

Singular points of curve families on surfaces

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1. **Introduction.** It is well known that there are two canonical bilinear forms on the tangent bundle of a smooth oriented surface which is immersed in the 3-dimensional Euclidian space. These are called the first fundamental form and the second fundamental form. The principal curvature of the surface are defined by comparing these two forms. And a point where two principal curvatures coincide is called a umbilic point. Except for umbilic points there exists a decomposition of the tangent space into two direct summands, each of which is tangent to the one of principal curvatures. And if a curve tangents to those tangent lines of direct summands at any points of it, it is called a curvature curve.

In general for a given symmetric bilinear form on a 2-dimensional Riemannian manifold, we define the corresponding curve family with singularities and its local topological classification in the section 10. And we show the structurally stable condition of curve families on 2-dimensional closed Riemannian manifolds in the section 15.

2. Let M be a 2-dimensional Riemannian manifold, and G a given Riemannian metric of M . For a given symmetric bilinear form Q on the tangent bundle of M , we call a real number λ the eigen value of Q at a point p of M (with respect to G), if the bilinear form $(Q - \lambda G)$ degenerates. The eigenvector space of the tangent space TM_p at the point p is defined by

$$L_\lambda = \{u \in TM_p \mid (Q - \lambda G)_p(u, v) = 0 \text{ for any } v \in TM_p\},$$

If these two eigen values are different at p , the eigen spaces are 1-dimensional and are mutually perpendicular with respect to G . Let S be the subset of M such that a point of S has multiple eigen values. And $L_\lambda = TM_p$ on S . On the open submanifold $M - S$, we have two tangent line fields. Let L be the tangent line field corresponding to the larger eigen value. Since L is a smooth line field, by integration there is curve family on M . We call it the curve family with singularities S defined by Q .

In general a curve family with singular points on M is defined by a 1-dimensional

foliation of $M - S$, where S is a closed subset of M . We call S the set of singular points. Given two curve families with singularities, if there exists a homeomorphism such that the singular set corresponds to the other singular set, and each curve of the foliation corresponds to the others, then we call these curve families are topologically equivalent. Similarly local topological equivalency is defined. Our purpose of this paper is the classification of generic topological types of the curve families with singular points defined by symmetric bilinear forms on a 2-dimensional Riemannian manifold.

3. Let M and N be Riemannian 2-manifolds and Q a symmetric bilinear form on M . And let f be a diffeomorphism from N to M . We have a induced bilinear form on N ,

$$f^*Q(u, v)_p = Q(f_*u, f_*v)_{f(p)},$$

where u and v are tangent vectors of N at p , and f_* denotes the differential map of f .

Proposition. if f is a conformal diffeomorphism from N to M then f is a topological equivalence from the curve family with singular points defined by f^*Q to the other defined by Q .

Proof. There exists a positive smooth function ϕ such that

$$f^*G_M = \phi G_N,$$

where G_M and G_N are given Riemannian metrics of M and N respectively.

$$\begin{aligned} \text{If} \quad & (f^*Q - \lambda \phi G_N)_p(u, v)_p = 0, \\ \text{then} \quad & (Q - \lambda G_M)_{f(p)}(f_*u, f_*v)_{f(p)} = 0. \end{aligned}$$

Therefore for any point p the eigen value is ϕ -times of the corresponding eigenvalue at $f(p)$ and each eigenvector space corresponds to the other defined by f_* . So f gives a required topological equivalence.

4. It is well known that on a 2-dimensional Riemannian manifold there exist isothermal local coordinates, $[B_E]$. By the last proposition it is enough to consider local topological types of curve families with singularities defined by symmetric bilinear forms, we may assume that M is a Euclidian 2-plane \mathbf{R}^2 . Except for singularities the local topological types are trivial. Now we consider the local topological types at the singular points.

Let A be a symmetric 2×2 matrix valued smooth function defined on some neighbourhood U of the origin which represents a symmetric bilinear form :

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix},$$

where a , b and c are some smooth functions on U .

