An Elementary Construction of Processes with Independent Increments

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(Received July 10, 1991)

1. Introduction

Let $X=\{X(t)\,;\ 0\leq t\leq T\}$ be an $\boldsymbol{R}^d$-valued stochastically continuous process with independent increments on a basic probability space $(\Omega, \mathscr{F}, \boldsymbol{P})$. We assume that X has no Gaussian component and $X(0)=0$ a. s. Then, for each t, the characteristic function of $X(t)$ is known to be of the form

$$\boldsymbol{E}[e^{i(z,X(t))}]=\exp\Big[i(z,a(t))+\int_{\boldsymbol{R}_0^d}\{e^{i(z,u)}-1-\frac{i(z,\boldsymbol{u})}{1+|\boldsymbol{u}|^2}\}\,dM_t(\boldsymbol{u})\Big],$$

where $a(t)$ is an $\boldsymbol{R}^d$-valued continuous function on $\boldsymbol{T}=[0,\ T]$ with $a(0)=0$, and $\{M_t\,;\ t\in\boldsymbol{T}\}$ is a set of Borel measures on $\boldsymbol{R}_0^d=\boldsymbol{R}^d\setminus\{0\}$ satisfying the conditions $\int_{\boldsymbol{R}_0^d}(1\wedge|\boldsymbol{u}|^2)\,dM_t\ (\boldsymbol{u})<\infty$ and $M_0(\boldsymbol{R}_0^d)=0$. The measures $\{M_t\}$ are called the Lévy spectral measures of X. We note that $M_t(B)$ is continuous and nondecreasing in t provided that $M_T(B)<\infty$. We shall simply write $X(t)\sim(a(t), M_t)$ to express the fact that the probability law of X is determined by the characteristic function of each $X(t)$ with the above form. There exists a process X with $X(t)\sim(a(t), M_t)$ for any pair $(a(t), M_t)$ satisfying the all conditions stated above. Indeed, this fact will be seen by the Kolmogorov's extension theorem.

To the Lévy spectral measures $\{M_t\}$ of X, there corresponds uniquely a Borel measure M on $\boldsymbol{S}=\boldsymbol{T}\times\boldsymbol{R}_0^d$, which is determined by the relation $M((s,\ t]\times B)=M_t(B)-M_s(B)$ for any $(s,\ t]\subset\boldsymbol{T}$ and any Borel subset $B\subset\boldsymbol{R}_0^d$ with $M_T(B)<\infty$. The measure M is called the time-jump measure of X. In the special case where M is finite, the characteristic function of $X(t)$ can be written in the form

$$\boldsymbol{E}[e^{i(z,X(t))}]=\exp\Big[i(z,\tilde{a}(t))+\int_{\boldsymbol{R}_0^d}\{e^{i(z,u)}-1\}\,dM_t(\boldsymbol{u})\Big].$$

Here $\tilde{a}(t)$ is an $\boldsymbol{R}^d$-valued continuous function on $\boldsymbol{T}$ given by

$$\tilde{a}(t)=a(t)-\int_{\boldsymbol{R}_0^d}\frac{\boldsymbol{u}}{1+|\boldsymbol{u}|^2}\,dM_t(\boldsymbol{u}).$$

The function $\tilde{a}(t)$ is called the drift component of X. If $\tilde{a}(t)=0$ on $\boldsymbol{T}$, the process X is realized as the step functions with only a finite number of jumps on $\boldsymbol{T}$.

Now the purpose of this note is to give a certain method to construct the process

$\boldsymbol{X}$ with $X(t) \sim (a(t), M_t)$ more directly from the given data $a(t)$ and M. When M is finite, this is accomplished by constructing an auxiliary weighted direct sum probability space $(\Omega^*, \mathscr{F}^*, \boldsymbol{P}^*)$ based on the Poisson distribution with intensity $M(S)$. In the general case, we consider the product probability space $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\boldsymbol{P}}) = \prod_{n=1}^{\infty} (\Omega^*, \mathscr{F}^*, \boldsymbol{P}^{*(n)})$, which is associated with certain decomposition $M = \sum_{n=1}^{\infty} M^{(n)}$ into finite Borel measures $M^{(n)}$ on $\boldsymbol{S}$. Then the desired process $\boldsymbol{X}$ will be defined on the space $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\boldsymbol{P}})$ by the series with independent summands, which are constructed from $M^{(n)}$ $(n \geq 1)$ respectively. We also note that the obtained process $\boldsymbol{X}$ is a Lévy process in the sense of K. Ito [3].

