphic to the group $E_{6(-26)}$. It is easy to verify that, for $\beta \in E_{6(-26)}$, the linear transformation α of $\mathfrak{P}$ defined by

$$\alpha = \begin{pmatrix} \beta & 0 & 0 & 0 \\ 0 & {}^t\beta^{-1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

belongs to $E_{7,1}$. Conversely suppose $\alpha \in E_{7,1}$ satisfies $\alpha 1=1$ and $\alpha \dot{1}=\dot{1}$. Then from the conditions $\{\alpha X, \alpha 1\} = \{\alpha X, \alpha \dot{1}\} = 0$ and $\{\alpha \dot{X}, \alpha 1\} = \{\alpha \dot{X}, \alpha \dot{1}\} = 0$, we see that α has the form

$$\alpha = \begin{pmatrix} \beta & \varepsilon & 0 & 0 \\ \delta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where β, γ, δ, ε are linear transformations of $\mathfrak{J}$. Since

$$\alpha \begin{pmatrix} X \\ \frac{1}{\eta}(X\times X) \\ \frac{1}{\eta^2}\det X \\ \eta \end{pmatrix} = \begin{pmatrix} \beta X + \frac{1}{\eta}\varepsilon(X\times X) \\ \delta X + \frac{1}{\eta}\gamma(X\times X) \\ \frac{1}{\eta^2}\det X \\ \eta \end{pmatrix} \in \mathfrak{M},$$

we have

$$(\beta X+\frac{1}{\eta}\varepsilon(X\times X))\times(\beta X+\frac{1}{\eta}\varepsilon(X\times X)) = \eta(\delta X+\frac{1}{\eta}\gamma(X\times X))$$

for all $0 \neq \eta \in \boldsymbol{R}$. Hence we have $\delta X = 0$ for all $X \in \mathfrak{J}$ as the coefficient of η, therefore $\delta=0$. Similarly $\varepsilon=0$. Thus

$$\alpha = \begin{pmatrix} \beta & 0 & 0 & 0 \\ 0 & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Again the condition $\alpha(X+(X\times X)^{\cdot}+\det X+\dot{1})=\beta X+(\gamma(X\times X))^{\cdot}+\det X+\dot{1}\in\mathfrak{M}$ implies

$$\begin{cases} \beta X\times\beta X = \gamma(X\times X), \\ (\beta X, \gamma(X\times X)) = 3\det X. \end{cases}$$

Hence $\det\beta X = \frac{1}{3}(\beta X, \beta X\times\beta X) = \frac{1}{3}(\beta X, \gamma(X\times X)) = \det X$, therefore $\beta \in E_{6(-26)}$ and $\gamma = {}^t\beta^{-1} \in E_{6(-26)}$. Thus Proposition 1 is proved.

The group $E_{7,1}$ contains also a subgroup

$$\boldsymbol{R}^* = \left\{ r = \begin{pmatrix} r^{-1}1 & 0 & 0 & 0 \\ 0 & r1 & 0 & 0 \\ 0 & 0 & r^3 & 0 \\ 0 & 0 & 0 & r^{-3} \end{pmatrix} \middle| \, 0 \neq r \in \boldsymbol{R} \right\}$$

(where 1 denotes the identity mapping of $\mathfrak{J}$) which is isomorphic to the group $\mathbf{R}^* = \{ r \in \boldsymbol{R} \mid r \neq 0 \}$.

From now on, we identify these groups $E_{6(-26)}$ and $E_{6,1}$, $\mathbf{R}^*$ and $\boldsymbol{R}^*$ under the above correspondences.

We consider the Lie algebra $\mathfrak{e}_{7,1}$ of the group $E_{7,1}$

$$\mathfrak{e}_{7,1} = \left\{ \Phi \in \mathrm{Hom}_{\boldsymbol{R}}(\mathfrak{P}, \mathfrak{P}) \,\middle|\, \begin{array}{c} \Phi L \times L = 0 \text{ for } L \in \mathfrak{M} \\ \{ \Phi P, Q \} + \{ P, \Phi Q \} = 0 \text{ for } P, Q \in \mathfrak{P} \end{array} \right\}.$$

H. Freudenthal proved in [2] the following

Theorem 2. *Any element Φ of the Lie algebra $\mathfrak{e}_{7,1}$ of the group $E_{7,1}$ is represented by the form*

$$\Phi = \Phi(\phi, A, B \rho) = \begin{pmatrix} \phi - \frac{1}{3}\rho 1 & 2B & 0 & A \\ 2A & \phi' + \frac{1}{3}\rho 1 & B & 0 \\ 0 & A & \rho & 0 \\ B & 0 & 0 & -\rho \end{pmatrix}$$

where $\phi \in \mathfrak{e}_{6,1} = \{ \phi \in \mathrm{Hom}_{\boldsymbol{R}}(\mathfrak{J}, \mathfrak{J}) \mid (\phi X, X, X) = 0 \}$ (which is the Lie algebra of the group $E_{6,1}$), ϕ' is the skew-transpose of ϕ with respect to the inner product (X, Y) : $(\phi X, Y) + (X, \phi' Y) = 0$, $A, B \in \mathfrak{J}$, $\rho \in \boldsymbol{R}$ and the action of Φ on $\mathfrak{P}$ is defined by

$$\Phi \begin{pmatrix} X \\ Y \\ \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \phi X - \frac{1}{3}\rho X + 2B \times Y + \eta A \\ 2A \times X + \phi' Y + \frac{1}{3}\rho Y + \xi B \\ (A, Y) + \rho\xi \\ (B, X) - \rho\eta \end{pmatrix}.$$

And the type of the Lie algebra $\mathfrak{e}_{7,1}$ is E_7.

We shall determine the Cartan index of the group $E_{7,1}$. For this purpose we use the following

Lemma 3 ([3] p. 345). *Let G be an algebraic subgroup of the general linear group $GL(n, \boldsymbol{R})$ such that the condition $A \in G$ implies ${}^tA \in G$. Then G is homeomorphic to the topological product of the group $G \cap O(n)$ (which is a maximal compact* *subgroup of* G) *and a Euclidean space* $\boldsymbol{R}^d$:

$$G \simeq (G \cap O(n)) \times \boldsymbol{R}^d$$

where $O(n)$ *is the orthogonal subgroup of* $GL(n, \boldsymbol{R})$. *In particular, the Cartan index of* G *is* $\dim G - 2\dim(G \cap O(n))$.

