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On Blowing Up of Solutions of Schrödinger Equations with Cubic Convolution

By SOH MATSUMOTO and KIYOSHI MOCHIZUKI

Department of Mathematics, Faculty of Science,
Shinshu University, Matsumoto 390, Japan
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Abstruct: Solutions to the Cauchy problem for Schrödinger equations with cubic convolution are considered. Conditions on the initial data and the potential are given so that the energy of solutions blows up in finite time.

Key Words: blow-up, Cauchy problem, Schrödinger equation with cubic convolution.

1. Introduction

In this note we shall consider the Cauchy problem

(1) $$i\partial_t u=\Delta u+f(u),\quad x\in \boldsymbol{R}^n \text{ and } t>0$$

(2) $$u(x,0)=\varphi(x),$$

where $i=\sqrt{-1}$, $\partial_t=\partial/\partial t$, $\Delta=\sum_{j=1}^{n}\partial_j^2$ $(\partial_j=\partial/\partial x_j)$, $f(u)$ represents a cubic convolution nonlinearity:

(3) $$f(u)=(V*|u|^2)u=\left(\int V(x-y)|u(y)|^2dy\right)u(x)$$

(all integrals are taken over $\boldsymbol{R}^n$), and $V(x)$, $\varphi(x)$ satisfy the following properties.

(A1) $V(x)$ is real valued and $V(-x)=V(x)$;

(A2) $|V(x)|\leq C|x|^{-\sigma}$ $(0<\sigma<n)$ or $V(x)\in L^p=L^p(\boldsymbol{R}^n)$ $(1\leq p\leq\infty)$;

(A3) $\varphi(x)\in H^r\cap\Sigma$, $r=\max\{3,\ [(n+1)/2]\}$.

Here $H^k=H^k(\boldsymbol{R}^n)$ ($k\geq 0$ integer) is the Sobolev space with norm

(4) $$||v||_{H^k}=\left(\sum_{|\alpha|\leq k}\int|D^\alpha v(x)|^2dx\right)^{1/2}$$

$\left(\alpha=(\alpha_1,\alpha_2,\cdots\cdots,\alpha_n)\right.$ being a multi-index and $\left.D^\alpha=\partial_1^{\alpha_1}\partial_2^{\alpha_2}\cdots\partial_n^{\alpha_n}\right)$, $\Sigma=\{\varphi;|x|\varphi(x)\in L^2\}$ and $[q]$ is the largest integer $\leq q$.

As we see in Ginibre-Velo [1], the above problem has a unique local solution in time, and since (A3) is rather strong, it belongs to $C^1([0,T_0);H^{r-2})\cap$

$C([0, T_0); H^r \cap \sum)$ for a $T_0>0$. (In[1] is treated only the case $V(x) \in L^p$. In case $|V(x)| \leq C|x|^{-\sigma}$, the Young inequality should be replaced by the generalized Young (Hardy-Littlewood-Sobolev) inequality to show e. g., Lemma 2.1 of [1].)

In this note we investigate conditions under which this solution may blow up in finite time. Such a blow up problem has been studied by Glassey [2] in case of the power nonlinearity $f(u)=g|u|^{p-1}u$ with $g>0$ and $p>1$. We shall show similar blow-up results requiring $x \cdot \nabla V(x) \equiv \sum_{j=1}^{n} x_j \partial_j V(x) \leq -cV(x)$ for some $c \geq 2$. In case $c>2$, his line of proof can be followed to our problem without any essential modification. On the other hand, the argument of Glassey-Schaeffer [3] can be applied to the critical case $c=2$.

2. The Blow-up Theorems

We begin with a lemma giving several identities which will be used in the proof of Theorems.

Lemma. *Let u be a solution of* (1), (2) *on an interual* $0 \leq t < T_0$. *Then*

$$||u(t)||^2 \equiv \left(\int |u(x, t)|^2 dx\right)^{1/2} = ||\varphi||^2; \tag{5}$$

$$\int \{|\nabla u(x, t)|^2 - \frac{1}{2}(V * |u|^2)|u(x, t)|^2\} dx = \text{const} \equiv E_0; \tag{6}$$

$$\frac{d}{dt}\int |x|^2 |u(x, t)|^2 dx = -4\text{Im}\int \overline{u(x, t)} x \cdot \nabla u(x, t) dx; \tag{7}$$

$$\begin{aligned} &\frac{d}{dt}\text{Im}\int \overline{(u(x, t)} x \cdot \nabla u(x, t) dx \\ &= -2\int \{|\nabla u(x, t)|^2 + \frac{1}{4}(x \cdot \nabla V * |u|^2)|u(x, t)|^2\} dx; \end{aligned} \tag{8}$$

$$\frac{n}{2}||u(t)||^2 + \text{Re}\int \overline{u(x, t)} x \cdot \nabla u(x, t) dx = 0. \tag{9}$$