The method given here is similar to those of P. L. Brockett and H. G. Tucker [1] and J. A. Veeh [4]. But our construction seems to be more elementary and direct. By employing our method in this note, we shall discuss elsewhere the problem on the equivalence and singularity of processes with independent increments ([2]).

2. Processes with a finite number of jumps: the case $M(S) < \infty$.

2.1. First we introduce the notion of the weighted direct sum of probability spaces, which plays an important role in the following. Let $(\Omega^{(\ell)}, \mathscr{F}^{(\ell)}, \boldsymbol{P}^{(\ell)})$, $\ell \geq 0$, be a sequence of probability spaces, and let (p_ℓ) be a probability distribution on the space of nonnegative integers. We assume that the sets $\Omega^{(\ell)}$, $\ell \geq 0$, are pairwise disjoint. Then we consider the probability space $(\Omega^+, \mathscr{F}^+, \boldsymbol{P}^+)$ given by $\Omega^+ = \bigcup_{\ell=0}^{\infty} \Omega^{(\ell)}$, $\mathscr{F}^+ = \{A = \bigcup_{\ell=0}^{\infty} A_\ell ; A_\ell \in \mathscr{F}^{(\ell)} \ (\ell \geq 0)\}$ and

$$\boldsymbol{P}^+(A) = \sum_{\ell=0}^{\infty} \boldsymbol{P}^{(\ell)}(A_\ell) p_\ell \qquad \text{for any } A = \bigcup_{\ell=0}^{\infty} A_\ell \in \mathscr{F}^+.$$

We call $(\Omega^+, \mathscr{F}^+, \boldsymbol{P}^+)$ the *weighted direct sum* of $(\Omega^{(\ell)}, \mathscr{F}^{(\ell)}, \boldsymbol{P}^{(\ell)})$, $\ell \geq 0$, based on (p_ℓ).

2.2. Now let $(S, \mathscr{B}(S), M)$ be a finite measure space satisfying the condition $M(\{t\} \times \boldsymbol{R}_0^d) = 0$ for each $t \in \boldsymbol{T}$, where $\mathscr{B}(S)$ is the Borel σ-algebra on $\boldsymbol{S} = \boldsymbol{T} \times \boldsymbol{R}_0^d$. Then we consider the set $\{M_t ; t \in \boldsymbol{T}\}$ of spectral measures on $\boldsymbol{R}_0^d$ associated with M, which are defined by $M_t(B) = M([0, t] \times B)$ for any Borel subset $B \subset \boldsymbol{R}_0^d$. If the set B is fixed, $\phi(t) = M_t(B)$ is a continuous nondecreasing function on $\boldsymbol{T}$ with $\phi(0) = 0$. For any $\ell \geq 1$, we denote by $(S^\ell, \mathscr{B}(S^\ell), M^\ell)$ the ℓ-fold product measure space of $(S, \mathscr{B}(S), M)$. Let us introduce the particular subset $\Omega_\ell \subset S^\ell$ given by

$$\Omega_\ell = \{(t_1, \boldsymbol{u}_1, \cdots, t_\ell, \boldsymbol{u}_\ell) \in S^\ell ; t_i < t_j \text{ if } i < j\}.$$

By setting $\boldsymbol{P}_\ell = \ell! \, M(S)^{-\ell} M^\ell$ on the Borel σ-algebra $\mathscr{B}(\Omega_\ell)$, we have a probability space $(\Omega_\ell, \mathscr{B}(\Omega_\ell), \boldsymbol{P}_\ell)$. For the sake of convenience, we also consider the trivial probability space $(\Omega_0, \mathscr{B}(\Omega_0), \boldsymbol{P}_0)$ given by $\Omega_0 = \{0\}$ and $\mathscr{B}(\Omega_0) = \{\phi, \Omega_0\}$. We denote by $(\Omega^*, \mathscr{F}^*, \boldsymbol{P}^*)$ the weighted direct sum of $(\Omega_\ell, \mathscr{B}(\Omega_\ell), \boldsymbol{P}_\ell)$, $\ell \geq 0$, based on the Poisson distribution with intensity $M(S)$. Precisely speaking, $(\Omega^*, \mathscr{F}^*, \boldsymbol{P}^*)$ is a probability space given by $\Omega^* = \bigcup_{\ell=0}^{\infty} \Omega_\ell$, $\mathscr{F}^* = \{A = \bigcup_{\ell=0}^{\infty} A_\ell ; A_\ell \in \mathscr{B}(\Omega_\ell) \ (\ell \geq 0)\}$ and

