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Strong Limits of M-subharmonic Functions

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1. Introduction

On the radial limits and strong limits of subharmonic functions in the unit disc U in $\mathbf{C}$, J.E. Littlewood studied thoroughly in [1], [2], [3]. He summed up his results in [3], §10, p.234:

Theorem A (Littlewood). *Suppose that u is a subharmonic function in U and satisfies*

(1)
$$\sup_{1/2\leqq r<1} \frac{1}{2\pi}\int_0^{2\pi} |u(re^{i\theta})|^p d\theta<\infty$$

for some p, $0<p<\infty$. When $p\geqq1$, there is almost always a radial limit

$$u^*(e^{i\theta})=\lim_{r\to1} u(re^{i\theta})$$

and u^* is a strong limit of u for any index $q<p$. When $p>1$, there need be no strong limit of u with index p, but u^* is one under the stronger hypothesis that u is harmonic. When $p=1$, there need be no strong limit of u with index $p(=1)$, even when u is harmonic. When $p<1$, there need be no strong limit for any index, nor need a radial limit exist almost everywhere (it need not exist outside a null set of θ).

D. Ullrich [6] has recently defined and developed the basic properties of the class of "M-subharmonic" functions in the unit ball B of $\mathbf{C}^n$. The main result of [6] is as follows:

Theorem B (Ullrich). *An M-subharmonic function in B satisfying an appropriate growth condition (analogous to (1) with $p=1$) has radial limits almost everywhere.*

In the case $n=1$ Theorem B is just the part of Theorem A, which relates to the radial limits. The purpose of the present paper is to prove that the part of Theorem A relating to the strong limits remains true for M-subharmonic functions. (See below, p. 7, Theorem 3. 4.)

2. Preliminaries

Throughout this paper n is a positive integer, $\mathbf{C}^n$ is the vector space of all

n-tuples $z=(z_1, \ldots, z_n)$ of complex numbers, with hermitian inner product $<z, w> =\sum z_j\bar{w}_j$, norm $|z|=<z, z>^{1/2}$, and corresponding unit ball

$$B=\{z\in\mathbf{C}^n : |z|<1\},$$

whose boundary is the sphere

$$S=\{\zeta\in\mathbf{C}^n : |\zeta|=1\}.$$

For z, $a\in B$, $a\neq 0$, define

$$\phi_a(z)=\frac{a-P_a z-(1-|a|^2)^{1/2}\, Q_a z}{1-<z, a>}$$

where $P_a z=<z, a>a/<a, a>$, and $P_a z+Q_a z=z$. By continuity, let $\phi_0(z)=-z$.

The group of biholomorphic mappings of B onto B will be denoted by M. Let $\mathfrak{U}=\mathfrak{U}(n)$ be the group of all unitary operators on the Hilbert space $\mathbf{C}^n$. Then $\mathfrak{U}$ is a compact group and $\mathfrak{U}\subset M$. Actually, $\mathfrak{U}=\{\psi\in M : \psi(0)=0\}$. Note that $\phi_a\in M$ for all $a\in B$.

Let σ denote the $\mathfrak{U}$-invariant positive Borel measure on S, normalized so that $\sigma(S)=1$. τ denotes the measure in B defined by

$$d\tau(z)=(1-|z|^2)^{-n-1}\, d\nu(z),$$

where ν is the Lebesgue measure on $\mathbf{C}^n=\mathbf{R}^{2n}$, normalized so that $\nu(B)=1$. Note that τ is M-invariant: For $f\in L^1(\tau)$ and $\psi\in M$, $\int_B f\circ\psi\, d\tau=\int_B f\, d\tau$ ([5], Theorem 2.2.6).

As usual, $C(X)$ stands for the class of all complex-valued continuous functions on X, where X is any topological space.

We shall also use the following notations:

$$rB=\{rz : z\in B\},\ r\bar{B}=\{rz : z\in\bar{B}\},\ rS=\{r\zeta : \zeta\in S\}$$

for $0<r<1$, where $\bar{B}=\{z\in\mathbf{C}^n : |z|\leq 1\}$.