If we subtract $(a+c)/2 \cdot E$ from A then each eigenvalue decreases by $(a+c)/2$

and it has the same eigenspaces as A . So we may assume that $a = -c$. The eigenvalues of A are $\pm (a^2 + b^2)^{1/2}$, and a point of U is a singular point if and only if $a(x) = b(x) = 0$. The eigenvector space which corresponds to $(a^2 + b^2)^{1/2}$ is

$$L = \{(u, v) \in U \mid au + bv = u(a^2 + b^2)^{1/2}\}.$$

Now let $D_1 = \det(A)$.

Generic condition I. For any singular point p , $D_1 \neq 0$.

It is easy to see that this condition is generic. And by the inverse function theorem, this condition implies that the singular points are isolated singular points.

5. For a smooth curve family on a neighbourhood U of the origin 0 in \mathbf{R}^2 we assume that 0 is an isolated singular point, and let S_ϵ^1 be a circle in U with the center 0 and a sufficiently small radius ϵ . For a point p of the unit circle S^1 we define a smooth map $f_\epsilon : S^1 \rightarrow P^1$ such that $f_\epsilon(p)$ is the tangent line of the curve at ϵp , where P^1 denotes the 1-dimensional real projective space which consists of all lines through 0 . P^1 is canonically oriented by the orientation of U , so we can define the mapping degree $\deg(f)$ of f_ϵ .

The mapping degree is homotopically invariant, and $\deg(f)$ is invariant for ϵ . We define index ι of the curve family at the isolated singular point by $\iota = \deg(f)/2$.

6. Now we calculate the index of the singular point of a curve family defined by A , a bilinear form with the generic condition I in section 4 at the isolated singular point.

The direction $[\cos(\theta) : \sin(\theta)]$ of the line defined by the eigenspace of A corresponding to the larger eigen value is given by

$$\begin{pmatrix} a & b \\ b & -a \end{pmatrix} \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} = (a^2 + b^2)^{1/2} \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}, \text{ where } A = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}.$$

That is

$$a^2 \cos^2 \theta + 2ab \cos \theta \sin \theta + b^2 \sin^2 \theta = a^2 \cos^2 \theta + b^2 \sin^2 \theta$$

and

$$a \sin(2\theta) = b \cos(2\theta).$$

Therefore the argument of (a, b) is 2θ . We have

Proposition. If D_1 of A at singular point 0 is positive then the index of the curve family at 0 is $1/2$. Similarly, if D_1 of A is negative then the index is $-1/2$.

Proof. In the case of positive D_1 , by the inverse function theorem (a, b) is the orientation preserving local diffeomorphism at 0 . A sufficiently small circle with center 0 is mapped to the neighbourhood of 0 with degree 1. By definition, we have the index $\iota = 1/2$.

7. At first we assume that A is linear, that is

$$A = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}, \quad a = a'x + a''y, \quad b = b'x + b''y,$$

where a' , a'' , b' and b'' are constant, and (x, y) is the canonical coordinates on U .

Then $D_1 = a'b'' - a''b'$, which is non zero by the generic condition I, $D_1 \neq 0$. Now we use the exponential map

$$\exp(r, \theta) = (e^r \cos \theta, e^r \sin \theta).$$

It is a well known conformal map, and is a local topological equivalence from the curve family defined by \exp^*A to the other defined by A ,

$$\exp^*A = {}^t(J(\exp))(A \exp)(J(\exp)),$$

where $J(\exp)$ is the Jacobian matrix of \exp . Therefore,

$$J(\exp)(r, \theta) = e^r \begin{pmatrix} c & -s \\ s & c \end{pmatrix}, \quad \text{where } c = \cos \theta \text{ and } s = \sin \theta,$$

and

$$\exp^*A = e^{3r} \begin{pmatrix} c & s \\ -s & c \end{pmatrix} \begin{pmatrix} a'c + a''s & b'c + b''s \\ b'c + b''s & -a'c - a''s \end{pmatrix} \begin{pmatrix} c & -s \\ s & c \end{pmatrix}.$$