$$P^*(A) = e^{-M(S)} \sum_{\ell=0}^{\infty} (\ell\,!)^{-1} M(S)^\ell P_\ell(A_\ell) \qquad \text{for any } A = \bigcup_{\ell=0}^{\infty} A_\ell \in \mathscr{F}^*.$$

We call $(\Omega^*, \mathscr{F}^*, \boldsymbol{P}^*)$ the *triangular probability space* associated with $(S, \mathscr{B}(S), M)$.

2. 3. As usual, let $D(\boldsymbol{T})$ be the space of $\boldsymbol{R}^d$-valued functions on $\boldsymbol{T}$ which are right-continuous and have left-hand limits everywhere. Let us introduce the particular subset $F(\boldsymbol{T}) \subset D(\boldsymbol{T})$ consisting of step functions $f(t)$ with $f(0)=0$ and with only a finite number of jumps on $\boldsymbol{T}$. Then we shall define a map $\Phi : \Omega^* \to F(\boldsymbol{T})$ in the following way. For any $\ell \geq 1$ and any $\omega^* = (t_1, \boldsymbol{u}_1, \cdots, t_\ell, \boldsymbol{u}_\ell) \in \Omega_\ell$, we set

$$[\Phi(\omega^*)](t) = \sum_{j=1}^{\ell} \boldsymbol{u}_j \,\theta(t_j \,|\, t) \qquad (t \in \boldsymbol{T}),$$

where we set $\theta(s|t) = 0$ $(t < s)$ and $=1$ $(s \leq t)$. For $\omega^* \in \Omega_0$, we set $[\Phi(\omega^*)](t) = 0$ $(t \in \boldsymbol{T})$. Clearly, the map Φ is bijective. Thus we obtain a probability space $(F(\boldsymbol{T}), \mathscr{F}(\boldsymbol{T}), [\boldsymbol{P}^*]_\Phi)$, where $\mathscr{F}(\boldsymbol{T})$ is the σ-algebra on $F(\boldsymbol{T})$ given by $\mathscr{F}(\boldsymbol{T}) = \{\Phi(A)\,;\, A \in \mathscr{F}^*\}$ and $[\boldsymbol{P}^*]_\Phi$ is the image measure of $\boldsymbol{P}^*$ induced by Φ. We note that the triangular probability space $(\Omega^*, \mathscr{F}^*, \boldsymbol{P}^*)$ faithfully describes the function probability space $(F(\boldsymbol{T}), \mathscr{F}(\boldsymbol{T}), [\boldsymbol{P}^*]_\Phi)$.

2. 4. Now we introduce an $\boldsymbol{R}^d$-valued process $\boldsymbol{Z} = \{Z(t)\,;\, 0 \leq t \leq T\}$ defined on the probability space $(\Omega^*, \mathscr{F}^*, \boldsymbol{P}^*)$, which is given by

$$Z(t, \omega^*) = [\Phi(\omega^*)](t) \qquad (t \in \boldsymbol{T},\ \omega^* \in \Omega^*).$$

Then we have the following

THEOREM 1. *Each sample function of the process* $\boldsymbol{Z}$ *belongs to the space* $F(\boldsymbol{T})$. *The probability law of* $\boldsymbol{Z}$ *is given by the relation* $Z(t) \sim (a(t), M_t)$, *where* $a(t)$ *is a continuous function on* $\boldsymbol{T}$ *defined by*

$$a(t) = \int_{\boldsymbol{R}_0^d} \frac{\boldsymbol{u}}{1+|\boldsymbol{u}|^2}\, dM_t(\boldsymbol{u}) \qquad (t \in \boldsymbol{T}).$$

Proof. For any $\ell \geq 1$, we denote by S_*^ℓ the subset of S^ℓ given by

$$S_*^\ell = \{(t_1, \boldsymbol{u}_1, \cdots, t_\ell, \boldsymbol{u}_\ell) \in S^\ell\,;\, t_i \neq t_j \text{ if } i \neq j\}.$$