2. 1. **Definition** ([6], §1.15). Suppose Ω is an open subset of B and u: $\Omega\to[-\infty, \infty)$ is upper semicontinious. Then u is *M-subharmonic* in Ω if (i) for every $a\in\Omega$ there exists $r(a)>0$ with $\phi_a(r(a)\bar{B})\subset\Omega$, such that

$$u(a)\leqq\int_S u(\phi_a(r\zeta))\, d\sigma(\zeta) \quad (0\leqq r\leqq r(a))$$

and (ii) none of the integrals in (i) are $-\infty$.

2. 2. **Definition** ([6], §1.1 and §1.5). Suppose $\Omega\subset B$ is open and $u\in C(\Omega)$. Then u is *M-harmonic* in Ω if

$$u(a)=\int_S u(\phi_a(r\zeta))\, d\sigma(\zeta) \tag{2}$$

for any $a\in\Omega$ and $r>0$ such that $\phi_a(r\bar{B})\subset\Omega$.

2. 3. Proposition ([6], §1.8). *If u is M-harmonic in Ω, an open subset of B, and $\phi_a(r\overline{B})\subset\Omega$, then*

$$u(a)=\frac{1}{\tau(\phi_a(rB))}\int_{\phi_a(rB)} u\, d\tau.$$

2. 4. Definition. For $0<t<1$, put

$$\rho(t)=\sup\left\{|\phi_z(w)| : z\in tS,\ w\in\frac{1}{2}B\right\}.$$

If $z\in tS$, then $z=\phi_z(0)\in\phi_z\left(\frac{1}{2}B\right)$, and so $t=|z|<\rho(t)<1$. We define

$$c(t)=\frac{\tau(\rho(t)B)}{\tau\left(\frac{1}{2}B\right)}$$

for $0<t<1 : 0<c(t)<\infty$.

2. 5. Lemma (cf. [6], §1.9). *Suppose $0<t<1$ and $\rho(t)<r<1$. If $u\geqq 0$ is an M-harmonic function in rB, then $u(z)\leqq c(t)u(0)$ for all $z\in tS$.*

Proof. Fix $z\in tS$. By the definition of the number $\rho(t)$,

$$\phi_z({}^1/_2\overline{B})\subset\rho(t)\overline{B}\subset rB.$$

It follows from Proposition 2.3 that

$$\begin{aligned}u(z)&=\frac{1}{\tau(\phi_z({}^1/_2B))}\int_{\phi_z({}^1/_2B)} u\, d\tau\\ &\leqq\frac{1}{\tau({}^1/_2B)}\int_{\rho(t)B} u\, d\tau\\ &=\frac{\tau(\rho(t)B)}{\tau({}^1/_2B)}u(0)=c(t)u(0).\end{aligned}$$

2. 6. Proposition ([6], §1. 11 and [5], Lemma 5. 5. 4). *Suppose $f\in C(S)$, $0<r<1$. Then there exists a unique $u\in C(r\overline{B})$ such that u is M-harmonic in rB and $u(r\zeta)=f(\zeta)$ $(\zeta\in S)$.*

2. 7. Definition ([6], §1.12). For $f\in C(S)$ and $0<r<1$ let $P_r[f]$ be the M-harmonic function in rB such that

$$\lim_{t\to r} P_r[f](t\zeta)=f(\zeta) \tag{3}$$

uniformly for $\zeta\in S$. (See Proposition 2.6.)

Note that $P_r : C(S)\to C(rB)$ is linear and positive ($P_r[f]\geqq 0$ for $f\geqq 0$) by the maximum principle ([5], Theorem 4.3.2).

2. 8. Lemma (cf. [6], § 1.13). *Suppose* $0<t<1$ *and* $\rho(t)<r<1$. *If* $f\in C(S)$ *and* $z\in tS$, *then*

$$|P_r[f](z)|\leqq 2c(t)\int_S |f|\ d\sigma. \tag{4}$$

Proof. Let Re $f=g$ and Im $f=h$. Then $g,\ h\in C(S)$ and $f=g+\sqrt{-1}h$. Let $g^+=\max(g,\ 0)$ and $g^-=\max(-g,\ 0)$. Then $g^+,\ g^-\in C(S)$ and $g^+\geqq 0,\ g^-\geqq 0,\ g=g^+-g^-$, $|g|=g^++g^-$.