And we assume that $(1, 0)$ is one of eigen vectors of \exp^*A . Then we have

$$(*) \quad (-s, c)A \begin{pmatrix} c \\ s \end{pmatrix} = 0.$$

And if the eigen value is positive,

$$(**) \quad (c, s)A \begin{pmatrix} c \\ s \end{pmatrix} > 0.$$

These conditions are invariant for r and are homogeneous of degree 3 with respect to c and s . Let θ be one of the solution of $(*)$, then $(\theta + \pi)$ is also a solution of $(*)$. But the signature of $(**)$ are different. Let D_2 be the discriminant of $(*)$. If D_2 is positive then $(*)$ has 3 solutions in $[0, \pi)$, and if D_2 is negative then $(*)$ has a unique solution in $[0, \pi)$. And these are corresponding to solutions $(*)$ and $(**)$ in $[0, 2\pi)$. Now we assume the following second generic condition :

Generic condition II. For any singularpoint with generic condition I, $D_2 \neq 0$.

8. There exist solutions of $(*)$ and $(**)$, and by rotating the coordinates, we may assume that one solution, say θ , is equal to 0. From these assumptions $(*)$ and $(**)$ imply

$$(*') \quad b' = 0,$$

$$(**') \quad a'' > 0.$$

the eigenvector spaces are invariant under multiplication of positive numbers to A .

Without loss of generalities, we may assume that $a' = 1$. By the condition (*'),

$$\begin{pmatrix} -s, c \end{pmatrix} \begin{pmatrix} c + a''s & b''s \\ b''s & -c - a''s \end{pmatrix} \begin{pmatrix} c \\ s \end{pmatrix} = 0, \\ (2 - b'')c^2s + 2a''cs^2 + b''s^3 = 0.$$

The discriminant is $D_2 = b''^2(a''^2 + (b'' - 1)^2 - 1)$.

And $D_1 = b$, we can classify singular points of symmetric bilinear form with generic conditions I and II, to the following three types.

- I. $D_1 < 0, D_2 > 0$.
- II. $D_1 > 0, D_2 > 0$.
- III. $D_1 > 0, D_2 < 0$.

9. Let $\eta(\theta)$ be the argument of the eigenvector space which corresponds to the positive eigenvalue. Its valuation is the followings. When θ increases to $\theta + 2\pi$, η decreases to $\eta - \pi$ in the case I, decreases to $\eta - 3\pi$ in the cases II and III, because the index depends on D_1 and is $1/2$ in the case I, $-1/2$ in the cases II and III, at a solution of (*) and (**) ($\eta \equiv 0 \pmod{\pi}$), the differential $\partial\eta/\partial\theta$ has following signature.

In the case I, $\partial\eta/\partial\theta < 0$ on three solutions in $[0, 2\pi)$.

In the case II, $\partial\eta/\partial\theta < 0$ on two solutions, and $\partial\eta/\partial\theta > 0$ at one solution.

In the case III, $\partial\eta/\partial\theta < 0$ at the unique solution.

By lemmas in the section 11, we can classify the topological types of the curve family defined by \exp^*A on the region between two solutions of (*) and (**). And lemma 4 gives the global topological equivalence for the representative curve families.

10. **Theorem I.** Topological types at the singular point 0 of curve families defined by linear symmetric bilinear forms are classified by D_1 and D_2 into three types I, II and III of the last section 9.

Before the proof of the theorem, we give some examples of curve families.

(1) Let $A = \begin{pmatrix} x & -y \\ -y & -x \end{pmatrix}$. The bilinear form A defines a curve family with a singular point of type I. Precisely the curve family is defined by $xdx + ydy = (x^2 + y^2)^{1/2} dx$, and solution curves are parabolic curves $y^2 = 2kx + k^2$ ($k > 0$), and $y = 0$ ($x > 0$). See Fig. I.