By the assumption on M, we see that $M^\ell(S^\ell \backslash S_*^\ell) = 0$. Then we have a measure space $(S_*^\ell, \mathscr{B}(S_*^\ell), M_*^\ell)$, where we set $M_*^\ell = M^\ell$ on the Borel σ-algebra $\mathscr{B}(S_*^\ell)$. Next we consider a surjective measurable map $\psi_\ell : S_*^\ell \to \Omega_\ell$ given by

$$\psi_\ell(t_1, \boldsymbol{u}_1, \cdots, t_\ell, \boldsymbol{u}_\ell) = (t_{\sigma(1)}, \boldsymbol{u}_{\sigma(1)}, \cdots, t_{\sigma(\ell)}, \boldsymbol{u}_{\sigma(\ell)}),$$

where σ is a permutation on $\{1, \cdots, \ell\}$ such that $t_{\sigma(1)} < \cdots < t_{\sigma(\ell)}$. Then we see that $[M_*^\ell]_{\psi_\ell} = \ell\,!\, M^\ell = M(S)^\ell \boldsymbol{P}_\ell$ on the measurable space $(\Omega_\ell, \mathscr{B}(\Omega_\ell))$, where $[M_*^\ell]_{\psi_\ell}$ is the image measure of M_*^ℓ induced by ψ_ℓ. We now compute

$$\boldsymbol{E}^*[\mathrm{e}^{\,i(z, Z(t))}] = \int_{\Omega^*} \exp[i(z, [\Phi(\omega^*)](t))]\, d\boldsymbol{P}^*(\omega^*)$$

$$
\begin{aligned}
&= \sum_{\ell=0}^{\infty} \int_{\Omega_\ell} \exp[i(z, [\Phi(\omega^*)](t))]\, e^{-M(S)} (\ell!)^{-1} M(S)^\ell \, dP_\ell(\omega^*) \\
&= e^{-M(S)} \sum_{\ell=0}^{\infty} (\ell!)^{-1} \int_{\Omega_\ell} \exp[i(z, [\Phi(\omega^*)](t))]\, d[M_*^\ell]_{\psi\ell}(\omega^*) \\
&= e^{-M(S)} \sum_{\ell=0}^{\infty} (\ell!)^{-1} \int\cdots\int_{S_*^\ell} \exp[i(z, \sum_{j=1}^{\ell} u_j \theta(t_j|t))]\, dM_*^\ell(t_1, u_1, \cdots, t_\ell, u_\ell) \\
&= e^{-M(S)} \sum_{\ell=0}^{\infty} (\ell!)^{-1} \int\cdots\int_{S^\ell} \prod_{j=1}^{\ell} \exp[i(z, u_j \theta(t_j|t))]\, dM^\ell(t_1, u_1, \cdots, t_\ell, u_\ell) \\
&= e^{-M(S)} \sum_{\ell=0}^{\infty} (\ell!)^{-1} \left[\iint_S \exp[i(z, u\theta(s|t))]\, dM(s, u)\right]^\ell \\
&= \exp\left[-M(S) + \iint_S \exp[i(z, u\theta(s|t))]\, dM(s, u)\right] \\
&= \exp\left[\iint_{[0,t]\times R_0^d} e^{i(z,u)}\, dM(s, u) - \iint_{[0,t]\times R_0^d} dM(s, u)\right] \\
&= \exp\left[\int_{R_0^d} \{e^{i(z,u)} - 1\}\, dM_t(u)\right].
\end{aligned}
$$

Thus we obtain the relation $Z(t) \sim (a(t), M_t)$. Q. E. D.

