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Non-compact simple Lie group $E_{8(-24)}$ of type E_8

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It is known that there exist three simple Lie groups of type E_8 up to local isomorphism, one of them is compact and the others are non-compact. We have shown in [7] that the group

$$E_8 = \{\ \alpha \in \mathrm{Iso}_{\boldsymbol{C}}(\mathfrak{e}_8{}^{\boldsymbol{C}}, \mathfrak{e}_8{}^{\boldsymbol{C}}) \mid \alpha[R_1,\ R_2] = [\alpha R_1,\ \alpha R_2], \langle \alpha R_1,\ \alpha R_2 \rangle = \langle R_1,\ R_2 \rangle \}$$

(where $\mathfrak{e}_8{}^{\boldsymbol{C}}$ is a simple Lie algebra over $\boldsymbol{C}$ of type E_8 and $\langle R_1,\ R_2 \rangle$ a positive definite Hermitian inner product in $\mathfrak{e}_8{}^{\boldsymbol{C}}$) is a simply connected compact simple Lie group of type E_8. In this paper, we consider one of the non-compact cases. Our results are as follows. The group

$$E_{8,\iota} = \{\ \alpha \in \mathrm{Iso}_{\boldsymbol{C}}(\mathfrak{e}_8{}^{\boldsymbol{C}}, \mathfrak{e}_8{}^{\boldsymbol{C}}) \mid \alpha[R_1,\ R_2] = [\alpha R_1,\ \alpha R_2],\ \langle \alpha R_1,\ \alpha R_2 \rangle_\iota = \langle R_1,\ R_2 \rangle_\iota \}$$

(where $\langle R_1,\ R_2 \rangle_\iota$ is another inner product in $\mathfrak{e}_8{}^{\boldsymbol{C}}$) is a connected non-compact simple Lie group of type E_8 and its center $z(E_{8,\iota})$ is trivial :

$$z(E_{8,\iota}) = \{1\}.$$

The group $E_{8,\iota}$ contains, as a subgroup, a special unitary group $SU(2)$ and a simply connected compact simple Lie group E_7 of type E_7 and the polar decomposition of $E_{8,\iota}$ is given by

$$E_{8,\iota} \simeq (SU(2) \times E_7)/\boldsymbol{Z}_2 \times \boldsymbol{R}^{112}.$$

The group $E_{8,\iota}$ contains also, as a subgroup, a special linear group $SL(2,\ \boldsymbol{R})$ and a connected non-compact simple Lie group $E_{7,1}$ of type $E_{7(-25)}$. In order to show this, we construct another group

$$E_{8,1} = \{\ \alpha \in \mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{e}_{8,1},\ \mathfrak{e}_{8,1}) \mid \alpha\mathfrak{T} = \mathfrak{T},\ \alpha[R_1,\ R_2] = [\alpha R_1,\ \alpha R_2]\}$$

(where $\mathfrak{e}_{8,1}$ is a simple Lie algebra of type $E_{8(-24)}$ and $\mathfrak{T}$ a submanifold of $\mathfrak{e}_{8,1}$)

which is isomorphic to $E_{8,\iota}$ and find subgroups $SL(2, \boldsymbol{R})$ and $E_{7,1}$ explicitly in this group $E_{8,1}$

I. Group $E_{8,\iota}$

1. Preliminaries.

Throughout this paper, we use the same notations as in [7]. However we arrange definitions and some properties of the exceptional Lie algebras $\mathfrak{e}_6{}^C$, $\mathfrak{e}_7{}^C$ and $\mathfrak{e}_8{}^C$.

1.1. Jordan algebra $\mathfrak{J}^C$ [1], [7].

Let $\mathfrak{C}^C$ denote the split Cayley algebra over the field of complex numbers $\boldsymbol{C}$ and $\mathfrak{J}^C$ the Jordan algebra of all 3×3 Hermitian matrices with entries in $\mathfrak{C}^C$ with respect to the multiplication $X\circ Y=\frac{1}{2}(XY+YX)$. In $\mathfrak{J}^C$, the inner product (X, Y), the positive definite Hermitian inner product $<X, Y>$, the crossed product $X\times Y$ and the cubic form (X, Y, Z) are defined respectively by

$$(X, Y)=\mathrm{tr}(X\circ Y), \qquad <X, Y>=(\overline{X}, Y),$$

$$X\times Y=\frac{1}{2}(2X\circ Y-\mathrm{tr}(X)Y-\mathrm{tr}(Y)X+(\mathrm{tr}(X)\mathrm{tr}(Y)-(X, Y))E),$$

$$(X, Y, Z)=(X, Y\times Z)$$

where $\overline{X}$ is the complex conjugate of X with respect to the field $\boldsymbol{C}$ and E the unit matrix.

1.2. Lie algebra $\mathfrak{e}_6{}^C$ [1], [7].

The exceptional Lie algebra $\mathfrak{e}_6{}^C$ over $\boldsymbol{C}$ of type E_6 is defined by

$$\mathfrak{e}_6{}^C=\{\ \phi\in \mathrm{Hom}_C(\mathfrak{J}^C, \mathfrak{J}^C)\ |\ (\phi X, X, X)=0\}.$$

For $A, B\in\mathfrak{J}^C$, we define $A\vee B\in\mathfrak{e}_6{}^C$ by

$$(A\vee B)X=\frac{1}{2}(B, X)A+\frac{1}{6}(A, B)X-2B\times(A\times X), \qquad X\in\mathfrak{J}^C,$$

then $\{A\vee B \mid A, B\in\mathfrak{J}^C\}$ generates $\mathfrak{e}_6{}^C$ additively. In $\mathfrak{e}_6{}^C$, we define a positive definite Hermitian inner product $<\phi_1, \phi_2>$ by

$$<\phi_1, \phi_2>=\sum_i<\phi_1\overline{B}_i, A_i>$$

where $\phi_2=\sum_i A_i\vee B_i$, $A_i, B_i\in\mathfrak{J}^C$. Finally, for $\phi\in\mathfrak{e}_6{}^C$, we denote the skew-transposes of ϕ by ϕ', ${}'\phi$ with respect to the inner products (X, Y), $<X, Y>$ in $\mathfrak{J}^C$ respectively :

$$(\phi X,\ Y)+(X,\ \phi' Y)=0, \qquad <\phi X,\ Y>+<X,\ '\phi Y>=0,$$

then ϕ', $'\phi\in\mathfrak{e}_6{}^C$.

1.3. Vector space $\mathfrak{P}^C$ [2], [7].

We define a 56 dimensional vector space $\mathfrak{P}^C$ by

$$\mathfrak{P}^C=\mathfrak{J}^C\oplus\mathfrak{J}^C\oplus C\oplus C.$$

In $\mathfrak{P}^C$, we define a positive definite Hermitian inner product $<P,\ Q>$ and a skew-symmetric inner product $\{P,\ Q\}$ respectively by

$$<P,\ Q>=<X,\ Z>+<Y,\ W>+\bar{\xi}\zeta+\bar{\eta}\omega,$$

$$\{P,\ Q\}=(X,\ W)-(Z,\ Y)+\xi\omega-\zeta\eta$$

for $P=(X,\ Y,\ \xi,\ \eta)$, $Q=(Z,\ W,\ \zeta,\ \omega)\in\mathfrak{P}^C$. Finally, for $P=(X,\ Y,\ \xi,\ \eta)\in\mathfrak{P}^C$, we define $\widehat{P}\in\mathfrak{P}^C$ by

$$\widehat{P}=(-\overline{Y},\ \overline{X},\ -\bar{\eta},\ \bar{\xi}).$$

1.4. Lie algebra $\mathfrak{e}_7{}^C$ [2], [4], [5], [7].

An exceptional Lie algebra $\mathfrak{e}_7{}^C$ over C of type E_7 is defined by

$$\mathfrak{e}_7{}^C=\{\ \Phi(\phi,\ A,\ B,\ \rho)\in\mathrm{Hom}_C(\mathfrak{P}^C,\ \mathfrak{P}^C)\ \mid\ \phi\in\mathfrak{e}_6{}^C,\ A,\ B\in\mathfrak{J}^C,\ \rho\in C\},$$

where $\Phi(\phi,\ A,\ B,\ \rho)$ is a linear transformation of $\mathfrak{P}^C$ defined by

$$\Phi(\phi,\ A,\ B,\ \rho)\begin{pmatrix}X\\Y\\\xi\\\eta\end{pmatrix}=\begin{pmatrix}\phi-\frac{1}{3}\rho 1 & 2B & 0 & A\\ 2A & \phi'+\frac{1}{3}\rho 1 & B & 0\\ 0 & A & \rho & 0\\ B & 0 & 0 & -\rho\end{pmatrix}\begin{pmatrix}X\\Y\\\xi\\\eta\end{pmatrix}$$

$$=\begin{pmatrix}\phi X-\frac{1}{3}\rho X+2B\times Y+\eta A\\ 2A\times X+\phi' Y+\frac{1}{3}\rho Y+\xi B\\ (A,\ Y)+\rho\xi\\ (B,\ X)-\rho\eta\end{pmatrix}.$$

The Lie bracket in $\mathfrak{e}_7{}^C$ is given by

$$[\Phi(\phi_1,\ A_1,\ B_1,\ \rho_1),\ \Phi(\phi_2,\ A_2,\ B_2,\ \rho_2)]=\Phi(\phi,\ A,\ B,\ \rho),$$

where

$$\begin{cases} \phi=[\phi_1, \ \phi_2]+2A_1\vee B_2-2A_2\vee B_1, \\ A=(\phi_1+\frac{2}{3}\rho_1 1)A_2-(\phi_2+\frac{2}{3}\rho_2 1)A_1, \\ B=(\phi_1{}'-\frac{2}{3}\rho_1 1)B_2-(\phi_2{}'-\frac{2}{3}\rho_2 1)B_1, \\ \rho=(A_1, \ B_2)-(B_1, \ A_2). \end{cases}$$