Since $P_r[g^+]\geqq 0$ is M-harmonic in rB, Lemma 2.5 and (2) show that

$$P_r[g^+](z)\leqq c(t)P_r[g^+](0) \tag{5}$$

and

$$\int_S P_r[g^+](s\zeta)\ d\sigma(\zeta)=P_r[g^+](0) \tag{6}$$

for all $s\in(0,\ r)$. Furthermore, by (3),

$$\lim_{s\to r}\int_S P_r[g^+](s\zeta)d\sigma(\zeta)=\int_S g^+(\zeta)d\sigma(\zeta). \tag{7}$$

By (5), (6) and (7), we have

$$P_r[g^+](z)\leqq c(t)\int_S g^+\ d\sigma. \tag{8}$$

Similarly, it holds that

$$P_r[g^-](z)\leqq c(t)\int_S g^-\ d\sigma. \tag{9}$$

In virtue of (8) and (9), linearity of P_r shows that

$$|P_r[g](z)|\leqq c(t)\int_S (g^++g^-)\ d\sigma=c(t)\int_S |g|\ d\sigma.$$

We also have

$$|P_r[h](z)|\leqq c(t)\int_S |h|\ d\sigma.$$

Since $f=g+\sqrt{-1}\,h$, linearity of P_r shows that

$$|P_r[f](z)|\leqq c(t)\int_S (|g|+|h|)\ d\sigma\leqq 2c(t)\int_S |f|\ d\sigma.$$

2. 9. Lemma (cf. [6], p. 504). *Suppose* $0<t<1$, $\rho(t)<r<1$ *and* $z\in tS$. *Then there exists a function* $P_{r,z}\in L^\infty(\sigma)$ *which satisfies the following conditions*:

i) $\|P_{r,z}\|_\infty\leqq 2c(t)$,

ii) $P_{r,z}(\zeta)\geqq 0$ *for almost all* $\zeta\in S$,

iii) $\int_S P_{r,z}d\sigma=1$,

iv) $P_r[f](z)=\int_S f\,P_{r,z}d\sigma$ *for all* $f\in C(S)$.

Proof. Since $C(S)$ is a dense subspace of $L^1(\sigma)$, in view of Lemma 2.8, there exists a bounded linear functional Λ on $L^1(\sigma)$ such that $\Lambda(f)=P_r[f](z)$ for all $f\in C(S)$, and $\|\Lambda\|\leqq 2c(t)$. Hence there exists a function $P_{r,z}\in L^\infty(\sigma)$ such that $\|P_{r,z}\|_\infty=\|\Lambda\|\leqq 2c(t)$ and

$$\Lambda(f)=\int_S fP_{r,z}d\sigma$$

for all $f\in L^1(\sigma)$.

Since $P_r[\cdot](z)$ is a positive linear functional on $C(S)$, it holds that $P_{r,z}(\zeta)\geqq 0$ for almost all $\zeta\in S$. For $f\in C(S)$, we have

$$P_r[f](z)=\Lambda(f)=\int_S fP_{r,z}d\sigma.$$

In the case $f\equiv 1$ this shows that

$$\int_S P_{r,z}d\sigma=1.$$

2. 10. Definition. Suppose $0<t<1$, $\rho(t)<r<1$ and $z\in tS$. For $f\in L^1(\sigma)$, we define

$$P_r[f](z)=\int_S fP_{r,z}d\sigma.$$

By Lemma 2.9, this is compatible with Definition 2.7.

2. 11. Lemma. *Suppose* $0<t<1$, $\rho(t)<r<1$ *and* $z\in tS$. *If* $U\in\mathfrak{U}$, *then* $P_{r,Uz}(\zeta)=P_{r,z}\circ U^{-1}(\zeta)$ *for almost all* $\zeta\in S$.