By using the expression

$$E^*[e^{i(z, Z(t))}] = \exp\left[\int_{R_0^d} \{e^{i(z,u)} - 1\}\, dM_t(u)\right],$$

we can further prove the following facts. When the support of M_t is bounded, we have

$$E^*[Z(t)] = \int_{R_0^d} u \, dM_t(u),$$

$$E^*\left[|Z(t) - E^*[Z(t)]|^2\right] = \int_{R_0^d} |u|^2 \, dM_t(u).$$

2. 5. Here we shall give another method of constructing processes with independent increments by using the technique of J. A. Veeh [4]. Let us consider the probability spaces $(S^\ell, \mathscr{B}(S^\ell), Q^{(\ell)})$, $\ell \geq 0$, where $Q^{(\ell)}$ is defined by $Q^{(\ell)} = M(S)^{-\ell} M^\ell$ for $\ell \geq 1$ and $(S^0, \mathscr{B}(S^0), Q^{(0)})$ stands for the trivial probability space. We denote by $(\Omega^+, \mathscr{F}^+, Q^+)$ the weighted direct sum of $(S^\ell, \mathscr{B}(S^\ell), Q^{(\ell)})$, $\ell \geq 0$, based on the Poisson distribution with intensity $M(S)$. We call $(\Omega^+, \mathscr{F}^+, Q^+)$ the *rectangular probability space* associated with $(S, \mathscr{B}(S), M)$. Then we shall define a map $\Psi : \Omega^+ \to F(T)$ in the following way. For any $\ell \geq 1$ and $\omega^+ = (t_1, u_1, \cdots, t_\ell, u_\ell) \in S^\ell$, we set

$$[\Psi(\omega^+)](t) = \sum_{j=1}^{\ell} u_j \theta(t_j|t) \qquad (t \in T).$$

For $\omega^+ \in S^0$, we set $[\Psi(\omega^+)](t) = 0$ $(t \in T)$. In this case the map Ψ is surjective but not injective. Now we introduce an R^d-valued process $Z = \{Z(t); 0 \leq t \leq T\}$ defined on the probability space $(\Omega^+, \mathscr{F}^+, Q^+)$, which is given by

$$Z(t, \omega^+) = [\Psi(\omega^+)](t) \qquad (t \in T, \omega^+ \in \Omega^+).$$

Then we can show that Theorem 1 is also true for this process Z.

3. Processes with infinitely many jumps: the case $M(S)=\infty$.

3. 1. Let M be a Borel measure on $\boldsymbol{S}=\boldsymbol{T}\times\boldsymbol{R}_0^d$ satisfying the conditions $\iint_S (1\wedge|\boldsymbol{u}|^2)\, dM(t,\boldsymbol{u})<\infty$ and $M(\{t\}\times\boldsymbol{R}_0^d)=0$ for each $t\in\boldsymbol{T}$. Let us introduce a decomposition $S=\bigcup_{n=1}^{\infty} S_n$, which is given by $S_n=\boldsymbol{T}\times B_n$ and $B_n=\left\{\frac{1}{n}\le|\boldsymbol{u}|<\frac{1}{n-1}\right\}$ $(1/0=\infty)$. Then we obtain a sequence $(\boldsymbol{S},\mathscr{B}(\boldsymbol{S}),M^{(n)})$ $(n\ge 1)$ of finite measure spaces, which are defined by $M^{(n)}(U)=M(U\cap S_n)$ for any $U\in\mathscr{B}(S)$. We denote by $\{M_t\}$ and $\{M_t^{(n)}\}$ the spectral measures associated with M and $M^{(n)}$ respectively.

3. 2. For each $n\ge 1$, we denote by $(\Omega^*,\mathscr{F}^*,\boldsymbol{P}^{*(n)})$ the triangular probability space associated with $(\boldsymbol{S},\mathscr{B}(\boldsymbol{S}),M^{(n)})$. Then we introduce the product probability space $(\tilde{\Omega},\tilde{\mathscr{F}},\tilde{\boldsymbol{P}})=\prod_{n=1}^{\infty}(\Omega^*,\mathscr{F}^*,\boldsymbol{P}^{*(n)})$ and the n-th projection $\pi_n:\tilde{\Omega}=(\Omega^*)^\infty\to\Omega^*$, which is given by $\pi_n(\tilde{\omega})=\omega_n^*$ for each $\tilde{\omega}=(\omega_1^*,\omega_2^*,\cdots)\in\tilde{\Omega}$.