Theorem 4. *The group* $E_{7,1}$ *is a simple Lie group of type* $E_{7(-25)}$.

Proof. We define in $\mathfrak{P}$ a positive definite symmetric inner product (P, Q) by

$$(P, Q) = (X, Z) + (Y, W) + \xi\zeta + \eta\omega$$

for $P = X + \dot{Y} + \dot{\xi} + \dot{\eta}$, $Q = Z + \dot{W} + \zeta + \dot{\omega} \in \mathfrak{P}$ and denote the transpose of Φ with respect to this inner product (P, Q) by ${}^t\Phi : (\Phi P, Q) = (P, {}^t\Phi Q)$. Then for

$$\Phi = \begin{pmatrix} \phi - \frac{1}{3}\rho 1 & 2B & 0 & A \\ 2A & \phi' + \frac{1}{3}\rho 1 & B & 0 \\ 0 & A & \rho & 0 \\ B & 0 & 0 & -\rho \end{pmatrix} \in \mathfrak{e}_{7,1},$$

we see easily that

$${}^t\Phi = \begin{pmatrix} -\phi' - \frac{1}{3}\rho 1 & 2A & 0 & B \\ 2B & -\phi + \frac{1}{3}\rho 1 & A & 0 \\ 0 & B & \rho & 0 \\ A & 0 & 0 & -\rho \end{pmatrix},$$

therefore ${}^t\Phi$ also belongs to $\mathfrak{e}_{7,1}$. Since $E_{7,1}$ is an algebraic subgroup of the general linear group $\mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{P}, \mathfrak{P}) = GL(56, \boldsymbol{R})$, from Lemma 3, the Lie algebra $\mathfrak{e}_{7,1} \cap \mathfrak{o}(\mathfrak{P})$ (where $\mathfrak{o}(\mathfrak{P}) = \mathfrak{o}(56) = \{\Phi \in \mathrm{Hom}_{\boldsymbol{R}}(\mathfrak{P}, \mathfrak{P}) \mid \Phi + {}^t\Phi = 0\}$) of the group $E_{7,1} \cap O(\mathfrak{P})$ (where $O(\mathfrak{P}) = O(56) = \{\alpha \in \mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{P}, \mathfrak{P}) \mid (\alpha P, \alpha Q) = (P, Q)\}$) is a maximal compact Lie subalgebra of $\mathfrak{e}_{7,1}$. Now if $\Phi \in \mathfrak{e}_{7,1}$ satisfies $\Phi + {}^t\Phi = 0$, then

$$\Phi = \begin{pmatrix} \delta & -2A & 0 & A \\ 2A & \delta & -A & 0 \\ 0 & A & 0 & 0 \\ -A & 0 & 0 & 0 \end{pmatrix}$$

where $\delta \in \mathfrak{f}_4 = \{\delta \in \mathfrak{e}_{6,1} \mid \delta' = \delta\}$ (which is the Lie algebra of F_4). Therefore $\dim(\mathfrak{e}_{7,1} \cap \mathfrak{o}(\mathfrak{P})) = \dim\mathfrak{f}_4 + \dim\mathfrak{J} = 52 + 27 = 79$. Hence

$$\text{The Cartan index of } \mathfrak{e}_{7,1} = \dim\mathfrak{e}_{7,1} - 2\dim(\mathfrak{e}_{7,1} \cap \mathfrak{o}(\mathfrak{P}))$$

$$= 133 - 2\times 79 = -25.$$

Thus we see that the type of the Lie algebra $\mathfrak{e}_{7,1}$ is $E_{7(-25)}$.

3. Connectedness of $E_{7,1}$.

We shall prove that the group $E_{7,1}$ is connected. We denote, for a while, the connected component of $E_{7,1}$ containing the identity 1 by $(E_{7,1})_0$.

Lemma 5, *For $A \in \mathfrak{J}$, the linear transformation* $\exp_1(A)$ *of* $\mathfrak{P}$ *defined by*

$$\exp_1(A) = \begin{pmatrix} 1 & 0 & 0 & A \\ 2A & 1 & 0 & A\times A \\ A\times A & A & 1 & \det A \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(*the action of* $\exp_1(A)$ *on* $\mathfrak{P}$ *is as similar to that of Theorem* 2) *belongs to* $(E_{7,1})_0$. *Similarly for* $B \in \mathfrak{J}$ *we can define*

$$\exp_2(B) = \begin{pmatrix} 1 & 2B & B\times B & 0 \\ 0 & 1 & B & 0 \\ 0 & 0 & 1 & 0 \\ B & B\times B & \det B & 1 \end{pmatrix} \in (E_{7,1})_0.$$

Proof.

$$\text{For } \Phi_1(A) = \begin{pmatrix} 0 & 0 & 0 & A \\ 2A & 0 & 0 & 0 \\ 0 & A & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \mathfrak{e}_{7,1}. \text{ we have } \exp_1(A) = \exp\Phi_1(A), \text{hence}$$

$\exp_1(A) \in (E_{7,1})_0$. Similarly $\exp_2(B) \in (E_{7,1})_0$.

Proposition 6. *The subgroup $G_1 = \{\alpha \in E_{7,1} \mid \alpha 1 = 1\}$ is the semi-direct product of the group $\exp_1(\mathfrak{J}) = \{\exp_1(A) \mid A \in \mathfrak{J}\}$ (which is an abelian group) and the group $E_{6,1}$:*

$$G_1 = \exp_1(\mathfrak{J})E_{6,1}, \quad \exp_1(\mathfrak{J}) \cap E_{6,1} = \{1\}.$$

Therefore G_1 is homeomorphic to

$$G_1 \simeq E_{6,1} \times \boldsymbol{R}^{27} \simeq F_4 \times \boldsymbol{R}^{53}.$$

In particular, the group G_1 is simply connected.

