Equivalence of Probability Laws for a Class of Infinitely Divisible Random Measures

By

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Abstract

This is the extended exposition of the previous paper [2]. Given an infinitely divisible (or **ID**) random measure Λ on a measurable space **T**, we provide a certain method to construct a version of Λ based on a Poisson random measure on the product space $\mathbf{S} = \mathbf{T} \times (\mathbf{R} \backslash \{0\})$. In particular, the present paper contains a new result about a class of **ID** random measures on **T** which are realized by **R**-valued signed measures on **T**. As an application we discuss the law equivalence of **ID** random measures on **T** by using our constructive method with Kakutani's theorem on the equivalence of infinite product probability measures.

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§1. Introduction.

In this paper we are concerned with our method to construct an infinitely divisible random measure on a measurable space **T** based on a Poisson random measure on $\mathbf{S} = \mathbf{T} \times \mathbf{R}_0$, where we put $\mathbf{R}_0 = \mathbf{R} \backslash \{0\}$. First we recall some basic definitions and notations. Let **T** be an arbitrary nonempty set and $\mathfrak{X}$ be a δ-ring of subsets of **T**. We assume that there exists an increasing sequence $\{\mathbf{T}_n ; n \geq 1\} \subset \mathfrak{X}$ with $\mathbf{T} = \bigcup_{n=1}^{\infty} \mathbf{T}_n$ and $\{t\} \in \mathfrak{X}$ for each $t \in \mathbf{T}$. Let $\Lambda = \{\Lambda(A) ; A \in \mathfrak{X}\}$ be an *infinitely divisible* (or **ID**) *random measure* on **T** with no Gaussian component, which is defined on a basic probability space $(\Omega, \mathfrak{F}, \mathbf{P})$ (see [8]). Precisely speaking, Λ is a real stochastic process with the following two properties.

(Λ.1)   Each $\Lambda(A)$ is an infinitely divisible random variable with no Gaussian component ;

(Λ.2)   For every sequence of disjoint sets A_n ($n \geq 1$) in $\mathfrak{X}$, the sequence $\{\Lambda(A_n) ; n \geq 1\}$ is independent and, whenever $\bigcup_{n=1}^{\infty} A_n \in \mathfrak{X}$, we have

$$\Lambda(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \Lambda(A_n) \qquad \mathbf{P}\text{-almost surely.}$$

Then the characteristic function of $\Lambda(A)$ can be written in the Lévy's canonical form

(1.1) $\mathbf{E}[\exp(iz\Lambda(A))] = \exp[izv(A) + \iint_{A \times \mathbf{R}_0} g(z, x) M(dtdx)] \quad (z \in \mathbf{R}, \ A \in \mathfrak{X}),$

$$g(z, x) = \exp(izx) - 1 - izx\mathbf{1}_J(x), \quad J = (-1, 1),$$

where $v : \mathfrak{T} \to \mathbf{R}$ is an $\mathbf{R}$-valued signed measure on $\mathbf{T}$ and M is a measure on $\mathbf{S}$ satisfying

$$(1.2) \qquad\qquad \iint_{A \times \mathbf{R}_0} (1 \wedge x^2) M(dtdx) < \infty \qquad (A \in \mathfrak{T}).$$

We denote by $\mathbf{P}_A$ the probability measure on a measurable space $(\mathbf{R}^{\mathfrak{T}}, \mathfrak{F}(\mathbf{R}^{\mathfrak{T}}))$ induced by the map $\Lambda : \Omega \ni \omega \longrightarrow \Lambda(\cdot, \omega) \in \mathbf{R}^{\mathfrak{T}}$, where $\mathbf{R}^{\mathfrak{T}}$ is the set of all $\mathbf{R}$-valued functions on $\mathfrak{T}$ and $\mathfrak{F}(\mathbf{R}^{\mathfrak{T}})$ is the σ-algebra on $\mathbf{R}^{\mathfrak{T}}$ generated by all coordinate functions. We mean by $\Lambda = {}^d[v, M]$ that the probability measure $\mathbf{P}_A$ is determined by parameters v and M. Let $(\mathbf{S}, \mathfrak{S})$ be the product measurable space given by $\mathfrak{S} = \sigma(\mathfrak{T}) \otimes \mathfrak{B}(\mathbf{R}_0)$, where $\sigma(\mathfrak{T})$ is the σ-algebra on $\mathbf{T}$ generated by $\mathfrak{T}$ and $\mathfrak{B}(\mathbf{R}_0)$ is the Borel σ-algebra on $\mathbf{R}_0$. Let $\mathcal{N} = \mathcal{N}(\mathbf{S})$ be the totality of nonnegative (possibly infinite) integer-valued measures on $(\mathbf{S}, \mathfrak{S})$. Let $\mathfrak{B}(\mathcal{N})$ be the σ-algebra on $\mathcal{N}$ generated by all functions f^* on $\mathcal{N}$ given by

$$f^*(\nu) = <\nu, f> = \int_{\mathbf{S}} f d\nu \qquad \text{for } f \in \mathscr{F}^+(\mathbf{S}) \text{ and } \nu \in \mathcal{N},$$

where $\mathscr{F}^+(\mathbf{S})$ is the set of all nonnegative measurable functions on $(\mathbf{S}, \mathfrak{S})$. An $\mathcal{N}$-valued random element ξ is called a *Poisson random measure* on $\mathbf{S}$ with intensity M if it is defined on $(\Omega, \mathfrak{F}, \mathbf{P})$ and its Laplace transform is given by

$$(1.3) \quad \mathbf{E}[\exp(-<\xi, f>)] = \exp\left[-\iint_{\mathbf{S}} \{1 - \exp(-f(t,x))\} M(dtdx) \right] \qquad \text{for } f \in \mathscr{F}^+(\mathbf{S}).$$