(2) Let $A = \begin{pmatrix} x & 3y \\ 3y & -x \end{pmatrix}$. The bilinear form A defines a curve family with a singular point of type II. See Fig. II.

(3) Let $A = \begin{pmatrix} x & y \\ y & -x \end{pmatrix}$. The bilinear form A defines a curve family with a singular point 0 of type III. See Fig. III.

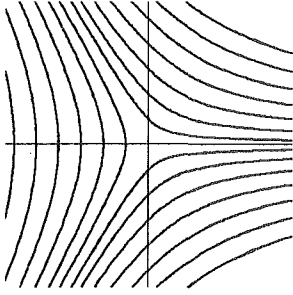


Fig. I

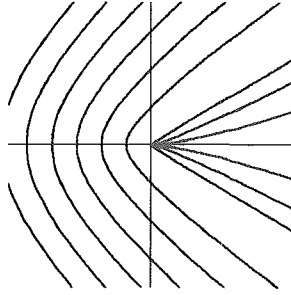


Fig. II

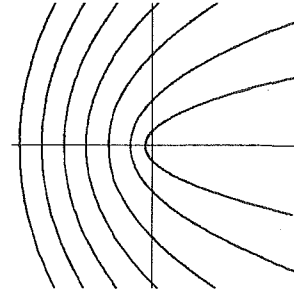


Fig. III

11. Let X be a smooth vector field defined on $\mathbf{R} \times [\theta_1, \theta_2]$.

Lemma 1. If X is represented by the form $(\eta, 1)$, then the integral curve family of X is topologically equivalent to the curve family $\{[\theta_1, \theta_2] \times k \subset [\theta_1, \theta_2] \times \mathbf{R} \mid k \in \mathbf{R}\}$, where vector field $(\eta, 1)$ means $\eta \partial / \partial \theta + \partial / \partial t$.

Proof. Let Φ be the integral flow of X , the following diffeomorphism f is the required topological equivalence.

$$f(t, \theta) = \Phi_{(\theta - \theta_1)}(t, \theta_1), \quad t \in \mathbf{R}, \theta \in [\theta_1, \theta_2].$$

The restriction of f to $(\theta_1 \times \mathbf{R})$ is the identity map and the restriction to $(\theta \times \mathbf{R})$ is an orientation preserving diffeomorphism onto itself for $\theta \in [\theta_1, \theta_2]$.

Lemma 2. Let X be a smooth vector field of type $(1, \xi)$, where ξ satisfies that $\xi(t, \theta_1) > 0$, $\xi(t, \theta_2) < 0$ and $\partial \xi / \partial \theta(t, \theta) < c$, $t \in \mathbf{R}$, $\theta \in [\theta_1, \theta_2]$, for some negative constant c . Then this integral curve family is equivalent to the integral curve family defined by the vector field $Y = (1, (\theta_1 + \theta_2)/2 - \theta)$.

Proof. We define a diffeomorphism f on $\mathbf{R} \times [\theta_1, \theta_2]$ such that

$$f(\Phi_s(t, \theta_1)) = \Psi_s(t, \theta_1), \quad t \geq 0,$$

and
$$f(\Phi_s(t, \theta_2)) = \Psi_s(t, \theta_2), \quad t \geq 0,$$

where Φ and Ψ are integral flows defined by X and Y respectively. If this map f is smoothly defined then f is the required topological equivalence.

Since $\{\Phi_s(t, \theta_1) \mid t \in \mathbf{R}, s \in [0, \infty)\}$, $(i=1, 2)$ are disjoint open sets of the region $\mathbf{R} \times [\theta_1, \theta_2]$, there exists an integral curve except for the curves starting from boundaries: $\theta = \theta_1$ or θ_2 . We show that this orbit uniquely exists. If two orbits through two points (t, θ_3) and (t, θ_4) , where $(\theta_3 < \theta_4)$, then these orbits $\Phi_s(t, \theta_3)$, $\Phi_s(t, \theta_4)$ are defined for any $s \in \mathbf{R}$, and $\Phi_s(t, \theta_1)$ is on $(t+s) \times [\theta_1, \theta_2]$.