Proof. Let $\alpha \in G$ and put $\alpha\dot{1} = M + \dot{N} + \mu + \dot{\nu}$. Then the conditions $\{\alpha 1, \alpha\dot{1}\} = 1$ and $\alpha\dot{1} \in \mathfrak{M}$ imply $\nu = 1$ and $N = M\times M$, $\mu = \det M$ respectively. Therefore we have

$$\exp_1(M)\dot{1} = M + (M\times M)^{\boldsymbol{\cdot}} + \det M + \dot{1} = \alpha\dot{1}, \ \exp_1(M)1 = 1 = \alpha 1.$$

so $(\exp_1(M))^{-1}\alpha \in E_{6,1}$, i. e.

$$\alpha \in \exp_1(\mathfrak{J})E_{6,1},$$

and conversely. Since the Lie subalgebra $\{\Phi_1(A)=\Phi(0, A, 0, 0)\in \mathfrak{e}_{7,1} \mid A\in\mathfrak{J}\}$ of $\mathfrak{e}_{7,1}$ is abelian, the group $\exp_1(\mathfrak{J})$ is also abelian. Moreover $\exp_1(\mathfrak{J})$ is a normal subgroup of G_1, because it holds that

$$\beta\exp_1(A)\beta^{-1}=\exp_1(\beta A) \qquad \text{for } \beta \in E_{6,1}, A \in \mathfrak{J}.$$

Therefore we have the following split exact sequence

$$1 \longrightarrow \exp_1(\mathfrak{J}) \longrightarrow G_1 \longrightarrow E_{6,1} \longrightarrow 1$$

Thus we see that G_1 is the semi-direct product of $\exp_1(\mathfrak{J})$ and $E_{6,1}$.

Theorem 7. *The group $E_{7,1}$ acts transitively on the manifold $\mathfrak{M}$ (which is connected) and the isotropy subgroup G_1 of $E_{7,1}$ at $1 \in \mathfrak{M}$ is* $\exp_1(\mathfrak{J})E_{6,1}$ (Proposition 6). *Therefore the homogeneous space $E_{7,1}/\exp_1(\mathfrak{J})E_{6,1}$ is homeomorphic* to $\mathfrak{M}$:

$$E_{7,1}/\exp_1(\mathfrak{J})E_{6,1} \simeq \mathfrak{M}.$$

In particular, the group $E_{7,1}$ is connected.

Proof. Obviously the group $E_{7,1}$ acts on $\mathfrak{M}$. We shall prove that the group $(E_{7,1})_0$ acts transitively on $\mathfrak{M}$. Since

$$\exp_1(-E)\exp_2(E)1 = \dot{1},\ \exp_1(E)\exp_2(-E)1=-\dot{1},\ \exp_2(-E)(\exp_1(E))^2\exp_2(-E)1=-1,$$

it is sufficient to show that any element $L\in\mathfrak{M}$ can be transformed in either of 1, -1, $\dot{1}$, $-\dot{1}$. Let $L = M + \dot{N} + \mu + \dot{\nu} \in \mathfrak{M}$. First assume $\mu>0$. Then $M = \dfrac{1}{\mu}(N\times N)$, $\nu = \dfrac{1}{\mu^2}\det N$. Choose $0< r \in \boldsymbol{R}$ such that $r^3=\mu$, then for

$$r = \begin{pmatrix} r^{-1}1 & 0 & 0 & 0 \\ 0 & r1 & 0 & 0 \\ 0 & 0 & r^3 & 0 \\ 0 & 0 & 0 & r^{-3} \end{pmatrix} \in (E_{7,1})_0$$

we have $r1=\mu$, and hence

$$\begin{aligned}\exp_2\Big(\frac{N}{\mu}\Big)r1 &= \mu\Big(\frac{N}{\mu}\times\frac{N}{\mu}\Big)+\mu\Big(\frac{N}{\mu}\Big)^{\cdot}+\mu+\mu\Big(\det\frac{N}{\mu}\Big)^{\cdot} \\ &= \frac{1}{\mu}(N\times N)+\dot{N}+\mu+\frac{1}{\mu^2}(\det N)^{\cdot}= L.\end{aligned}$$

If $\mu<0$. L can be transformed in -1. Similarly in the case $\nu\neq 0$ the statement is also valid. Next we consider the case $L = M + \dot{N} \in \mathfrak{M}$, $N\neq 0$. Then $M\times M = N$

$\times N=0$, $\det M=0$, $(N, N)\neq 0$ and so

$$\exp_1(N)L= * + \dot{*} + (N, N) + \dot{*}.$$

So we can reduce to the first case $\mu\neq 0$. In the case of $M\neq 0$, the statement is also valid. Thus the transitivity of $(E_{7,1})_0$ on $\mathfrak{M}$ is proved. Therefore we have $\mathfrak{M}=(E_{7,1})_0 1$, hence $\mathfrak{M}$ is connected. Since the group $E_{7,1}$ acts transitively on $\mathfrak{M}$ and the isotropy subgroup of $E_{7,1}$ is $\exp_1(\mathfrak{J})E_{6,1}$, we have the following homeomorphism

$$E_{7,1}/\exp_1(\mathfrak{J})E_{6,1}\simeq\mathfrak{M}.$$

Since $\exp_1(\mathfrak{J})E_{6,1}$ is connected, $E_{7,1}$ is also connected. Thus the proof of Theorem 7 is completed.

