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Realization of automorphisms σ of order 3 and G^{σ} of compact exceptional Lie groups G, I, $G=G_2, F_4, E_6$

Dedicated to Professor Nagayoshi Iwahori on his 60th birthday

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J. A. Wolf and A. Gray [1] classified automorphisms σ of order 3 and the fixed subgroups G^{σ} of connected compact simple Lie groups G of centerfree. In this paper, we find these automorphisms σ and realize G^{σ} for simply connected compact exceptional Lie groups $G=G_2$, F_4 and E_6. (As for E_7 and E_8, they will appear in the next issue). Our result is the following second column. The first column is the chart of involutive automorphisms and the fixed subgroups which are connected our cases.

G_2	γ	$(Sp(1)\times Sp(1))/\boldsymbol{Z}_2$	γ_3	$(U(1)\times Sp(1))/\boldsymbol{Z}_2$
	———		w	$SU(3)$
F_4	γ	$(Sp(1)\times Sp(3))/\boldsymbol{Z}_2$	γ_3	$(U(1)\times Sp(3))/\boldsymbol{Z}_2$
	σ	$Spin(9)$	σ_3	$(U(1)\times Spin(7))/\boldsymbol{Z}_2$
	———		w	$(SU(3)\times SU(3))/\boldsymbol{Z}_3$
E_6	γ	$(Sp(1)\times \mathrm{SU}(6))/\boldsymbol{Z}_2$	γ_3	$(U(1)\times \mathrm{SU}(6))/\boldsymbol{Z}_2$
			γ_3'	$(Sp(1)\times \mathrm{S}(\mathrm{U}(1)\times \mathrm{U}(5)))/\boldsymbol{Z}_2$
	σ	$(\mathrm{U}(1)\times Spin(10))/\boldsymbol{Z}_4$	σ_3	$(\mathrm{U}(1)\times U(1)\times Spin(8))/(\boldsymbol{Z}_4\times \boldsymbol{Z}_2)$
			σ_3'	$(\mathrm{U}(1)\times Spin(10))/\boldsymbol{Z}_4$
	———		w	$(SU(3)\times SU(3)\times SU(3))/\boldsymbol{Z}_3$

Notations. (1) Let G be a group and σ an automorphism of G. G^{σ} denotes $\{g \in G \mid \sigma g=g\}$. If σ is an inner automorphism $\mathrm{Ad}s$ induced by $s \in G$, $G^{\mathrm{Ad}s}$ is briefly denoted by G^s: $G^s=\{g \in G \mid sg=gs\}$. Moreover, for a subset S of G, the centralizer of S in G is denoted by G^S: $G^S=\{g \in G \mid sg=gs$ for all $s \in S\}$.

(2) When two groups G, G' are isomorphic: $G \cong G'$, we often identify these groups: $G=G'$.

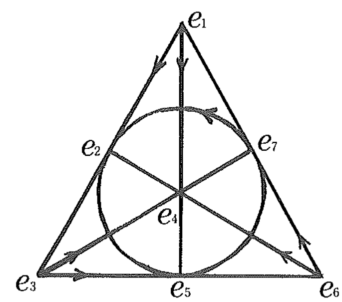
(3) For an $\boldsymbol{R}$-vector space V, its complexification $\{u+iv \mid u, v \in V\}$ is denoted by V^{C}. The complex conjugation in V^{C} is denoted by τ: $\tau(u+iv)=u-iv$.

(4) The definitions of classical Lie groups $U(n)$, $SU(n)$ and $Sp(n)$, $n=1, 3$

appeared in this paper are usual ones: $U(n)=\{A \in M(n, \mathbf{C}) \mid A^*A=E\}$, $SU(n)=\{A \in U(n) \mid \det A=1\}$ and $Sp(n)=\{A \in M(n, \boldsymbol{H}) \mid A^*A=E\}$.

1. The group G_2

Let $\mathfrak{C}=\sum_{i=0}^{7} \boldsymbol{R}e_i$ be the Cayley division algebra with the multiplication such that $e_0=1$ is the unit, $e_i{}^2=-1$, $1 \leqq i \leqq 7$, $e_ie_j=-e_je_i, 1 \leqq i \neq j \leqq 7$ and $e_1e_2=e_3$, $e_3e_5=e_6$, $e_2e_5=e_7$ etc. .... In $\mathfrak{C}$, the conjugation $\bar{x}$, the inner product (x, y) and the length $|x|$ are naturally defined. The Cayley algebra $\mathfrak{C}$ contains the field of real numbers $\boldsymbol{R}$ naturally, furthermore the fields of complex numbers $\boldsymbol{C}$, $\boldsymbol{C}_1$ and quaternions $\boldsymbol{H}$:



$$\boldsymbol{C}=\{\xi+\eta e_4 \mid \xi, \eta \in \boldsymbol{R}\}, \quad \boldsymbol{C}_1=\{\xi+\eta e_1 \mid \xi, \eta \in \boldsymbol{R}\},$$
$$\boldsymbol{H}=\{\xi+\xi_1e_1+\xi_2e_2+\xi_3e_3 \mid \xi, \xi_i \in \boldsymbol{R}\}.$$

Hereafter e_4 is briefly denoted by e.

The automorphism group G_2 of the Cayley algebra $\mathfrak{C}$,

$$G_2=\{\alpha \in \mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{C}, \mathfrak{C}) \mid \alpha(xy)=(\alpha x)(\alpha y)\}$$

is a simply connected compact simple Lie group of type G_2 [8]. To find some subgroups of G_2, we will give alternative definitions of the Cayley algebra $\mathfrak{C}$.

1. In $\mathfrak{C}=\boldsymbol{H} \oplus \boldsymbol{H}e$, we define a multiplication, a conjugation $\overline{}$ and an inner product $(\ ,\ )$ respectively by

$$(a+be)(c+de)=(ac-\bar{d}b)+(b\bar{c}+da)e,$$
$$\overline{a+be}=\bar{a}-be,$$
$$(a+be, c+de)=(a, c)+(b, d).$$

2. In $\mathfrak{C}=\boldsymbol{C} \oplus \boldsymbol{C}^3$, we define a multiplication etc. by

$$(a+\boldsymbol{m})(b+\boldsymbol{n})=(ab-\overline{\boldsymbol{m}^*\boldsymbol{n}})+(a\boldsymbol{n}+\bar{b}\boldsymbol{m}+\overline{\boldsymbol{m}\times\boldsymbol{n}}),$$
$$\overline{a+\boldsymbol{m}}=\bar{a}-\boldsymbol{m},$$
$$(a+\boldsymbol{m}, b+\boldsymbol{n})=(a, b)+(\boldsymbol{m}, \boldsymbol{n})$$

where $\boldsymbol{m}\times\boldsymbol{n} \in \boldsymbol{C}^3$ is the exterior product of $\boldsymbol{m}, \boldsymbol{n} \in \boldsymbol{C}^3$ and $(\boldsymbol{m}, \boldsymbol{n})=\frac{1}{2}(\boldsymbol{m}^*\boldsymbol{n}+\boldsymbol{n}^*\boldsymbol{m})$.

1.1. Automorphism γ_3 of order 3 and subgroup $(U(1)\times Sp(1))/\boldsymbol{Z}_2$ of G_2

We define an $\boldsymbol{R}$-linear transformation γ of $\mathfrak{C}$ by

$$\gamma(a+be)=a-be, \quad a+be \in \boldsymbol{H} \oplus \boldsymbol{H}e=\mathfrak{C}.$$

Then we have $\gamma \in G_2$ and $\gamma^2=1$.

Known result 1.1 [2]. *The group $(G_2)^\gamma$ is isomorphic to the group $(Sp(1)\times Sp$*

$(1))/\boldsymbol{Z}_2$ $(\cong SO(4))$ *by an isomorphism induced from the homomorphism* $\psi: Sp(1)\times Sp(1)\to(G_2)^{\gamma}$,

$$\psi(p, q)(a+be)=qa\bar{q}+(pb\bar{q})e, \quad a+be \in \boldsymbol{H}\oplus\boldsymbol{H}e=\mathfrak{C}$$

with $\mathrm{Ker}\,\psi=\boldsymbol{Z}_2=\{(1, 1), (-1, -1)\}$.

