J. FAC. SCI., SHINSHU UNIVERSITY, Vol. 27, No. 1, 1992

Generic curve families on 2-dimensional manifold

Hisao KAMIYA

Department of Mathematics, Faculty of Science, Shinshu University
Matsumoto 390, Japan
(Received July 28, 1992)

1. Abstract.

In the last paper [1] we show the local classification of generic curve families defined by symmetric bilinear forms on a compact 2-dimensional Riemannian manifold M. Here we give the definition of generic symmetric bilinear forms similarly to the definition of a Morse-Smale vector field in flows on M. The Morse-Smale theory is shown in [2],[3],[4],[5]. And we show that generic symmetric bilinear forms make open, dense and structural stable subset in bilinear forms on M.

2. Definitions.

Let M be a compact 2-dimensional manifold with a Riemannian metric G. And Q be a smooth symmetric bilinear form of the tangent bundle of M. For a point p of M, if$(Q-\lambda G)$ degenerates, λ is called an eigen value of Q at p. Let $W_\lambda=\{x \in T_pM \mid (Q-\lambda G)\ (x,*)=0\}$. If dim $W_\lambda=2$ *i.e.* an eigen value of Q is multiple, we call this point p a singular point. Let S be the set of singular points. S is a closed subset of M. In the open submanifold M$-$S which consists of regular points, the eigen space W_λ of the larger eigen value λ gives a smooth line field L on M$-$S. And an integral curve family is defined.

For a 2-dimensional Riemannian manifold, it is known that a local chart (h, U) can be chosen as a conformal diffeomorphism. $h_*G=\rho E$(where ρ is a positive smooth function on the Euclidian plane.) shows that the h_*-image of an eigen vector of Q is also an eigen value of h_*Q. And the line field L corresponds to the line field defined by h_*Q. Therefore the local theory of a curve family is similar to the one of the Euclidian plane.

3. Local theory.

We show the following result in the last paper[1]. We define C^r-topology on the space of all symmetric bilinear forms, $(r \geq 3)$. On the Euclidian plane, generic singular points are isolated and local topological types are classified into 3 types. Typical

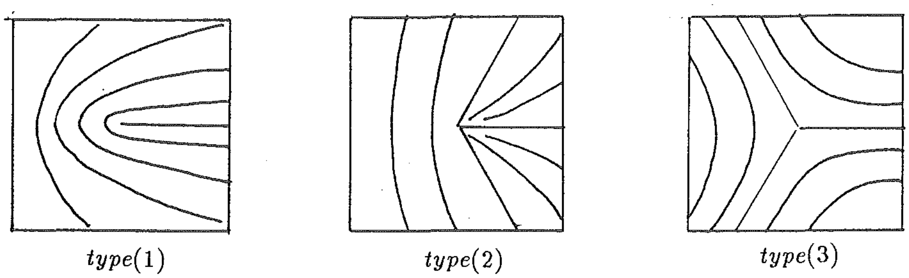

Fig. 1

singularities of 3 types are given by the following 3 symmetric bilinear forms. See fig. 1

1. $Q=\begin{pmatrix} x & y \\ y & -x \end{pmatrix}$. This singular point has 1 separatrix.
2. $Q=\begin{pmatrix} x & 3y \\ 3y & -x \end{pmatrix}$. This singular point has 2 separatrix.
3. $Q=\begin{pmatrix} x & -y \\ -y & -x \end{pmatrix}$. This singular point has 3 separatrix.

4. On vector fields.

For $p \in$ M$-$S, there is a neighbourhood V of p such that line field L is trivial on V. We can choose a unit vector field X in the line filed on V. The image of an integral flow of X is a curve of the curve family.

On the other hand, for vector field on M gives an integral flow. Image of the flow is equal to the curve family defined by the symmetric bilinear form $Q_p(x, y)=G(X_p, x) G(X_p, y)$. For the Euclidian plane, this means $Q={}^tXX$.

These results shows that the local behavior of curve families at regular points is quite similar to the one of flows. But around singular points situations are quite different.

5. MS-like symmetric bilinear forms.

We define Morse-Smale like symmetric bilinear forms MS in symmetric bilinear forms as follows. $Q \in MS$ iff Q satisfies the following 3 conditions.

1. The curve family has finite singular points of generic type and finite circles which is hyperbolic type as closed orbit of unit velocity vector field as in section 4.
2. Any curve does not connect two separatrices. (This condition corresponds to the non-existence of saddle connections in the Morse-Smale flow.)

3. The one-sided limit set of a curve C is a singular point or a circle. One side limit set means $\bigcup\{\overline{C_1} \mid C_1$ is a common side of cuts of the curve $C = C_1 \bigcup point \bigcup C_2.\}$

Theorem. *MS* is open, dense and structural stable subset of symmetric bilinear forms with C^r-topology, $(r \geq 3)$. The structural stability means that for any enough near symmetric bilinear forms Q_1, Q_2, corresponding curve families are topologically equivalent. And if there exists a homeomorphism from M to itself such that any curve of a curve family F_1 corresponds to a some curve of an other curve family F_2 then we say that the curve families F_1, F_2 are topologically equivalent. And this homeomorphism is called topological equivalence.

