Non-symmetry of the Freudenthal's magic square

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In the theory of simple Lie algebras, the following chart is called the Freudenthal's magic square [1].

B_1	A_2	C_3	F_4
A_2	$A_2 \oplus A_2$	A_5	E_6
C_3	A_5	D_6	E_7
F_4	E_6	E_7	E_8

One of the meaning is as follows. To define exceptional Lie algebras F_4, E_6, E_7 and E_8 of the last column, we use usually the Caylay algebra $\mathfrak{C}$. If we replace $\mathfrak{C}$ with the fields of real numbers $\boldsymbol{R}$, complex numbers $\boldsymbol{C}$ and quaternions $\boldsymbol{H}$, then the first, second and third columns are obtained, respectively. The beuaty of this chart is in its symmetry.

We have constructed simply connected compact exceptional Lie groups F_4, E_6, E_7 and E_8 ([2], [3], [4]). Of course we used the Cayley algebra $\mathfrak{C}$ in these constructions. Now, we do the same replacement as above, then we have the following chart.

$SO(3)$	$(SU(3)/\boldsymbol{Z}_3)\cdot\boldsymbol{Z}_2$	$Sp(3)/\boldsymbol{Z}_2$	F_4
$SU(3)$	$((SU(3)\times SU(3))/\boldsymbol{Z}_3)\cdot\boldsymbol{Z}_2$	$SU(6)/\boldsymbol{Z}_2$	E_6
$Sp(3)$	$(SU(6)/\boldsymbol{Z}_3)\cdot\boldsymbol{Z}_2$	$Ss(12)$	E_7
F_4	$(E_6/\boldsymbol{Z}_3)\cdot\boldsymbol{Z}_2$	$E_7/\boldsymbol{Z}_2$	E_8

We can see a slight non-symmetry in this chart.

References

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