Let $\omega_1=-\frac{1}{2}+\frac{\sqrt{3}}{2}e_1 \in Sp(1) \subset \boldsymbol{H} \subset \mathfrak{C}$. Denote $\psi(\omega_1, 1)$ by γ_3:

$$\gamma_3(a+be)=a+(\omega_1 b)e, \quad a+be \in \boldsymbol{H}\oplus\boldsymbol{H}e=\mathfrak{C}.$$

Of course $\gamma_3 \in G_2$ and $\gamma_3{}^3=1$.

Theorem 1. 2. *The group* $(G_2)^{\gamma_3}$ *is isomorphic to the group* $(U(1)\times Sp(1))/\boldsymbol{Z}_2$ $(\cong U(2))$ *where* $\boldsymbol{Z}_2=\{(1, 1), (-1, -1)\}$.

Proof. Let $U(1)=\{s \in \boldsymbol{C}_1 \mid |s|=1\} \subset Sp(1) \subset \boldsymbol{H} \subset \mathfrak{C}$. We define a homomorphism $\psi: U(1)\times Sp(1)\to(G_2)^{\gamma_3}$ by the restriction of ψ of Known result 1. 1. Clearly $\gamma_3\psi(s, q)=\psi(s, q)\gamma_3$ for $(s, q) \in U(1)\times Sp(1)$, so ψ is well-defined. We shall show that ψ is onto. Let $\alpha \in(G_2)^{\gamma_3}$. Since α commutes with γ_3, $\mathfrak{C}_{\gamma_3}=\{x \in \mathfrak{C} \mid \gamma_3 x=x\}=\boldsymbol{H}$ is invariant under α. So α also commutes with γ: $\alpha \in (G_2)^{\gamma}$. Hence, from Known result 1. 1, there exist $s, q \in Sp(1)$ such that $\alpha=\psi(s, q)$. From the commutativity $\gamma_3\alpha=\alpha\gamma_3$, that is, $\psi(\omega_1 s, q)=\psi(s\omega_1, q)$, we have $(\omega_1 s, q)=\pm(s\omega_1, q)$, so $\omega_1 s=s\omega_1$, therefore $s \in U(1)$. Hence ψ is onto. Obviously $\mathrm{Ker}\,\psi=\boldsymbol{Z}_2$. Thus we have the isomorphism $(U(1)\times Sp(1))/\boldsymbol{Z}_2\cong(G_2)^{\gamma_3}$.

Corollary 1. 3. $(G_2)^{\gamma_3}=(G_2)^S$ *where* $S=\psi(U(1), 1)$. *In particular, the manifold* $G_2/(G_2)^{\gamma_3}$ *has a homogeneous complex structure.*

1. 2. Automorphism w of order 3 and subgroup $SU(3)$ of G_2

Let $\omega=-\frac{1}{2}+\frac{\sqrt{3}}{2}e \in \boldsymbol{C} \subset \mathfrak{C}$. We define an $\boldsymbol{R}$-linear transformation w of $\mathfrak{C}$ by

$$w(a+\boldsymbol{m})=a+\omega\boldsymbol{m}, \quad a+\boldsymbol{m} \in \boldsymbol{C}\oplus\boldsymbol{C}^3=\mathfrak{C}.$$

Then we have $w \in G_2$ and $w^3=1$.

Remark. We have the following

Proposition 1. 4. *For* $a\in\mathfrak{C}$ *such that* $|a|=1$, *the condition that the mapping* α_a : $\mathfrak{C}\to\mathfrak{C}$, $\alpha_a x=ax\bar{a}$ *belongs to the group* G_2 *is* $a^3=\pm 1$.

Now, w is nothing but the mapping $\alpha_{\bar{\omega}}$: $wx=\bar{\omega}x\omega$, $x \in \mathfrak{C}$.

Known result 1. 5 [7], [8]. *The group* $(G_2)_e=\{\alpha \in G_2 \mid \alpha e=e\}$ *is isomorphic to the group* $SU(3)$ *by the isomorphism* $\psi: SU(3)\to(G_2)_e$,

$$\psi(A)(a+\boldsymbol{m})=a+A\boldsymbol{m}, \quad a+\boldsymbol{m} \in \boldsymbol{C}\oplus\boldsymbol{C}^3=\mathfrak{C}.$$

Theorem 1. 6. *The group* $(G_2)^w$ *coincides with the group* $(G_2)_e$, *so it is isomorphic to the group* $SU(3)$.

Proof. We shall show $(G_2)^w=(G_2)_e$. Clearly $(G_2)_e=\psi(SU(3)) \subset (G_2)^w$. Conversely, let $\alpha \in(G_2)^w$. Since α commutes with w, $\mathfrak{C}_w=\{x \in \mathfrak{C} \mid wx=x\}=\boldsymbol{C}$ is invariant under α. So α induces an automorphism of $\boldsymbol{C}$, hence

$$\alpha e = e \quad \text{or} \quad \alpha e = -e.$$

In the latter case, consider a mapping $\gamma: \mathfrak{C} \to \mathfrak{C}$, $\gamma(a+\boldsymbol{m}) = \bar{a} + \overline{\boldsymbol{m}}$. Then $\gamma \in G_2$ and $\gamma e = -e$. (This γ is the same one as γ of the preceding section 1. 1). Put $\beta = \gamma\alpha$. Since $\beta e = e$, we have $\beta \in (G_2)_e \subset (G_2)^w$. Therefore $\gamma = \beta\alpha^{-1} \in (G_2)^w$. However this is a contradiction. In fact, $\omega\overline{\boldsymbol{m}} = w\overline{\boldsymbol{m}} = w(\gamma\boldsymbol{m}) = \gamma(\omega\boldsymbol{m}) = \overline{\omega\boldsymbol{m}} = \overline{\omega}\,\overline{\boldsymbol{m}}$ for all $\boldsymbol{m} \in \boldsymbol{C}^3$ which is false. Hence $\alpha e = e$, so $\alpha \in (G_2)_e$. Thus we have $(G_2)^w \subset (G_2)_e$.

2. The group F_4

Let $\mathfrak{J} = \{X \in M(3, \mathfrak{C}) \mid X^* = X\}$ be the exceptional Jordan algebra with the Jordan multiplication

$$X \circ Y = \frac{1}{2}(XY + YX).$$

In $\mathfrak{J}$, we define a positive definite inner product (X, Y) by $\mathrm{tr}(X \circ Y)$. Moreover, in $\mathfrak{J}$, we define a multiplication $X \times Y$ called the Freudenthal multiplication, a trilinear form (X, Y, Z) and the determinant $\det X$ respectively by

$$X \times Y = \frac{1}{2}(2X \circ Y - \mathrm{tr}(X)Y - \mathrm{tr}(Y)X + (\mathrm{tr}(X)\mathrm{tr}(Y) - (X, Y))E),$$
$$(X, Y, Z) = (X, Y \times Z), \quad \det X = \frac{1}{3}(X, X, X).$$

The algebra $\mathfrak{J}$ with the multiplication $X \times Y$ and the inner product (X, Y) will be called the Freudenthal algebra.