4. Center $z(E_{7,1})$ of $E_{7,1}$.

Theorem 8. *The center $z(E_{7,1})$ of the group $E_{7,1}$ is isomorphic to the cyclic group Z_2 of order* 2 :

$$z(E_{7,1})=\{1, -1\}\cong\boldsymbol{Z}_2.$$

Proof. Let $\alpha\in z(E_{7,1})$. From the commutativity with $\beta\in E_{6,1}\subset E_{7,1}$. we have $\beta\alpha 1=\alpha\beta 1=\alpha 1$. If we denote $\alpha 1=M+\dot{N}+\mu+\dot{\nu}$, then $\beta M+({}^t\beta^{-1}N)^{\cdot}+\mu+\dot{\nu}=M+\dot{N}+\mu+\dot{\nu}$, hence

$$\beta M=M,\quad {}^t\beta^{-1}N=N\qquad\text{for all }\beta\in E_{6,1}.$$

Therefore $M=N=0$, so $\alpha 1=\mu+\dot{\nu}$, where $\mu\nu=0$ (since $\alpha 1\in\mathfrak{M}$). Suppose that $\mu=0$, i.e. $\alpha 1=\dot{\nu}\neq 0$, then from the commutativity with

$$r=\begin{pmatrix} r^{-1}1 & 0 & 0 & 0\\ 0 & r1 & 0 & 0\\ 0 & 0 & r^3 & 0\\ 0 & 0 & 0 & r^{-3}\end{pmatrix}\in\boldsymbol{R}^*\subset E_{7,1},$$

we have

$$(r^{-3}\nu)^{\cdot}=r\dot{\nu}=r\alpha 1=\alpha r1=\alpha r^3=(r^3\nu)^{\cdot}\quad\text{for all } r\in\boldsymbol{R}^*.$$

This is contradiction. Hence $\alpha 1=\mu$. Similarly $\alpha\dot{1}=\dot{\lambda}$. The condition $\{\alpha 1, \alpha\dot{1}\}=1$ implies $\mu\lambda=1$, hence

$$\alpha 1=\mu,\qquad \alpha\dot{1}=(\mu^{-1})^{\cdot}.$$

Next note that

$$\iota' = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

belongs to $E_{7,1}$. Then the commutativity condition $\iota'\alpha = \alpha\iota'$ implies

$$\dot{\mu} = \iota'\mu = \iota'\alpha 1 = \alpha\iota' 1 = \alpha\dot{1} = (\mu^{-1})^{\cdot},$$

hence $\mu = \mu^{-1}$, i.e. $\mu = \pm 1$. In the case of $\mu = 1$, $\alpha \in E_{6,1}$ so $\alpha \in z(E_{6,1}) = \{1\}$ [5] i.e. $\alpha = 1$. In the case of $\mu = -1$, $-\alpha \in z(E_{6,1}) = \{1\}$, i.e. $\alpha = -1$. Thus we see that $z(E_{7,1}) = \{1, -1\}$.

5. Group $E_{7,\iota}$ and its Lie algebra $e_{7,\iota}$.

We construct another simple Lie group of type $E_{7(-25)}$. Let C denote the field of complex numbers and $\mathfrak{J}^C$ the complexification of $\mathfrak{J}$. In $\mathfrak{J}^C$ also, the inner product (X, Y), crossed product $X \times Y$, the cubic form (X, Y, Z) and the determinant $\det X$ are defined as similar in $\mathfrak{P}$. Let $\mathfrak{P}^C$ be also the complexification of $\mathfrak{P}$:

$$\mathfrak{P}^C = \mathfrak{J}^C \oplus \mathfrak{J}^C \oplus C \oplus C.$$

We define a mapping $\times : \mathfrak{P}^C \times \mathfrak{P}^C \longrightarrow \mathfrak{J}^C \oplus \mathfrak{J}^C \oplus C$ as similar as the case $\mathfrak{P}$ and a space $\mathfrak{M}^C$ by

$$\mathfrak{M}^C = \{ L \in \mathfrak{P}^C \mid L \times L = 0, \ L \neq 0 \}.$$

Finally in $\mathfrak{J}^C$, $\mathfrak{P}^C$, positive definite Hermitian inner products $\langle X, Y\rangle$, $\langle P, Q\rangle$ and the inner product $\langle P, Q\rangle_\iota$, the skew-symmetric inner product $\{P, Q\}$ are defined respectively by

$$\begin{aligned} &\langle X, Y\rangle = (\tau X, Y) = (\overline{X}, Y), \\ &\langle P, Q\rangle = \langle X, Z\rangle + \langle Y, W\rangle + \overline{\xi}\zeta + \overline{\eta}\omega, \\ &\langle P, Q\rangle_\iota = \langle X, Z\rangle - \langle Y, W\rangle + \overline{\xi}\zeta - \overline{\eta}\omega, \\ &\{ P, Q \} = (X, W) - (Z, Y) + \xi\omega - \zeta\eta, \end{aligned}$$

where $\tau : \mathfrak{J}^C \longrightarrow \mathfrak{J}^C$ is the complex conjugate (τX is also denoted by $\overline{X}$) and $P = X + \dot{Y} + \xi + \dot{\eta}$, $Q = Z + \dot{W} + \zeta + \dot{\omega} \in \mathfrak{P}^C$.

Now the group $E_{7,\iota}$ is defined to be the group of linear isomorphisms of $\mathfrak{P}^C$ leaving the space $\mathfrak{M}^C$, some skew-symmetric inner product $\{P, Q\}$ and the inner product $\langle P, Q\rangle_\iota$ invariant:

$$E_{7,\iota} = \left\{ \alpha \in \mathrm{Iso}_C(\mathfrak{P}^C, \mathfrak{P}^C) \;\middle|\; \begin{array}{c} \alpha\mathfrak{M}^C = \mathfrak{M}^C, \ \{\alpha 1, \alpha\dot{1}\} = 1 \\ \langle \alpha P, \alpha Q\rangle_\iota = \langle P, Q\rangle_\iota \text{ for } P, Q \in \mathfrak{P}^C \end{array} \right\}.$$

We define a subgroup E_6 of $E_{7,\iota}$ by

$$E_6=\{\,\alpha\in E_{7,\iota}\mid \alpha 1=1,\ \alpha\dot{1}=\dot{1}\,\}.$$

Proposition 9. *The group E_6 is a simply connected compact simple Lie group of type E_6 and isomorphic to the group*

$$\begin{aligned}E_{6(-78)}&=\{\,\beta\in \mathrm{Iso}_C(\mathfrak{J}^C,\ \mathfrak{J}^C)\mid \det\beta X=\det X,\ \langle\beta X,\ \beta Y\rangle=\langle X,\ Y\rangle\,\}\\&=\{\,\beta\in \mathrm{Iso}_C(\mathfrak{J}^C,\ \mathfrak{J}^C)\mid \beta X\times\beta Y=\tau\beta\tau(X\times Y),\ \langle\beta X,\ \beta Y\rangle=\langle X,\ Y\rangle\,\}\end{aligned}$$

(see [7]) *by the correspondence*

$$E_{6(-78)}\in\beta\longrightarrow\begin{pmatrix}\beta&0&0&0\\0&\tau\beta\tau&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\in E_{7,\iota}.$$

Proof. It is seen by the analogous proof of Proposition 1 (or see [4] Proposition 2).