The purpose of this paper is to construct a version of $\Lambda = {}^d[v, M]$ based on a Poisson random measure on $\mathbf{S}$ with intensity M. Our construction will be applied to the problem of law equivalence for **ID** random measures on $\mathbf{T}$. In Section 2 we introduce the notion of canonical probability space $(\tilde{\Omega}, \tilde{\mathfrak{F}}, \tilde{\mathbf{P}})$ corresponding to the measure space $(\mathbf{S}, \mathfrak{S}, M)$. Then we have a probability space $(\mathcal{N}, \mathfrak{B}(\mathcal{N}), \mathbf{Q}^M)$ such that the identity map I on $\mathcal{N}$ is a Poisson random measure on $\mathbf{S}$ with intensity M. In this connection, we obtain a criterion for the integrability of $f(t,x)$ with respect to a Poisson random measure on $\mathbf{S}$ when we impose on $\mathfrak{T}$ a certain additional condition (Theorem 3). Furthermore we obtain a class of **ID** random measures on $\mathbf{T}$ which are realized by $\mathbf{R}$-valued signed measures defined on $\mathfrak{T}$ (Theorem 4). In particular, when M satisfies the condition

$$(1.4) \qquad\qquad \iint_{A \times J} |x| M(dtdx) \equiv m(A) < \infty \quad (A \in \mathfrak{T}),$$

we see that sample functions of Λ are realized by $\mathbf{R}$-valued signed measures on $\mathbf{T}$ and also that Λ is expressed as a difference of independent nonnegative **ID** random measures on $\mathbf{T}$ (Theorem 5).

In order to discuss the problem of law equivalence, we shall introduce some notations. Given σ-finite measures μ and ν on a measurable space $(\mathbf{E}, \mathfrak{G})$, we mean by $\mu \ll \nu$ that μ is absolutely continuous with respect to ν. We mean by $\mu \sim \nu$ that μ and ν are equivalent, i.e., mutually absolutely continuous. The Hellinger-Kakutani distance and inner product are defined respectively by

$$\text{dist}(\mu, \nu) = \left[\int_E (\sqrt{d\mu} - \sqrt{d\nu})^2 \right]^{1/2} \quad \text{and} \quad \rho(\mu, \nu) = \int_E \sqrt{d\mu d\nu}.$$

Now given **ID** random measures Λ_1 and Λ_2 on **T**, we establish a sufficient condition for $\mathbf{P}_{\Lambda_1} \sim \mathbf{P}_{\Lambda_2}$ in terms of the parameters associated with Λ_1 and Λ_2.

Theorem 1. *Assume* $\Lambda_j = {}^d [v_j, M^{(j)}]$ $(j = 1, 2)$. *Then* $\mathbf{P}_{\Lambda_1} \sim \mathbf{P}_{\Lambda_2}$ *if the following three conditions hold simultaneously:*

(E.1) $M^{(1)} \sim M^{(2)}$,

(E.2) $\text{dist}(M^{(1)}, M^{(2)}) < \infty$,

(E.3) $\qquad v_1(A) - v_2(A) = \iint_{A \times J} x\{M^{(1)} - M^{(2)}\}(dtdx) \quad (A \in \mathfrak{T}).$

This theorem is closely related to the previous results stated in [1], [2], [3] and [4]. We shall prove Theorem 1 in Section 4. For this purpose we construct the versions of Λ_j $(j = 1, 2)$ along the procedure stated in Section 2. They are defined on the canonical probability spaces corresponding to $(\mathbf{S}, \mathfrak{S}, M^{(j)})$ $(j = 1, 2)$ respectively. The proof is then reduced to the Kakutani's theorem on the equivalence of infinite product probability measures. We note that (E.1) and (E.2) guarantee the law equivalence of Poisson random measures ξ_1 and ξ_2 on **S** with intensities $M^{(1)}$ and $M^{(2)}$ respectively (see [9]).

§2. A Construction of Infinitely Divisible Random Measures.

Let $\Lambda = {}^d [v, M]$ be an **ID** random measure on **T** stated in Section 1. The aim of this section is to construct a version of Λ based on a Poisson random measure on **S** with intensity M. For simplicity we may assume $M(\mathbf{S}) > 0$. We begin with the case that M is a finite measure on $(\mathbf{S}, \mathfrak{S})$.

Case (I): $M(\mathbf{S}) < \infty$. For each $k \geq 1$, let $(\mathbf{S}^k, \mathfrak{S}^k, \mathbf{P}_k)$ be a probability space given by $\mathbf{P}_k = M(\mathbf{S})^{-k} M^k$, where we mean by $(\mathbf{S}^k, \mathfrak{S}^k, M^k)$ the k-fold product measure space of $(\mathbf{S}, \mathfrak{S}, M)$. Then we consider a probability space $(\Omega^*, \mathfrak{F}^*, \mathbf{P}^*)$ defined by

(2.1) $\qquad \Omega^* = \bigcup_{k=0}^{\infty} \mathbf{S}^k, \qquad \mathfrak{F}^* = \left\{ A^* = \bigcup_{k=0}^{\infty} A_k; \ A_k \in \mathfrak{S}^k \ (k \geq 0) \right\},$