By using the map $\Phi:\Omega^*\to F(\boldsymbol{T})$ stated in Section 2, we set

$$Z_n(t,\tilde{\omega})=[\Phi(\pi_n(\tilde{\omega}))](t) \qquad (t\in\boldsymbol{T},\ \tilde{\omega}\in\tilde{\Omega}).$$

Then we have $\boldsymbol{R}^d$-valued processes $\boldsymbol{Z}_n=\{Z_n(t)\,;\,0\le t\le T\}$ $(n\ge 1)$ defined on $(\tilde{\Omega},\tilde{\mathscr{F}},\tilde{\boldsymbol{P}})$. We note that these processes are independent and the probability law of each $\boldsymbol{Z}_n$ can be described by Theorem 1. In other words, putting $a_n(t)=\int_{\boldsymbol{R}_0^d}\frac{\boldsymbol{u}}{1+|\boldsymbol{u}|^2}\,dM_t^{(n)}(\boldsymbol{u})$, we have the relation $Z_n(t)\sim(a_n(t),M_t^{(n)})$.

3. 3. Let $a(t)$ be an arbitrary $\boldsymbol{R}^d$-valued continuous function on $\boldsymbol{T}$ with $a(0)=0$. We now introduce $\boldsymbol{R}^d$-valued processes $\boldsymbol{W}_n=\{W_n(t)\,;\,0\le t\le T\}$ $(n\ge 1)$ defined on $(\tilde{\Omega},\tilde{\mathscr{F}},\tilde{\boldsymbol{P}})$, which are given by

$$W_n(t,\tilde{\omega})=a(t)+\sum_{k=1}^{n}\{Z_k(t,\tilde{\omega})-a_k(t)\} \qquad (t\in\boldsymbol{T},\ \tilde{\omega}\in\tilde{\Omega}).$$

Then we have the following

THEOREM 2. *There exists an $\boldsymbol{R}^d$-valued process $\boldsymbol{W}=\{W(t)\,;\,0\le t\le T\}$ defined on $(\tilde{\Omega},\tilde{\mathscr{F}},\tilde{\boldsymbol{P}})$, for which the following conditions hold:*

(i) *Almost all sample functions of $\boldsymbol{W}$ belong to the space $D(\boldsymbol{T})$;*

(ii) *For a. a. $\tilde{\omega}\in\tilde{\Omega}$, sample functions $W_n(t,\tilde{\omega})$ converges uniformly on $\boldsymbol{T}$ to $W(t,\tilde{\omega})$ as $n\to\infty$;*

(iii) *The probability law of $\boldsymbol{W}$ is given by the relation $W(t)\sim(a(t),M_t)$.*

Proof. Let us write $W_n(t,\tilde{\omega})$ $(n\ge 2)$ in the form

$$W_n(t,\tilde{\omega})=a(t)+\{Z_1(t,\tilde{\omega})-a_1(t)\}+Y_n(t,\tilde{\omega})+b_n(t),$$

where we set

$$Y_n(t,\tilde{\omega})=\sum_{k=2}^{n}\{Z_k(t,\tilde{\omega})-\tilde{\boldsymbol{E}}[Z_k(t)]\},$$
$$b_n(t)=\sum_{k=2}^{n}\{\tilde{\boldsymbol{E}}[Z_k(t)]-a_k(t)\}.$$

By means of the independence of $\{Z_k\}$, we have, for any $m > n \geq 2$,

$$\begin{aligned}\tilde{E}\left[\,|\,Y_m(T) - Y_n(T)\,|^2\right] &= \sum_{k=n+1}^{m} \tilde{E}\left[\,|\,Z_k(T) - \tilde{E}[Z_k(T)]\,|^2\right] \\ &= \sum_{k=n+1}^{m} \int_{R_0^d} |\,\boldsymbol{u}\,|^2\, dM_T^{(k)}(\boldsymbol{u}) \\ &= \int_{B(n,\,m)} |\,\boldsymbol{u}\,|^2\, dM_T(\boldsymbol{u}), \qquad B(n,\,m) = \{\frac{1}{m} \leq |\,\boldsymbol{u}\,| < \frac{1}{n}\}.\end{aligned}$$

By using the Kolmogorov's inequality, we can show the following:

$$\tilde{P}\{\sup_{t \in T} |\,Y_m(t) - Y_n(t)\,| > \varepsilon\} \leq \varepsilon^{-2} \int_{B(n,\,m)} |\,\boldsymbol{u}\,|^2\, dM_T(\boldsymbol{u})$$

for any $\varepsilon > 0$ and any $m > n \geq 2$. Therefore it follows from the assumption on M that

$$\lim_{m,n\to\infty} \tilde{P}\{\sup_{t \in T} |\,Y_m(t) - Y_n(t)\,| > \varepsilon\} = 0.$$

