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Note on Relative Stiefel Manifolds

By HIDEYUKI KACHI

Department of Mathematics, Faculty of Science
Shinshu University
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1. Introduction

Using the standard embedding of the quaternionic Stiefel manifold $X_{n,k}$ in the complex Stiefel manifold $W_{2n,2k}$, we write

$$X'_{n,k} = (W_{2n,2k}, X_{n,k}) \quad (k=1, 2, \cdots).$$

There is the natural projection

$$p' : X'_{n,k} \longrightarrow X'_{n,1}$$

induced by the natural projection

$$p : W_{2n,2k} \longrightarrow W_{2n,2}.$$

Note that the inclusion

$$i : (W_{2n-1,1}, e) \longrightarrow X'_{n,1}$$

induces an isomorphism

$$i_* : \pi_r(W_{2n-1,1}, e) \longrightarrow \pi_r(X'_{n,1})$$

for all values of r.

We say that a relative cross-section of $X'_{n,k}$ is an element $\alpha \in \pi_{4n-3}(X'_{n,k})$ such that $p'_*(\alpha)$ generates $\pi_{4n-3}(X'_{n,1}) \cong Z$.

Then we shall prove the following theorem :

Theorem. *The relative Stiefel manifold $X'_{n,k}$ admits a relative cross-section if and only if* $k = 1$, or

$k = 2$ *and* $n \equiv 2$ *mod* 24.

For the pair of spaces $W'_{n,k} = (V_{2n,2k}, W_{n,k})$, James [1], [2] proved that the relative Stiefel manifold $W'_{n,k}$ admits a relative cross-section if and only if either $k = 1$, or

$k = 2$ and $n \equiv 0 \bmod 2$, or
$k = 3$ or 4 and $n \equiv 4 \bmod 24$.

2. Preliminary

Consider the factor space $X_n = U(2n)/Sp(n)$ $(n = 1, 2, \cdots)$, with the obvious embeddings $X_1 \subset X_2 \subset X_3 \subset \cdots$. The triad homotopy groups

$$\pi_r(U(2n)\ ;\ Sp(n),\ U(2n\text{-}2k))$$

can be identified with the relative homotopy group

$$\pi_r(X_n, X_{n-k})$$

on the one hand, or with

$$\pi_r(W_{2n,2k}, X_{n,k})$$

on the other. Thus we can identify

(2.1) $\pi_r(X'_{n,k}) = \pi_r(X_n, X_{n-k})$.

The homotopy exact sequence of the triple (X_n, X_{n-1}, X_{n-k}) can be written in the form

(2.2) $\cdots \longrightarrow \pi_r(X'_{n-1,k-1}) \xrightarrow{j_*'} \pi_r(X'_{n,k}) \xrightarrow{p_*'} \pi_r(X'_{n,1}) \xrightarrow{\partial'} \pi_{r-1}(X'_{n-1,k-1}) \longrightarrow \cdots\cdots,$

where j' denotes the inclusion $X'_{n-1,k-1} \longrightarrow X'_{n,k}$ and ∂' the boundary homomorphism.

The image of the generator (ι_{4m-3}) of $\pi_{4n-3}(W_{2n-1,1}) \cong Z$ by i_* will be denoted by $[\iota_{4n-3}] \in \pi_{4n-3}(X'_{n,1})$. Equivalently, by a relative cross-section of $X'_{n,k}$ we mean an element of $\pi_{4n-3}(X'_{n,k})$ (or the representative of such an element) which projects into $[\iota_{4n-3}]$ under

$$p_*' : \pi_{4n-3}(X'_{n,k}) \longrightarrow \pi_{4n-3}(X'_{n,1}).$$

Thus we have

(2.3) $X'_{n,k}$ admits a relative cross-section if and only if the homomorphism ∂' : $\pi_{4n-3}(X'_{n,1}) \longrightarrow \pi_{4n-4}(X'_{n-1,k-1})$ is trivial.

For example, take $n = k$. Then the relative Stiefel manifold $X'_{n,n} = (U(2n), Sp(n))$ admits a relative cross-section if and only if the fibration $X_n \longrightarrow S^{4n-3}$ admits a cross-section in the ordinary sense, i.e., if and only if $n = 2$ ([3]).

Clearly

(2.4) $X'_{n,1}$ admits a relative cross-section for all values of n.

Also

(2.5) $X'_{n,k-1}$ admits a relative cross-section if $X'_{n,k}$ does.

3. Proof of Theorem

Let (a,b) be the g. c. d. of a and b.