For $P=(X, Y, \xi, \eta)$, $Q=(Z, W, \zeta, \omega)\in\mathfrak{P}^C$, we define $P\times Q\in\mathfrak{e}_7{}^C$ by

$$P\times Q=\Phi(\phi, \ A, \ B, \ \rho), \quad \begin{cases} \phi=-\frac{1}{2}(X\vee W+Z\vee Y), \\ A=-\frac{1}{4}(2Y\times W-\xi Z-\zeta X), \\ B=\frac{1}{4}(2X\times Z-\eta W-\omega Y), \\ \rho=\frac{1}{8}((X, \ W)+(Z, \ Y)-3(\xi\omega+\zeta\eta)). \end{cases}$$

Then $\{ \ P\times Q \mid P, \ Q\in\mathfrak{P}^C \ \}$ generates $\mathfrak{e}_7{}^C$ additively. In $\mathfrak{e}_7{}^C$, we define a positive definite Hermitian inner product $\langle\Phi_1, \ \Phi_2\rangle$ by

$$\langle\Phi_1, \ \Phi_2\rangle=2\langle\phi_1, \ \phi_2\rangle+4\langle A_1, \ A_2\rangle+4\langle B_1, \ B_2\rangle+\frac{8}{3}\overline{\rho_1}\rho_2$$

where $\Phi_i=\Phi(\phi_i, \ A_i, \ B_i, \ \rho_i)\in\mathfrak{e}_7{}^C$, $i=1, \ 2$. Finally, for $\Phi=\Phi(\phi, \ A, \ B, \ \rho)\in\mathfrak{e}_7{}^C$, we denote the skew-transpose of Φ by $'\Phi$ with respect to the inner product $\langle P, \ Q\rangle$ in $\mathfrak{P}^C$: $\langle\Phi P, \ Q\rangle+\langle P, \ '\Phi Q\rangle=0$, then

$$'\Phi=\Phi('\phi, \ -\overline{B}, \ -\overline{A}, \ -\overline{\rho}).$$

In particular, $'\Phi\in\mathfrak{e}_7{}^C$. And the Lie algebra

$$\mathfrak{e}_7=\{ \ \Phi\in\mathfrak{e}_7{}^C \mid \Phi={}'\Phi \ \}$$

is a compact Lie algebra of type E_7.

1.5. Lie algebra $\mathfrak{e}_8{}^C$ [2], [7].

An exceptional Lie algebra $\mathfrak{e}_8{}^C$ is defined as follows. In a 248 dimensional vector space

$$\mathfrak{e}_8{}^C=\mathfrak{e}_7{}^C\oplus\mathfrak{P}^C\oplus\mathfrak{P}^C\oplus C\oplus C\oplus C,$$

we define a Lie bracket $[R_1, \ R_2]$ by

$$[(\Phi_1,\ P_1,\ Q_1,\ r_1,\ s_1,\ t_1),\ (\Phi_2,\ P_2,\ Q_2,\ r_2,\ s_2,\ t_2)] = (\Phi,\ P,\ Q,\ r,\ s,\ t)$$

where

$$\begin{cases} \Phi = [\Phi_1,\ \Phi_2] + P_1 \times Q_2 - P_2 \times Q_1, \\ P = \Phi_1 P_2 - \Phi_2 P_1 + r_1 P_2 - r_2 P_1 + s_1 Q_2 - s_2 Q_1, \\ Q = \Phi_1 Q_2 - \Phi_2 Q_1 - r_1 Q_2 + r_2 Q_1 + t_1 P_2 - t_2 P_1, \\ r = -\dfrac{1}{8}\{P_1,\ Q_2\} + \dfrac{1}{8}\{P_2,\ Q_1\} + s_1 t_2 - s_2 t_1, \\ s = \dfrac{1}{4}\{P_1,\ P_2\} + 2r_1 s_2 - 2r_2 s_1, \\ t = -\dfrac{1}{4}\{Q_1,\ Q_2\} - 2r_1 t_2 + 2r_2 t_1. \end{cases}$$

Then $\mathfrak{e}_8{}^C$ becomes a simple Lie algebra over C of type E_8. In $\mathfrak{e}_8{}^C$, we use notations

$$(\Phi,\ 0,\ 0,\ 0,\ 0,\ 0) = \Phi, \qquad (0,\ P,\ 0,\ 0,\ 0,\ 0) = \overline{P},$$
$$(0,\ 0,\ Q,\ 0,\ 0,\ 0) = \underline{Q}, \qquad (0,\ 0,\ 0,\ 1,\ 0,\ 0) = 1,$$
$$(0,\ 0,\ 0,\ 0,\ 1,\ 0) = \overline{1}, \qquad (0,\ 0,\ 0,\ 0,\ 0,\ 1) = \underline{1}.$$

Then the table of the Lie bracket among them is given as follows :

	Φ_2	$\overline{P}_2$	$\underline{Q}_2$	1	$\overline{1}$	$\underline{1}$
Φ_1	$[\Phi_1,\ \Phi_2]$	$(\Phi_1 P_2)^{-}$	$(\Phi_1 Q_2)_{-}$	0	0	0
$\overline{P}_1$	$-(\Phi_2 P_1)^{-}$	$\frac{1}{4}\{P_1,\ P_2\}\overline{1}$	$P_1 \times Q_2 - \frac{1}{8}\{P_1,\ Q_2\}1$	$-\overline{P}_1$	0	$-\underline{P}_1$
$\underline{Q}_1$	$-(\Phi_2 Q_1)_{-}$	$-P_2 \times Q_1 + \frac{1}{8}\{P_2,\ Q_1\}1$	$-\frac{1}{4}\{Q_1,\ Q_2\}\underline{1}$	$\underline{Q}_1$	$-\overline{Q}_1$	0
1	0	$\overline{P}_2$	$-\underline{Q}_2$	0	$2\overline{1}$	$-2\underline{1}$
$\overline{1}$	0	0	$\overline{Q}_2$	$-2\overline{1}$	0	1
$\underline{1}$	0	$\underline{P}_2$	0	$2\underline{1}$	-1	0

For $R = (\Phi,\ P,\ Q,\ r,\ s,\ t) \in \mathfrak{e}_8{}^C$, we denote the adjoint transformation $\mathrm{ad}R$ of $\mathfrak{e}_8{}^C$ by $\Theta(\Phi,\ P,\ Q,\ r,\ s,\ t)$:

$$\Theta(\Phi, P, Q, r, s, t)\begin{pmatrix} \Phi_1 \\ P_1 \\ Q_1 \\ r_1 \\ s_1 \\ t_1 \end{pmatrix} = \begin{pmatrix} \mathrm{ad}\Phi & -Q & P & 0 & 0 & 0 \\ -P & \Phi+r1 & s & -P & -Q & 0 \\ -Q & t & \Phi-r1 & Q & 0 & P \\ 0 & -\frac{1}{8}Q & -\frac{1}{8}P & 0 & -t & s \\ 0 & \frac{1}{4}P & 0 & -2s & 2r & 0 \\ 0 & 0 & -\frac{1}{4}Q & 2t & 0 & -2r \end{pmatrix}\begin{pmatrix} \Phi_1 \\ P_1 \\ Q_1 \\ r_1 \\ s_1 \\ t_1 \end{pmatrix}$$

$$=[(\Phi, P, Q, r, s, t), (\Phi_1, P_1, Q_1, r_1, s_1, t_1)]=[R, R_1]=(\mathrm{ad}R)R_1.$$

Since $\mathfrak{e}_8{}^C$ is simple, the Lie algebra $\mathrm{Der}(\mathfrak{e}_8{}^C)$ of all derivations of $\mathfrak{e}_8{}^C$ consists of $\mathrm{ad}R$, $R\in\mathfrak{e}_8{}^C$:

$$\mathrm{Der}(\mathfrak{e}_8{}^C)=\{\ \Theta(\Phi, P, Q, r, s, t)\ |\ \Phi\in\mathfrak{e}_7{}^C,\ P,\ Q\in\mathfrak{P}^C,\ r,\ s,\ t\in C\ \}$$

and it is also isomorphic to the Lie algebra $\mathfrak{e}_8{}^C$.

In $\mathfrak{e}_8{}^C$, we define a positive definite Hermitian inner product $\langle R_1, R_2\rangle$ by

$$\langle R_1, R_2\rangle=\langle\Phi_1, \Phi_2\rangle+\langle P_1, P_2\rangle+\langle Q_1, Q_2\rangle+8\bar{r}_1r_2+4\bar{s}_1s_2+4\bar{t}_1t_2$$

where $R_i=(\Phi_i, P_i, Q_i, r_i, s_i, t_i)\in\mathfrak{e}_8{}^C$, $i=1, 2$. Finally, for $\Theta=\Theta(\Phi, P, Q, r, s, t)\in\mathrm{Der}(\mathfrak{e}_8{}^C)$, we denote the skew-transpose of Θ by ${}'\Theta$ with respect to the inner product $\langle R_1, R_2\rangle$: $\langle\Theta R_1, R_2\rangle+\langle R_1, {}'\Theta R_2\rangle=0$, then

$${}'\Theta=\Theta({}'\Phi, -\hat{Q}, \hat{P}, -\bar{r}, -\bar{t}, -\bar{s}).$$

2. Group $E_{8,\iota}$.

In $\mathfrak{e}_8{}^C$, we define another inner product $\langle R_1, R_2\rangle_\iota$ by

$$\langle R_1, R_2\rangle_\iota=\langle\Phi_1, \Phi_2\rangle-\langle P_1, P_2\rangle-\langle Q_1, Q_2\rangle+8\bar{r}_1r_2+4\bar{s}_1s_2+4\bar{t}_1t_2$$

where $R_i=(\Phi_i, P_i, Q_i, r_i, s_i, t_i)\in\mathfrak{e}_8{}^C$, $i=1, 2$.

