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Curve families on the plane defined by linear matrix fields

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Abstract : We give the topological classification of curve families on R^2 that is given by the set of integral curves of the eigen space field of a matrix field where component functions of the matrix are linear functions.

In the paper [2], we showed the local topological classification of generic curve families defined by symmetric bilinear forms on a 2-dimensional Riemannian manifold. In [3] we discussed the topological classification of global generic curve families defined by symmetric bilinear forms on a compact 2-dimensional Riemannian manifold.

In this paper, we give the topological classification of curve families on the Euclidian plane R^2 defined by fields of 2-2 matrices such that their components are linear functions. A given matrix field on R^2 defines an eigen space field of the larger eigen value on the tangent bendle of R^2. It gives a set of integral curves. We call this a curve family. We consider the case that each component of the matrix is a linear function of R^2. The origin of R^2 is a singular point of the curve family. But this singular point is not generic. Its codimension of the origin $(=3)$ is greater than the dimension of the plane $(=2)$.

The similar topological classification of linear flows on the Euclidian space R^n is well-known. Details of this fact are found in [1].

Let $M=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a field of 2-2 matrices on R^2, where a, b, c and d are smooth functions of the coordinates x, y of R^2. For a point p of R^2, let λ be an eigen value of M at p, and $W_\lambda=\{x \in R^2 \mid (M-\lambda E)x=0\}$ be the eigen space of λ. If dim $W_\lambda=2$ or there is no eigen value, we call this point p a singular point. We distinguish these 2 tpes singular points. The eigen spaces of the matrix M are equal to the one of the matrix $\begin{pmatrix} (a-d)/2 & b \\ c & (-a+d)/2 \end{pmatrix}$. From now we assume that $a=-d$. In this case $tr\ M$ is always 0.

The condition of singular points is $a=b=c=0$ or $a^2+bc>0$. Let S be the set of singular points. Eigen spaces W_λ of the larger eigen values λ give a smooth line field L on R^2-S. An integral curve family is defined by L. We call this the curve family defined by M. The curve family defined by $-M$ is the same one defined by the eigen space field defined by smaller eigen values.

Let C_1 and C_2 be smooth curve family on R^2 with singular points set S_1 and S_2. If there exists a homeomorphism h from R^2 to itself such that h maps S_1 onto S_2 preserving a type of singular point, and for any curve of C_1 onto a curve of C_2. We call h a topological equivalence of C_1 and C_2, and we call that C_1 is topological equivalent to C_2.

Now we consider the particular case of M that a, b and c are linear functions of x, y.

The definition of the eigen vector (u, v) and the eigen value λ is $\begin{pmatrix} a & b \\ c & -a \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}=\lambda\begin{pmatrix} u \\ v \end{pmatrix}$.

This follows $au+bv=\lambda u$ and $cu-av=\lambda v$. Let $\Phi(x, y, u, v)=cu^2-2auv-bv^2$. This polynomial is a homogeneous polynomdial of degree 1 with respect to x and y, and a homogeneous polynomial of degree 2 with respect to u and v. And the eigen vector condition $\Phi=0$ gives a graph Γ in the torus T^2. At first we give some elementary properties of it.

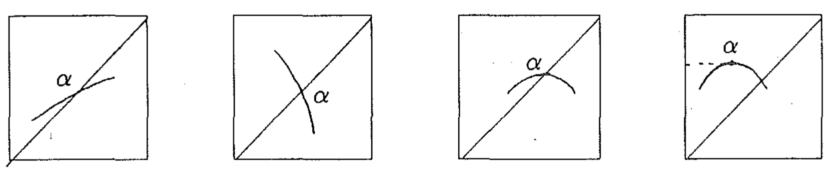
Proposition. 1. *Amlost lines $l=\{x : y$ is constant$\}$ intersect in 2 points or does not intersect with Γ. If it intersects in 3 or more points then l is contained in Γ.*

Proposition. 2. *Almost lines $l=\{u : v$ is constant$\}$ intersect in only 1 point with Γ. If it intersects in 2 or more points then l is contained in Γ.*

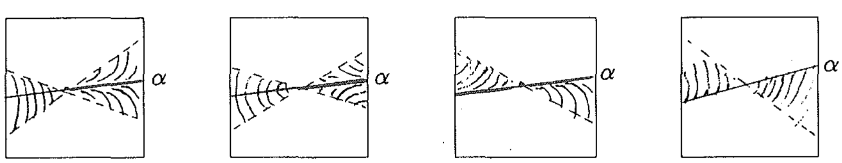
Proposition. 3. *If Γ includes a line of the type in Prop. 1 or Prop. 2, then Φ is factorized to 2 polynomials.*

These results are followed from elementary properties of homogeneous polynomial Φ.