The group $E_{7,\iota}$ contains also a subgroup

$$U(1)=\left\{\theta=\begin{pmatrix}\theta^{-1}1&0&0&0\\0&\theta 1&0&0\\0&0&\theta^3&0\\0&0&0&\theta^{-3}\end{pmatrix}\;\middle|\;\theta\in C,\ |\theta|=1\right\}$$

which is isomorphic to the unitary group $\mathrm{U}(1)=\{\,\theta\in C\mid|\theta|=1\,\}$.

From now on, we identify these group $E_{6(-78)}$ and E_6, $\mathrm{U}(1)$ and $U(1)$ under the above correspondences.

We consider the Lie algebra $\mathfrak{e}_{7,\iota}$ of the group $E_{7,\iota}$:

$$\mathfrak{e}_{7,\iota}=\left\{\Phi\in \mathrm{Hom}_C(\mathfrak{P}^C,\ \mathfrak{P}^C)\;\middle|\;\begin{array}{c}\Phi L\times L=0 \text{ for } L\in\mathfrak{M}^C\\ \{\Phi 1,\ \dot{1}\}+\{1,\ \Phi\dot{1}\}=0\\ \langle\Phi P,\ Q\rangle_\iota+\langle P,\ \Phi Q\rangle_\iota=0 \text{ for } P,\ Q\in\mathfrak{P}^C\end{array}\right\}.$$

Theorem 10. *Any element Φ of the Lie algebra $\mathfrak{e}_{7,\iota}$ is represented by the form*

$$\Phi=\begin{pmatrix}\phi-\frac{1}{3}\rho 1&2A&0&\overline{A}\\2\overline{A}&\tau\phi\tau+\frac{1}{3}\rho 1&A&0\\0&\overline{A}&\rho&0\\A&0&0&-\rho\end{pmatrix}$$

where $\phi \in \mathfrak{e}_6 = \{\phi \in \mathrm{Hom}_C(\mathfrak{J}^C, \mathfrak{J}^C) \mid (\phi X, X, X) = 0, <\phi X, Y> + <X, \phi Y> = 0\}$ *(which is the Lie algebra of the group* E_6*),* $A \in \mathfrak{J}^C$, $\rho \in \boldsymbol{C}$ *such that* $\rho + \bar{\rho} = 0$ *and the action of* Φ *on* $\mathfrak{P}^C$ *is defined by*

$$\Phi \begin{pmatrix} X \\ Y \\ \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \phi X - \frac{1}{3}\rho X + 2A \times Y + \eta \bar{A} \\ 2\bar{A} \times X + \tau\phi\tau Y + \frac{1}{3}\rho Y + \xi A \\ <A, Y> + \rho\xi \\ (A, X) - \rho\eta \end{pmatrix}.$$

In particular, the type of the Lie group $E_{7,\iota}$ *is* E_7 [2].

Proof. It is obtained by the analogous argument as Theorem 3 of [4].

6. Involutive automorphism $\boldsymbol{\iota}$ **and subgroup** $(U(1) \times \boldsymbol{E}_6)/\boldsymbol{Z}_3$**.**

We define an involutive linear isomorphism ι of $\mathfrak{P}^C$ by

$$\iota = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Then two inner products $<P, Q>$, $<P, Q>_\iota$ in $\mathfrak{P}^C$ are combined with relations

$$<P, Q>_\iota = <\iota P, Q> = <P, \iota Q>, \qquad <P, Q> = <\iota P, Q>_\iota = <P, \iota Q>_\iota.$$

The following Lemma is easily verified.

Lemma 11. *For* $\alpha \in E_{7,\iota}$, *we have* $\iota\alpha\iota \in E_{7,\iota}$.

Therefore we can define an automorphism $\boldsymbol{\iota} : E_{7,\iota} \longrightarrow E_{7,\iota}$ by

$$\boldsymbol{\iota}\alpha = \iota\alpha\iota \qquad \alpha \in E_{7,\iota}.$$

Proposition 12. *The subgroup* $\{\alpha \in E_{7,\iota} \mid \iota\alpha\iota = \alpha\}$ *of the group* $E_{7,\iota}$ *is isomorphic to the group* $(U(1) \times E_6)/\boldsymbol{Z}_3$:

$$\{\alpha \in E_{7,\iota} \mid \iota\alpha\iota = \alpha\} \cong (U(1) \times E_6)/\boldsymbol{Z}_3$$

where $\boldsymbol{Z}_3 = \{(1, 1), (\omega, \omega 1), (\omega^2, \omega^2 1)\}$, $\omega \in \boldsymbol{C}$, $\omega^3 = 1$, $\omega \neq 1$, *and*

$$\omega = \begin{pmatrix} \omega^{-1}1 & 0 & 0 & 0 \\ 0 & \omega 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in U(1), \qquad \omega 1 = \begin{pmatrix} \omega 1 & 0 & 0 & 0 \\ 0 & \omega^{-1}1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in E_6.$$

Proof. We define a mapping $\phi : U(1)\times E_{6(-78)} \longrightarrow \{\alpha \in E_{7,\iota} \mid \iota\alpha\iota = \alpha\}$ by

$$\phi(\theta,\ \beta)= \theta\beta = \begin{pmatrix} \theta^{-1}1 & 0 & 0 & 0 \\ 0 & \theta 1 & 0 & 0 \\ 0 & 0 & \theta^3 & 0 \\ 0 & 0 & 0 & \theta^{-3} \end{pmatrix}\begin{pmatrix} \beta & 0 & 0 & 0 \\ 0 & \tau\beta\tau & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \beta\theta.$$