6. Example 1.

If Q has no singular points, the curve family is orientable. And the above theorem is similar to the Morse-Smale theorem for the corresponding unit vector field.

Example 2.

As in fig. 2 we define a bilinear form on a disk D^2. This curve families are symmetric for horizontal line and has 2 singular points of type (1). And we assume that any curve are perpendicularly transverse to the boundary. We paste 2 copies of this curve families at boundaries rotated for angle θ which is algebraically independent of π. Thus we get a curve family on the sphere S^2. The curve of this curve family which is transverse at a boundary point of angle x also is transverse at boundary points $x + 2\theta$, $x + 4\theta$, $x + 6\theta$,... and so on. Any orbit on the boundary is dense. So we see that any curve is dense in S^2. This behavior does not happen in the case of vector fields. Poincare Bendixon's theorem shows that any orbit of flow in S^2 is not dense.

7. Proof. The open and the structural stability

For singular points, condition (1) is given by transversality theory. If Q regularly transverse to the codimension-2 submanifold which consists of multiple eigen valued symmetric bilinear forms, then singular points are isolated. Further more if Q is

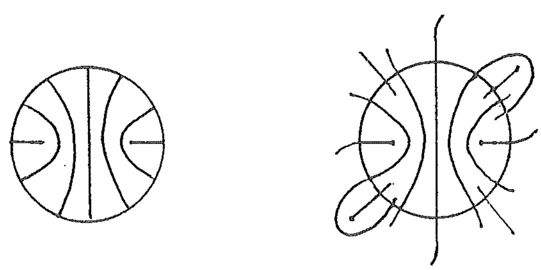

Fig. 2

regularly transverse to the discriminant of the separatrix equation [1], then all singular points are of generic type of section 3. And on some neighbourhood of a circle, the curve family is regarded as closed orbits of the unit vector field in section 4. From the Morse-Smale Theory, in the neighbourhood of a hyperbolic closed orbit the curve family is open and structural stable. The conditions (2) and (3) show that any curve's one-sided limit set is a circle or a type (2) singular point where the curve is not separatrix. These conditions show that structure of the one-sided curve family through some neighbourhood is trivial. Therefore local perturbations do not change topological structure of the curve family.

8. Proof. The denseness.

The denseness of the first condition of MS follows from Thom's transver-sality theory. Singular points are finite and all of them are generic, therefore separatrices consist of curves of finite number. The condition of circles is given similarly from the Morse-Smale theory.

These are measure 0 set of M. And if a separatrix connection exists, by a small perturbation on a neighbourhood of a point of the separatrix connection, the separatrix connection are vanished.

If the one-sided limit set of a curve C in the curve family is C, then C is a circle. If the one-sided limit set of a curve C is a circle, a small perturbation gives its hyperbolicity. These results are shown by the Morse-Smale theory.

Let p is a point of the one-sided limit set of a curve C. And let T be a orthogonal trajectory of the curve family through p, and c_1, c_2 be enough near points of p in T such that C is transverse to T at c_1 and c_2, and other points of the curve segment from c_1 to c_2, C don't intersect T. Let T' be a orthogonal trajectory translated along the curve family. As in fig. 3 a small perturbation on the region from T to T' gives a new curve family which has new circles as the one-sided limit set of every curve near by C. For this purpose, we classify behaviors of curve C near p into 3 cases.

Case 1. We assume that C is transverse to T at c_1 and c_2 in the same curve direction, and a neighbourhood of T at c_1 is orientation preserving homeomorphic to a neighbourhood of T at c_2 by the first return map of the curve family. Let a_1, b_1, d_1 be points of the neighbourhood of c_1, and a_2, b_2, d_2 points of the neighbourhood of c_2 such that a_1, b_1, c_1, d_1 correspond to a_2, b_2, c_2, d_2 by the fist return map. Now we perturb the curve family such thas a_1, b_2, c_2, d_2, in T are correspond to a_1, b_2, c_2, d_2, in T' by the new curve family. Then the curves through the point b_2 in T and c_2 in T are circles, and they are hyperbolic in generic and C and a curve near to it has an above circle as the one-sided limit set.