The group $E_{8,\iota}$ is defined to be the group of automorphisms of $\mathfrak{e}_8{}^C$ leaving the inner product $\langle R_1, R_2\rangle_\iota$ invariant :

$$E_{8,\iota}=\{\ \alpha\in\mathrm{Iso}_C(\mathfrak{e}_8{}^C, \mathfrak{e}_8{}^C)\ |\ \alpha[R_1, R_2]=[\alpha R_1, \alpha R_2],\ \langle\alpha R_1, \alpha R_2\rangle_\iota=\langle R_1, R_2\rangle_\iota\}.$$

The Lie algebra $\mathfrak{e}_{8,\iota}$ of the group $E_{8,\iota}$ is

$$\mathfrak{e}_{8,\iota}=\{\ \Theta\in\mathrm{Der}(\mathfrak{e}_8{}^C)\ |\langle\Theta R_1, R_2\rangle_\iota+\langle R_1, \Theta R_2\rangle_\iota=0\ \}.$$

We define an involutive automorphism ι of $\mathfrak{e}_8{}^C$ by

$$\iota = \begin{pmatrix} 1 & & & & & \\ & -1 & & & & \\ & & -1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix}.$$

Then $\iota \in E_{8,\iota}$. And the two inner products $<R_1, R_2>$, $<R_1, R_2>_\iota$ in $\mathfrak{e}_8{}^C$ are combined with relations

$$<R_1, R_2>_\iota = <\iota R_1, R_2> = <R_1, \iota R_2>,$$
$$<R_1, R_2> = <\iota R_1, R_2>_\iota = <R_1, \iota R_2>_\iota.$$

We can define an automorphism ι of $E_{8,\iota}$ by

$$\iota\alpha = \iota\alpha\iota, \qquad \alpha \in E_{8,\iota}.$$

And for $\Theta = \Theta(\Phi, P, Q, r, s, t) \in \mathfrak{e}_{8,\iota}$ we have $\iota\Theta\iota \in \mathfrak{e}_{8,\iota}$, more explicitly

$$\iota\Theta\iota = \Theta(\Phi, -P, -Q, r, s, t).$$

Theorem 1. *Any element Θ of the Lie algebra $\mathfrak{e}_{8,\iota}$ is represented by the form*

$$\Theta = \Theta(\Phi, P, -\widehat{P}, r, s, -\bar{s}), \ \Phi \in \mathfrak{e}_7, \ P \in \mathfrak{P}^C, \ r, \ s \in \boldsymbol{C}, \ \bar{r} + r = 0.$$

In particular, the type of the group $E_{8,\iota}$ is E_8.

Proof. Put $\Theta = \Theta(\Phi, P, Q, r, s, t) \in \mathfrak{e}_8{}^C$, $\Phi \in \mathfrak{e}_7{}^C$, P, $Q \in \mathfrak{P}^C$, $r, s, t \in \boldsymbol{C}$. From the condition $<\Theta R_1, R_2>_\iota + <R_1, \Theta R_2>_\iota = 0$, that is,

$$<\Theta R_1, R_2> + <R_1, \iota\Theta\iota R_2> = 0, \qquad R_1, R_2 \in \mathfrak{e}_8{}^C,$$

we have $\iota\Theta\iota = {}'\Theta$, i.e.,

$$\Theta(\Phi, -P, -Q, r, s, t) = \Theta({}'\Phi, -\widehat{Q}, \widehat{P}, -\bar{r}, -\bar{t}, -\bar{s}),$$

hence $\Phi = {}'\Phi$, $Q = -\widehat{P}$, $r = -\bar{r}$, $t = -\bar{s}$. Therefore we see that the complexification of $\mathfrak{e}_{8,\iota}$ is $\mathfrak{e}_8{}^C$, so the Lie algebra $\mathfrak{e}_{8,\iota}$ is also of type E_8.

3. Subgroups E_7 and $SU(2)$ of $E_{8,\iota}$.

We have proved in [4], [6] that the group

$$E_{7(-133)} = \{ \beta \in \mathrm{Iso}_C(\mathfrak{P}^C, \mathfrak{P}^C) \mid \beta(P \times Q)\beta^{-1} = \beta P \times \beta Q, \ <\beta P, \beta Q> = <P, Q> \}$$

is a simply connected compact simple Lie group of type E_7. Now, we shall show that the group $E_{8,\iota}$ contains compact subgroups of type E_7 and A_2.

Theorem 2. *The group $E_{8,\iota}$ contains a subgroup*

$$E_7=\{\ \alpha\in E_{8,\iota}\ \mid\ \alpha 1=1,\ \alpha\overline{1}=\overline{1},\ \alpha\underline{1}=\underline{1}\ \}$$

which is a simply connected compact simple Lie group of type E_7.

Proof. The mapping

$$E_{7(-133)}\ni\beta\longrightarrow\beta=\begin{pmatrix}\mathrm{Ad}\beta & & & & & \\ & \beta & & & & \\ & & \beta & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1\end{pmatrix}\in E_7\subset E_{8,\iota},$$

(where $\mathrm{Ad}\beta:\mathfrak{e}_7{}^{C}\longrightarrow\mathfrak{e}_7{}^{C}$ is defined by $(\mathrm{Ad}\beta)\Phi=\beta\Phi\beta^{-1}$) gives an isomorphism between $E_{7(-133)}$ and E_7. The analogy of this proof is in [7] Theorem 25, so we omitt here. (This Theorem follows also from the following Theorem 4).

Theorem 3. *The group $E_{8,\iota}$ contains a subgroup*

$$SU(2)=\left\{A=\begin{pmatrix}1&0&0&0&0&0\\0&a1&-\bar{b}1&0&0&0\\0&b1&\bar{a}1&0&0&0\\0&0&0&|a|^2-|b|^2&-ab&-\overline{ab}\\0&0&0&2a\bar{b}&a^2&-\bar{b}^2\\0&0&0&2\bar{a}b&-b^2&\bar{a}^2\end{pmatrix}\ \middle|\ \begin{pmatrix}a&-\bar{b}\\b&\bar{a}\end{pmatrix}\in\mathrm{SU}(2)\right\}$$

which is isomorphic to the special unitary group $\mathrm{SU}(2)=\{A\in M(2,\ \boldsymbol{C})\,|\,A^*A=E,\ \det A=1\}$.

Proof. It is easy to verify that $SU(2)$ is a subgroup of $E_{8,\iota}$ (or see the following Theorem 4).

In the followings, we identify these groups $E_{7(-133)}$ with E_7, $SU(2)$ with $\mathrm{SU}(2)$ under the above correspondences.

4. Involutive automorphism ι and subgroup $(SU(2)\times E_7)/Z_2$ of $E_{8,\iota}$.

Theorem 4. *The subgroup $\{\ \alpha\in E_{8,\iota}\ \mid\ \iota\alpha\iota=\alpha\ \}$ of the group $E_{8,\iota}$ is isomorphic to the group $(SU(2)\times E_7)/\boldsymbol{Z}_2$, where $\boldsymbol{Z}_2=\{(E,\ 1),\ (-E,\ \iota)\}$.*

Proof. We define a mapping $\phi: SU(2)\times E_7\longrightarrow\{\ \alpha\in E_{8,\iota}\ \mid\ \iota\alpha\iota=\alpha\ \}$ by

$$\phi(A,\ \beta)=A\beta=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & a1 & -\bar{b}1 & 0 & 0 & 0 \\ 0 & b1 & \bar{a}1 & 0 & 0 & 0 \\ 0 & 0 & 0 & |a|^2-|b|^2 & -ab & -\bar{a}\bar{b} \\ 0 & 0 & 0 & 2a\bar{b} & a^2 & -\bar{b}^2 \\ 0 & 0 & 0 & 2\bar{a}b & -b^2 & \bar{a}^2 \end{pmatrix}\begin{pmatrix} \mathrm{Ad}\beta & 0 & 0 & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Since $A\in SU(2)$ and $\beta\in E_7$ commute in $E_{8,\iota}$: $A\beta=\beta A$, obviously ϕ is a homomorphism. We shall prove that ϕ is onto. If $\alpha\in E_{8,\iota}$ satisfies $\iota\alpha\iota=\alpha$, then α has the form

$$\alpha=\begin{pmatrix} \beta_1 & 0 & 0 & \Psi_1 & \Psi_2 & \Psi_3 \\ 0 & \beta_2 & \beta_{23} & 0 & 0 & 0 \\ 0 & \beta_{32} & \beta_3 & 0 & 0 & 0 \\ l_1 & 0 & 0 & r_1 & r_2 & r_3 \\ l_2 & 0 & 0 & s_1 & s_2 & s_3 \\ l_3 & 0 & 0 & t_1 & t_2 & t_3 \end{pmatrix}$$

where $\beta_1 : \mathfrak{e}_7{}^C \longrightarrow \mathfrak{e}_7{}^C$, β_2, β_3, β_{23}, $\beta_{32} : \mathfrak{P}^C \longrightarrow \mathfrak{P}^C$, $l_i : \mathfrak{e}_7{}^C \longrightarrow C$ are linear mappings, $\Psi_i\in\mathfrak{e}_7{}^C$ and r_i, s_i, $t_i\in C$, $i=1, 2, 3$.