The automorphism group F_4 of the Jordan algebra $\mathfrak{J}$,

$$\begin{aligned} F_4 &= \{\alpha \in \mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{J}, \mathfrak{J}) \mid \alpha(X \circ Y) = \alpha X \circ \alpha Y\} \\ &= \{\alpha \in \mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{J}, \mathfrak{J}) \mid \det\alpha X = \det X, (\alpha X, \alpha Y) = (X, Y)\} \\ &= \{\alpha \in \mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{J}, \mathfrak{J}) \mid \alpha(X \times Y) = \alpha X \times \alpha Y\} \end{aligned}$$

is a simply connected compact simple Lie group of type F_4 [3], [8]. The group F_4 contains G_2 as a subgroup naturally, that is, any $\alpha \in G_2$ is regarded as $\alpha \in F_4$ by

$$\alpha \begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix} = \begin{pmatrix} \xi_1 & \alpha x_3 & \overline{\alpha x_2} \\ \overline{\alpha x_3} & \xi_2 & \alpha x_1 \\ \alpha x_2 & \overline{\alpha x_1} & \xi_3 \end{pmatrix}.$$

To find some subgroups of F_4, we will give alternative definitions of Freudenthal algebra $\mathfrak{J}$. For $K = \boldsymbol{R}, \boldsymbol{C}$, let $\mathfrak{J}_K = \mathfrak{J}(3, K) = \{X \in M(3, K) \mid X^* = X\}$ be the Freudenthal algebra with the multiplication $X \times Y$ and the inner product (X, Y) as analogous to ones in $\mathfrak{J}$.

1. In $\mathfrak{J} = \mathfrak{J}(3, \boldsymbol{H}) \oplus \boldsymbol{H}^3$ (where $\boldsymbol{H}^3 = \{(a_1, a_2, a_3)$ "row vector" $\mid a_i \in \boldsymbol{H}\}$), we define a multiplication and an inner product respectively by

$$(X + \boldsymbol{a}) \times (Y + \boldsymbol{b}) = (X \times Y - \frac{1}{2}(\boldsymbol{a}^*\boldsymbol{b} + \boldsymbol{b}^*\boldsymbol{a})) - \frac{1}{2}(\boldsymbol{a}Y + \boldsymbol{b}X),$$
$$(X + \boldsymbol{a}, Y + \boldsymbol{b}) = (X, Y) + 2(\boldsymbol{a}, \boldsymbol{b})$$

where $(\boldsymbol{a}, \boldsymbol{b})=\frac{1}{2}(\boldsymbol{a}\boldsymbol{b}^*+\boldsymbol{b}\boldsymbol{a}^*)=\frac{1}{2}\operatorname{tr}(\boldsymbol{a}^*\boldsymbol{b}+\boldsymbol{b}^*\boldsymbol{a})$.

2. In $\mathfrak{J}=\mathfrak{J}(3, \boldsymbol{C})\oplus M(3, \boldsymbol{C})$, we define a multiplication etc. by

$$(X+M)\times(Y+N)=(X\times Y-\frac{1}{2}(M^*N+N^*M))-\frac{1}{2}(MY+NX+\overline{M\times N})$$
$$(X+M, Y+N)=(X, Y)+2(M, N)$$

where, for $M=(\boldsymbol{m}_1, \boldsymbol{m}_2, \boldsymbol{m}_3)$, $N=(\boldsymbol{n}_1, \boldsymbol{n}_2, \boldsymbol{n}_3)\in M(3, \boldsymbol{C})$, $M\times N\in M(3, \boldsymbol{C})$ is defined by

$$M\times N=\begin{pmatrix} \boldsymbol{m}_2\times\boldsymbol{n}_3 & \boldsymbol{m}_3\times\boldsymbol{n}_1 & \boldsymbol{m}_1\times\boldsymbol{n}_2 \\ + & + & + \\ \boldsymbol{n}_2\times\boldsymbol{m}_3 & \boldsymbol{n}_3\times\boldsymbol{m}_1 & \boldsymbol{n}_1\times\boldsymbol{m}_2 \end{pmatrix}$$

and $(M, N)=\frac{1}{2}\operatorname{tr}(M^*N+N^*M)$.

2. 1. Automorphism γ_3 of order 3 and subgroup $(U(1)\times Sp(3))/\boldsymbol{Z}_2$ of F_4

We consider $\boldsymbol{R}$-linear transformations γ, γ_3 of $\mathfrak{J}$ which are extensions of $\gamma, \gamma_3\in G_2$ to F_4 respectively. Of course $\gamma, \gamma_3\in F_4$ and $\gamma^2=1$, $\gamma_3{}^3=1$.

Known result 2. 1 [5]. *The group $(F_4)^\gamma$ is isomorphic to the group $(Sp(1)\times Sp(3))/\boldsymbol{Z}_2$ by an isomorphism induced from the homomorphism $\psi: Sp(1)\times Sp(3)\to(F_4)^\gamma$,*

$$\psi(p, A)(X+\boldsymbol{a})=AXA^*+p\boldsymbol{a}A^*, \quad X+\boldsymbol{a}\in\mathfrak{J}(3, \boldsymbol{H})\oplus\boldsymbol{H}^3=\mathfrak{J}$$

with $\operatorname{Ker}\psi=\boldsymbol{Z}_2=\{(1, E), (-1, -E)\}$.

Theorem 2. 2. *The group $(F_4)^{\gamma_3}$ is isomorphic to the group $(U(1)\times Sp(3))/\boldsymbol{Z}_2$ where $\boldsymbol{Z}_2=\{(1, E), (-1, -E)\}$.*

Proof. Let $U(1)=\{s\in C_1 \mid |s|=1\}\subset Sp(1)\subset\boldsymbol{H}\subset\mathfrak{C}$. We define a homomorphism $\psi: U(1)\times Sp(3)\to(F_4)^{\gamma_3}$ by the restriction of ψ of Known result 2. 1. Then ψ induces an isomorphism $(U(1)\times Sp(3))/\boldsymbol{Z}_2\cong(F_4)^{\gamma_3}$ whose proof is similar to Theorem 1. 2.

Corollary 2. 3. $(F_4)^{\gamma_3}=(F_4)^S$ *where* $S=\psi(U(1), 1)$. *In particular, the manifold $F_4/(F_4)^{\gamma_3}$ has a homogeneous complex structure.*

2. 2. Automorphism σ_3 of order 3 and subgroup $(U(1)\times Spin(7))/\boldsymbol{Z}_2$ of F_4

Let $U(1)=\{a\in\boldsymbol{C} \mid |a|=1\}$. For $a\in U(1)$, we define an $\boldsymbol{R}$-linear transformation D_a of $\mathfrak{J}$ by

$$D_a\begin{pmatrix} \xi_1 & x_3 & \overline{x}_2 \\ \overline{x}_3 & \xi_2 & x_1 \\ x_2 & \overline{x}_1 & \xi_3 \end{pmatrix}=\begin{pmatrix} \xi_1 & x_3a & \overline{ax_2} \\ \overline{x_3a} & \xi_2 & \overline{a}x_1\overline{a} \\ ax_2 & a\overline{x}_1a & \xi_3 \end{pmatrix}$$

Then we have $D_a\in F_4$. Denote D_{-1} by σ. Of course $\sigma\in F_4$ and $\sigma^2=1$.