Proof. Let $U\in\mathfrak{U}$ and $f\in C(S)$. Then $f\circ U\in C(S)$. By Definition 2.7, it holds that

$$\begin{aligned}\lim_{s\to r} P_r[f\circ U](s\zeta)&=f\circ U(\zeta)=f(U\zeta)=\lim_{s\to r} P_r[f]\,(sU\zeta)\\&=\lim_{s\to r} P_r[f](Us\zeta)=\lim_{s\to r} P_r[f]\circ U(s\zeta).\end{aligned}$$

Since $P_r[f\circ U]$ and $P_r[f]\circ U$ are both M-harmonic in rB, it follows from the maximum principle ([5], Theorem 4.3.2) that $P_r[f\circ U]=P_r[f]\circ U$ in rB. By Definition 2.10 and the $\mathfrak{U}$-invariance of the measure σ, therefore we have

$$\begin{aligned}\int_S fP_{r,Uz}d\sigma&=P_r[f](Uz)=P_r[f\circ U]\,(z)\\&=\int_S(f\circ U)P_{r,z}\,d\sigma=\int_S f(P_{r,z}\circ U^{-1})\,d\sigma.\end{aligned}$$

Since this holds for all $f\in C(S)$, we can conclude that $P_{r,Uz}(\zeta)=P_{r,z}\circ U^{-1}(\zeta)$ for almost all $\zeta\in S$.

2. 12. Lemma (cf. [6], pp. 504–505). *Suppose* $0<t<1$ *and* $\rho(t)<r<1$. *If* $1\leqq p<\infty$ *and* $f\in L^p(\sigma)$, *then we have*

$$\int_S |P_r[f](t\zeta)|^p\, d\sigma(\zeta)\leqq \int_S |f|^p\, d\sigma.$$

Proof. Let dU denote Haar measure on $\mathfrak{U}$. By [5], Proposition 1.4.7,

$$I\equiv\int_S |P_r[f](t\zeta)|^p\, d\sigma(\zeta)=\int_{\mathfrak{U}} |P_r[f]\,(tUe_1)|^p\, dU$$
$$=\int_{\mathfrak{U}} |P_r[f](Ute_1)|^p\, dU$$

where $e_1=(1,0,\dots,0)\in S$. It follows from Definition 2.10 that

$$I=\int_{\mathfrak{U}} \left| \int_S f(\zeta)P_{r,Ute_1}(\zeta)\, d\sigma(\zeta)\right|^p dU.$$

By Lemma 2.9, ii), iii), Hölder's inequality and Fubini's theorem, we have

$$I\leqq\int_{\mathfrak{U}} \left\{\int_S |f(\zeta)|^p\, P_{r,Ute_1}(\zeta)\, d\sigma(\zeta)\right\} dU \tag{10}$$
$$=\int_S\left\{|f(\zeta)|^p\int_{\mathfrak{U}} P_{r,Ute_1}(\zeta)\, dU\right\} d\sigma(\zeta).$$

Lemma 2.11 shows that

$$\int_{\mathfrak{U}} P_{r,Ute_1}(\zeta)\, dU=\int_{\mathfrak{U}} P_{r,te_1}\circ U^{-1}(\zeta)\, dU. \tag{11}$$

Since $\mathfrak{U}$ is a compact group, it is unimodular (see e. g. [4], p. 117). Hence it follows from [5], Proposition 1.4.7 and Lemma 2.9, iii) that

$$\int_{\mathfrak{U}} P_{r,te_1}\circ U^{-1}(\zeta)\, dU=\int_{\mathfrak{U}} P_{r,te_1}\circ U(\zeta)\, dU \tag{12}$$
$$=\int_S P_{r,te_1}\, d\sigma=1.$$

By (10), (11) and (12), we have

$$I\leqq\int_S |f(\zeta)|^p\, d\sigma(\zeta).$$

3. Strong limits of *M*-subharmonic functions

In addition to the preliminaries described in §2, we need further three theorems to prove our main result (Theorem 3.4.).

3. 1. Theorem ([6], §3.1). *Suppose u is an M-subharmonic function in B and*

$$\sup_{1/2\leqq r<1}\int_S |u_r|\, d\sigma<\infty$$

where $u_r(\zeta)=u(r\zeta)$ *for* $\zeta\in S$, $0<r<1$. *Then*

$$u^*(\zeta)=\lim_{r\to 1} u(r\zeta)$$

exists for almost all $\zeta\in S$.