I. $[1,\ \bar{\bar{1}}]=2\bar{\bar{1}}$ implies $[\alpha 1,\ \alpha\bar{\bar{1}}]=2\alpha\bar{\bar{1}}$, that is,

$$\begin{aligned}&[(\Psi_1,\ 0,\ 0,\ r_1,\ s_1,\ t_1),\ (\Psi_2, 0,\ 0,\ r_2,\ s_2,\ t_2)]\\ &\quad=([\Psi_1,\ \Psi_2],\ 0,\ 0,\ s_1t_2-s_2t_1,\ 2r_1s_2-2r_2s_1,\ -2r_1t_2+2r_2t_1)\\ &\quad=2(\Psi_2,\ 0,\ 0,\ r_2,\ s_2,\ t_2).\end{aligned}$$

Hence we have

(1) $[\Psi_1,\ \Psi_2]=2\Psi_2$, (2) $s_1t_2-s_2t_1=2r_2$,

(3) $r_1s_2-r_2s_1=s_2$, (4) $-r_1t_2+r_2t_1=t_2$.

Similarly, from $[1,\ \underline{\underline{1}}]=-2\underline{\underline{1}}$, $[\bar{\bar{1}},\ \underline{\underline{1}}]=1$, we have

(5) $[\Psi_1,\ \Psi_3]=-2\Psi_3$, (6) $s_1t_3-s_3t_1=-2r_3$,

(7) $r_1s_3-r_3s_1=-s_3$, (8) $-r_1t_3+r_3t_1=-t_3$,

(9) $[\Psi_2,\ \Psi_3]=\Psi_1$, (10) $s_2t_3-s_3t_2=r_1$,

(11) $2r_2s_3-2r_3s_2=s_1$, (12) $-2r_2t_3+2r_3t_2=t_3$.

$[\Phi, 1]=0$ implies $[\alpha\Phi, \alpha 1]=0$, that is,

$$[(\beta_1\Phi, 0, 0, l_1\Phi, l_2\Phi, l_3\Phi), (\Psi_1, 0, 0, r_1, s_1, t_1)]$$
$$=([\beta_1\Phi, \Psi_1], 0, 0, -s_1l_3\Phi+t_1l_2\Phi, -2r_1l_2\Phi+2s_1l_1\Phi, 2r_1l_3\Phi-2t_1l_1\Phi)=0.$$

Hence we have

(13) $[\beta_1\Phi, \Psi_1]=0,$ (14) $s_1l_3=t_1l_2,$

(15) $r_1l_2=s_1l_1,$ (16) $r_1l_3=t_1l_1.$

Similarly, from $[\Phi, \bar{1}]=0$, $[\Phi, \underline{1}]=0$, we have

(17) $[\beta_1\Phi, \Psi_2]=0,$ (18) $s_2l_3=t_2l_2,$

(19) $r_2l_2=s_2l_1,$ (20) $r_2l_3=t_2l_1,$

(21) $[\beta_1\Phi, \Psi_3]=0,$ (22) $s_3l_3=t_3l_2,$

(23) $r_3l_2=s_3l_1,$ (24) $r_3l_3=t_3l_1.$

And $\alpha[\Phi_1, \Phi_2]=[\alpha\Phi_1, \alpha\Phi_2]$ implies

(25) $\beta_1[\Phi_1, \Phi_2]=[\beta_1\Phi_1, \beta_1\Phi_2].$

We shall prove that $\Psi_1=\Psi_2=\Psi_3=0$ and $l_1=l_2=l_3=0$.

Case (i) : $\begin{pmatrix} r_1 & r_2 & r_3 \\ s_1 & s_2 & s_3 \\ t_1 & t_2 & t_3 \end{pmatrix}$ is not zero. For example, assume $r_1\neq 0$. First we show that β_1 is non-degenerate. Suppose β_1 is degenerate, then there exists $0\neq\Phi_0\in\mathfrak{e}_7{}^C$ such that $\beta_1\Phi_0=0$. From $\langle\alpha\Phi_0, \alpha 1\rangle=\langle\Phi_0, 1\rangle=0$, we have

$$\langle\beta_1\Phi_0, \Psi_1\rangle+8\overline{l_1\Phi_0}r_1+4\overline{l_2\Phi_0}s_2+4\overline{l_3\Phi_0}t_1=0.$$

Since $l_2=\dfrac{s_1}{r_1}l_1$, $l_3=\dfrac{t_1}{r_1}l_1$ from (15), (16), we have

$$\overline{l_1\Phi_0}(8|r_1|^2+4|s_1|^2+4|t_1|^2)=0.$$

Therefore $l_1\Phi_0=0$, and hence $l_2\Phi_0=l_3\Phi_0=0$. Therefore $\alpha\Phi_0=0$ for $\Phi_0\neq 0$. This contradicts to the non-degeneracy of α. Thus we see that β_1 is non–degenerate, so $\beta_1\mathfrak{e}_7{}^C=\mathfrak{e}_7{}^C$. Hence (15) shows that Ψ_1 is a central element of $\beta_1\mathfrak{e}_7{}^C=\mathfrak{e}_7{}^C$. Since the Lie algebra $\mathfrak{e}_7{}^C$ is simple, we have

$$\Psi_1=0, \qquad \text{and hence} \qquad \Psi_2=\Psi_3=0$$

from (1), (5). Again using $\langle\alpha\Phi, \alpha 1\rangle=\langle\Phi, 1\rangle=0$, that is, $\overline{l_1\Phi}(8|r_1|^2+4|s_1|^2+4|t_1|^2)$

$=0$, we have $l_1\Phi=0$ for all $\Phi\in\mathfrak{e}_7{}^C$. Hence

$$l_1=0, \qquad \text{and hence} \qquad l_2=l_3=0$$

from (15), (16).

case (ii). $r_i=s_i=t_i=0$, $i=1, 2, 3$ (which doesn't occur). In this case, $\Psi_1\neq 0$, $\Psi_2\neq 0$, $\Psi_3\neq 0$ from the non-degeneracy of α. $133=\dim\mathfrak{e}_7{}^C=\dim(\beta_1\mathfrak{e}_7{}^C+l_1\mathfrak{e}_7{}^C+l_2\mathfrak{e}_7{}^C+l_3\mathfrak{e}_7{}^C)$ implies $\dim\beta_1\mathfrak{e}_7{}^C\geqq 130$, and from $<\beta_1\Phi, \Psi_i>=0$, $i=1, 2, 3$, $<\Psi_i, \Psi_j>=0$, $i\neq j$, we see that $\dim\beta_1\mathfrak{e}_7{}^C$ is just 130, so

$$\mathfrak{e}_7{}^C=\beta_1\mathfrak{e}_7{}^C\oplus C\Psi_1\oplus C\Psi_2\oplus C\Psi_3.$$

However (13), (17), (21), (25) show that $\beta_1\mathfrak{e}_7{}^C$ is an ideal of $\mathfrak{e}_7{}^C$. So $\beta_1\mathfrak{e}_7{}^C=\mathfrak{e}_7{}^C$ from the simplicity of the Lie algebra $\mathfrak{e}_7{}^C$. This contradicts to $\dim\mathfrak{e}_7{}^C=\dim\beta_1\mathfrak{e}_7{}^C=130$ $<133=\dim\mathfrak{e}_7{}^C$.

Thus α has the form

$$\alpha=\begin{pmatrix}\beta_1 & 0 & 0 & 0 & 0 & 0\\ 0 & \beta_2 & \beta_{23} & 0 & 0 & 0\\ 0 & \beta_{32} & \beta_3 & 0 & 0 & 0\\ 0 & 0 & 0 & r_1 & r_2 & r_3\\ 0 & 0 & 0 & s_1 & s_2 & s_3\\ 0 & 0 & 0 & t_1 & t_2 & t_3\end{pmatrix}.$$

II. $[\tilde{P}, 1]=-\tilde{P}$ implies $[\alpha\tilde{P}, \alpha 1]=-\alpha\tilde{P}$, that is,

$$\begin{aligned}&[(0, \beta_2P, \beta_{32}P, 0, 0, 0), (0, 0, 0, r_1, s_1, t_1)]\\ &\quad=(0, -r_1\beta_2P-s_1\beta_{32}P, r_1\beta_{32}P-t_1\beta_2P, 0, 0, 0)\\ &\quad=-(0, \beta_2P, \beta_{32}P, 0, 0, 0).\end{aligned}$$

Hence we have

(26) $(1-r_1)\beta_2=s_1\beta_{32}$, (27) $(1+r_1)\beta_{32}=t_1\beta_2$.

Similarly, from $[\tilde{P}, \tilde{1}]=0$, $[\tilde{P}, \underset{\sim}{1}]=-\underset{\sim}{P}$, we have

(28) $r_2\beta_2=-s_2\beta_{32}$, (29) $r_2\beta_{32}=t_2\beta_2$,

(30) $r_3\beta_2+s_3\beta_{32}=\beta_{23}$, (31) $r_3\beta_{32}-t_3\beta_2=\beta_3$.

And from $[\underset{\sim}{Q}, 1]=\underset{\sim}{Q}$, $[\underset{\sim}{Q}, \tilde{1}]=-\tilde{Q}$, $[\underset{\sim}{Q}, \underset{\sim}{1}]=0$, we have

(32) $(1+r_1)\beta_{23}=-s_1\beta_3$, (33) $(1-r_1)\beta_3=-t_1\beta_{23}$,

(34) $r_2\beta_{23}+s_2\beta_3=\beta_2$, (35) $r_2\beta_3-t_2\beta_{23}=-\beta_{32}$,

(36) $r_3\beta_{23}=-s_3\beta_3$, (37) $r_3\beta_3=t_3\beta_{23}$.