Hereafter we use the following notations in $\mathfrak{J}$ [6].

$$E_1=\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix},\quad E_2=\begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix},\quad E_3=\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix},$$
$$F_1(x)=\begin{pmatrix}0&0&0\\0&0&x\\0&\bar{x}&0\end{pmatrix},\quad F_2(x)=\begin{pmatrix}0&0&\bar{x}\\0&0&0\\x&0&0\end{pmatrix},\quad F_3(x)=\begin{pmatrix}0&x&0\\\bar{x}&0&0\\0&0&0\end{pmatrix}.$$

Known result 2.4. [4], [8]. *The group $(F_4)^{\sigma}$ coincides with the group $(F_4)_{E_1}=\{\alpha \in F_4 \mid \alpha E_1=E_1\}$, so it is isomorphic to the group $Spin(9)$ which is the universal covering group of $SO(9)=SO(V^9)$ where $V^9=\{X \in \mathfrak{J} \mid E_1\circ X=0,\ \mathrm{tr}(X)=0\}$.*

Let $\omega=-\frac{1}{2}+\frac{\sqrt{3}}{2}e \in U(1) \subset \boldsymbol{C} \subset \mathfrak{C}$ and denote D_ω by σ_3. Of course $\sigma_3 \in F_4$ and $\sigma_3{}^3=1$. To investigate the group $(F_4)^{\sigma_3}$, we consider $\boldsymbol{R}$-vector subspaces $\mathfrak{J}_{\sigma_3}$, $(\mathfrak{J}_{\sigma_3})^{\perp}$ of $\mathfrak{J}$:

$$\mathfrak{J}_{\sigma_3}=\{X \in \mathfrak{J} \mid \sigma_3X=X\}=\{\xi_1E_1+\xi_2E_2+\xi_3E_3+F_1(t) \mid \xi_i \in \boldsymbol{R},\ t \in \boldsymbol{C}^{\perp}\},$$
$$(\mathfrak{J}_{\sigma_3})^{\perp}=\text{the orthogonal complement of } \mathfrak{J}_{\sigma_3} \text{ in } \mathfrak{J}$$
$$=\{F_1(s)+F_2(x_2)+F_3(x_3) \mid s \in \boldsymbol{C},\ x_i \in \mathfrak{C}\}$$

where $\boldsymbol{C}^{\perp}$ is the orthogonal complement of $\boldsymbol{C}$ in $\mathfrak{J}$. Then $\mathfrak{J}=\mathfrak{J}_{\sigma_3}\oplus(\mathfrak{J}_{\sigma_3})^{\perp}$ and $\mathfrak{J}_{\sigma_3}$, $(\mathfrak{J}_{\sigma_3})^{\perp}$ are invariant under the group $(F_4)^{\sigma_3}$.

Lemma 2.5. *For $\alpha \in (F_4)^{\sigma_3}$, we have $\alpha E_1=E_1$. Hence $(F_4)^{\sigma_3}$ is a subgroup of $(F_4)_{E_1}=Spin(9)$.*

Proof is similar to [6, Lemma 9], however we need some modifications. To show $\alpha E_2 \in \mathfrak{J}(2, \mathfrak{C})=\{\xi_2E_2+\xi_3E_3+F_1(x) \mid \xi_i \in \boldsymbol{R},\ x \in \mathfrak{C}\}$, put $\alpha E_2=\xi_1E_1+\xi_2E_2+\xi_3E_3+F_1(t)$, $\xi_i \in \boldsymbol{R}$, $t \in \boldsymbol{C}^{\perp}$ and suppose $\xi_1 \neq 0$. From $\alpha E_2 \times \alpha E_2=0$, we see that $\xi_2=\xi_3=t=0$, that is, $\alpha E_2=\xi_1E_1$. Next use $\alpha E_2 \times \alpha F_1(1)=0$, then we see that $\alpha F_1(1)=\eta E_1$ for some $0 \neq \eta \in \boldsymbol{R}$ which contradicts to $\alpha E_2=\xi_1E_1$. Hence $\xi_1=0$. Thus we have $\alpha E_2 \in \mathfrak{C}(2, \mathfrak{J})$. Similarly $\alpha E_3 \in \mathfrak{J}(2, \mathfrak{C})$. Therefore $\alpha E_1 \notin \mathfrak{J}(2, \mathfrak{C})$ moreover $\alpha E_1=\xi E_1$ by the same argument of [6, Lemma 9]. Finally from the relation $\alpha E_1 \circ \alpha E_1=\alpha E_1$, ξ must be 1. Thus we have $\alpha E_1=E_1$.

From Lemma 2.5, we see that $\boldsymbol{R}$-vector subspaces

$$\{\xi_2E_2+\xi_3E_3+F_1(t) \mid \xi_i \in \boldsymbol{R},\ t \in \boldsymbol{C}^{\perp}\},\ \{F_2(x_2)+F_3(x_3) \mid x_i \in \mathfrak{C}\},\quad \{F_1(s) \mid s \in \boldsymbol{C}\}$$

of $\mathfrak{J}$ are invariant under the group $(F_4)^{\sigma_3}$.

We define a subgroup $(F_4)_{E_1, F_1(s)}$ of the group F_4 by

$$(F_4)_{E_1, F_1(s)}=\{\alpha \in F_4 \mid \alpha E_1=E_1,\ \alpha F_1(s)=F_1(s) \text{ for all } s \in \boldsymbol{C}\}$$
$$=\{\alpha \in Spin(9) \mid \alpha F_1(1)=F_1(1),\ \alpha F_1(e)=F_1(e)\}.$$

This group $(F_4)_{E_1, F_1(s)}$ is isomorphic to the group $Spin(7)$ which is the universal covering group of $SO(7)=SO(V^7)$ where $V^7=\{\xi(E_2-E_3)+F_1(t) \mid \xi \in \boldsymbol{R},\ t \in \boldsymbol{C}^{\perp}\}$. Furthermore we use the following notation.

$$(F_4)^{U(1)} = \{\alpha \in F_4 \mid D_a\alpha = \alpha D_a \text{ for all } a \in U(1)\}.$$

Lemma 2.6. *$Spin(7) = (F_4)_{F_1, F_1(s)}$ is a subgroup of $(F_4)^{U(1)}$.*

Proof. Let $\beta \in Spin(7)$. Then for D_a, $a \in U(1)$ we have

$$\begin{aligned}\beta D_a F_1(z) &= \beta F_1(\bar{a}z\bar{a}) = \beta F_1(\bar{a}^2 s + t) \quad (z = s+t,\ s \in \boldsymbol{C},\ t \in \boldsymbol{C}^{\perp}) \\ &= F_1(\bar{a}^2 s) + \beta F_1(t) = F_1(\bar{a}^2 s) + (\xi_2 E_2 + \xi_3 E_3 + F_1(t'))\end{aligned}$$

(for some $\xi_i \in \boldsymbol{R}$, $t' \in \boldsymbol{C}^{\perp}$). On the other hand,

$$\begin{aligned}D_a\beta F_1(z) &= D_a\beta F_1(s+t) = D_a(F_1(s) + \beta F_1(t)) \\ &= D_a(F_1(s) + \xi_2 E_2 + \xi_3 E_3 + F_1(t')) = F_1(\bar{a}^2 s) + \xi_2 E_2 + \xi_3 E_3 + F_1(t')).\end{aligned}$$

Thus we have $\beta D_a F_1(z) = D_a \beta F_1(z)$, $z \in \mathfrak{C}$. Next, for $z \in \mathfrak{C}$,

$$\begin{aligned}\beta D_a F_2(z) &= \beta F_2(az) = 4\beta(F_1(1) \times F_2(z)) \times F_1(\bar{a}) = 4(F_1(1) \times F_2(z)) \times F_1(\bar{a}) \\ &= 4(F_1(1) \times (F_2(x_2) + F_3(x_3))) \times F_1(\bar{a}) \quad (\text{for some } x_i \in \mathfrak{C}) \\ &= F_2(ax_2) + F_3(x_3 a) = D_a(F_2(x_2) + F_3(x_3)) = D_a\beta F_2(z).\end{aligned}$$

Similarly $\beta D_a F_3(z) = D_a \beta F_3(z)$. Clearly $D_a\beta = \beta D_a$ on E_1. Finally

$$D_a\beta E_2 = D_a(\xi_2 E_2 + \xi_3 E_3 + F_1(t)) = \xi_2 E_2 + \xi_3 E_3 + F_1(t) = \beta E_2 = \beta D_a E_2$$

(for some $\xi_i \in \boldsymbol{R}$, $t \in \boldsymbol{C}^{\perp}$). Similarly $D_a\beta E_3 = \beta D_a E_3$. Thus we have $D_a\beta = \beta D_a$, that is, $\beta \in (F_4)^{U(1)}$.