Now by employing the Lévy's equivalence theorem for sums of independent random variables, we can show the existence of an $\boldsymbol{R}^d$-valued process $\boldsymbol{Y} = \{Y(t)\,;\, 0 \leq t \leq T\}$ with the following properties: Almost all sample functions of $\boldsymbol{Y}$ belong to the space $D(\boldsymbol{T})$ and, for a. a. $\tilde{\omega} \in \tilde{\Omega}$, sample functions $Y_n(t,\,\tilde{\omega})$ converges uniformly on $\boldsymbol{T}$ to $Y(t,\,\tilde{\omega})$ as $n \to \infty$. On the other hand, it follows from the assumptions on M that functions $a_n(t)$ and $b_n(t)$ are continuous on $\boldsymbol{T}$. Further $b_n(t)$ converges uniformly on $\boldsymbol{T}$ to a continuous function $b(t) = \int_{\{0<|\boldsymbol{u}|<1\}} \frac{|\,\boldsymbol{u}\,|^2}{1+|\,\boldsymbol{u}\,|^2}\, \boldsymbol{u}\, dM_t(\boldsymbol{u})$ as $n \to \infty$. Thus setting

$$W(t,\,\tilde{\omega}) = a(t) + \{Z_1(t,\,\tilde{\omega}) - a_1(t)\} + Y(t,\,\tilde{\omega}) + b(t),$$

we obtain the process $\boldsymbol{W} = \{W(t)\,;\, 0 \leq t \leq T\}$, for which the conditions (i) and (ii) hold. We now proceed to the condition (iii). First we note that

$$\tilde{E}\,[e^{i(z,\,Z_k(t))}] = \exp\left[\,i\,(z,\,a_k(t)) + \int_{R_0^d} \{e^{i(z,u)} - 1 - \frac{i\,(z,\,\boldsymbol{u})}{1+|\,\boldsymbol{u}\,|^2}\}\, dM_t^{(k)}(\boldsymbol{u})\right].$$

It follows that

$$\begin{aligned}\tilde{E}\,[e^{i(z,\,W_n(t))}] &= \exp\left[\,i\,(z,\,a(t)) + \sum_{k=1}^{n} \int_{R_0^d} \{e^{i(z,u)} - 1 - \frac{i\,(z,\,\boldsymbol{u})}{1+|\,\boldsymbol{u}\,|^2}\}\, dM_t^{(k)}(\boldsymbol{u})\right] \\ &= \exp\left[\,i\,(z,\,a(t)) + \int_{\{|u|\geq 1/n\}} \{e^{i(z,u)} - 1 - \frac{i\,(z,\,\boldsymbol{u})}{1+|\,\boldsymbol{u}\,|^2}\}\, dM_t(\boldsymbol{u})\right].\end{aligned}$$

Therefore we see by the condition (ii) that

$$\begin{aligned}\tilde{E}\,[e^{i(z,\,W(t))}] &= \lim_{n\to\infty} \tilde{E}\,[e^{i(z,\,W_n(t))}] \\ &= \exp\left[\,i\,(z,\,a(t)) + \int_{R_0^d} \{e^{i(z,u)} - 1 - \frac{i\,(z,\,\boldsymbol{u})}{1+|\,\boldsymbol{u}\,|^2}\}\, dM_t(\boldsymbol{u})\right].\end{aligned}$$

Thus we obtain the relation $W(t) \sim (a(t),\,M_t)$. Q. E. D.

3. 4. By modifying the process $\boldsymbol{W}$ stated in Theorem 2, we immediately obtain a Lévy process in the sense of K. Ito [3]. For each $U \in \mathscr{B}(S)$, we denote by $J_W(U,\,\tilde{\omega})$ the number of points $t \in \boldsymbol{T}$, for which $(t,\; W(t,\,\tilde{\omega}) - W(t-0,\,\tilde{\omega})) \in U$ holds. Then $\{J_W(U)\,;\, U \in \mathscr{B}(S)\}$ is a Poisson random measure on S with intensity measure M. In the case $M(S)=\infty$, almost all sample functions of $\boldsymbol{W}$ have infinitely many jumps on $\boldsymbol{T}$. Additionally we note that the discussion in Section 3 also covers the case $M(S)<\infty$.

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