We shall prove that there exist $a, b, c, d\in \boldsymbol{C}$ and $\beta, \gamma\in \mathrm{Iso}_C(\mathfrak{C}_7{}^C, \mathfrak{C}_7{}^C)$ such that

$$\begin{cases}\beta_2=a\beta, & \beta_{23}=c\gamma,\\ \beta_{32}=b\beta, & \beta_3=d\gamma,\end{cases}\qquad \begin{cases}r_2=-ab, & r_3=cd,\\ s_2=a^2, & t_3=d^2.\end{cases}\tag{38}$$

Case (i) : $s_2\neq 0$. $s_2\neq 0$ implies $t_3\neq 0$. In fact, suppose $t_3=0$. Then we have $s_3t_1=2r_3$, $r_3t_1=0$ from (6), (8), hence $r_3=0$. So $s_3\neq 0$ (because α is non-degenerate) and hence $t_1=0$. Hence $r_1=-1$ from (7). From $\langle\alpha 1, \alpha 1\rangle=\langle 1, 1\rangle=8$, that is, $8+4|s_1|^2=8$, hence $s_1=0$. And $r_2=0$ from (2) and finally $s_2=0$. This contradicts to the hypothesis $s_2\neq 0$. Now, choose $a, d\in\boldsymbol{C}$ such that

$$a^2=s_2,\qquad d^2=t_3$$

and put

$$b=-\frac{r_2}{a},\qquad c=\frac{r_3}{d},$$

$$\beta=\frac{1}{a}\beta_2,\qquad \gamma=\frac{1}{d}\beta_3.$$

Then $\beta_{32}=-\dfrac{r_2}{s_2}\beta_2=b\beta$ from (28) and $\beta_{23}=\dfrac{r_3}{t_3}\beta_3=c\gamma$ from (37). Obviously $\beta, \gamma\in\mathrm{Iso}_C(\mathfrak{C}_7{}^C, \mathfrak{C}_7{}^C)$, because α is non-degenerate.

Case (ii) : $s_2=0$. $s_2=0$ implies $t_3=0$ and $r_2=r_3=0$, $t_2\neq 0$, $s_3\neq 0$ from the same arguments as Case (i). Hence $\beta_2=\beta_3=0$ from (29), (36). Now, choose $b, c\in\boldsymbol{C}$ such that

$$-b^2=t_2,\qquad -c^2=s_3$$

and put

$$a=0,\qquad d=0,$$

$$\beta=\frac{1}{b}\beta_{32},\qquad \gamma=\frac{1}{c}\beta_{23}.$$

Then (38) is also valid in this case.

III. $[\bar{\bar{P}}, \bar{\bar{Q}}]=\dfrac{1}{4}\{P, Q\}\bar{\bar{1}}$ implies $[\alpha\bar{\bar{P}}, \alpha\bar{\bar{Q}}]=\dfrac{1}{4}\{P, Q\}\alpha\bar{\bar{1}}$, that is,

$$[(0,\ \beta_2P,\ \beta_{32}P,\ 0,\ 0,\ 0),\ (0,\ \beta_2Q,\ \beta_{32}Q,\ 0,\ 0,\ 0)]$$
$$=(\beta_2P\times\beta_{32}Q-\beta_{32}P\times\beta_2Q,\ 0,\ 0,\ -\frac{1}{8}\{\beta_{32}P,\ \beta_2Q\}-\frac{1}{8}\{\beta_2P,\ \beta_{32}Q\},$$
$$\frac{1}{4}\{\beta_2P,\ \beta_2Q\}-\frac{1}{4}\{\beta_{32}P,\ \beta_{32}Q\})$$
$$=(0,\ 0,\ 0,\ \frac{1}{4}\{P,\ Q\}r_2,\ \frac{1}{4}\{P,\ Q\}s_2,\ \frac{1}{4}\{P,\ Q\}t_2)$$

Hence we have

(39) $\beta_2P\times\beta_{32}Q=\beta_{32}P\times\beta_2Q$,

(40) $\{\beta_2P,\ \beta_{32}Q\}+\{\beta_{32}P,\ \beta_2Q\}=-2r_2\{P,\ Q\}$,

(41) $\{\beta_2P,\ \beta_2Q\}=s_2\{P,\ Q\}$, (42) $\{\beta_{32}P,\ \beta_{32}Q\}=-t_2\{P,\ Q\}$.

Similarly, from $[\overline{\overline{P}},\ \underline{\underline{Q}}]=P\times Q-\frac{1}{8}\{P,\ Q\}1$, $[\underline{P},\ \underline{\underline{Q}}]=-\frac{1}{4}\{P,\ Q\}\underline{1}$, we have

(43) $\beta_1(P\times Q)=\beta_2P\times\beta_3Q-\beta_{32}P\times\beta_{23}Q$,

(44) $\{\beta_2P,\ \beta_3Q\}+\{\beta_{32}P,\ \beta_{23}Q\}=r_1\{P,\ Q\}$,

(45) $2\{\beta_2P,\ \beta_{23}Q\}=-s_1\{P,\ Q\}$, (46) $2\{\beta_{32}P,\ \beta_3Q\}=t_1\{P,\ Q\}$,

(47) $\beta_{23}P\times\beta_3Q=\beta_3P\times\beta_{23}Q$,

(48) $\{\beta_3P,\ \beta_{23}Q\}+\{\beta_{23}P,\ \beta_3Q\}=2r_3\{P,\ Q\}$,

(49) $\{\beta_{23}P,\ \beta_{23}Q\}=-s_3\{P,\ Q\}$, (50) $\{\beta_3P,\ \beta_3Q\}=t_3\{P,\ Q\}$.

From either one of (41), (42) and either one of (49), (50), we have

$$\{\beta P,\ \beta Q\}=\{P,\ Q\},\qquad \{\gamma P,\ \gamma Q\}=\{P,\ Q\}, \tag{51}$$

Since there exists $\lambda\in \boldsymbol{C}$ such that $\gamma=\lambda\beta$ from (31), so $\lambda^2=1$ from (51). If $\lambda=-1$, then by considering $-b$ instead of b, we may assume that

$$\beta=\gamma. \tag{52}$$

Now, from (44), (45), (46), (49), we have

$$r_1=ad+bc,\qquad (\ r_2=-ab\ ),\qquad (\ r_3=cd\ ),$$
$$s_1=-2ac,\qquad (s_2=a^2),\qquad s_3=-c^2,$$
$$t_1=2bd,\qquad t_2=-b^2,\qquad (t_3=d^2).$$

IV. $[\Phi,\ \overline{\overline{P}}]=(\Phi P)^{=}$ implies $[\alpha\Phi,\ \alpha\overline{\overline{P}}]=\alpha(\Phi P)^{=}$, that is,

$$\begin{aligned}&[(\beta_1\Phi,\ 0,\ 0,\ 0,\ 0,\ 0),\ (0,\ \beta_2 P,\ \beta_{32}P,\ 0,\ 0,\ 0)]\\ &\quad=(0,\ (\beta_1\Phi)(\beta_2 P),\ (\beta_1\Phi)(\beta_{32}P),\ 0,\ 0,\ 0)\\ &\quad=(0,\ \beta_2(\Phi P),\ \beta_{32}(\Phi P),\ 0,\ 0,\ 0).\end{aligned}$$

Hence we have

(53) $\beta_1\Phi\beta_2=\beta_2\Phi$, (54) $\beta_1\Phi\beta_{32}=\beta_{32}\Phi$.

Similarly, from $[\Phi,\ \underset{\dot{}}{Q}]=(\Phi Q)_{\dot{}}$, we have

(55) $\beta_1\Phi\beta_{23}=\beta_{23}\Phi$, (56) $\beta_1\Phi\beta_3=\beta_3\Phi$.

Now, from either one of (53), (54), we have $\beta_1\Phi=\beta\Phi\beta^{-1}$, in particular,

$$\beta_1(P\times Q)=\beta(P\times Q)\beta^{-1}. \tag{57}$$

From (43) we have

$$\beta_1(P\times Q)=(ad-bc)\beta P\times\beta Q. \tag{58}$$

Since $ab-bc\neq 0$, choose $p\in \boldsymbol{C}$ such that $p^2=ad-bc$ and rewrite again

$$\frac{1}{p}\beta\longrightarrow\beta,\qquad pa\longrightarrow a,\qquad pb\longrightarrow b,\qquad pc\longrightarrow c,\qquad pd\longrightarrow d.$$

Then, with respect to these new β, a, b, c, d, the above statements (especially (38)) are also valid and from (57), (58) we have

$$\beta(P\times Q)\beta^{-1}=\beta P\times\beta Q. \tag{59}$$

Finally, we have

$$\begin{cases}|a|^2+|b|^2=1 & \text{from } \langle\alpha\dot{1},\ \alpha\dot{1}\rangle=\langle\dot{1},\ \dot{1}\rangle,\\ |c|^2+|d|^2=1 & \text{from } \langle\alpha\underset{\dot{}}{1},\ \alpha\underset{\dot{}}{1}\rangle=\langle\underset{\dot{}}{1},\ \underset{\dot{}}{1}\rangle,\\ ac+bd=0 & \text{from } \langle\alpha\dot{1},\ \alpha\underset{\dot{}}{1}\rangle=\langle\dot{1},\ \underset{\dot{}}{1}\rangle=0,\\ ad-bc=1.\end{cases}$$

So $\begin{pmatrix}a & c\\ b & d\end{pmatrix}=\begin{pmatrix}a & -\bar{b}\\ b & \bar{a}\end{pmatrix}\in SU(2)$. And from $\langle\alpha\dot{P},\ \alpha\dot{Q}\rangle=\langle\dot{P},\ \dot{Q}\rangle$, that is, $\langle\beta_2 P, \beta_2 Q\rangle$ $+\langle\beta_{23}P,\ \beta_{23}Q\rangle=\langle P,\ Q\rangle$, i. e., $(|a|^2+|b|^2)\langle\alpha P,\ \alpha Q\rangle=\langle P,\ Q\rangle$, hence we have

$$\langle\beta P,\ \beta Q\rangle=\langle P,\ Q\rangle. \tag{60}$$

So $\beta\in E_7$ from (59), (60) and $\beta_1=\mathrm{Ad}\beta$ from (57). Thus

$$\alpha=\begin{pmatrix} \mathrm{Ad}\beta & 0 & 0 & 0 & 0 & 0 \\ 0 & a\beta & -\bar{b}\beta & 0 & 0 & 0 \\ 0 & b\beta & \bar{a}\beta & |a|^2-|b|^2 & -ab & \overline{ab} \\ 0 & 0 & 0 & 2\bar{a}b & a^2 & -\bar{b}^2 \\ 0 & 0 & 0 & 2a\bar{b} & -b^2 & \bar{a}^2 \end{pmatrix}$$

$$=\phi(\begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix},\ \beta)\in\phi(SU(2)\times E_7).$$

Hence ϕ is onto. It is easy to verify that $\ker\phi=\{(E,\ 1)\ (-E,\ \iota)\}$. Thus the proof of Theorem 4 is completed.