Theorem 2.7. *The group $(F_4)^{\sigma_3}$ is isomorphic to the group $(U(1) \times Spin(7))/\boldsymbol{Z}_2$ where $\boldsymbol{Z}_2 = \{(1, 1), (-1, -1)\}$.*

Proof. We define a mapping $\psi : U(1) \times Spin(7) \to (F_4)^{\sigma_3}$ by

$$\psi(a, \beta) = D_a\beta.$$

Obviously ψ is well-defined: $\psi(a, \beta) \in (F_4)^{\sigma_3}$ (Lemma 2.6). Since $D_a (a \in U(1))$ and $\beta \in Spin(7)$ commute (Lemma 2.6), ψ is a homomorphism. We shall show that ψ is onto. Let $\alpha \in (F_4)^{\sigma_3}$. Put $\alpha F_1(1) = F_1(s_0)$, $s_0 \in \boldsymbol{C}$. Then we have

$$\alpha F_1(\omega) = \alpha F_1(\bar{\omega} 1 \bar{\omega}) = \alpha D_{\omega} F_1(1) = D_{\omega}\alpha F_1(1) = D_{\omega} F_1(s_0) = F_1(\omega s_0), \tag{1}$$

$$\alpha F_1(\bar{\omega}) = \alpha D_{\omega} D_{\omega} F_1(1) = D_{\omega} D_{\omega}\alpha F_1(1) = D_{\omega} D_{\omega} F_1(s_0) = F_1(\bar{\omega} s_0). \tag{2}$$

Taking (1)$-$(2), we have $\alpha F_1(e) = F_1(es_0)$. Now, choose $a_0 \in \boldsymbol{C}$ such that $\bar{a}_0{}^2 = s_0$. Then

$$\alpha F_1(1) = F_1(s_0) = F_1(\bar{a}_0{}^2) = D_{a_0} F_1(1), \qquad \alpha F_1(e) = F_1(es_0) = F_1(\bar{a}_0{}^2 e) = D_{a_0} F_1(e).$$

Put $\beta = D_{a_0}{}^{-1}\alpha$, then $\beta F_1(1) = F_1(1)$, $\beta F_1(e) = F_1(e)$ and $\beta E_1 = E_1$ (Lemma 2.5), so $\beta \in Spin(7)$. Thus we have

$$\alpha = D_{a_0}\beta, \quad D_{a_0} \in U(1),\ \beta \in Spin(7),$$

that is, ψ is onto. Obviously $\mathrm{Ker}\,\psi = \boldsymbol{Z}_2$. Thus we have the isomorphism $(U(1) \times Spin(7))/\boldsymbol{Z}_2 \cong (F_4)^{\sigma_3}$.

Corollary 2.8. *$(F_4)^{\sigma_3}=(F_4)^S$ where $S=\psi(U(1), 1)$. In particular, the manifold $F_4/(F_4)^{\sigma_3}$ has a homogeneous complex structure.*

2.3. Automorphism w of order 3 and subgroup $(SU(3)\times SU(3))/Z_3$ of F_4

Let $\omega=-\frac{1}{2}+\frac{\sqrt{3}}{2}e \in C \subset \mathfrak{C}$ and we define an R-linear transformation w of $\mathfrak{J}$ by

$$w(X+M)=X+\omega M, \quad X+M \in \mathfrak{J}(3, C)\oplus M(3, C) = \mathfrak{J}.$$

This w is the same one as $w \in G_2 \subset F_4$. Of course $w^3=1$.

Theorem 2.9. *The group $(F_4)^w$ is isomorphic to the group $(SU(3)\times SU(3))/Z_3$ where $Z_3=\{(E, E), (\omega E, \omega E), (\omega^2 E, \omega^2 E)\}$.*

Proof. We define a mapping $\psi: SU(3)\times SU(3)\to(F_4)^w$ by

$$\psi(P, A)(X+M)=AXA^*+PMA^*, \quad X+M \in \mathfrak{J}(3, C)\oplus M(3, C)=\mathfrak{J}.$$

ψ is well-defined: $\psi(P, A) \in F_4$ [6] moreover $\in(F_4)^w$. Obviously ψ is a homomorphism. We shall show that ψ is onto. Let $\alpha \in(F_4)^w$. Since the restriction α' of α to $\mathfrak{J}_w=\{X\in \mathfrak{J} \mid wX=X\}= \mathfrak{J}(3, C)$ belongs to the group $F_{4,C}=\{\alpha \in \mathrm{Iso}_R(\mathfrak{J}_C, \mathfrak{J}_C) \mid \alpha(X \circ Y)=\alpha X \circ \alpha Y\}$, there exists $A \in SU(3)$ such that

$$\alpha X=AXA^* \quad \text{or} \quad \alpha X=A\bar{X}A^*, \quad X \in \mathfrak{J}(3, C)$$

[7]. In the former case, put $\beta=\psi(E, A)^{-1}\alpha$, then $\beta \mid \mathfrak{J}(3, C)=1$. Hence $\beta \in G_2$, moreover $\beta \in(G_2)_e=(G_2)^w$ (Theorem 1.1)$=SU(3)$. Hence there exists $P \in SU(3)$ such that

$$\beta(X+M)=X+PM=\psi(P, E)(X+M), \quad X+M\in \mathfrak{J}_C\oplus M(3, C)=\mathfrak{J}.$$

Therefore we have $\alpha=\psi(E, A)\beta=\psi(E, A)\psi(P, E)=\psi(P, A)$. In the latter case, consider the mapping $\gamma: \mathfrak{J}\to \mathfrak{J}$, $\gamma(X+M)=\bar{X}+\bar{M}$, $X+M \in \mathfrak{J}$ and recall $\gamma \in G_2 \subset F_4$. Put $\beta=\alpha^{-1}\psi(E, A)\gamma$, then $\beta \in F_4$ and $\beta \mid \mathfrak{J}_C=1$. Hence $\beta \in(G_2)_e=(G_2)^w \subset (F_4)^w$. Since β, α, $\psi(E, A) \in(F_4)^w$, γ also $\in(F_4)^w$, so $\gamma \in(G_2)^w$ which is a contradiction (Theorem 1.6). Thus we see that ψ is onto. $\mathrm{Ker}\,\psi=Z_3$ is easily obtained. Thus we have the isomorphism $(SU(3)\times SU(3))/Z_3 \cong(F_4)^w$.

3. The group E_6

Let $\mathfrak{J}^C=\{X_1+iX_2 \mid X_i \in \mathfrak{J}\}$(called the complex exceptional Jordan algebra) be the complexification of $\mathfrak{J}$. As in $\mathfrak{J}$, in $\mathfrak{J}^C$ also, we define multiplications $X \circ Y$, $X\times Y$, the inner product (X, Y), the trilinear form (X, Y, Z) and the determinant $\det X$. Finally, in $\mathfrak{J}^C$, we define a positive definite Hermitian inner pruduct $<X, Y>$ by $(\tau X, Y)$.

The group

$$E_6=\{\alpha \in \mathrm{Iso}_C(\mathfrak{J}^C, \mathfrak{J}^C) \mid \det\alpha X=\det X, <\alpha X, \alpha Y>=<X, Y>\}$$
$$=\{\alpha \in \mathrm{Iso}_C(\mathfrak{J}^C, \mathfrak{J}^C) \mid (\alpha X, \alpha Y, \alpha Z)=(X, Y, Z), <\alpha X, \alpha Y>=<X, Y>\}$$

$$=\{\alpha \in \mathrm{Iso}_{C}(\mathfrak{J}^C, \mathfrak{J}^C) \mid \alpha X \times \alpha Y = \tau\alpha\tau(X\times Y),\ <\alpha X, \alpha Y> = <X, Y>\}$$

is a simply connected compact simple Lie group of type E_6 [6]. For $\alpha \in F_4$, its complexification $\alpha^C : \mathfrak{J}^C \to \mathfrak{J}^C$ belongs to E_6, so we can regard F_4 as a subgroup of E_6 under the complexification.