$\mathbf{P}^*(A^*) = \exp(-M(\mathbf{S})) \sum_{k=0}^{\infty} (k!)^{-1} M(\mathbf{S})^k \mathbf{P}_k(A_k) \qquad \text{for} \qquad A^* = \bigcup_{k=0}^{\infty} A_k \in \mathfrak{F}^*,$

where $(\mathbf{S}^0, \mathfrak{S}^0, \mathbf{P}_0)$ is the trivial probability space given by $\mathbf{S}^0 = \{0\}$ and $\mathfrak{S}^0 = \{\emptyset, \mathbf{S}^0\}$. We call $(\Omega^*, \mathfrak{F}^*, \mathbf{P}^*)$ the *basic canonical probability space* associated with $(\mathbf{S}, \mathfrak{S}, M)$. Let $\Phi : \Omega^* \longrightarrow \mathcal{N}$ be an $\mathfrak{F}^*/\mathfrak{B}(\mathcal{N})$-measurable map given by $\Phi(0) = 0$ and

(2.2) $\qquad <\Phi(\omega^*), f> = \sum_{i=1}^{k} f(p_i(\omega^*)) \qquad \text{for} \quad f \in \mathscr{F}^+(\mathbf{S})$

when $\omega^* = (p_1(\omega^*), \cdots, p_k(\omega^*)) \in \mathbf{S}^k$ $(k \geq 1)$. Then we obtain a Poisson random measure Φ on **S** with intensity M, which is defined on $(\Omega^*, \mathfrak{F}^*, \mathbf{P}^*)$. We define

(2.3) $\qquad \Gamma(A, \omega^*) = \iint_{A \times \mathbf{R}_0} x \Phi(dtdx, \omega^*) \quad (A \in \mathfrak{T}, \ \omega^* \in \Omega^*),$

(2.4) $\Lambda^*(A, \omega^*) = v(A) + \iint_{A \times \mathbf{R}_0} x\Phi(dtdx, \omega^*) - \iint_{A \times J} xM(dtdx)$ $(A \in \mathfrak{T}, \; \omega^* \in \Omega^*),$

where we put $\Phi(U, \omega^*) = [\Phi(\omega^*)](U)$ for $U \in \mathfrak{S}$ and $\omega^* \in \Omega^*$. Then we easily see

(2.5) $\mathbf{E}^*[\exp(iz\Gamma(A))] = \exp\left[\iint_{A \times \mathbf{R}_0} \{\exp(izx) - 1\}M(dtdx)\right]$ $(z \in \mathbf{R}, \; A \in \mathfrak{T}),$

where $\mathbf{E}^*[\,\cdot\,]$ stands for the expectation with respect to $\mathbf{P}^*$. Further we have

Proposition 1.   *The process $\Lambda^* = \{\Lambda^*(A); \; A \in \mathfrak{T}\}$ is an **ID** random measure on* **T** *which is defined on $(\Omega^*, \mathfrak{F}^*, \mathbf{P}^*)$ and characterized by $\Lambda^* =^d [v, M]$.*

Case (II): $M(\mathbf{S}) = \infty$. On account of (1.2) we can choose a sequence $\{\mathbf{S}_n; \; n \geq 1\} \subset \mathfrak{S}$ of disjoint subsets of **S** satisfying $\mathbf{S} = \bigcup_{n=1}^{\infty} \mathbf{S}_n$ and $0 < M(\mathbf{S}_n) < \infty \; (n \geq 1)$. Let $\{M_n; \; n \geq 1\}$ be a sequence of finite measures on $(\mathbf{S}, \mathfrak{S})$ defined by $M_n(U) = M(U \cap \mathbf{S}_n)$ for $U \in \mathfrak{S}$. Let us introduce an infinite product probability space

(2.6) $(\tilde{\Omega}, \tilde{\mathfrak{F}}, \tilde{\mathbf{P}}) = \prod_{n=1}^{\infty} (\Omega^*, \mathfrak{F}^*, \mathbf{P}_n^*),$

where $(\Omega^*, \mathfrak{F}^*, \mathbf{P}_n^*)$ is the basic canonical probability space associated with $(\mathbf{S}, \mathfrak{S}, M_n)$. We call $(\tilde{\Omega}, \tilde{\mathfrak{F}}, \tilde{\mathbf{P}})$ the *canonical probability space* associated with decomposition $M = \sum_{n=1}^{\infty} M_n$ on $(\mathbf{S}, \mathfrak{S})$. Let Ψ and Ψ_n be $\mathcal{N}$-valued random elements defined on $(\tilde{\Omega}, \tilde{\mathfrak{F}}, \tilde{\mathbf{P}})$ which are given by

(2.7) $\Psi = \sum_{n=1}^{\infty} (\Phi \circ \pi_n)$ and $\Psi_n = \sum_{i=1}^{n} (\Phi \circ \pi_i)$ $(n \geq 1).$

Here π_n denotes the n-th projection map from $\tilde{\Omega} = (\Omega^*)^{\infty}$ onto Ω^*. Then we have Poisson random measures Ψ and Ψ_n on **S** with intensities M and $M_{(n)} = \sum_{i=1}^{n} M_i$ respectively. Inspired by (2.4), we define, for each $n \geq 1$,

(2.8) $\tilde{\Lambda}_n(A, \tilde{\omega}) = v(A) + \iint_{A \times \mathbf{R}_0} x\Psi_n(dtdx, \tilde{\omega}) - \iint_{A \times J} xM_{(n)}(dtdx)$ $(A \in \mathfrak{T}, \; \tilde{\omega} \in \tilde{\Omega}).$

Then we have an **ID** random measure $\tilde{\Lambda}_n = \{\tilde{\Lambda}_n(A); \; A \in \mathfrak{T}\}$ on **T**, which is defined on $(\tilde{\Omega}, \tilde{\mathfrak{F}}, \tilde{\mathbf{P}})$ and characterized by $\tilde{\Lambda}_n =^d [v, M_{(n)}]$. Further we see the sequence $\{\tilde{\Lambda}_n(A); \; n \geq 1\}$ converges in law to $\Lambda(A)$ for each fixed $A \in \mathfrak{T}$. On account of the Lévy's equivalence theorem on the convergence of series with independent summands, we can find a random variable $\tilde{\Lambda}_{\infty}(A)$ defined on $(\tilde{\Omega}, \tilde{\mathfrak{F}}, \tilde{\mathbf{P}})$ to which $\{\tilde{\Lambda}_n(A)\}$ converges almost surely as $n \longrightarrow \infty$. Thus we have