5. Polar decomposition of $E_{8,\iota}$.

In order to give a polar decomposition of the group $E_{8,\iota}$, we use the following

Lemma 5 ([3] p. 345). *Let G be a pseudoalgebraic subgroup of the general linear group $GL(n,\ C)$ such that the condition $A\in G$ implies $A^*\in G$. Then G is homeomorphic to the topological product of the group $G\cap U(n)$ and a Euclidean space R^d :*

$$G\simeq(G\cap U(n))\times R^d$$

where $U(n)$ is the unitary subgroup of $GL(n,\ C)$.

Lemma 6. *$E_{8,\iota}$ is a pseudoalgebraic subgroup of the general linear group $GL(248,\ C)=\mathrm{Iso}_C(\mathfrak{e}_8{}^C,\ \mathfrak{e}_8{}^C)$, and satisfies the condition $\alpha\in E_{8,\iota}$ implies $\alpha^*\in E_{8,\iota}$, where α^* is the transpose of α with respect to the inner product $\langle R_1,\ R_2\rangle$: $\langle\alpha R_1,\ R_2\rangle=\langle R_1,\ \alpha^*R_2\rangle$.*

Proof. Since $\langle\alpha^*R_1,\ R_2\rangle=\langle R_1,\ \alpha R_2\rangle=\langle\iota R_1,\ \alpha R_2\rangle_\iota=\langle\alpha^{-1}\iota R_1,\ R_2\rangle_\iota=\langle\iota\alpha^{-1}\iota R_1,\ R_2\rangle$ for $\alpha\in E_{8,\iota}$, we have

$$\alpha^*=\iota\alpha^{-1}\iota\in E_{8,\iota}.$$

And it is obvious that $E_{8,\iota}$ is pseudoalgebraic, because $E_{8,\iota}$ is defined by pseudoalgebraic relations $\alpha[R_1,\ R_2]=[\alpha R_1,\ \alpha R_2]$ and $\langle\alpha R_1,\ \alpha R_2\rangle_\iota=\langle R_1,\ R_2\rangle_\iota$.

Let $U(248)=U(\mathfrak{e}_8{}^C)=\{\ \alpha\in\mathrm{Iso}_C(\mathfrak{e}_8{}^C,\ \mathfrak{e}_8{}^C)\ |\langle\alpha R_1,\ \alpha R_2\rangle=\langle R_1,\ R_2\rangle\}$ denote the unitary subgroup of the general linear group $GL(248,\ C)=\mathrm{Iso}_C(\mathfrak{e}_8{}^C,\ \mathfrak{e}_8{}^C)$, then we have

$$E_{8,\iota}\cap U(\mathfrak{e}_8{}^C)=\{\ \alpha\in E_{8,\iota}\ |\ \iota\alpha\iota=\alpha\ \}$$
$$\cong(SU(2)\times E_7)/Z_2 \qquad \text{(Theorem 4)}$$

Since $E_{8,\iota}$ is a simple Lie group of type E_8, the dimension of $E_{8,\iota}$ is 248. Hence the dimension d of the Euclidean part of $E_{8,\iota}$ is

$$d = \dim E_{8,\iota} - \dim(SU(2)\times E_7) = 248-(3+133) = 112.$$

Thus we get the following

Theorem 7. *The group $E_{8,\iota}$ is homeomorphic to the topological product of the group $(SU(2)\times E_7)/\boldsymbol{Z}_2$ and a 112 dimensional Euclidean space $\boldsymbol{R}^{112}$:*

$$E_{8,\iota} \simeq (SU(2)\times E_7)/\boldsymbol{Z}_2 \times \boldsymbol{R}^{112}.$$

In particular, the group $E_{8,\iota}$ is a connected non-compact simple Lie group of type $E_{8(-24)}$.

6. Center $z(E_{8,\iota})$ of $E_{8,\iota}$.

Theorem 8. *The center $z(E_{8,\iota})$ of the group $E_{8,\iota}$ is trivial* : $z(E_{8,\iota})=\{1\}$.

Proof. Let $\alpha\in z(E_{8,\iota})$. From the commutativity with $\iota\in E_{8,\iota}$, α has the form

$$\alpha = A\beta, \qquad A\in SU(2),\ \beta\in E_7$$

from Theorem 4. Furthermore, from the commutativity with all $A\in SU(2)$, we see $A\in z(SU(2))=\{E, -E\}$. Similarly we see $\beta\in z(E_7)=\{1,\ \iota\}$ [4]. Hence $\alpha=1$ or ι. However $\iota\notin z(E_{8,\iota})$ from Theorem 4. Thus $z(E_{8,\iota})=\{1\}$.

II. Group $E_{8,1}$

In order to investigate the group $E_{8,\iota}$ more detail, we shall construct one more group $E_{8,1}$ which is isomorphic to $E_{8,\iota}$.

7. Preliminaries.

We consider the real restriction of the preceding chapter. The statements are similar to the complex cases. In the real case, the inner products $<\ ,\ >$ will be denoted by $(\ ,\ )$.

7.1. Jordan algebra $\mathfrak{J}$ [1].

Let $\mathfrak{C}$ denote the non-split Cayley algebra over the field of real numbers $\boldsymbol{R}$ and $\mathfrak{J}=\mathfrak{J}(3, \mathfrak{C})$ the Jordan algebra consisting of all 3×3 Hermitian matrices with entries in $\mathfrak{C}$ with respect to the multiplication $X\circ Y=\dfrac{1}{2}(XY+YX)$.

7.2. Lie algebra $\mathfrak{e}_{6,1}$ [1].

The Lie algebra $\mathfrak{e}_{6,1}$ is defined by

$$\mathfrak{e}_{6,1}=\{\ \phi\in \mathrm{Hom}_{\boldsymbol{R}}(\mathfrak{J}, \mathfrak{J})\ |\ (\phi X, X, X)=0\ \}.$$

Then $\mathfrak{e}_{6,1}$ is a simple Lie algebra of type $E_{6(-26)}$. This $\mathfrak{e}_{6,1}$ is the Lie algebra of a Lie group

$$E_{6,1}=\{\ \alpha\in \mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{J},\ \mathfrak{J})\ \mid\ \det\alpha X=\det X\ \}$$

which is a simply connected non-compact simple Lie group of type $E_{6(-26)}$.

7.3. Lie algebra $\mathfrak{e}_{7,1}$ [2], [5], [6].

We define a vector space $\mathfrak{P}$ by

$$\mathfrak{P}=\mathfrak{J}\oplus\mathfrak{J}\oplus\boldsymbol{R}\oplus\boldsymbol{R}.$$

And the Lie algebra $\mathfrak{e}_{7,1}$ is defined by

$$\mathfrak{e}_{7,1}=\{\ \Phi\in\mathrm{Hom}_{\boldsymbol{R}}(\mathfrak{P},\ \mathfrak{P})\ \mid\ \Phi=\Phi(\phi,\ A,\ B,\ \rho),\ \phi\in\mathfrak{e}_{6,1},\ A,\ B\in\mathfrak{J},\ \rho\in\boldsymbol{R}\ \}$$

as in I. 1. 2. Then $\mathfrak{e}_{7,1}$ is a simple Lie algebra of type $E_{7(-25)}$. This $\mathfrak{e}_{7,1}$ is the Lie algebra of a Lie group

$$\begin{aligned}E_{7(-25)}&=\{\ \alpha\in\mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{P},\ \mathfrak{P})\ \mid\ \alpha(P\times Q)\alpha^{-1}=\alpha P\times\alpha Q\ \}\\&=\{\ \alpha\in\mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{P},\ \mathfrak{P})\ \mid\ \alpha\mathfrak{M}=\mathfrak{M},\ \{\alpha P,\ \alpha Q\}=\{P,\ Q\}\}\end{aligned}$$

(where $\mathfrak{M}=\{\ P\in\mathfrak{P}\ \mid\ P\times P=0, P\neq 0\ \}$) which is a connected non-compact simple Lie group of type $E_{7(-25)}$.

8. Lie algebra $\mathfrak{e}_{8,1}$.

We define a Lie algebra

$$\mathfrak{e}_{8,1}=\mathfrak{e}_{7,1}\oplus\mathfrak{P}\oplus\mathfrak{P}\oplus\boldsymbol{R}\oplus\boldsymbol{R}\oplus\boldsymbol{R}$$

as in I. 2.