By the assumption

$$|\xi(t, \theta_3) - \xi(t, \theta_4)| > c |\theta_3 - \theta_4|,$$

and

$$|\Phi_s(t, \theta_3) - \Phi_s(t, \theta_4)| > -c |\theta_3 - \theta_4| s.$$

For sufficiently small $s < 0$, the left side member is greater than $|\theta_2 - \theta_1|$. Then one of the orbit starts from a point of boundary.

Similarly for Ψ , and we may define f by the following

$$f(\Phi_s(t, \theta_3)) = \Psi_s(t, \theta_3).$$

Then f is defined on $\mathbf{R} \times [\theta_1, \theta_2]$ and it is easy to see that f is a homeomorphism onto itself. The mapping f is a required topological equivalence.

Lemma 3. Let X be a smooth vector field of type $(1, \xi)$, where $\xi(t, \theta_1) < 0$, $\xi(t, \theta_2) > 0$, $\partial\xi/\partial\theta(t, \theta) > c$, for some positive constant c , and $t \in \mathbf{R}$, $\theta \in [\theta_1, \theta_2]$. The integral curve family defined by X is topologically equivalent to the integral curve family defined by the vector field $Y = (1, -(\theta_1 + \theta_2)/2 + \theta)$.

The proof is similar to the proof of lemma 2.

Lemma 4. Let $Y = (1, -\theta)$ be a vector field defined on $\mathbf{R} \times [0, 1]$, and g be an orientation preserving diffeomorphism of \mathbf{R} such that $\{g(x) - x \mid x \in \mathbf{R}\}$ is bounded. There exists a topological equivalence f of the curve family defined by Y to itself such that the homeomorphism f coincides with the given diffeomorphism g on $\mathbf{R} \times \{1\}$.

Proof. At first step we make a smooth isotopy from the identity map of \mathbf{R} to g , $\varphi : \mathbf{R} \times [0, 1] \rightarrow \mathbf{R}$ such that $\partial\varphi/\partial t(t) > -1/t$. Let M be an upper bound of $(x - g(x))$, and $Y = (-1/(t + \delta), 1)$ a smooth vector field on $\mathbf{R} \times [0, 1]$ where δ is a positive constant. For sufficiently small δ (precisely $\delta < \min\{\exp(-M-1), 0.1\}$), the integral flow $\Phi_t(x, 0)$ satisfies that

$$\begin{aligned} \partial\Phi/\partial t &= -1/(t + \delta) - 1/t, \\ x - \Phi_{0.5}(x, 0) &< M. \end{aligned}$$

On the other hand we give an isotopy $\Psi : \mathbf{R} \times [1/2, 1] \rightarrow \mathbf{R}$

$$\Psi(x, t) = (2t - 1)g(x) + (2 - 2t)\Phi_{0.5}(x, 0).$$

Then

$$\begin{aligned} \partial\Psi/\partial t &= 2(g(x) - \Phi_{0.5}(x, 0)) \\ &> 2(x - M + M - x) = 0. \end{aligned}$$

Let $\{\mu_1, \mu_2\}$ be a partition of unity on $[0, 1]$ for the covering $\{[0, 1], (0, 1]\}$. We define the required isotopy

$$\Phi(x, t) = \mu_1(t)\Phi_t(x, 0) + \mu_2(t)\Psi(x, t).$$

Next we define the topological equivalence f . Let (x, t) be a point on the integral curve of Y starting at $(x_0, 1)$, we define $f(x, t)$ as the intersection point of the integral orbit of Y starting at $(g(x_0), 1)$ and the curve $\{(\Phi(x, s), s) \mid s \in [0, 1]\}$. Let (y, s) be an intersection point, we show that it is uniquely and smoothly determined. Since $(y,$

s) is on the integral orbit of Y starting at $(g(x_0), 1)$,

$$y - g(x_0) = -\log(s)$$

and $\Phi(x, s) = y$.

$$\text{Let } \alpha(s) = \Phi(x, s) - g(x_0) + \log(s),$$

then $\partial\alpha/\partial s = \partial\Phi/\partial s + 1/s > 0$,

$$\alpha(1) = g(x) - f(x_0) \geq 0,$$

and $\lim_{s \rightarrow \infty} \alpha(s) = -\infty$.