Proposition 2.   *The process $\tilde{\Lambda}_{\infty} = \{\tilde{\Lambda}_{\infty}(A); \; A \in \mathfrak{T}\}$ is an **ID** random measure on* **T** *which is defined on $(\tilde{\Omega}, \tilde{\mathfrak{F}}, \tilde{\mathbf{P}})$ and characterized by $\tilde{\Lambda}_{\infty} =^d [v, M]$.*

Furthermore we shall provide an explicit representation of $\tilde{\Lambda}_{\infty}(A)$. For this purpose we replace (2.8) by the expression

(2.9) $\qquad \tilde{\Lambda}_n(A, \tilde{\omega}) = v(A) + \iint_{(A \times J) \cap R_n} x\{\Psi(dtdx, \tilde{\omega}) - M(dtdx)\}$

$$+ \iint_{(A \times J^c) \cap R_n} x\Psi(dtdx, \tilde{\omega}), \quad (A \in \mathfrak{T}, \ \tilde{\omega} \in \tilde{\Omega}),$$

where we put $R_n = \bigcup_{i=1}^n \mathbf{S}_i$ $(n \geq 1)$. Noting that $\tilde{\mathbf{E}}[\Psi(A \times J^c)] = M(A \times J^c) < \infty$, we see the second integral of (2.9) converges almost surely as $n \longrightarrow \infty$. Therefore taking the limit of (2.9) in probability, we immediately obtain

(2.10) $\qquad \tilde{\Lambda}_\infty(A, \tilde{\omega}) = v(A) + \iint_{A \times J} x\{\Psi(dtdx, \tilde{\omega}) - M(dtdx)\} + \iint_{A \times J^c} x\Psi(dtdx, \tilde{\omega})$

$\tilde{\mathbf{P}}$-almost surely for each $A \in \mathfrak{T}$.

In the rest of this section we are concerned with a realization of Λ based on the space $\mathcal{N} = \mathcal{N}(\mathbf{S})$ of nonnegative integer-valued measures on $(\mathbf{S}, \mathfrak{S})$. We mean by $(\mathcal{N}, \mathfrak{B}(\mathcal{N}), \mathbf{Q}^M)$ a probability space given by

(2.11) $\qquad \mathbf{Q}^M = [\mathbf{P}^*]_\Phi \quad$ in Case (I) $\quad$ and $\quad \mathbf{Q}^M = [\tilde{\mathbf{P}}]_\Psi \quad$ in Case (II),

where $[\mathbf{P}^*]_\Phi$ and $[\tilde{\mathbf{P}}]_\Psi$ stand for the images of $\mathbf{P}^*$ and $\tilde{\mathbf{P}}$ induced by Φ and Ψ respectively. Then the identity map I on $(\mathcal{N}, \mathfrak{B}(\mathcal{N}), \mathbf{Q}^M)$ is considered as a Poisson random measure on $\mathbf{S}$ with intensity M. Consequently we introduce a process $\mathbf{G} = \{G(A); A \in \mathfrak{T}\}$ which is defined on $(\mathcal{N}, \mathfrak{B}(\mathcal{N}), \mathbf{Q}^M)$ and expressed in the form

(2.12) $\quad G(A, \nu) = v(A) + \iint_{A \times J} x\{\nu(dtdx) - M(dtdx)\} + \iint_{A \times J^c} x\nu(dtdx) \ (A \in \mathfrak{T}, \ \nu \in \mathcal{N})$.

Precisely speaking, for each $A \in \mathfrak{T}$, the first integral of (2.12) is defined as the limit in probability of the sequence $\left\{\iint_{(A \times J) \cap R_n} x\{\nu(dtdx) - M(dtdx)\}; \ n \geq 1\right\}$ in Case (II). We note that the random variable $G(A)$ is well defined by the above discussion. Then Propositions 1 and 2 yield immediately the following

Theorem 2. *The process* $\mathbf{G} = \{G(A); A \in \mathfrak{T}\}$ *is an* **ID** *random measure on* $\mathbf{T}$ *which is defined on* $(\mathcal{N}, \mathfrak{B}(\mathcal{N}), \mathbf{Q}^M)$ *and characterized by* $\mathbf{G} = {}^d[v, M]$.

§3. A Subclass of Infinitely Divisible Random Measures.

The purpose of this section is to provide a class of **ID** random measures on $\mathbf{T}$ which are realized by $\mathbf{R}$-valued signed measures on $\mathbf{T}$. First we consider the integrability of measurable functions with respect to the Poisson random measure on $\mathbf{S}$ which is associated with $\Lambda = {}^d[v, M]$. Let $(\mathcal{N}, \mathfrak{B}(\mathcal{N}), \mathbf{Q}^M)$ be the probability space given by (2.11). Let $\psi(t, x)$ be an $\mathbf{R}$-valued measurable function on $(\mathbf{S}, \mathfrak{S})$. We put

(3.1) $\qquad m_\psi(A) = \iint_{A \times J} |\psi(t, x)| M(dtdx) \qquad (A \in \mathfrak{T})$,

(3.2) $\qquad V_\psi(A, \nu) = \iint_{A \times \mathbf{R}_0} |\psi(t, x)| \nu(dtdx) \qquad (A \in \mathfrak{T}, \ \nu \in \mathcal{N})$.