Then obviously ϕ is a homomorphism. We shall prove that ϕ is onto. If $\alpha\in E_{7,\iota}$ satisfies $\iota\alpha\iota=\alpha$, then α has the form

$$\alpha = \begin{pmatrix} \beta & 0 & M & 0 \\ 0 & \gamma & 0 & N \\ a & 0 & \mu & 0 \\ 0 & b & 0 & \nu \end{pmatrix}$$

where β, γ are linear transformations of $\mathfrak{J}^C$, a, b linear functionals of $\mathfrak{J}^C$, M, $N\in \mathfrak{J}^C$ and μ, $\nu \in \boldsymbol{C}$. The conditions $\alpha\dot{1}$, $\alpha 1 \in \mathfrak{M}^C$ imply

$$\mu M = 0, \qquad \nu N = 0$$

respectively. We shall show that $M = N = 0$. Assume $M \neq 0$, then $\mu = 0$, so a is not identically 0. And then from $\{\alpha 1,\ \alpha\dot{1}\}=1$, we have

$$(M,\ N)=1, \tag{i}$$

hence $N \neq 0$, so $\nu = 0$. Furthermore the condition

$$\alpha\begin{pmatrix} X \\ \frac{1}{\eta}(X \times X) \\ \frac{1}{\eta^2}\det X \\ \eta \end{pmatrix} = \begin{pmatrix} \beta X + \frac{1}{\eta^2}(\det X)M \\ \frac{1}{\eta}\gamma(X\times X)+\eta N \\ a(X) \\ \frac{1}{\eta}b(X\times X) \end{pmatrix} \in \mathfrak{M}^C$$

implies

$$\begin{cases} (\frac{1}{\eta}\gamma(X \times X)+\eta N)\times(\frac{1}{\eta}\gamma(X\times X)+\eta N)= a(X)(\beta X+\frac{1}{\eta^2}(\det X)M), \\ (\beta X+\frac{1}{\eta^2}(\det X)M,\ \frac{1}{\eta}\gamma(X\times X)+\eta N)= 3a(X)\frac{1}{\eta}b(X\times X) \end{cases}$$

for all $0 \neq \eta \in \boldsymbol{C}$. Hence we have

$$\begin{cases} 2\gamma(X\times X)\times N = a(X)\beta X, & \text{(ii)} \\ \gamma(X\times X)\times\gamma(X\times X) = a(X)(\det X)M, & \text{(iii)} \\ (\beta X,\ \gamma(X\times X)) + \det X = 3a(X)b(X\times X). & \text{(iv)} \end{cases}$$

Therefore

$$a(X)\det X \underset{\text{(i)}}{=} a(X)(\det X)(M,\ N) \underset{\text{(iii)}}{=} (\gamma(X\times X)\times\gamma(X\times X),\ N)$$

$$= (\gamma(X\times X),\ \gamma(X\times X)\times N) \underset{\text{(ii)}}{=} \frac{1}{2}a(X)(\gamma(X\times X),\ \beta X)$$

$$\underset{\text{(iv)}}{=} \frac{1}{2}a(X)(3a(X)b(X\times X) - \det X).$$

Hence

$$a(X)\det X = (a(X))^2 b(X\times X).$$

Thus we have

$$\det X = a(X)b(X\times X)$$

(since $a : \mathfrak{J}^C \longrightarrow \boldsymbol{C}$ is a linear functional and $\det X - a(X)b(X\times X)$ is continuous with respect to X, even if for X such that $a(X)=0$). This contradicts to the irreducibility of the determinant $\det X$ with respect to the variables of its components. Thus we have $M=0$. Similarly $N=0$. So

$$\alpha 1 = \mu, \qquad \alpha\dot{1} = (\mu^{-1})^{\cdot} \qquad\qquad \mu \in \boldsymbol{C},\ |\mu| = 1.$$

Choose $\theta \in \boldsymbol{C}$ such that $\theta^3 = \mu$ and put $\beta = \theta^{-1}\alpha$, then $\beta 1 = 1$ and $\beta\dot{1} = \dot{1}$, therefore $\beta \in E_6$. Thus we have

$$\alpha = \theta\beta \qquad\qquad \theta \in U(1),\ \beta \in E_6.$$

So ϕ is onto. $\operatorname{Ker}\phi = \{(1,\ 1),\ (\omega,\ \omega 1),\ (\omega^2,\ \omega^2 1)\}$, $\omega \in \boldsymbol{C}$, $\omega^3 = 1$, $\omega \neq 1$, is easily obtained. Thus the proof of Proposition 12 is completed.

7. Polar decomposition of $E_{7,\iota}$.

In order to give a polar decomposition of the group $E_{7,\iota}$, we use the following

Lemma 13 ([3] p. 345). *Let G be a pseudoalgebraic subgroup of the general linear group $GL(n,\ \boldsymbol{C})$ such that the condition $A \in G$ implies $A^* \in G$. Then G is homeomorphic to the topological product of the group $G \cap U(n)$ (which is a maximal compact subgroup of G) and a Euclidean space $\boldsymbol{R}^d$:*

$$G \simeq (G \cap U(n)) \times \boldsymbol{R}^d$$

where $U(n)$ is the unitary subgroup of $GL(n,\ \boldsymbol{C})$.