Proposition. 4. *Let $\Delta=\{x : y=u : u\}$ be the diagonal line in T^2. At an intersection point of Γ and Δ, one of the half lines starting from the origin directed to $x : y$ is a curve of the curve family.*



inner halfline *outer halfline* *critical halfline* *critical*



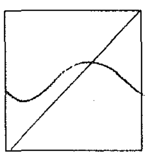
Proposition. 5. *At the intersection point p, let $1<d(u:v)/d(x:y)(p)$ where $d(u:v)$ means the differential of the standard coordinate of S^1. This means that the growth of $(x:y)$ is superior to the growth of $(u:v)$. So every curve of the curve family tends to 0 with the moving to the direction to the half line. We call this half line an inner half line of the curve family.*

Proposition. 6. *On the other hand if $d(u:v)/d(x:y)(p)<1$ then the growth of $(x:y)$ is inferior to the growth of $(u:v)$. So every curve of the curve family tends to ∞ with the moving to the direction to the half line. We call this half line an outer half line of the curve family.*

These two results are useful to determine the topological type of the curve family.

Proposition. 7. *If there exists a point p with $d(x:y)/d(u:v)=0$, then Γ contains the line $\{x:y \text{ is constant}\} \ni p$ or it is a simple critical point $(d^2(x:y)/d(u:v)^2 \neq 0)$. In the second case, the one outside region of the half line, there exists no eigen space. And on the half line, eigen values are multiple.*

Using these facts, we get the following classification of the polynomial Φ. Each following case defines a topological type of curve families.



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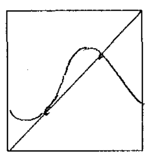

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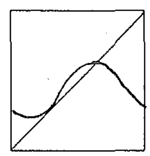

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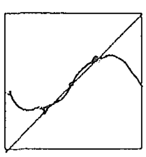

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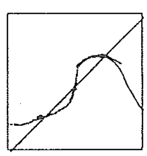

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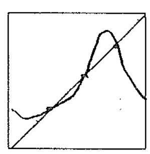

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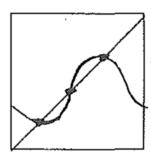

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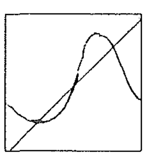

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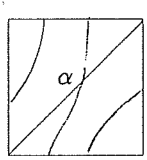

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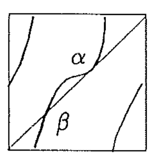

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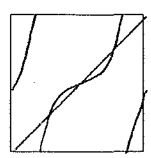

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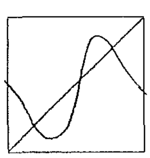

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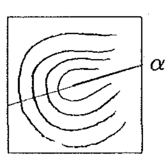
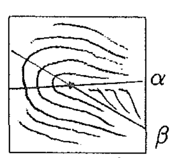
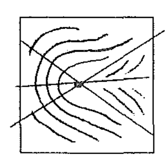






Case 1. Trivial case. $a=b=c=0$. All points of R^2 are singular.

Case 2. We assume that Φ is factorized to $(\alpha x+\beta y)\Psi(u, v)$.

Case 2.1. If $\Psi=0$ has no solution in S^1. then $(\alpha x+\beta y)=0$ is the set of singular points that has a 2 dimensional eigen space. And other points are singular points with no eigen spaces.

Case 2.2. If $\Psi=0$ has only one solution in S^1 and the solution is not equal to the direction of the singular line, the $(\alpha x+\beta y)=0$ is the set of singular points that has a 2 dimensional eigen space. And other points have constant 1 dimensional eigen spaces with a different direction to the singular line.

Case 2.3. If $\Psi=0$ has only one solution in S^1 and the solution is equal to the direction of the singular line, then the corresponding curve family is the set of parallel lines, one of that is the singular line.

Case 2.4. If $\Psi=0$ has 2 solutions in S^1 and solutions are different from the direction of the singular line, then $(\alpha x+\beta y)=0$ is the set of singular points that has a 2 dimensional eigen space. And each outside region of the singular line has the curve family of parallel lines with a different direction.

Case 2.5. If $\Psi=0$ has 2 solutions in S^1 and one solution is the same direction of the singular line, then $(\alpha x+\beta y)=0$ is the set of singular points that has a 2 dimensional eigen space. And each outside region of the singular line has the curve family of parallel lines with a different direction. One side of the parallel lines is parallel to the singular line.