Proposition 3. *Suppose* $m_\psi(A) < \infty$. *Then there exists a set* $\mathcal{N}_0 \in \mathfrak{B}(\mathcal{N})$ *satisfying*

(3.3) $\mathbf{Q}^M(\mathcal{N}_0) = 1$ *and* $V_\psi(A, \nu) < \infty$ $(\nu \in \mathcal{N}_0)$.

Proof. By applying (1.3) to $f(t, x) = \mathbf{1}_A(t) |\psi(t, x)|$, we have

(3.4) $\mathbf{E}_{\mathbf{Q}^M}[\exp(-V_\psi(A, \cdot))] = \exp\left[-\iint_{A \times \mathbf{R}_0} \{1 - \exp(-|\psi(t, x)|)\} M(dtdx)\right]$,

where $\mathbf{E}_{\mathbf{Q}^M}[\cdot]$ stands for the expectation with respect to $\mathbf{Q}^M$. Putting

$$m_j(A) = \iint_{A \times I(j)} \{1 - \exp(-|\psi(t, x)|)\} M(dtdx) \quad (j = 1, 2)$$

with $I(1) = J$ and $I(2) = J^c$, we have $m_1(A) \le m_\psi(A)$, $0 \le m_2(A) < \infty$ and
$$-\log \mathbf{E}_{\mathbf{Q}^M}[\exp(-V_\psi(A, \cdot))] = m_1(A) + m_2(A) \le m_\psi(A) + m_2(A) < \infty.$$
Then we have $\mathbf{E}_{\mathbf{Q}^M}[\exp(-V_\psi(A, \cdot))] > 0$. Further putting $\mathcal{N}_0 = \{\nu \in \mathcal{N}; V_\psi(A, \nu) < \infty\}$, we see this implies $\mathbf{Q}^M(\mathcal{N}_0) > 0$. On the other hand we can apply the Kolmogorov's 0-1 law to the expression

(3.5) $V_\psi(A, \nu) = \sum_{n=1}^\infty \iint_{A \times B_n} |\psi(t, x)| \nu(dtdx)$ $(\nu \in \mathcal{N})$,

where $B_n = \{x \in \mathbf{R}_0; 1/n \le |x| < 1/(n-1)\}$ $(n \ge 1)$ with $1/0 = \infty$. Thus we obtain $\mathbf{Q}^M(\mathcal{N}_0) = 1$.

Remark. Whenever $\psi(t, x)$ is bounded on $A \times J$, we can show the condition $m_\psi(A) < \infty$ is necessary in order to obtain $\mathcal{N}_0 \in \mathfrak{B}(\mathcal{N})$ satisfying (3.3).

For further investigation we need to introduce the following condition on $\mathfrak{T}$.
(3.6) For each $A \in \mathfrak{T}$, there exists $n \ge 1$ such that $A \subset \mathbf{T}_n$.

Theorem 3. *Assume* (3.6) *and*
(3.7) $m_\psi(A) < \infty$ *for each $A \in \mathfrak{T}$.*
Then there exists a set $\mathcal{N}_0 \in \mathfrak{B}(\mathcal{N})$ satisfying
(3.8) $\mathbf{Q}^M(\mathcal{N}_0) = 1$ *and* $V_\psi(A, \nu) < \infty$ *for each $A \in \mathfrak{T}$ and $\nu \in \mathcal{N}_0$.*

Proof. It follows from (3.7) that Proposition 3 yields $\mathcal{N}_n \in \mathfrak{B}(\mathcal{N})$ $(n \ge 1)$ satisfying
(3.9) $\mathbf{Q}^M(\mathcal{N}_n) = 1$ and $V_\psi(\mathbf{T}_n, \nu) < \infty$ $(\nu \in \mathcal{N}_n)$.
Now putting $\mathcal{N}_0 = \bigcap_{n=1}^\infty \mathcal{N}_n$, we obtain (3.8) by (3.6) and (3.9).

We now provide here a class of **ID** random measures on **T** which are realized by **R**-valued signed measures on **T**. Given an **R**-valued measurable function $\psi(t, x)$ on **S**, we introduce a measure M_ψ on $(\mathbf{S}, \mathfrak{S})$ defined by

(3.10) $M_\psi(U) = M(\{(t, x) \in \mathbf{S}; (t, \psi(t, x)) \in U\})$ for $U \in \mathfrak{S}$.

For $A \in \mathfrak{T}$ and $\nu \in \mathcal{N}$, we put

$$(3.11) \qquad H_\psi(A, \nu) = v(A) + \iint_{A \times \mathbf{R}_0} \psi(t, x)\nu(dtdx) - \iint_{A \times J} \psi(t, x)M(dtdx)$$

provided the integrals in the right hand side are finite. Otherwise we put $H_\psi(A, \nu) = 0$. Furthermore, if we assume (3.7) and also

$$(3.12) \qquad \iint_{A \times \mathbf{R}_0} |\psi(t, x)|\mathbf{1}_J(\psi(t, x))M(dtdx) < \infty \qquad (A \in \mathfrak{T}),$$

we have an $\mathbf{R}$-valued signed measure $\hat{m}_\psi$ on $\mathbf{T}$ defined by

$$(3.13) \qquad \hat{m}_\psi(A) = \iint_{A \times \mathbf{R}_0} \psi(t, x)\{\mathbf{1}_J(\psi(t, x)) - \mathbf{1}_J(x)\}M(dtdx) \qquad (A \in \mathfrak{T}).$$

Theorem 4. *Assume* (3.6), (3.7) *and also*

$$(3.14) \qquad \iint_{A \times \mathbf{R}_0} (1 \wedge |\psi(t, x)|)M(dtdx) < \infty \qquad (A \in \mathfrak{T}).$$

Then the process $\mathbf{H}_\psi = \{H_\psi(A); A \in \mathfrak{T}\}$ *is an* **ID** *random measure on* $\mathbf{T}$ *which is defined on* $(\mathcal{N}, \mathfrak{B}(\mathcal{N}), \mathbf{Q}^M)$ *and characterized by* $\mathbf{H}_\psi =^d [v + \hat{m}_\psi, M_\psi]$.