Therefore the intersection point is uniquely and smoothly determined. So $f(x, t) = (y, s)$ defines a diffeomorphism of $\mathbf{R} \times [0, 1]$, and we get the topological equivalence of the curve families.

12. In general let A be a smooth symmetric bilinear form on a neighbourhood of an isolated singular point $0 \in \mathbf{R}^2$, say

$$A = \begin{pmatrix} a & b \\ b & -a \end{pmatrix},$$

where a, b are smooth functions defined on some 0-neighbourhood and $a(0) = b(0) = 0$.

Let L be a linear approximation of A . That is

$$L = \begin{pmatrix} a'x + a''y & b'x + b''y \\ b'x + b''y & -a'x - a''y \end{pmatrix},$$

where $a' = \partial a / \partial x(0)$, $a'' = \partial a / \partial y(0)$, $b' = \partial b / \partial x(0)$ and $b'' = \partial b / \partial y(0)$.

$$\exp^* A = \exp^*(L + H),$$

where H is a bilinear form of higher order, and

$$\exp^* A = e^{3r} \begin{pmatrix} c & s \\ -s & c \end{pmatrix} L \begin{pmatrix} c & -s \\ s & c \end{pmatrix} + e^{4r} H'$$

where H' is a some smooth matrix.

For a sufficiently small r , eigenvector spaces of $\exp^* A$ are approximated by those of $\begin{pmatrix} c & s \\ -s & c \end{pmatrix} L \begin{pmatrix} c & -s \\ s & c \end{pmatrix}$. Since a simple root of a polynomial equation depends on its coefficients smoothly, if θ_0 is a root of (*) and (**) of $\exp^* L$ and if it satisfies one of the conditions of lemmas 1, 2, and 3, on some neighbourhood of θ , then $\exp^* A$ satisfies the same condition on $(-\infty, r_0) \times [\theta_1, \theta_2]$ for some r_0 . Therefore on the region, the curve family defined by $\exp^* A$ is topologically equivalent to the curve family defined by $\exp^* L$. Similarly to the linear case, the curve family is defined on the common isolated singular point 0.

Theorem II. Let M be a 2-dimensional Riemannian manifold and Q a smooth

symmetric bilinear form of M . If the linear part of Q satisfies generic conditions I and II in the section 5, at the isolated singular points of the curve family defined by Q , the curve family is locally equivalent to the one of the examples in section 10.

13. Let M be a compact 2-dimensional Riemannian manifold. For a given curve family without non-isolated singular points, the following index theorem is known.

Theorem III. [B_A] The index sum of all singular points of the curve family coincides with the Euler characteristic number $\chi(M)$:

$$\sum_{p \in S} \iota_p = \chi(M),$$

where S is the singular point set of the curve family.

Proof. Let S_0 be the subset of S which consists of all singular points with integral indices, and S_1 be that with nonintegral indices. Since a curve family is locally orientable on $M-S$, we may construct a covering space $\overline{M-S}$ for its orientations. On the neighbourhood of an isolated singular point with an integral index, the curve family is orientable, and $\overline{M-S}$ is extendable to $\overline{M_0} = \overline{M-S} \cup S_0 \cup S_0'$, where S_0' is a copy of S_0 . Now for some coordinates neighbourhood of a singular point p of S_1 , where p corresponds to 0. We give the following branched covering $\pi : \mathbf{R}^2 \longrightarrow \mathbf{R}^2$, $\pi(x, y) = (x^2 - y^2, 2xy)$.