Case 2. We assume that C is transverse to T at c_1 and c_2 in the same curve direction, and a neighbourhood of T at c_1 is orientation reversing homeomorphic to a neighbourhood of T at c_2 by the first return map of the curve family. The curve returns to c_3 of T by the first return map orientation reversingly. We can assume that c_2 is between c_1 and c_3. If not, for c_1 and c_3 we can apply the case 1. Let a_1,b_1,d_1 be points of the neighbourhood of c_1, and a_2, b_2, d_2 points of the neighbourhood of c_2 such that a_1, b_1, c_1, d_1 correspond to a_2, b_2, c_2, d_2 by the first return map. And a_3, b_3, d_3 are points of the neighbourhood of c_3 such that a_2, b_2, c_2, $d_2$correspond to a_3, b_3, c_3, d_3 by the first return map.

Now we perturb the curve family such that a_1, b_3, c_3, d_3 in T are correspond to a_1, b_1, c_1, d_3 in T' by the new curve family. The segment a_1b_3 is translated to a_2b_2 in T by the first return map. By the fixed point theory this translation has a hyperbolic fixed point in generic. The curve through this point is hyperbolic circle. C and a curve near to it have the above circle as the one-sided limit set.

Case 3. We assume that C is transverse to T at c_1 and c_2 in the differnt curve direction, and a neighbourhood of T at c_1 is orientation reversing homeomorphic to a neighbourhood of T at c_2 by the first return map of the curve family. The curve returns from c_2 to c_3 of T by the first return map in the different curve direction and orientation reversing. We can assume that c_2 is between c_1 and c_3. If not, for c_1 and c_3 we can apply the case 1. and 2. Let a_1, b_1 be points of the neighbourhood of c_1, and a_2, d_2 points of the neighbourhood of c_2 such that a_1, c_1, d_1 correspond to a_2, c_2, d_2 by the first return map. And a_3, d_3 is points of the neighbourhood of c_3 such that a_2, c_2, d_2

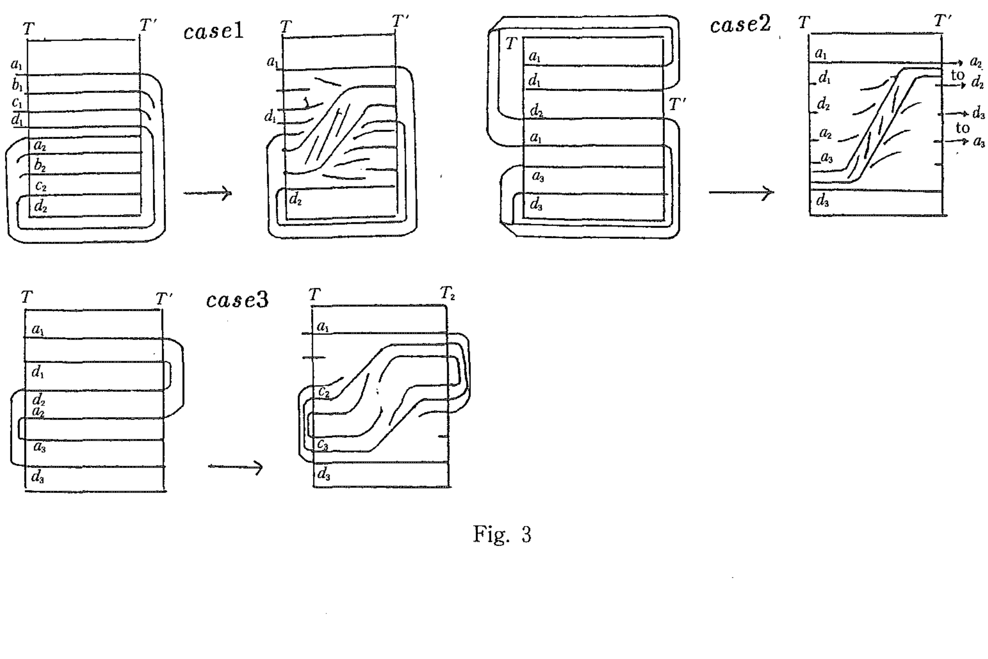

Fig. 3

correspond to a_3, c_3, d_3 by the first return map.

Now we perturb the curve family such that a_1, c_2, c_3, d_3 in T correspond to a_1, c_1, c_2, d_3 in T' by the new curve family.

Then the curve through the point c_2 in T is a circle, and it is hyperbolic in generic and C and a curve near to it has the above circle as the one-sided limit set.

This result can be proved in the case of orientation preserving or the case of orientation preserving and reversing by the quite same way.

These local perturbations make a MS type curve family enough close to the given curve family. This shows the denseness of MS.

References

[1] H. Kamiya Singular Points of curve families on surfaces. 1986. Jour. of faculity of sciences, Shinshu University.

[2] Palis and de Melo Geometric Theory of Dynamical Systems 1982. Springer-Verlag

[3] J. Palis and S. Smale, Structural stability theorem In: Global Analysis. Proc. Symp. in Pure Math vol.XIV A. M. S.

[4] J. Palis, On Morse-Smale bynamical system Topology 8. 1969

[5] M. Peixoto, Structural stability on two dimensional manifolds Topology 1. 1962