Proof. First we introduce a map $I_\psi: \mathcal{N} \longrightarrow \mathcal{N}$ given by $I_\psi(\nu) = \nu_\psi$, where we put

$$(3.15) \qquad \nu_\psi(U) = \nu(\{(t, x) \in \mathbf{S}; (t, \psi(t, x)) \in U\}) \qquad (U \in \mathfrak{S}, \nu \in \mathcal{N}).$$

Then we see that I_ψ is a Poisson random measure on $\mathbf{S}$ with intensity M_ψ which is defined on $(\mathcal{N}, \mathfrak{B}(\mathcal{N}), \mathbf{Q}^M)$. On the other hand, Theorem 3 guarantees the existence of a set $\mathcal{N}_0 \in \mathfrak{B}(\mathcal{N})$ with $\mathbf{Q}^M(\mathcal{N}_0) = 1$ such that $H_\psi(A, \nu)$ is expressed by (3.11) for each $A \in \mathfrak{T}$ and $\nu \in \mathcal{N}_0$. On account of (3.14), $H_\psi(A, \nu)$ can be expressed as follows: For each $A \in \mathfrak{T}$ and $\nu \in \mathcal{N}_0$, we have

$$H_\psi(A, \nu) = v(A) + \hat{m}_\psi(A) + \iint_{A \times \mathbf{R}_0} \psi(t, x)\nu(dtdx) - \iint_{A \times \mathbf{R}_0} \psi(t, x)\mathbf{1}_J(\psi(t, x))M(dtdx)$$

$$= v(A) + \hat{m}_\psi(A) + \iint_{A \times \mathbf{R}_0} x\nu_\psi(dtdx) - \iint_{A \times J} xM_\psi(dtdx).$$

Therefore we immediately obtain the conclusion by Theorem 2.

By applying Theorem 4 to $\psi(t, x) = x$, we can realize $\Lambda =^d [v, M]$ in the space of $\mathbf{R}$-valued signed measures on $\mathbf{T}$ whenever both (3.6) and

$$(3.16) \qquad \iint_{A \times J} |x|M(dtdx) \equiv m(A) < \infty \qquad (A \in \mathfrak{T})$$

are satisfied. In detail, let $\mathbf{H}^+ = \{H^+(A); A \in \mathfrak{T}\}$, $\mathbf{H}^- = \{H^-(A); A \in \mathfrak{T}\}$ and $\mathbf{H} = \{H(A); A \in \mathfrak{T}\}$ be **ID** random measures on $\mathbf{T}$, which are defined on $(\mathcal{N}, \mathfrak{B}(\mathcal{N}), \mathbf{Q}^M)$ and expressed as follows:

$$(3.17) \qquad H^\pm(A, \nu) = v^\pm(A) + m(A) + \iint_{A \times \mathbf{R}_0} x^\pm\nu(dtdx) - \iint_{A \times J} x^\pm M(dtdx),$$

$$(3.18) \qquad H(A, \nu) = v(A) + \iint_{A \times \mathbf{R}_0} x\nu(dtdx) - \iint_{A \times J} xM(dtdx).$$

Here $v = v^+ - v^-$ stands for the Jordan decomposition of v. We put $\mathbf{R}_\pm = \{\pm x > 0\}$ and

$M_\pm(U) = M(U \cap (\mathbf{T} \times \mathbf{R}_\pm))$ for $U \in \mathfrak{S}$ respectively. Then we have

Theorem 5. *Assume* (3.6) *and* (3.16). *Then* $\mathbf{H}^\pm$ *and* $\mathbf{H}$ *are characterized by* $\mathbf{H}^\pm =^d$ $[v^\pm + m, M_\pm]$ *and* $\mathbf{H} =^d [v, M]$ *respectively. Furthermore* $\mathbf{H}^+$ *and* $\mathbf{H}^-$ *are independent and also there exists a set* $\mathcal{N}_0 \in \mathfrak{B}(\mathcal{N})$ *with* $\mathbf{Q}^M(\mathcal{N}_0) = 1$ *satisfying*

(3.19) $H(A, \nu) = H^+(A, \nu) - H^-(A, \nu)$ *and* $0 \le H^\pm(A, \nu) < \infty$ $(A \in \mathfrak{T}, \nu \in \mathcal{N}_0)$.

§4. The Proof of Theorem 1.

On account of (E.1) and (E.2) we may assume that $M^{(1)}(\mathbf{S})$ and $M^{(2)}(\mathbf{S})$ are simultaneously finite or infinite. Indeed, (E.1) yields

$$M^{(1)}(\mathbf{S}) = \int_{\{\phi \ge 16\}} \phi dM^{(2)} + \int_{\{\phi < 16\}} \phi dM^{(2)} \le 2 \int_{\mathbf{S}} (\sqrt{\phi} - 1)^2 dM^{(2)} + 16 M^{(2)}(\{\phi < 16\}),$$

where $\phi = dM^{(1)}/dM^{(2)}$. This implies that

$$M^{(i)}(\mathbf{S}) \le 2\mathrm{dist}(M^{(i)}, M^{(j)})^2 + 16 M^{(j)}(\mathbf{S})   (i, j = 1, 2).$$

Therefore combining (E.2) with these inequalities yields the assertion.