3. 1. Automorphism γ_3 of order 3 and subgroup $(U(1)\times SU(6))/Z_2$ of E_6

We consider C-linear transformations γ, γ_3 of $\mathfrak{J}^C$ which are the complexifications of γ, $\gamma_3 \in G_2 \subset F_4$, respectively. Of course γ, $\gamma_3 \in E_6$ and $\gamma^2=1$, $\gamma_3{}^3=1$.

Let $C=R^C=\{\xi_0+i\xi_1 \mid \xi_i \in R\}$ and we define an R-linear mapping $k : H \to M(2, C)$ by

$$k((\xi_0+\xi_1 e_1)+e_2(\xi_2+\xi_3 e_1)) = \begin{pmatrix} \xi_0+i\xi_1 & -\xi_2+i\xi_3 \\ \xi_2+i\xi_3 & \xi_0-i\xi_1 \end{pmatrix}, \quad \xi_i \in R.$$

This k is naturally extended to R-linear mappings

$$k : M(3,H) \to M(6, C), \quad k : H^3 \to M(2, 6, C).$$

Moreover these k are extended to C-linear isomorphisms $k : M(3, H)^C \to M(6, C)$, $k : (H^3)^C \to M(2, 6, C)$ respectively by

$$k(X_1+iX_2) = k(X_1)+ik(X_2), \quad X_i \in M(3, H),$$
$$k(a_1+ia_2) = k(a_1)+ik(a_2), \quad a_i \in H^3.$$

Finally, we define a C-vector space $\mathfrak{S}(6, C)$ by

$$\mathfrak{S}(6, C)=\{S \in M(6, C) \mid {}^tS=-S\}$$

and a C-linear isomorphism $k_J : \mathfrak{J}(3, H)^C \to \mathfrak{S}(6, C)$ by

$$k_J(X_1+iX_2)=k(X_1)J+ik(X_2)J, \quad X_i \in \mathfrak{J}(3, H)$$

where $J=\begin{pmatrix} J' & 0 & 0 \\ 0 & J' & 0 \\ 0 & 0 & J' \end{pmatrix}$, $J'=\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Known result 3. 1. [6]. *The group $(E_6)^\gamma$ is isomorphic to the group $(Sp(1)\times SU(6))/Z_2$ by an isomorphism induced from the homomorphism $\psi : Sp(1)\times SU(6) \to (E_6)^\gamma$,*

$$\psi(p, A)(X+a)=k_J{}^{-1}(Ak_J(X){}^tA)+pk^{-1}(k(a)A^*), \quad X+a \in \mathfrak{J}_H{}^C \oplus (H^3)^C = \mathfrak{J}^C$$

with $\mathrm{Ker}\,\psi = Z_2 = \{(1, E), (-1, -E)\}$.

Theorem 3. 2. *The group $(E_6)^{\gamma_3}$ is isomorphic to the group $(U(1)\times SU(6))/Z_2$ where $Z_2=\{(1, E), (-1, -E)\}$.*

Proof. Let $U(1)=\{s \in C_1 \mid |s|=1\} \subset Sp(1) \subset H \subset \mathfrak{C}$. We define a homomorphism $\psi : U(1)\times SU(6) \to (E_6)^{\gamma_3}$ by the restriction of ψ of Known result 3. 1. Then ψ induces an isomorphism $(U(1)\times SU(6))/Z_2 \cong (E_6)^{\gamma_3}$ whose proof is similar to Theorems 1. 2, 2. 2.

Corollary 3.3. $(E_6)^{\gamma_3}=(E_6)^S$ *where* $S=\psi(U(1),1)$. *In particular, the manifold* $E_6/(E_6)^{\gamma_3}$ *has a homogeneous complex structure.*

3.2. Automorphism γ_3' of order 3 and subgroup $(Sp(1)\times S(U(1)\times U(5)))/Z_2$ of E_6

Let $v=\exp\dfrac{2\pi i}{9}\in C$ and put $A_v=\begin{pmatrix} v^5 & & & \\ & v^{-1} & & \\ & & \ddots & \\ & & & v^{-1}\end{pmatrix}\in SU(6)\subset M(6,C)$. Put $\gamma'=\psi(1,A_v)$ where ψ is the mapping $\psi: Sp(1)\times SU(6)\to(E_6)^\gamma$ defined in Known result 3.1. Of course $\gamma'\in E_6$ and $\gamma'^9=1$. Since $A_v{}^3=v^6E\in z(SU(6))$ (the center of SU(6)) and $\psi(1,A_v{}^3)=\omega 1$ (where $\omega=-\dfrac{1}{2}+i\dfrac{\sqrt{3}}{2}\in C$) $\in z(E_6)$ (the center of E_6), γ' induces an automorphism γ_3' of E_6 of order 3,

$$\gamma_3'(\alpha)=\gamma'\alpha\gamma'^{-1},\quad \alpha\in E_6.$$

In order to investigate the group $(E_6)^{\gamma_3'}$, we consider C-eigen vector subspaces $(\mathfrak{J}^C)_{v^i}$, $i=0,1,\dots,8$ of $\mathfrak{J}^C$ with respect to γ':

$$(\mathfrak{J}^C)_v=\{X+\boldsymbol{a}\in\mathfrak{J}_H{}^C\oplus(\boldsymbol{H}^3)^C\mid \gamma'(X+\boldsymbol{a})=v(X+\boldsymbol{a})\}$$
$$=\{0+(a_1(e_1-i),a_2,a_3)\mid a_1\in\boldsymbol{H},\ a_2,a_3\in\boldsymbol{H}^C\},$$
$$(\mathfrak{J}^C)_{v^4}=\{X+\boldsymbol{a}\in\mathfrak{J}_H{}^C\oplus(\boldsymbol{H}^3)^C\mid \gamma'(X+\boldsymbol{a})=v^4(X+\boldsymbol{a})\}$$
$$=\left\{\begin{pmatrix}\xi_1 & (e_1+i)a_3 & \overline{a_2(e_1-i)}\\ \overline{(e_1+i)a_3} & 0 & 0\\ a_2(e_1-i) & 0 & 0\end{pmatrix}+(a_1(e_1+i),0,0)\ \middle|\ \begin{matrix}\xi_1\in\boldsymbol{R}\\ a_i\in\boldsymbol{H}\end{matrix}\right\},$$
$$(\mathfrak{J}^C)_{v^7}=\{X+\boldsymbol{a}\in\mathfrak{J}_H{}^C\oplus(\boldsymbol{H}^3)^C\mid \gamma'(X+\boldsymbol{a})=v^7(X+\boldsymbol{a})\}$$
$$=\left\{\begin{pmatrix}0 & (e_1-i)a_3 & \overline{a_2(e_1+i)}\\ \overline{(e_1-i)a_3} & \xi_2 & a_1\\ a_2(e_1+i) & \overline{a_1} & \xi_3\end{pmatrix}+0\ \middle|\ \begin{matrix}\xi_2,\xi_3\in\boldsymbol{R}\\ a_1\in\boldsymbol{H}^C,\ a_2,a_3\in\boldsymbol{H}\end{matrix}\right\},$$
$$(\mathfrak{J}^C)_{v^i}=\{X+\boldsymbol{a}\in\mathfrak{J}_H{}^C\oplus(\boldsymbol{H}^3)^C\mid \gamma'(X+\boldsymbol{a})=v^i(X+\boldsymbol{a})\}$$
$$=\{0\},\quad i=0,2,3,5,6,8.$$

These spaces are invariant under the group $(E_6)^{\gamma_3'}$.

Theorem 3.4. *The group* $(E_6)^{\gamma_3'}$ *is isomorphic to the group* $(Sp(1)\times S(U(1)\times U(5)))/\boldsymbol{Z}_2$ *where* $\boldsymbol{Z}_2=\{(1,(1,E)),(1,(-1,-E))\}$.