Lemma 14. *$E_{7,\iota}$ is a pseudoalgebraic subgroup of the general linear group $GL(56, \boldsymbol{C}) = \mathrm{Iso}_{\boldsymbol{C}}(\mathfrak{P}^C, \mathfrak{P}^C)$ and satisfies the condition $\alpha \in E_{7,\iota}$ implies $\alpha^* \in E_{7,\iota}$ where α^* is the transpose of α with respect to the inner product $\langle P, Q\rangle : \langle \alpha P, Q\rangle = \langle P, \alpha^* Q\rangle$.*

Proof. Since $\langle \alpha^* P, Q\rangle = \langle P, \alpha Q\rangle = \langle \iota P, \alpha Q\rangle_\iota = \langle \alpha^{-1}\iota P, Q\rangle_\iota = \langle \iota\alpha^{-1}\iota P, Q\rangle$ for $\alpha \in E_{7,\iota}$, we have

$$\alpha^* = \iota\alpha^{-1}\iota \in E_{7,\iota} \qquad \text{(Lemma 11)}.$$

And it is obvious that $E_{7,\iota}$ is pseudoalgebraic, because $E_{7,\iota}$ is defined by pseudoalgebraic relations $\alpha\mathfrak{M}^C = \mathfrak{M}^C$, $\{\alpha \dot{1}, \alpha \underset{\cdot}{1}\} = 1$ and $\langle \alpha P, \alpha Q\rangle_\iota = \langle P, Q\rangle_\iota$.

Let $U(56) = U(\mathfrak{P}^C) = \{\alpha \in \mathrm{Iso}_{\boldsymbol{C}}(\mathfrak{P}^C, \mathfrak{P}^C) \mid \langle \alpha P, \alpha Q\rangle = \langle P, Q\rangle\}$ denote the unitary subgroup of the general linear group $GL(56, \boldsymbol{C}) = \mathrm{Iso}_{\boldsymbol{C}}(\mathfrak{P}^C, \mathfrak{P}^C)$, then we have

$$\begin{aligned} E_{7,\iota} \cap U(\mathfrak{P}^C) &= \{\alpha \in E_{7,\iota} \mid \iota\alpha\iota = \alpha\} \\ &\cong (U(1) \times E_6)/\boldsymbol{Z}_3 \qquad \text{(Proposition 12)}. \end{aligned}$$

Since $E_{7,\iota}$ is a simple Lie group of type E_7, the dimension of $E_{7,\iota}$ is 133. Hence the dimension d of the Euclidean part of $E_{7,\iota}$ and the Cartan index i are calculated as follows :

$$\begin{aligned} d &= \dim E_{7,\iota} - \dim(U(1) \times E_6) = 133 - (1 + 78) = 54, \\ i &= \dim E_{7,\iota} - 2\dim(U(1) \times E_6) = 133 - 2(1 + 78) = -25. \end{aligned}$$

Thus we get the following

Theorem 15. *The group $E_{7,\iota}$ is homeomorphic to the topological product of the group $(U(1) \times E_6)/\boldsymbol{Z}_3$ and a 54 dimensional Euclidean space $\boldsymbol{R}^{54}$:*

$$E_{7,\iota} \simeq (U(1) \times E_6)/\boldsymbol{Z}_3 \times \boldsymbol{R}^{54}.$$

In particular, the group $E_{7,\iota}$ is a connected non-compact simple Lie group of type $E_{7(-25)}$.

8. Center $z(E_{7,\iota})$ of $E_{7,\iota}$.

Lemma 16. *For $a \in \boldsymbol{C}$, the transformation of $\mathfrak{P}^C$ defined by*

$$\alpha_1(a) = \begin{pmatrix} 1 + (\cosh|a| - 1)p_1 & 2a\dfrac{\sinh|a|}{|a|}E_1 & 0 & \bar{a}\dfrac{\sinh|a|}{|a|}E_1 \\ 2\bar{a}\dfrac{\sinh|a|}{|a|}E_1 & 1 + (\cosh|a| - 1)p_1 & a\dfrac{\sinh|a|}{|a|}E_1 & 0 \\ 0 & \bar{a}\dfrac{\sinh|a|}{|a|}E_1 & \cosh|a| & 0 \\ a\dfrac{\sinh|a|}{|a|}E_1 & 0 & 0 & \cosh|a| \end{pmatrix}$$

(if $a=0$, then $a\dfrac{\sinh|a|}{|a|}$ means 0) belongs to $E_{7,\iota}$, where $E_1=\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}\in\mathfrak{J}^C$,

the mapping $p_1:\mathfrak{J}^C\longrightarrow\mathfrak{J}^C$ is defined by

$$p_1\begin{pmatrix}\xi_1&x_3&\bar{x}_2\\\bar{x}_3&\xi_2&x_1\\x_2&\bar{x}_1&\xi_3\end{pmatrix}=\begin{pmatrix}\xi_1&0&0\\0&\xi_2&x_1\\0&\bar{x}_1&\xi_3\end{pmatrix}$$

and the action of $\alpha_1(a)$ on $\mathfrak{P}^C$ is defined as similar to that of Theorem 10.

Proof.

For $\Phi_1(a)=\begin{pmatrix}0&2aE_1&0&\bar{a}E_1\\2\bar{a}E_1&0&aE_1&0\\0&\bar{a}E_1&0&0\\aE_1&0&0&0\end{pmatrix}\in\mathfrak{e}_{7,\iota}$, we have $\alpha_1(a)=\exp\Phi_1(a)$, hence

$\alpha_1(a)\in E_{7,\iota}$.

Theorem 17. *The center $z(E_{7,\iota})$ of the group $E_{7,\iota}$ is isomorphic to the cyclic group of order* 2 :

$$z(E_{7,\iota})=\{1,\ -1\}.$$

Proof. Let $\alpha\in z(E_{7,\iota})$. From the commutativity with $\beta\in E_6\subset E_{7,\iota}$, we have $\beta\alpha 1=\alpha\beta 1=\alpha 1$. If we denote $\alpha 1=M+\dot{N}+\mu+\dot{\nu}$, then $\beta M+(\tau\beta\tau M)^{\cdot}+\mu+\dot{\nu}=M+\dot{N}+\mu+\dot{\nu}$, hence we have

$$\beta M=M,\qquad \tau\beta\tau N=N\qquad\text{for all }\beta\in E_6.$$

Therefore $M=N=0$, so $\alpha 1=\mu+\dot{\nu}$. Similarly $\alpha\dot{1}=\lambda+\dot{\kappa}$. The conditions $\alpha\dot{1}$, $\alpha 1\in\mathfrak{M}^C$, $\{\alpha 1,\ \alpha\dot{1}\}=1$, $\langle\alpha 1,\ \alpha 1\rangle_\iota=1$ imply

$$\mu\nu=0,\quad \lambda\kappa=0,\quad \mu\kappa-\lambda\nu=1,\quad |\mu|^2-|\nu|^2=1$$

respectively, hence

$$\alpha 1=\mu,\quad \alpha\dot{1}=(\mu^{-1})^{\cdot}\qquad \mu\in\boldsymbol{C},\ |\mu|=1.$$