Proposition 9. *$\mathfrak{e}_{8,1}$ is a simple Lie algebra of type $E_{8(-24)}$.*

Proof. Since the complexification Lie algebra of $\mathfrak{e}_{8,1}$ is $\mathfrak{e}_8{}^C$, $\mathfrak{e}_{8,1}$ is a simple Lie algebra of type E_8. A maximal compact subalgebra of $\mathfrak{e}_{8,1}$ is

$$\begin{aligned}\mathfrak{k}&=\{\ \Theta\in\mathfrak{e}_{8,1}\ \mid\ {}'\Theta=\Theta\ \}\\&=\{\ \Theta(\Phi,\ P,\ \hat{P},\ 0,\ s,-s)\in\mathfrak{e}_{8,1}\ \mid\ \Phi\in\mathfrak{e}_{7,1},\ {}'\Phi=\Phi,\ P\in\mathfrak{P},\ s\in\boldsymbol{R}\ \}.\end{aligned}$$

Hence the Cartan index of $\mathfrak{e}_{8,1}$ is

$$\dim\mathfrak{e}_{8,1}-2\dim\mathfrak{k}=248-2(79+56+1)=-24,$$

that is, the type of $\mathfrak{e}_{8,1}$ is $E_{8(-24)}$.

9. Manifold $\mathfrak{X}$ and group $E_{8,1}$.

We define a subspace of $\mathfrak{e}_{8,1}$ by

$$\mathfrak{T}=\left\{\begin{pmatrix}\Phi\\P\\Q\\r\\s\\t\end{pmatrix}\in\mathfrak{e}_{8,1}\;\middle|\;\begin{array}{l}2t\Phi+Q\times Q=0\\ t^2P-trQ+\dfrac{1}{6}(Q\times Q)Q=0\\ st^3+r^2t^2-\dfrac{1}{96}\{Q,\ (Q\times Q)Q\}=0\\ t>0\end{array}\right\}.$$

Now, the group $E_{8,1}$ is defined to be the group of all automorphisms of the Lie algebra $\mathfrak{e}_{8,1}$ leaving $\mathfrak{T}$ invariant :

$$E_{8,1}=\{\ \alpha\in\mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{e}_{8,1},\ \mathfrak{e}_{8,1})\ \mid\ \alpha\mathfrak{T}=\mathfrak{T},\ \alpha[R_1,\ R_2]=[\alpha R_1,\ \alpha R_2]\ \}.$$

Proposition 10. $\mathfrak{T}=\{\ \exp(\Theta(0,\ P_1,\ 0,\ r_1,\ s_1,\ 0))\underline{1}\ \mid\ P_1\in\mathfrak{P},\ r_1,\ s_1\in\boldsymbol{R}\ \}$. *In particular, $\mathfrak{T}$ is connected.*

Proof is the same as [7] Proposition 27.

Theorem 11. *$E_{8,1}$ is a Lie group of type $E_{8(-24)}$.*

Proof. The Lie algebra $\mathfrak{e}_{8,1}$ of $E_{8,1}$ is the derivation Lie algebra Der $(\mathfrak{e}_{8,1})$ (its proof is the same as [7] Proposition 28) which is isomorphic to $\mathfrak{e}_{8,1}$. Hence the type of the group $E_{8,1}$ is $E_{8(-24)}$ from Proposition 9.

10. Subgroups $E_{7,1}$ and $SL(2,\ R)$ of $E_{8,1}$.

We shall show that the group $E_{8,1}$ contains non-compact subgroups of type E_7 and A_2.

Theorem 12. *The group $E_{8,1}$ contains a subgroup*

$$E_{7,1}=\{\ \alpha\in E_{8,1}\ \mid\ \alpha 1=1,\ \alpha\bar{1}=\bar{1},\ \alpha\underline{1}=\underline{1}\}$$

which is a connected non-compact simple Lie group of type $E_{7(-25)}$.

Proof. The mapping

$$E_{7(-25)}\ni\beta\longrightarrow\beta=\begin{pmatrix}\mathrm{Ad}\beta&&&&&\\&\beta&&&&\\&&\beta&&&\\&&&1&&\\&&&&1&\\&&&&&1\end{pmatrix}\in E_{7,1}\subset E_{8,1}$$

gives an isomorphism between $E_{7(-25)}$ and $E_{7,1}$. Its proof is analogous to [7] Theorem 25 (in [7], in order to prove that $\alpha\in E_{7,1}$ is a digonal form, we used the properties of the inner product $<\ ,\ >$, but it follows only from the condition $\alpha[R_1,\ R_2]=[\alpha R_1,\ \alpha R_2]$).

Proposition 13. *The group $E_{8,1}$ contains a subgroup*

$$SL(2,\ \boldsymbol{R})=\left\{ A=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & a1 & c1 & 0 & 0 & 0 \\ 0 & b1 & d1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1+2bc & -ab & cd \\ 0 & 0 & 0 & -2ac & a^2 & -c^2 \\ 0 & 0 & 0 & 2bd & -b^2 & d^2 \end{pmatrix} \ \middle|\ \begin{pmatrix} a & c \\ b & d \end{pmatrix} \in \mathrm{SL}(2,\ \boldsymbol{R}) \right\}$$

which is isomorphic to the special linear group $\mathrm{SL}(2,\ \boldsymbol{R})=\{A\in M(2,\ \boldsymbol{R})\ |\ \det A=1\}$.

We identify these groups $E_{7(-25)}$ with $E_{7,1}$, $\mathrm{SL}(2,\ \boldsymbol{R})$ with $SL(2,\ \boldsymbol{R})$ under the above correspondences.

11. Connectedness of $E_{8,1}$.

We shall prove that the group $E_{8,1}$ is connected.

Proposition 14. *The isotropy subgroup $G_{\underline{1}}=\{\ \alpha\in E_{8,1}\ |\ \alpha\underline{1}=\underline{1}\ \}$ of the group $E_{8,1}$ at $\underline{1}\in\mathfrak{T}$ is the semi-direct product of groups $\exp(\underline{\mathfrak{P}})\exp(\underline{\boldsymbol{R}})$ and $E_{7,1}$*:

$$G_{\underline{1}}=(\exp(\underline{\mathfrak{P}})\exp(\underline{\boldsymbol{R}}))E_{7,1}, \qquad (\exp(\underline{\mathfrak{P}})\exp(\underline{\boldsymbol{R}}))\cap E_{7,1}=\{1\},$$

where

$$\exp(\underline{\mathfrak{P}})\exp(\underline{\boldsymbol{R}})=\{\ \exp(\Theta(0,\ 0,\ Q,\ 0,\ 0,\ t))\ |\ Q\in\mathfrak{P},\ t\in\boldsymbol{R}\ \}.$$

In particular, $G_{\underline{1}}$ is connected.

Proof. First, note that $\underline{\mathfrak{P}}\oplus\underline{\boldsymbol{R}}=\{\ \underline{Q}+\underline{t}=(0,\ 0,\ Q,\ 0,\ 0,\ t)\ |\ Q\in\mathfrak{P},\ \ t\in\boldsymbol{R}\ \}$ is a subalgebra of $\mathfrak{e}_{8,1}$ and $[\underline{Q},\ \underline{t}]=0$, so $\exp(\underline{\mathfrak{P}})\exp(\underline{\boldsymbol{R}})$ is a connected subgroup of $E_{8,1}$ and $\exp(\underline{Q}))=\exp(\Theta(0,\ 0,\ Q,\ 0,\ 0,\ 0))$, $\exp(\underline{t})=\exp(\Theta(0,\ 0,\ 0,\ 0,\ 0,\ t))$ commute to each other. Now, let $\alpha\in G_{\underline{1}}$ and put

$$\alpha 1=(\Phi,\ P,\ Q,\ r,\ s,\ t), \qquad \alpha\overline{1}=(\Phi_1,\ P_1,\ Q_1,\ r_1,\ s_1,\ t_1).$$

Then, $[1,\ \underline{1}]=-2\underline{1}$, $[\overline{1},\ \underline{1}]=1$ implies $[\alpha 1,\ \underline{1}]=-2\underline{1}$, $[\alpha\overline{1},\ \underline{1}]=\alpha 1$, that is,

$$(0,\ 0,-P,\ s,\ 0,\ -2r)=(0,\ 0,\ 0,\ 0,\ 0\ -2),$$

$$(0,\ 0,\ -P_1,\ s_1,\ 0,\ -2r_1)=(\Phi,\ P,\ Q,\ r,\ s,\ t)$$

respectively. Hence we have

$$P=0, \qquad s=0, \qquad r=1,$$

$$\Phi=0, \qquad P_1=-Q, \qquad s_1=1, \qquad r_1=-\frac{t}{2}.$$

Furthermore $[1, \tilde{1}]=2\tilde{1}$ implies $[\alpha 1, \alpha\tilde{1}]=2\alpha\tilde{1}$, that is,

$$[(0,\ 0,\ Q,\ 1,\ 0,\ t),\ (\Phi_1, -Q,\ Q_1,\ -\frac{t}{2},\ 1,\ t_1)]$$

$$=(Q\times Q,\ -2Q,\ -\Phi_1 Q-Q_1-\frac{3}{2}tQ,\ -t,\ 2,\ -\frac{1}{4}\{Q,\ Q_1\}-t^2-2t).$$

$$=2(\Phi_1,\ -Q,\ Q_1,\ -\frac{t}{2},\ 1,\ t).$$

Hence we have

$$\Phi_1=\frac{1}{2}Q\times Q, \qquad Q_1=-\frac{t}{2}Q-\frac{1}{3}\Phi_1 Q, \qquad t_1=-\frac{t^2}{4}-\frac{1}{16}\{Q,\ Q_1\}.$$

Thus we see that α has the form

$$\alpha=\begin{pmatrix} * & * & * & 0 & \frac{1}{2}Q\times Q & 0 \\ * & * & * & 0 & -Q & 0 \\ * & * & * & Q & -\frac{t}{2}Q-\frac{1}{6}(Q\times Q)Q & 0 \\ * & * & * & 1 & -\frac{t}{2} & 0 \\ * & * & * & 0 & 1 & 0 \\ * & * & * & t & -\frac{t^2}{4}+\frac{1}{96}\{Q,\ (Q\times Q)Q\} & 1 \end{pmatrix}.$$