Proof. First we shall show that $(\boldsymbol{H}^3)^C$ is invariant under the group $(E_6)^{\gamma_3'}$. From the form of $(\mathfrak{J}^C)_{v^i}$, it is sufficient to show that we have $\alpha\boldsymbol{a}\in(\boldsymbol{H}^3)^C$ for $\alpha\in(E_6)^{\gamma_3'}$, $\boldsymbol{a}=(a(e_1+i),0,0)=F_1((a(e_1+i)))$, $a\in\boldsymbol{H}$. Now, in fact,

$$\alpha F_1((a(e_1+i))e)=-4\alpha((F_1(1)\times F_3((e_1-i)\bar{a}))\times F_3(e))$$
$$=-4((\alpha F_1(1)\times\alpha F_3((e_1-i)\bar{a}))\times\tau\alpha\tau F_3(e)$$
$$\subset-4(\mathfrak{J}_H{}^C\times\mathfrak{J}_H{}^C)\times(\boldsymbol{H}^3)^C\subset\mathfrak{J}_H{}^C\times(\boldsymbol{H}^3)^C\subset(\boldsymbol{H}^3)^C.$$

Thus we see that $(H^3)^C$ is invariant under the group $(E_6)^{\gamma_3'}$, hence $\mathfrak{J}_H{}^C=((H^3)^C)^{\perp}=\{X \in \mathfrak{J}^C \mid \langle X, Y\rangle=0$ for all $Y \in (H^3)^C\}$ is also invariant under $(E_6)^{\gamma_3'}$. Consequently, $\alpha \in (E_6)^{\gamma_3'}$ commutes with γ: $(E_6)^{\gamma_3'} \subset (E_6)^{\gamma}$. Now, we define a homomorphism $\psi: Sp(1)\times S(U(1)\times U(5))\to(E_6)^{\gamma_3'}$ by the restriction of ψ of Known result 3. 1. Clearly ψ is well-defined. We shall show that ψ is onto. Let $\alpha \in (E_6)^{\gamma_3'}$. Since $(E_6)^{\gamma_3'} \subset (E_6)^{\gamma}$, from Known result 3. 1, there exist $p \in Sp(1)$, $A \in SU(6)$ such that $\alpha=\psi(p, A)$. From the commutativity $\gamma_3'\alpha=\alpha\gamma_3'$, that is, $\psi(p, A_v A)=\psi(p, AA_v)$, we have $A_v A=AA_v$. Hence $A \in S(U(1)\times U(5))(\cong U(5))$. Thus ψ is onto. Obviously $\mathrm{Ker}\,\psi=Z_2$. Thus we have the isomorphism $(Sp(1)\times S(U(1)\times U(5)))/Z_2\cong(E_6)^{\gamma_3'}$.

Corollary 3. 5. $(E_6)^{\gamma_3'}=(E_6)^S$ *where* $S=\{\psi(1, A)\mid A=\begin{pmatrix} a^5 & & & \\ & a & & \\ & & \ddots & \\ & & & a\end{pmatrix} \in SU(6), a \in U(1)\}$. *In particular, the manifold* $E_6/(E_6)^{\gamma_3'}$ *has a homogeneous complex structure.*

3. 3. Automorphism σ_3 of order 3 and subgroup $(U(1)\times U(1)\times Spin(8))/(Z_4\times Z_2)$ of E_6

Let $U(1)=\{\theta \in C \mid |\theta|=1\}$ and we define an imbedding $\phi: U(1)\to E_6$ by

$$\phi(\theta)\begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3\end{pmatrix}=\begin{pmatrix} \theta^4\xi_1 & \theta x_3 & \theta\bar{x}_2 \\ \theta\bar{x}_3 & \theta^{-2}\xi_2 & \theta^{-2}x_1 \\ \theta x_2 & \theta^{-2}\bar{x}_1 & \theta^{-2}\xi_3\end{pmatrix}.$$

Now, we regard σ, $\sigma_3 \in F_4$ as elements of E_6. Of course $\sigma^2=1$, $\sigma_3{}^3=1$.

Known result 3. 6 [6]. (1) *The group* $(E_6)_{E_1}=\{\alpha \in E_6\mid \alpha E_1=E_1\}$ *is isomorphic to the group* $Spin(10)$ *which is the universal covering group of* $SO(10)=SO(V^{10})$ *where* $V^{10}=\{X \in \mathfrak{J}^C\mid 2E_1\times X=-\tau X\}$.

(2) *The group* $(E_6)^{\sigma}$ *is isomorphic to the group* $(U(1)\times Spin(10))/Z_4$ *by an isomorphism induced from the homomorphism* $\psi: U(1)\times Spin(10)\to(E_6)^{\sigma}$,

$$\psi(\theta, \beta)=\phi(\theta)\beta$$

with $\mathrm{Ker}\,\psi=Z_4=\{(1, \phi(1)), (-1, \phi(-1)), (i, \phi(i)), (-i, \phi(-i))\}$.

Lemma 3. 7. *For* $\alpha \in (E_6)^{\sigma_3}$, *there exists* $\xi \in U(1)$ *such that* $\alpha E_1=\xi E_1$.

Proof is similar to Lemma 2. 5 and see [6, Lemma 9].

We define a subgroup $(E_6)_{E_1, F_1(s)}$ of the group E_6 by

$$\begin{aligned}(E_6)_{E_1, F_1(s)}&=\{\alpha \in E_6\mid \alpha E_1=E_1,\ \alpha F_1(s)=F_1(s) \text{ for all } s \in C\}\\ &=\{\alpha \in Spin(10)\mid \alpha F_1(1)=F_1(1),\ \alpha F_1(e)=F_1(e)\}.\end{aligned}$$

This group $(E_6)_{E_1, F_1(s)}$ is isomorphic to the group $Spin(8)$ which is the universal covering group of $SO(8)=SO(V^8)$ where $V^8=\{\xi E_2-\tau\xi E_3+F_1(t)\mid \xi \in C,\ t \in C^{\perp}\}$. Furthermore we use the following notation.

$$(E_6)^{U(1)}=\{\alpha \in E_6\mid D_a\alpha=\alpha D_a \text{ for all } a \in U(1)\}.$$

Lemma 3. 8. *$Spin(8)=(E_6)_{E_1, F_1(s)}$ is a subgroup of $(E_6)^{U(1)}$.*

Proof is similar to Lemma 2. 6.

Theorem 3. 9. *The group $(E_6)^{\sigma_3}$ is isomorphic to the group $(\mathrm{U}(1)\times U(1)\times Spin(8))/(\boldsymbol{Z}_4\times\boldsymbol{Z}_2)$ where $\boldsymbol{Z}_4=\{(1, 1, 1), (\mathrm{i}, e, \phi(\mathrm{i})D_e), (-1, -1, 1), (-\mathrm{i}, -e, \phi(\mathrm{i})D_e)\}$ and $\boldsymbol{Z}_2=\{(1, 1, 1), (1, -1, \sigma)\}$.*

Proof. We define a mapping $\psi : \mathrm{U}(1)\times U(1)\times Spin(8)\to(E_6)^{\sigma_3}$ by

$$\psi(\theta, a, \beta)=\phi(\theta)D_a\beta.$$

Obviously ψ is well-defined : $\psi(\theta, a, \beta)\in(E_6)^{\sigma_3}$(Lemma 3. 7). Since $\phi(\theta)(\theta\in\mathrm{U}(1))$, $D_a\in U(1)$ and $\beta\in Spin(8)$ commute with one another (Lemma 3. 8), ψ is a homomorphism. We shall show that ψ is onto. Let $\alpha\in(E_6)^{\sigma_3}$. From Lemma 3. 7, there exists $\theta\in\mathrm{U}(1)$ such that

$$\alpha E_1=\theta^4E_1=\phi(\theta)E_1.$$

Put $\beta=\phi(\theta)^{-1}\alpha$, then $\beta E_1=E_1$, that is, $\beta\in((E_6)^{\sigma_3})_{E_1}=\{\alpha\in(E_6)^{\sigma_3}\mid \alpha E_1=E_1\}$. From Lemma 3. 7, we see that the vector space

$$\{F_1(s)\mid s\in\boldsymbol{C}\}=\{X\in((\mathfrak{J}^C)_{\sigma_3})^{\perp}\mid E_1\times X=0, <E_1, X>=0, 2E_1\times X=-\tau X\}$$

is invariant under the group $((E_6)^{\sigma_3})_{E_1}$. So we can put $\beta F_1(1)=F_1(s_0)$, $s_0\in\boldsymbol{C}$. Then we have also $\beta F_1(e)=F_1(es_0)$ (cf. Theorem 2. 7). Choose $a_0\in\boldsymbol{C}$ such that $\bar{a}_0{}^2=s_0$. Then $\beta F_1(1)=D_{a_0}F_1(1)$, $\beta F_1(e)=D_{a_0}F_1(e)$. Put $\delta=D_{a_0}{}^{-1}\beta$, then $\delta\in Spin(8)$. Hence we have

$$\alpha=\phi(\theta)D_{a_0}\beta,\quad \theta\in\mathrm{U}(1),\ a_0\in U(1),\ \delta\in Spin(8).$$