Except for 0, it is a local diffeomorphism, and the pullback of the curve family exists and it is the orientation double covering of the foliation without 0. At the singular point $\bar{0}$, the index of the covering curve family is 2-times of the index at 0 minus 1. Therefore there exists a branched covering space for orientations, $\overline{M} = \overline{M_0} \cup S_1$ with the oriented curve family. The curve family has two times number of singular points of integral indices in \overline{M} , and has the same number of singular points of nonintegral indices as in M .

Then the total index of singular points on \overline{M} is

$$\iota(\overline{M}) = 2 \sum_{p \in S_0} \iota(p) + \sum_{p \in S_1} (2\iota(p) - 1).$$

And the Euler characteristic number of \overline{M} is

$$\chi(\overline{M}) = 2\chi(M) - \#(S_1).$$

By the index theorem of Hopf $\iota(\overline{M}) = \chi(\overline{M})$, we have

$$\sum_{p \in M} \iota(p) = \chi(M).$$

14. **Remark.** Let M be a compact 2-dimensional Riemannian manifold. For a given flow X on M , by forgetting the orientation of integral curve family of the flow, we get a curve family with singular points. This curve family is equal to the curve family defined by the following symmetric bilinear form :

$$A(u, v) = \langle u, X \rangle \langle v, X \rangle.$$

Eigenvalues of this bilinear form are 0 and $|X|^2$, so if $X \neq 0$ then X is a tangent vector to the corresponding curve family. But all the singular points of the curve family do not satisfy the generic condition I.

For example, let X be a following vector field with singular point 0 defined by

$$X = x\partial/\partial x - \partial/\partial y,$$

then the related bilinear form is

$$A = \begin{pmatrix} x^2 - y^2 & -2xy \\ -2xy & -x^2 + y^2 \end{pmatrix}.$$

D_1 of A is 0 at 0, but by deforming this matrix for sufficiently small $\varepsilon > 0$, we get

$$A + \varepsilon E = \begin{pmatrix} x^2 - y^2 + \varepsilon & -2xy \\ -2xy & -x^2 + y^2 + \varepsilon \end{pmatrix},$$

which defines a curve family with two generic singular points of index $-1/2$.

15. Peixoto shows in [P_E] the condition for the structural stability of a flow on a 2-dimensional closed manifold. Analogously we define the structural stability of a curve family. Let $Q(M)$ be the set of all smooth bilinear forms on a 2-dimensional closed Riemannian manifold M , which has the compact-open topology.

For $Q \in Q(M)$, if there exists the Q -neighbourhood U_Q in $Q(M)$ and any bilinear form on U_Q defines a curve family which is topologically equivalent to the curve family defined by Q , we call it a structurally stable curve family. Except for singular points, the curve family is locally orientable, and the same method holds and the same properties are given as in [P_E]. We define a singular curve (corresponds to the separatrix in [P_E]) of curve family. If some one side closure of the curve contains the singular point p , and there exists a enoughly near curve of curve family such that the same one side closure do not contain p , then we call a curve in the curve family a separatrix of the singular point p . In examples of the section 11, a singular point of type I, II and III has 3, 2 and 1 separatrices, respectively. We may modify the condition of having no saddle connection to the condition 3 in the following theorem. The conditions of generic singular points are given in last sections. Therefore we get the following theorem.

Theorem IV. If the curve family defined by a bilinear form on a 2-dimensional closed Riemannian manifold, satisfies following conditions, then the curve family is structurally stable.

1. All singular points of the curve family are generic in the sense of generic conditions I and II.

2. Any one side limit set of a curve is a closed curve or a singular point.

3. Separatrices in common with double sides don't exist in the curve family.
4. The number of closed curves is finite, and by giving some orientation for its neighbourhood, closed curves are stable periodic orbits in the sense of flow in [P_E].

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