Proof. We multiply (1) by $2\bar{u}$ and take the imaginary part. Then since $V(x)$ is real, the same proof of [2; Lemma] yields (5) and (7). Next we multiply (1) by $2\partial_t \bar{u}$ and integrate the real part of this identity. Then since $V(x-y)=V(y-x)$ ((A1)), we have

$$\int (V * |u|^2) \partial_t |u|^2 dx = \frac{1}{2}\partial_t \int (V * |u|^2)|u|^2 dx$$

and (6) follows. To derive (8), we multiply (1) by $2x \cdot \nabla \bar{u}$ and integrate the real part of this identity. Then noting the equality

$$\int x\cdot(\nabla V*|u|^2)|u|^2dx=\frac{1}{2}\int(x\cdot\nabla V*|u|^2)|u|^2dx$$

which also results from (A1), we can follow the proof of [2; Lemma]. Finally, (9) is easily obtained if we integrate the identity

$$\frac{1}{2}\nabla\cdot(x|u|^2)=\frac{n}{2}|u|^2+\mathrm{Re}\bar{u}(x\cdot\nabla u). \square$$

With these identities we can establish our blow-up theorems.

Theorem 1. *Let* u *be a solution of* (1), (2). *Assume that*

(A4) $E_0\leq 0$;

(A5) $\mathrm{Im}\int\overline{\varphi(x)}x\cdot\nabla\varphi(x)dx>0$;

(A6) *There exists a* $c>2$ *such that* $x\cdot\nabla V(x)\leq -cV(x)$.

Then there exists a finite time T *such that*

(10) $$\lim_{t\to T-}||\nabla u(t)||=+\infty.$$

Proof. We briefly repeat the proof of [2; Theorem]. Put

(11) $$y(t)=\mathrm{Im}\int\bar{u}(x\cdot\nabla u)dx.$$

Then by (6), (8) of Lemma and the above (A4), (A6) we have

(12) $$y'(t)\geq(c-2)\int|\nabla u|^2dx-cE_0\geq(c-2)||\nabla u(t)||^2\geq 0.$$

Moreover, by (A5) we see $y(0)>0$ and $y'(0)>0$. So the function $y(t)$ is positive and increasing whenever u exists. It follows from (7) of Lemma that $\left(\int|x|^2|u|^2dx\right)'(t)=-4y(t)<0$. Therefore,

$$\int|x|^2|u|^2dx\leq\int|x|^2|\varphi|^2dx\equiv d_0^2<\infty,$$

and the Schwarz inequality applied to (11) yields $y(t)\leq d_0||\nabla u(t)||$. Using this and (12), we obtain the differential inequality

$$y'(t)\geq(c-2)d_0^{-2}y(t)^2,\quad y(0)>0.$$

Integrating gives the estimate

$$||\nabla u(t)||\geq d_0^{-1}y(t)\geq y(0)d_0/\{d_0^2-(c-2)y(0)t\},$$

which implies (10).

Theorem 2. *Let* u *be the solution of* (1), (2). *Assume that*

$$(\mathrm{A4})'\quad\begin{cases}E_0<0, \text{ or}\\ E_0=0 \text{ and } \mathrm{Im}\int\overline{\varphi(x)}x\cdot\nabla\varphi(x)dx>0, \text{ or}\\ E_0>0 \text{ and } \mathrm{Im}\int\varphi(x)\,x\nabla\varphi(x)\,dx\geq\sqrt{E_0}||x\varphi||,\end{cases}$$

(A6)′ $x\cdot\nabla V(x)\leq -2V(x)$.

Then there exists a finite time T verifying (10).

Proof (cf., [3; Theorem III]). Integrate (7) over $(0, t)$. Then we have noting (11) and the inequality $y'(t)\geq -2E_0$ ((12) with $C=2$),

$$||xu(t)||^2=||x\varphi||^2-4\int_0^t y(s)ds\leq ||x\varphi||^2-4y(0)t+4E_0t^2.$$

Here $||x\varphi||+|y(0)|<\infty$ by (A3) and

$$\begin{cases} E_0<0, \text{ or} \\ E_0=0 \text{ and } y(0)>0, \text{ or} \\ y(0)\geq\sqrt{E_0}||x\varphi|| \end{cases}$$

by(A4)′. Thus, there exists a $\tilde{T}<\infty$ such that

(13) $$\lim_{t\to\tilde{T}-} ||xu(t)||=0.$$

On the other hand, it follows from (5) and (9) that

(14) $$0<||\varphi||^2=||u(t)||^2\leq 2\, n\, ||\nabla u(t)||\, ||xu(t)||.$$

(13) and (14) show the theorem. ▯

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