Thus ψ is onto. Finally we shall determine $\mathrm{Ker}\,\psi$. Let $\phi(\theta)D_a\delta=1$, $\theta\in\mathrm{U}(1)$, $a\in U(1)$, $\delta\in Spin(8)$. From $\phi(\theta)D_a\delta E_1=E_1$, we have $\theta^4=1$. Hence $\theta=\pm1, \pm\mathrm{i}$. In the case of $\theta=1$, from $D_a\delta F_1(1)=F_1(1)$, we have $F_1(\bar{a}^2)=F_1(1)$. so $a^2=1$. Therefore $a=1$, $\delta=1$ or $a=-1$, $\delta=D_{-1}=\sigma$. So $(1, 1, 1), (1, -1, \sigma)\in\mathrm{Ker}\,\psi$. In other cases of θ, we can similarly determine elements of $\mathrm{Ker}\,\psi$. Thus

$$\begin{aligned}\mathrm{Ker}\,\psi=\{&(1, 1, 1), (\mathrm{i}, e, \phi(\mathrm{i})D_e), (-1, -1, 1), (-\mathrm{i}, -e, \phi(\mathrm{i})D_e),\\ &(1, -1, \sigma), (-\mathrm{i}, -e, \phi(\mathrm{i})D_e), (-1, 1, \sigma), (-\mathrm{i}, e, \phi(-\mathrm{i})D_e)\}\\ =&<(\mathrm{i}, e, \phi(\mathrm{i})D_e)>\times<(1, -1, \sigma)>=\boldsymbol{Z}_4\times\boldsymbol{Z}_2.\end{aligned}$$

Thus we have the isomorphism $(\mathrm{U}(1)\times U(1)\times Spin(8))/(\boldsymbol{Z}_4\times\boldsymbol{Z}_2)\cong(E_6)^{\sigma_3}$.

Corollary 3. 10. *$(E_6)^{\sigma_3}=(E_6)^{S_1}$ where $S_1=\psi(1, U(1), 1)$*
$=(E_6)^{S_2}$ where $S_2=\psi(\mathrm{U}(1), U(1), 1)$.

In particular, the manifold $E_6/(E_6)^{\sigma_3}$ has a homogeneous complex structure.

3. 4. Automorphism σ_3' of order 3 and subgroup $(\mathrm{U}(1)\times Spin(10))/\boldsymbol{Z}_4$ of E_6

Let $\phi : \mathrm{U}(1)\to E_6$ be the imbedding defined in Known result 3. 6. Now, let $\nu=\exp$

$\frac{2\pi i}{9} \in \mathbf{C}$ and denote $\phi(v)$ by σ'. Of course $\sigma' \in E_6$ and $\sigma'^9=1$. Since $\sigma'^3=\omega 1 \in z(E_6)$, σ' induces an automorphism σ_3' of E_6 of order 3,

$$\sigma_3'(\alpha)=\sigma'\alpha\sigma'^{-1}, \quad \alpha \in E_6.$$

Theorem 3.11. *The group $(E_6)^{\sigma_3'}$ coincides with the group $(E_6)^{\sigma}$, so it is isomorphic to the group* $(\mathrm{U}(1)\times Spin(10))/\boldsymbol{Z}_4$.

Proof. Since

$$\sigma_3' \begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix} = \begin{pmatrix} v^4\xi_1 & vx_3 & v\bar{x}_2 \\ v\bar{x}_3 & v^{-2}\xi_2 & v^{-2}x_1 \\ vx_2 & v^{-2}\bar{x}_1 & v^{-2}\xi_3 \end{pmatrix},$$

C-vector subspaces $\{\xi E_1 \mid \xi \in \mathbf{C}\}$, $\{F_2(x_2)+F_3(x_3) \mid x_i \in \mathfrak{C}^C\}$ and $\{\xi_2E_2+\xi_3E_3+F_1(x) \mid \xi_i \in \mathbf{C},\ x \in \mathfrak{C}^C\}$ of $\mathfrak{J}^C$ are invariant under the group $(E_6)^{\sigma_3'}$. In particular, $\alpha \in (E_6)^{\sigma_3'}$ commutes with σ: $(E_6)^{\sigma_3'} \subset (E_6)^{\sigma}$. The converse inclusion $(E_6)^{\sigma} \subset (E_6)^{\sigma_3'}$ is clear because $(E_6)^{\sigma}=\phi(\mathrm{U}(1))Spin(10)$. Thus we have $(E_6)^{\sigma_3'}=(E_6)^{\sigma}\cong(\mathrm{U}(1)\times Spin(10))/\boldsymbol{Z}_4$.

Corollary 3.12. $(E_6)^{\sigma_3'}=(E_6)^{S}$ *where* $S=\psi(\mathrm{U}(1), 1)$. *In particular, the manifold $E_6/(E_6)^{\sigma_3'}$ has a homogeneous complex structure.*

3.5. Automorphism w of order 3 and subgroup $(SU(3)\times SU(3)\times SU(3))/\boldsymbol{Z}_3$ of E_6

Let $\omega=-\frac{1}{2}+\frac{\sqrt{3}}{2}e \in \boldsymbol{C} \subset \mathfrak{J}$ and we define a C-linear transformation w of $\mathfrak{J}^C$ by

$$w(X+M)=X+\omega M, \quad X+M \in \mathfrak{J}(3, \boldsymbol{C})^C \oplus M(3, \boldsymbol{C})^C=\mathfrak{J}^C.$$

This w is the same one as $w \in G_2 \subset F_4 \subset E_6$. Of course $w^3=1$.

Theorem 3.5. *The group $(E_6)^w$ is isomorphic to the group $(SU(3)\times SU(3)\times SU(3))/\boldsymbol{Z}_3$ where $\boldsymbol{Z}_3=\{(1, E, E), (\omega 1, \omega E, \omega E), (\omega^2 1, \omega^2 E, \omega^2 E)\}$.*

Proof. We define a mapping $\psi: SU(3)\times SU(3)\times SU(3)\to(E_6)^w$ by

$$\psi(P, A, B)(X+M)=h(A, B)Xh(A, B)^*+PM\tau h(A, B)^*,$$
$$X+M \in \mathfrak{J}(3, \boldsymbol{C})^C \oplus M(3, \boldsymbol{C})^C=\mathfrak{J}^C$$

where $h: M(3, \boldsymbol{C})\times M(3, \boldsymbol{C})\to M(3, \boldsymbol{C})^C$ is the mapping defined by $h(A, B)=\frac{A+B}{2}+i\frac{(A-B)e}{2}$. ψ is well-defined: $\psi(P, A, B) \in E_6$ [7] moreover $\in (E_6)^w$. Obviously ψ is a homomorphism. The proof that ψ is onto is similar to Theorem 2.9. Thus we have the isomorphism $(SU(3)\times SU(3)\times SU(3))/\boldsymbol{Z}_3\cong(E_6)^w$.

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