Choose $\theta\in\boldsymbol{C}$ such that $\theta^3=\mu$ and then put $\beta=\theta^{-1}\alpha$, where

$$\theta=\begin{pmatrix}\theta^{-1}1&0&0&0\\0&\theta 1&0&0\\0&0&\theta^3&0\\0&0&0&\theta^{-3}\end{pmatrix}\in U(1).$$

Then $\beta 1 = \theta^{-1}\alpha 1 = \theta^{-1}\mu = \theta^{-3}\theta^3 = 1$, similarly $\beta\dot{1} = \dot{1}$, hence $\beta \in E_6$. Moreover $\beta \in z(E_6)$ (which denotes the center of E_6), in fact, $\beta\beta' = \theta^{-1}\alpha\beta' = \theta^{-1}\beta'\alpha = \beta'\theta^{-1}\alpha = \beta'\beta$ for all $\beta' \in E_6$. Thus we have

$$\alpha = \theta\beta \qquad \theta \in U(1),\ \beta \in z(E_6).$$

Since $z(E_6) = \{1,\ \omega 1,\ \omega^2 1\}$, $\omega \in \boldsymbol{C}$, $\omega^3 = 1$, $\omega \neq 1$ [7], we have

$$\alpha = \begin{pmatrix} \theta^{-1}\omega 1 & 0 & 0 & 0 \\ 0 & \theta\omega^{-1}1 & 0 & 0 \\ 0 & 0 & \theta^3 & 0 \\ 0 & 0 & 0 & \theta^{-3} \end{pmatrix} \qquad \omega \in \boldsymbol{C},\ \omega^3 = 1.$$

Again from the commutativity with $\alpha_1(a)$ of Lemma 16 : $\alpha_1(1)\alpha = \alpha\alpha_1(1)$, we have

$$\begin{aligned} \theta\omega^{-1}\cosh 1 E_2 + \theta^{-1}\omega(\sinh 1 E_3)^{\cdot} &= \alpha_1(1)(\theta^{-1}\omega E_2) = \alpha_1(1)\alpha E_2 \\ &= \alpha\alpha_1(1)E_2 = \alpha(\cosh 1 E_2 + (\sinh 1 E_3)^{\cdot}) \\ &= \theta^{-1}\omega\cosh 1 E_2 + (\theta\omega^{-1}\sinh 1 E_3)^{\cdot} \end{aligned}$$

where $E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $E_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, hence $\theta^{-1}\omega = \theta\omega^{-1}$, i.e. $\theta^{-1}\omega = \pm 1$.

Therefore $\alpha = \pm 1$, i.e. $z(E_{7,\iota}) = \{1,\ -1\}$. Thus the proof of Theorem 17 is completed.

9. Isomorphism $E_{7,1} \cong E_{7,\iota}$.

From Theorems 4, 7 and 15, we see that the groups $E_{7,1}$ and $E_{7,\iota}$ are both connected and their Lie algebras have the same type $E_{7(-25)}$. Therefore there exist central normal subgroups N_1, N_ι of the simply connected simple Lie group $\widetilde{E}_{7(-25)}$ of type $E_{7(-25)}$ such that

$$E_{7,1} \cong \widetilde{E}_{7(-25)}/N_1, \qquad E_{7,\iota} \cong \widetilde{E}_{7(-25)}/N_\iota.$$

We shall show $N_1 = N_\iota$. From the general theory of Lie groups, we know that the center $z(\widetilde{E}_{7(-25)})$ of $\widetilde{E}_{7(-25)}$ is the infinite cyclic group $\boldsymbol{Z}$ [6]. Now assume that $N_1 \neq N_\iota$. Since the centers of $E_{7,1}$ and $E_{7,\iota}$ are both $\boldsymbol{Z}_2$ (Theorems 8, 17), we may assume that $2\boldsymbol{Z} = N_1 \subset N_\iota = \boldsymbol{Z}$ without loss of generality. Consider the natural homomorphism

$$f : E_{7,1} \cong \widetilde{E}_{7(-25)}/N_1 \longrightarrow \widetilde{E}_{7(-25)}/N_\iota \cong E_{7,\iota}.$$

Then $f^{-1}(z(E_{7,\iota})) = f^{-1}(\boldsymbol{Z}_2)$ is a discrete (because $E_{7,1}$ is simple Lie group) normal subgroup, therefore $f^{-1}(z(E_{7,\iota}))$ is a central (because $E_{7,1}$ is connected) normal subgroup of $E_{7,1}$: $f^{-1}(z(E_{7,\iota})) \subset z(E_{7,1})$ and the order of $f^{-1}(z(E_{7,\iota}))$ is not less than 4. This contradicts to $z(E_{7,1}) = \boldsymbol{Z}_2$. Therefore $N_1 = N_\iota$ and we see that the groups $E_{7,1}$ and $E_{7,\iota}$

are isomorphic :

$$E_{7,1} \cong E_{7,\iota}.$$

Thus from the preceding arguments we have the following main

Theorem 18. *The group* $E_{7,1} = \{\alpha \in \mathrm{Iso}_{R}(\mathfrak{P}, \mathfrak{P}) \mid \alpha\mathfrak{M} = \mathfrak{M},\ \{\alpha P, \alpha Q\} = \{P, Q\}\}$ *is a connected non-compact simple Lie group of type* E_7, *its center* $z(E_{7,1})$ *is the cyclic group of order* 2 :

$$z(E_{7,1}) = \{1, -1\}$$

and the polar decompsition is given by

$$E_{7,1} \cong (U(1) \times E_6)/\boldsymbol{Z}_3 \times \boldsymbol{R}^{54}.$$

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