3. 2. Theorem ([5], Theorem 3.3.4 and Theorem 4.3.3). *Suppose* u *is an* M-*harmonic function in* B *and*

$$\sup_{0<r<1}\int_S |u_r|^p d\sigma<\infty$$

for some $p\in(1,\infty)$. *Then*

$$\lim_{r\to 1}\int_S |u_r-u^*|^p d\sigma=0.$$

3. 3. Theorem ([6], § 1. 22). *Suppose* u *is an* M-*subharmonic function in* B *and*

$$\sup_{0<r<1}\int_S u_r\, d\sigma<\infty.$$

Then there exist an M-*harmonic function* h *in* B *and an* M-*subharmonic function* v *such that*

$$u=h+v,\quad u\leqq h,\ v\leqq 0$$

in B *and*

$$h(0)=\lim_{r\to 1}\int_S u_r\, d\sigma. \tag{13}$$

3. 4. Theorem. *Suppose* u *is an* M-*subharmonic function in* B *and* $1\leqq p<\infty$. *If* u *satisfies*

$$\sup_{1/2\leqq r<1}\int_S |u_r|^p\, d\sigma<\infty, \tag{14}$$

then we have

$$\lim_{r\to 1}\int_S |u_r-u^*|^q\, d\sigma=0 \tag{15}$$

for all $q\in(0, p)$.

Proof. First we consider the case $1<p<\infty$. Since $1<p<\infty$ and $\sigma(S)=1$, we have

$$\sup_{1/2\leqq r<1}\int_S |u_r|\, d\sigma\leqq \{\sup_{1/2\leqq r<1}\int_S |u_r|^p\, d\sigma\}^{1/p}<\infty, \tag{16}$$

by (14). It follows from Theorem 3.1 that the radial limit

$$u^*(\zeta)=\lim_{r\to 1} u(r\zeta)$$

exists for almost all $\zeta\in S$. In view of (16), Theorem 3.3 shows that

$$u=h+v,\quad h\geqq u,\quad v\leqq 0$$

in B, where h is an M-harmonic function in B and v is an M-subharmonic function in B. h satisfies (13). Hence we have

$$\lim_{r\to 1}\int_S |v_r|\,d\sigma=0. \tag{17}$$

Let $z\in B$ and $\rho(|z|)<r<1$. Then

$$-h(z)+P_r[u_r](z)=P_r[v_r](z). \tag{18}$$

By Definition 2.10 and Lemma 2.9,

$$\begin{aligned}|P_r[v_r](z)|&=\left|\int_S v_r P_{r,z}\,d\sigma\right|\\ &\leqq \|P_{r,z}\|_\infty\int_S |v_r|\,d\sigma\leqq 2c(|z|)\int_S |v_r|\,d\sigma.\end{aligned}$$

It follows from (17) and (18) that

$$\lim_{r\to 1} P_r[u_r](z)=h(z). \tag{19}$$

Suppose $1/2\leqq t<1$. If $\rho(t)<r<1$, then

$$\int_S |P_r[u_r](t\zeta)|^p\,d\sigma(\zeta)\leqq\int_S |u_r|^p\,d\sigma, \tag{20}$$

by Lemma 2.12. In view of (19), (20), Fatou's lemma, we have

$$\begin{aligned}\int_S |h(t\zeta)|^p\,d\sigma(\zeta)&=\int_S \lim_{r\to 1} |P_r[u_r](t\zeta)|^p\,d\sigma(\zeta)\\ &\leqq \liminf_{r\to 1}\int_S |P_r[u_r](t\zeta)|^p\,d\sigma(\zeta)\\ &\leqq \liminf_{r\to 1}\int_S |u_r|^p\,d\sigma\\ &\leqq \sup_{1/2\leqq r<1}\int_S |u_r|^p\,d\sigma.\end{aligned}$$

Hence

$$\sup_{1/2\leqq r<1}\int_S |h_r|^p\,d\sigma\leqq \sup_{1/2\leqq r<1}\int_S |u_r|^p\,d\sigma<\infty. \tag{21}$$