Case (I): $M^{(j)}(\mathbf{S}) < \infty$ ($j = 1, 2$). We construct the basic canonical probability space $(\Omega^*, \mathfrak{F}^*, \mathbf{P}^{*(j)})$ associated with $(\mathbf{S}, \mathfrak{S}, M^{(j)})$ for each $j = 1, 2$. It is obvious by the construction that $M^{(1)} \sim M^{(2)}$ implies $\mathbf{P}^{*(1)} \sim \mathbf{P}^{*(2)}$. Let us consider a family of random variables $\Xi(A, \omega^*)$ $(A \in \mathfrak{T})$ defined by

(4.1) $\Xi(A, \omega^*) = v_1(A) + \iint_{A \times \mathbf{R}_0} x \Phi(dtdx, \omega^*) - \iint_{A \times J} x M^{(1)}(dtdx)$ $(A \in \mathfrak{T}, \omega^* \in \Omega^*)$.

Then we see by (E.3) an alternative expression

(4.2) $\Xi(A, \omega^*) = v_2(A) + \iint_{A \times \mathbf{R}_0} x \Phi(dtdx, \omega^*) - \iint_{A \times J} x M^{(2)}(dtdx)$ $(A \in \mathfrak{T}, \omega^* \in \Omega^*)$.

Therefore we see by Proposition 1 that $\Xi = \{\Xi(A); A \in \mathfrak{T}\}$ is an **ID** random measure on $\mathbf{T}$, which is defined on $(\Omega^*, \mathfrak{F}^*, \mathbf{P}^{*(j)})$ and characterized by

(4.3) $\Xi =^d [v_j, M^{(j)}]$ with respect to $\mathbf{P}^{*(j)}$ $(j = 1, 2)$.

This implies that

(4.4) $\mathbf{P}_{A_j} = [\mathbf{P}^{*(j)}]_\Xi$ on $(\mathbf{R}^{\mathfrak{T}}, \mathfrak{F}(\mathbf{R}^{\mathfrak{T}}))$ $(j = 1, 2)$,

where $[\mathbf{P}^{*(j)}]_\Xi$ stands for the image of $\mathbf{P}^{*(j)}$ induced by the map $\Xi : \Omega^* \ni \omega^* \longrightarrow \Xi(\cdot, \omega^*) \in \mathbf{R}^{\mathfrak{T}}$. Thus combining (4.4) with $\mathbf{P}^{*(1)} \sim \mathbf{P}^{*(2)}$ yields the relation $\mathbf{P}_{A_1} \sim \mathbf{P}_{A_2}$.

Remark. In Case (I), we have the identity

(4.5) $\rho(\mathbf{P}^{*(1)}, \mathbf{P}^{*(2)}) = \exp[-(1/2)\mathrm{dist}(M^{(1)}, M^{(2)})^2]$.

Indeed, putting $M = M^{(1)} + M^{(2)}$, we construct the basic canonical probability space $(\Omega^*, \mathfrak{F}^*, \mathbf{P}^*)$ associated with $(\mathbf{S}, \mathfrak{S}, M)$. Then we have $\mathbf{P}^{*(j)} \ll \mathbf{P}^*$ $(j = 1, 2)$ and

$$\rho(\mathbf{P}^{*(1)}, \mathbf{P}^{*(2)}) = \int_{\Omega^*} \sqrt{d\mathbf{P}^{*(1)}/d\mathbf{P}^*} \sqrt{d\mathbf{P}^{*(2)}/d\mathbf{P}^*} \, d\mathbf{P}^*$$

$$= \sum_{k=0}^{\infty} \int_{\mathbf{S}^k} \sqrt{\exp{(M^{(2)}(\mathbf{S}))}dM^{(1)k}/dM^k} \sqrt{\exp{(M^{(1)}(\mathbf{S}))}dM^{(2)k}/dM^k} \; d\mathbf{P}^*$$

$$= \exp{[(1/2)M(\mathbf{S})]} \sum_{k=0}^{\infty} \exp{[-M(\mathbf{S})]}(k!)^{-1} \int_{\mathbf{S}^k} \sqrt{dM^{(1)k}/dM^k} \sqrt{dM^{(2)k}/dM^k} \; dM^k$$

$$= \exp{[-(1/2)M(\mathbf{S})]} \sum_{k=0}^{\infty} (k!)^{-1} \left[\int_{\mathbf{S}} \sqrt{dM^{(1)}/dM} \sqrt{dM^{(2)}/dM} \; dM \right]^k$$

$$= \exp{\left[-(1/2)\{M^{(1)}(\mathbf{S}) + M^{(2)}(\mathbf{S})\} + \int_{\mathbf{S}} \sqrt{dM^{(1)}/dM} \sqrt{dM^{(2)}/dM} \; dM \right]}$$

$$= \exp{\left[-(1/2) \int_{\mathbf{S}} \{\sqrt{dM^{(1)}/dM} - \sqrt{dM^{(2)}/dM}\}^2 dM \right]}$$

$$= \exp{[-(1/2)\mathrm{dist}(M^{(1)}, M^{(2)})^2]}.$$

Case (II): $M^{(j)}(\mathbf{S}) = \infty$ $(j=1, 2)$. First we note that the existence of the integral in (E.3) is guaranteed by (E.2). Indeed, putting $M = M^{(1)} + M^{(2)}$, we have

$$(4.6) \qquad \iint_{A \times J} |x| \cdot \|M^{(1)} - M^{(2)}\|(dtdx) = \iint_{A \times J} |x| \cdot |dM^{(1)}/dM - dM^{(2)}/dM| dM$$

$$= \iint_{A \times J} |\sqrt{dM^{(1)}/dM} - \sqrt{dM^{(2)}/dM}| \cdot |x|\{\sqrt{dM^{(1)}/dM} + \sqrt{dM^{(2)}/dM}\} dM$$

$$\leq \left[\iint_{A \times J} \{\sqrt{dM^{(1)}/dM} - \sqrt{dM^{(2)}/dM}\}^2 dM \right]^{1/2} .$$

$$\left[\iint_{A \times J} |x|^2 \{\sqrt{dM^{(1)}/dM} + \sqrt{dM^{(2)}/dM}\}^2 dM \right]^{1/2}$$

$$\leq \mathrm{dist}(M^{(1)}, M^{(2)}) \left[2 \iint_{A \times J} |x|^2 \{dM^{(1)}/dM + dM^{(2)}/dM\} dM \right]^{1/2}$$

$$= \mathrm{dist}(M^{(1)}, M^{(2)}) \left[2 \iint_{A \times J} |x|^2 \; dM \right]^{1/2} < \infty.$$