Case 3. Φ is factorized to $(\alpha u+\beta v)\Psi(u, v, x, y)$.

Case 3.1. $\Psi=0$ defines the linear transformation ψ from S^1 to itself. We assume that the mapping degree of ψ is -1 then $\Psi=0$ and Δ intersects only 2 points of outer type. Moreover we assume that the intersection of $\Psi=0$, Δ and the singular line is empty.

Case 3.2. We assume the mapping degree of ψ is -1 then $\Psi=0$ and Δ intersect only 2 points of outer type. Moreover we assume that the intersection of $\Psi=0$, Δ and the singular line is not empty. Then the half line is parallel to constant lines l : $\alpha u+\beta v=0$.

Case 3.3. We assume that the mapping degree of ψ is 1 and $\Psi=0$ does not intersect to Δ.

Case 3.4. We assume that the mapping degree of ψ is 1 and $\Psi=0$ intersects to Δ. at only 1 point with a different direction to l.

Case 3.5. We assume that the mapping degree of ψ is 1 and $\Psi=0$ intersects to Δ. at only 1 point with the same direction to l.

Case 3.6. We assume that the mapping degree of ψ is 1 and $\Psi=0$ intersects to Δ. at 2 points with a different direction to l.

Case 3.7. We assume that the mapping degree of ψ is 1 and $\Psi=0$ intersects to Δ. at 2 points. And one of the points has the same direction to l.

Case 3.8. $\Psi=0$ coincides with Δ.

Case 4. $\Phi=0$ is represented to the explicit function $(x:y)=\phi(u:v)$. And mapping degree of ϕ is -2.

Case 5. $\Phi=0$ is represented to the explicit funhtion $(x:y)=\phi(u:v)$. And mapping degree of ϕ is 0.

Case 5.1. $\Phi=0$ intersects to Δ in 1 point with the type of outer half line.

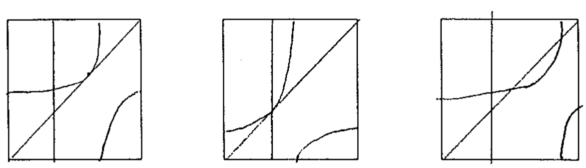
Case 5.2. $\Phi=0$ intersects to Δ in 1 point with the type of inner half line.

Case 5.3. $\Phi=0$ intersects to Δ in only a critical half line. Near the half line, all curves are contact to the half line.

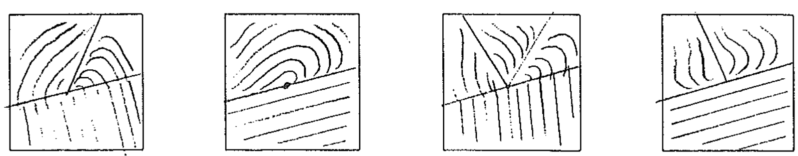
Case 5.4. $\Phi=0$ internects to Δ in 2 points. One of them is a tangent point and the other is the type of outer half line.

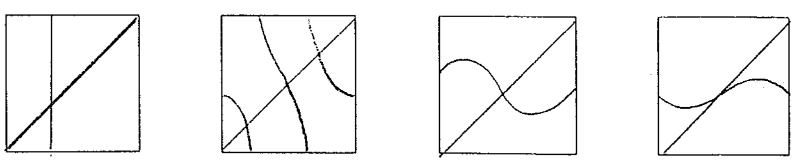
Case 5.5. $\Phi=0$ intersects to Δ in 2 points. One of them is a tangent point and the other is the type of inner half line.

Case 5.6. $\Phi=0$ intersects to Δ in 2 points. One of them is a tangent point and the

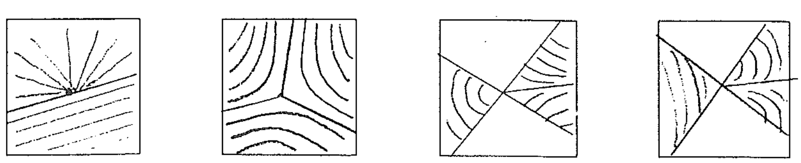


3.4 3.5 3.6 3.7





3.8 4. 5.1 5.2



other is a critical half line.

Case 5.7. $\Phi=0$ intersects to Δ in 3 points. Types of intersection points are inner, outer, inner. respectively.

Case 5.8. $\Phi=0$ intersects to Δ in 3 points. Types of intersection points are inner, outer, critical. respectively.

Case 5.9. $\Phi=0$ intersects to Δ in 3 points. Types of intersection points are inner, outer, outer.