Since $v=u-h$, this also gives

(22) $$\sup_{1/2\leq r<1}\int_S |v_r|^p d\sigma<\infty.$$

Now we shall prove (15). If (15) holds for one value of q, so does it for any smaller value, and we may assume that $1<q<p$. Let $\alpha=(p-1)/(p-q)$, $\beta=(p-1)/(q-1)$, then $1/\alpha+1/\beta=1$ and $q=1/\alpha+p/\beta$, so that, Hölder's inequality shows

$$\int_S |v_r|^q d\sigma=\int_S |v_r|^{1/\alpha}|v_r|^{p/\beta}d\sigma$$
$$\leq\Big(\int_S |v_r|d\sigma\Big)^{1/\alpha}\Big(\int_S |v_r|^p d\sigma\Big)^{1/\beta}.$$

It follows (17) and (22) that

(23) $$\lim_{r\to 1}\int_S |v_r|^q\, d\sigma=0.$$

In view of (21), (22) and Theorem 3.1, the radial limits

$$h^*(\zeta)=\lim_{r\to 1} h(r\zeta),\ v^*(\zeta)=\lim_{r\to 1} v(r\zeta)$$

exist for almost all $\zeta\in S$. Since $v\leq 0$, (17) and Fatou's lemma show that $v^*(\zeta)=0$, and so, $u^*(\zeta)=h^*(\zeta)$ for almost all $\zeta\in S$. Hence

(24) $$\int_S |u_r-u^*|^q d\sigma=\int_S |h_r+v_r-h^*|^q d\sigma$$
$$\leq 2^{q-1}\Big\{\int_S |h_r-h^*|^q d\sigma+\int_S |v_r|^q\, d\sigma\Big\}.$$

In view of (21), Theorem 3.2 gives

(25) $$\lim_{r\to 1}\int_S |h_r-h^*|^q d\sigma=0.$$

By (23), (24) and (25), we have

$$\lim_{r\to 1}\int_S |u_r-u^*|^q d\sigma=0.$$

Secondly we consider the case $p=1$. (cf. [3], p. 229) Suppose $0<q<p=1$. Theorem 3.1 shows that the radial limit

$$u^*(\zeta)=\lim_{r\to 1} u_r(\zeta)$$

exists for almost all $\zeta\in S$. It follows from Egoroff's theorem that for an arbitrarily small number $\varepsilon>0$ there exists a closed subset E of S such that $\sigma(S\setminus E)<\varepsilon$ and

(26) $$u^*(\zeta)=\lim_{r\to 1} u_r(\zeta)$$

uniformly for $\zeta\in E$. Since $1/q>1$, Hölder's inequality gives

$$\int_{S\setminus E} |u_r - u^*|^q d\sigma \leqq (\sigma(S\setminus E))^{1-q} \left(\int_{S\setminus E} |u_r - u^*| d\sigma\right)^q \tag{27}$$
$$\leqq \varepsilon^{1-q} \left(\int_S |u_r - u^*| d\sigma\right)^q .$$

By Fatou's lemma, we have

$$\int_S |u^*| d\sigma = \int_S \lim_{r\to 1} |u_r| d\sigma \leqq \liminf_{r\to 1} \int_S |u_r| d\sigma$$
$$\leqq \sup_{1/2\leqq r<1} \int_S |u_r| d\sigma \equiv K < \infty ,$$

and so, for $1/2 \leqq r < 1$,

$$\int_S |u_r - u^*| d\sigma \leqq \int_S |u_r| d\sigma + \int_S |u^*| d\sigma \leqq 2K .$$

It follows from (27) that

$$\int_{S\setminus E} |u_r - u^*|^q d\sigma \leqq (2K)^q \ \varepsilon^{1-q} . \tag{28}$$

On the other hand, by (26),

$$\lim_{r\to 1} \int_E |u_r - u^*|^q d\sigma = 0 . \tag{29}$$

(28) and (29) give

$$\lim_{r\to 1} \int_S |u_r - u^*|^q \ d\sigma = 0 .$$

This completes the proof.

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