Lemma 3.1. $\pi_{4n-4}(X'_{n,2}) \cong Z_{(n-2,24)}$.

proof. From (2.1) and the homotopy exact sequence of the pair (X_n, X_{n-2}), we have the exact sequence

$$\pi_{4n-4}(X_n) \longrightarrow \pi_{4n-4}(X'_{n,2}) \longrightarrow \pi_{4n-5}(X_{n-2}) \longrightarrow \pi_{4n-5}(X_n).$$

This sequence is as follows ([4]);

$$0 \longrightarrow \pi_{4n-4}(X'_{n,2}) \longrightarrow Z_{(n-2,24)} \longrightarrow 0 \qquad \text{for } n \text{ even},$$

$$0 \longrightarrow \pi_{4n-4}(X'_{n,2}) \longrightarrow Z_{(n-2,24)} \oplus Z_2 \longrightarrow Z_2 \longrightarrow 0 \qquad \text{for } n \text{ odd}.$$

Thus we have Lemma.

Lemma 3.2. *The relative Stiefel manifold $X'_{n,2}$ admits a relative cross-section if and only if $n \equiv 2 \bmod 24$.*

Proof. Consider the exact sequence

$$\pi_{4n-3}(X'_{n,2}) \longrightarrow \pi_{4n-3}(X'_{n,1}) \xrightarrow{\partial'} \pi_{4n-4}(X'_{n-1,1})$$
$$\longrightarrow \pi_{4n-4}(X'_{n,2}) \longrightarrow \pi_{4n-4}(X'_{n,1}) = 0$$

of (2.2). Let $[\nu_{4n-7}]$ be the generator of $\pi_{4n-4}(X'_{n-1,1}) \cong Z_{24}$ for $n \geqq 3$. From the exactness of above sequence and Lemma 3.1, we have

$$\partial'([\iota_{4n-3}]) = (n-2,\ 24)\,[\nu_{4n-7}].$$

Thus, from (2.3), we have Lemma.

Lemma 3.3. *The relative Stiefel manifold $X'_{n,3}$ does not admit a relative cross-section for all $n \geqq 3$.*

Proof. Suppose that $X'_{n,3}$ admits a relative cross-section.
Then, from (2.5) and Lemma 3.2, $n \equiv 2 \bmod 24$ and $\partial' : \pi_{4n-3}(X'_{n,1}) \longrightarrow \pi_{4n-4}(X'_{n-1,2})$ is trivial by (2.3).

Consider the commutative diagram

$$\begin{array}{ccc}
\pi_{4n-3}(X'_{n,1}) & \xrightarrow{\quad\partial'\quad} & \pi_{4n-4}(X'_{n-1,2}) \\
\uparrow \cong & & \uparrow \\
\pi_{4n-3}(W_{2n,2}) & \longrightarrow & \pi_{4n-4}(W_{2n-2,4}) \\
\nwarrow \cong & \partial_5 \nearrow & \uparrow j_* \\
& \pi_{4n-3}(W_{2n-1,1}) & \\
& & \pi_{4n-4}(X_{n-1,2})
\end{array}$$

The right hand column of sequence is the homotopy exact sequence of the pair $(W_{2n-2,4},\ X_{n-1,2})$ and ∂_5 is the boundary homomorphism associated the fibration $W_{2n-1,5} \longrightarrow W_{2n-1,1} = S^{4n-3}$.

Then, from commutativity of the diagram,

$$\partial_5((\iota_{4n-3})) \in \text{Image of } j_*.$$

Let $b_{2n-1,5}$ denote the order of $\partial_5((\iota_{4n-3}))$ in $\pi_{4n-4}(W_{2n-2,4})$.
Then $b_{2n-1,5}$ is 2 at most, since $\pi_{4n-4}(X_{n-1,2}) \cong Z_2$ ([6]).

By Walker [7],

$$\frac{(2n-6)!}{(2n-2)!} M(n-1,\ n-3)\, b_{2n-1,5} \in Z$$

where $M(n-1,\ n-3) = (n-3)(n-2)(2n-5)(10n^3-57n^2+95n-48)/2^3 3^2 5$.

$$\frac{(2n-6)!}{(2n-2)!} M(n-1,\ n-3) = \frac{(n-3)(10n^3-57n^2+95n-48)}{2^5 3^2 5(n-1)(2n-3)}$$

If $n \equiv 2 \bmod 24$, then

$$10n^3-57n^2+95n-48 \not\equiv 0 \bmod 4,$$
$$10n^3-57n^2+95n-48 \equiv 0 \bmod 2.$$

This shows that $b_{2n-1,5}$ is a multiple of 2^4. Thus, we have a contradiction.

From (2.5), Lemmas 3.2 and 3.3, the proof of Theorem is complete.

References

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