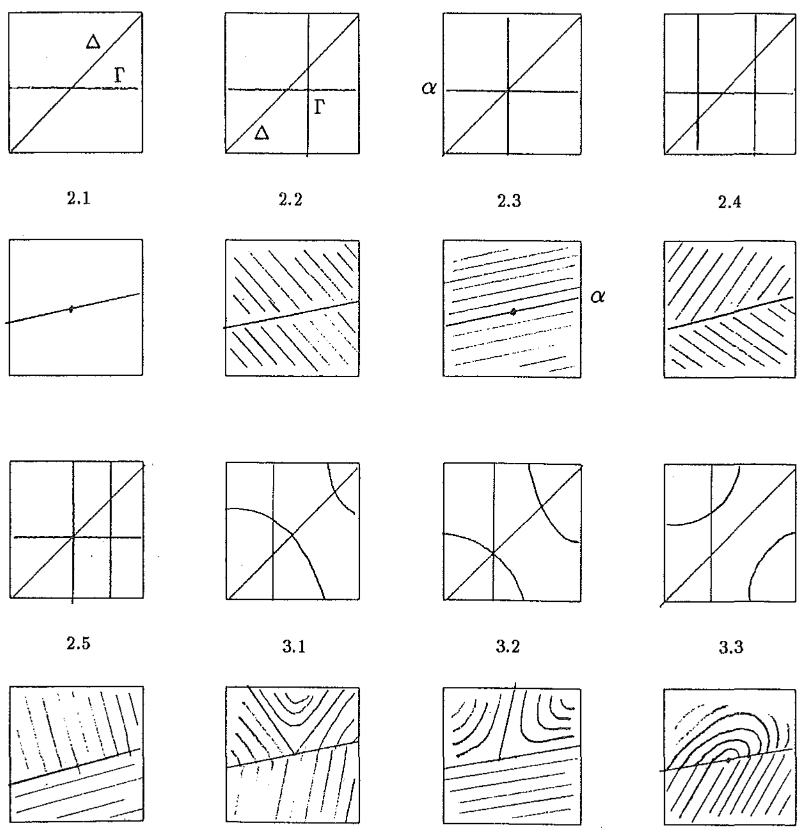
Case 5.10. $\Phi=0$ intersects to Δ in 3 points. Types of intersection points are critical, outer, critical. respectively.

Case 5.11. $\Phi=0$ intersects to Δ in 3 points. Types of intersection points are critical, outer, outer. respectively.

Case 5.12. $\Phi=0$ intersects to Δ in 3 points. Types of intersection points are outer, outer, outer. respectively.

Case 6. $\Phi=0$ is represented to the explicit function $(x:y)=\phi(u:v)$. And the mapping degree of ϕ is 2.

Case 6.1. The mapping degree of ϕ is 2 and $\Psi=0$ intersects to Δ in 1 point.



2.1 2.2 2.3 2.4

2.5 3.1 3.2 3.3

Case 6.2. The mapping degree of ϕ is 2 and $\Psi=0$ intersects to Δ in 2 points.

Case 6.3. The mapping degree of ϕ is 2 and $\Psi=0$ intersects to Δ in 3 points.

In cases of 5.3–5.12, we show only the graph of Γ, we abbreviate figures of the curve families. It is easy to give an example of the matrix field in each case.

Theorem 1. *Topological types of these cases are classified to following 18 types.*

Type 1.	*cases* 1,	*Type* 2.	*case* 2.1
Type 3.	*cases* 2.2	*Type* 4.	*case* 2.3, 2.5
Type 5.	*cases* 2.4	*Type* 6.	*cases* 4, 3.1, 3.2
Type 7.	*cases* 6.1, 3.3	*Type* 8.	*cases* 6.2, 6.3, 3.4–3.8
Type 9.	*cases* 5.1	*Type* 10.	*cases* 5.2
Type 11.	*case* 5.3	*Type* 12.	*case* 5.4
Type 13.	*case* 5.5, 5.7	*Type* 14.	*case* 5.6, 5.8
Type 15.	*case* 5.9	*Type* 16.	*case* 5.10
Type 17.	*case* 5.11	*Type* 18.	*case* 5.12

Theorem 2. *We restrict the matrix to the symmetric case ($b=c$), topological types are classified to following 6 types. In this condition, two eigen spaces are orthogonal. This follows that there exists only following types.*

Type 1.	*case* 1.	*Type* 2.	*case* 2.4
Type 3.	*case* 2.5	*Type* 4.	*case* 4
Type 5.	*case* 6.1	*Type* 6.	*cases* 6.2, 6.3

Three types 4,5 and 6 appear in generic. See [2], [3].

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