On the other hand, $\exp(\underset{=}{\frac{t}{2}})\exp(\underset{=}{Q})\tilde{1}$

$$=\exp\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{t}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{t}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & t & 0 & 0 \end{pmatrix}\exp\begin{pmatrix} 0 & -Q & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -Q & 0 \\ -Q & 0 & 0 & Q & 0 & 0 \\ 0 & -\frac{1}{8}Q & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{4}Q & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{t}{2} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -\frac{t}{2} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & t & -\frac{t^2}{4} & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2}Q\times Q \\ -Q \\ -\frac{1}{6}(Q\times Q)Q \\ 0 \\ 1 \\ \frac{1}{96}\{Q,\ (Q\times Q)\,Q\} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2}Q\times Q \\ -Q \\ -\frac{t}{2}Q-\frac{1}{6}(Q\times Q)Q \\ -\frac{t}{2} \\ 1 \\ -\frac{t^2}{4}+\frac{1}{96}\{Q,\ (Q\times Q)\,Q\} \end{pmatrix} = \alpha\overline{1},$$

and

$$\exp\left(\underline{\frac{t}{2}}\right)\exp\,(\underline{Q})1 = \exp\left(\underline{\frac{t}{2}}\right)(0,\ 0,\ Q,\ 1,\ 0,\ 0) = (0,\ 0,\ Q,\ 1,\ 0,\ t) = \alpha 1,$$

$$\exp\left(\underline{\frac{t}{2}}\right)\exp\,(\underline{Q})\underline{1} = \underline{1} = \alpha\underline{1}.$$

Therefore $\exp(-\underline{Q})\exp\left(-\underline{\frac{t}{2}}\right)\alpha\in E_{7,1}$, hence we have

$$G_{\underline{1}} = (\exp\,(\underline{\mathfrak{P}})\,\exp(\underline{R}))E_{7,1}.$$

Next, for $\beta\in E_{7,1}$, we have

$$\beta(\exp(\underline{Q}))\beta^{-1} = \exp(\underline{\beta Q}), \qquad \beta(\exp(\underline{t}))\beta^{-1} = \exp(\underline{t}).$$

In fact,

$$\beta(\exp(\underline{Q}))\beta^{-1}R = \beta(\exp(\underline{Q}))\beta^{-1}(\Phi_1,\ P_1,\ Q_1,\ r_1,\ s_1,\ t_1)$$

$$= \beta(\exp(\underline{Q}))(\beta^{-1}\Phi_1\beta,\ \beta^{-1}P_1,\ \beta^{-1}Q_1,\ r_1,\ s_1,\ t_1)$$

$$=\beta\begin{pmatrix} \beta^{-1}\Phi_1\beta-Q\times\beta^{-1}P_1+\frac{1}{2}s_1Q\times Q \\ \beta^{-1}P_1-s_1Q \\ \beta^{-1}Q_1-\beta^{-1}\Phi_1\beta Q+r_1Q+\frac{1}{2}(Q\times\beta^{-1}P_1)Q-\frac{1}{16}\{Q,\ \beta^{-1}P_1\}Q-\frac{1}{6}s_1(Q\times Q)Q \\ r_1-\frac{1}{8}\{Q,\ \beta^{-1}Q_1\} \\ s_1 \\ t_1-\frac{1}{4}\{Q,\ \beta^{-1}Q_1\}+\frac{1}{8}\{Q,\ \beta^{-1}\Phi_1\beta Q\}-\frac{1}{24}\{Q,(Q\times\beta^{-1}P_1)Q\}+\frac{1}{96}s_1\{Q,\ (Q\times Q)Q\} \end{pmatrix}$$

$$=\begin{pmatrix} \Phi_1-\beta Q\times P_1+\frac{1}{2}s_1\ (\beta Q\times\beta Q) \\ P_1-s_1\beta Q \\ Q_1-\Phi_1\beta Q+r_1\beta Q+\frac{1}{2}(\beta Q\times P_1)\beta Q-\frac{1}{16}\{\beta Q,\ P_1\}\beta Q-\frac{1}{6}s_1(\beta Q\times\beta Q)\beta Q \\ r_1-\frac{1}{8}\{\beta Q,\ Q_1\} \\ s_1 \\ t_1-\frac{1}{4}\{\beta Q,\ Q_1\}+\frac{1}{8}\{\beta Q,\Phi_1\beta Q\}-\frac{1}{24}\{\beta Q,\ (\beta Q\times P_1)\beta Q\}+\frac{1}{96}s_1\{\beta Q,\ (\beta Q\times\beta Q)\beta Q\} \end{pmatrix}$$

$$=\exp(\beta\underline{Q})R,$$

an similarly $\beta(\exp(\underline{t}))\beta^{-1}=\exp(\underline{t})$. This shows that $\exp(\underline{\mathfrak{P}})\exp(\underline{\boldsymbol{R}})$ is a nomal subgroup of $G_{\underline{1}}$. Thus we have a split exact sequence

$$1\longrightarrow\exp(\underline{\mathfrak{P}})\exp(\underline{\boldsymbol{R}})\longrightarrow G_{\underline{1}}\longrightarrow E_{7,1}\longrightarrow 1.$$

Hence $G_{\underline{1}}$ is the semi-direct product of $\exp(\underline{\mathfrak{P}})\exp(\underline{\boldsymbol{R}})$ and $E_{7,1}$.

Theorem 15. *The group $E_{8,1}$ acts on $\mathfrak{T}$ transitively and the isotropy subgroup at $\underline{1}\in\mathfrak{T}$ of $E_{8,1}$ is the semi-direct product of subgroups $\exp(\underline{\mathfrak{P}})\exp(\underline{\boldsymbol{R}})$ and $E_{7,1}$. Therefore we have the following homeomorphism*

$$E_{8,1}/(\exp(\underline{\mathfrak{P}})\exp(\underline{\boldsymbol{R}}))E_{7,1}\simeq\mathfrak{T}.$$

In particular, the group $E_{8,1}$ is connected.

Proof is the direct consequence of Propositions 10, 14.

From the above Theorem we have

Theorem 16. *The group $E_{8,1}$ is the connected component containing the identity of the automorphismgroup* $\mathrm{Aut}(\mathfrak{e}_{8,1})=\{\ \alpha\in\mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{e}_{8,1},\ \mathfrak{e}_{8,1})\ |\ [\alpha R_1,\ R_2]=[\alpha R_1,\ \alpha R_2]\ \}$.

12. Center of $E_{8,1}$.

Theorem 17. *The center $z(E_{8,1})$ of the group $E_{8,1}$ is trivial* : $z(E_{8,1})=\{1\}$.

Proof. Let $\alpha\in z(E_{8,1})$. From the commutativity with $\beta\in E_{7,1}$,

$$\alpha\beta 1=\alpha 1,\qquad \alpha\beta\dot{1}=\alpha\dot{1},\qquad \alpha\beta\underset{\cdot}{1}=\alpha\underset{\cdot}{1}.$$

From this we see that α has the form

$$\alpha=\begin{pmatrix}\beta & 0\\ l & B\end{pmatrix},\qquad B\in M(3,\ \boldsymbol{R}).$$

Next, from the commutativity with $A\in SL(2,\ \boldsymbol{R})$,

$$B\begin{pmatrix}1+2bc & -ab & cd\\ -2ac & a^2 & -c^2\\ 2bd & -b^2 & d^2\end{pmatrix}=\begin{pmatrix}1+2bc & -ab & cd\\ -2ac & a^2 & -c^2\\ 2bd & -b^2 & d^2\end{pmatrix}B$$

where $\begin{pmatrix}a & c\\ b & d\end{pmatrix}\in SL(2,\ \boldsymbol{R})$, we see $B=\begin{pmatrix}r & 0 & 0\\ 0 & r & 0\\ 0 & 0 & r\end{pmatrix}$, $r\neq 0$. Furthermore, from $[\alpha\dot{1},\ \alpha\underset{\cdot}{1}]=\alpha 1$, we have $r^2=r$, hence $r=1$, so $B=E$. Hence $\alpha\in E_{7,1}$, moreover $\alpha\in z(E_{7,1})$ which is $\{1,\ \iota\}$[5]. And we see easily

$$\iota\exp(\underset{\cdot}{Q})\neq\exp(\underset{\cdot}{Q})\iota,\qquad \text{for } Q\in\mathfrak{P}$$

(see Proposition 14). Therefore $\alpha=1$. Thus we have $z(E_{8,1})=\{1\}$.

13. Isomorphism $E_{8,\iota}\cong E_{8,1}$.

From Theorems 7, 11, 15, we see that the groups $E_{8,\iota}$ and $E_{8,1}$ are both connected and their Lie algebras have the same type $E_{8(-24)}$. Therefore there exist central normal subgroups N_ι, N_1 of the simply connected simple Lie group $E_{8(-24)}$ of type $E_{8(-24)}$ such that

$$E_{8,\iota}\cong E_{8(-24)}/N_\iota,\qquad E_{8,1}\cong E_{8(-24)}/N_1.$$

From Theorem 7, we know that the center of the group $E_{8(-24)}$ is the cyclic group of order 2 : $z(E_{8(-24)})=\boldsymbol{Z}_2$. And the centers of $E_{8,\iota}$, $E_{8,1}$, are both trivial (Theorems 8, 17). Hence it must be $N_\iota=N_1=\boldsymbol{Z}_2$. Therefore the groups $E_{8,\iota}$ and $E_{8,1}$ are isomorphic :

$$E_{8,\iota}\cong E_{8,1}.$$

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