Here $\|w\|$ stands for the total variation measure of a signed measure w. According to the procedure stated in Section 2, we shall construct versions of Λ_1 and Λ_2 based on Poisson random measures on $\mathbf{S}$. We can find a sequence $\{\mathbf{S}_n; n \geq 1\}$ of disjoint subsets of $\mathbf{S}$ in $\mathfrak{S}$ satisfying $\mathbf{S} = \bigcup_{n=1}^{\infty} \mathbf{S}_n$ and $0 < M^{(j)}(\mathbf{S}_n) < \infty$ $(n \geq 1, j = 1, 2)$. For each $j = 1, 2$, we construct the canonical probability space

$$(\tilde{\Omega}, \tilde{\mathfrak{F}}, \tilde{\mathbf{P}}^{(j)}) = \prod_{n=1}^{\infty} (\Omega^*, \mathfrak{F}^*, \mathbf{P}_n^{*(j)})$$

associated with decomposition $M^{(j)} = \sum_{n=0}^{\infty} M_n^{(j)}$ on $(\mathbf{S}, \mathfrak{S})$, where we put $M_n^{(j)}(U) = M^{(j)}(U \cap \mathbf{S}_n)$ for $U \in \mathfrak{S}$. Now (E.1) implies $M_n^{(1)} \sim M_n^{(2)}$ and also $\mathbf{P}_n^{*(1)} \sim \mathbf{P}_n^{*(2)}$ for each $n \geq 1$. Further (E.2) with (4.5) implies

$$(4.7) \qquad \prod_{n=1}^{\infty} \rho(\mathbf{P}_n^{*(1)}, \mathbf{P}_n^{*(2)}) = \exp{\left[-(1/2)\sum_{n=1}^{\infty} \mathrm{dist}(M_n^{(1)}, M_n^{(2)})^2 \right]}$$

$$= \exp{\left[-(1/2)\mathrm{dist}(M^{(1)}, M^{(2)})^2 \right]} > 0.$$

Therefore we obtain $\tilde{\mathbf{P}}^{(1)} \sim \tilde{\mathbf{P}}^{(2)}$ by the Kakutani's theorem on the equivalence of infinite product probability measures (see [5]). By applying Proposition 2, we obtain stochastic processes $\tilde{\Lambda}_{\infty}^{(j)} = \{\tilde{\Lambda}_{\infty}^{(j)}(A); A \in \mathfrak{T}\}$ $(j = 1, 2)$ satisfying the following two conditions.

(4.8) $\tilde{\Lambda}_\infty^{(j)}$ is defined on $(\tilde{\Omega}, \tilde{\mathfrak{F}}, \tilde{\mathbf{P}}^{(j)})$ and characterized by $\tilde{\Lambda}_\infty^{(j)} = {}^d[v_j, M^{(j)}]$;

(4.9) For each $A \in \mathfrak{T}$, the sequence $\{\tilde{\Lambda}_n^{(j)}(A); n \geq 1\}$ converges almost surely to $\tilde{\Lambda}_\infty^{(j)}(A)$ with respect to $\tilde{\mathbf{P}}^{(j)}$ as $n \longrightarrow \infty$, where we put $M_{(n)}^{(j)} = \sum_{i=1}^n M_i^{(j)}$ and

$$(4.10) \quad \tilde{\Lambda}_n^{(j)}(A, \tilde{\omega}) = v_j(A) + \iint_{A \times \mathbf{R}_0} x \Psi_n(dtdx, \tilde{\omega}) - \iint_{A \times J} x M_{(n)}^{(j)}(dtdx) \quad (A \in \mathfrak{T}, \tilde{\omega} \in \tilde{\Omega}).$$

On account of (E.2) and (4.10) we have the following equations:

$$(4.11) \qquad \lim_{n \to \infty} \iint_{A \times J} x \{M_{(n)}^{(1)} - M_{(n)}^{(2)}\}(dtdx) = \iint_{A \times J} x\{M^{(1)} - M^{(2)}\}(dtdx),$$

$$(4.12) \qquad \tilde{\Lambda}_n^{(1)}(A, \tilde{\omega}) - \tilde{\Lambda}_n^{(2)}(A, \tilde{\omega}) = v_1(A) - v_2(A) - \iint_{A \times J} x\{M_{(n)}^{(1)} - M_{(n)}^{(2)}\}(dtdx)$$

for $A \in \mathfrak{T}$, $\tilde{\omega} \in \tilde{\Omega}$ and $n \geq 1$. Therefore combining (E.3) with $\tilde{\mathbf{P}}^{(1)} \sim \tilde{\mathbf{P}}^{(2)}$ yields that $\tilde{\Lambda}_\infty^{(1)}(A) = \tilde{\Lambda}_\infty^{(2)}(A)$ almost surely with respect to $\tilde{\mathbf{P}}^{(1)}$ and $\tilde{\mathbf{P}}^{(2)}$. Now putting $\Theta(A, \tilde{\omega}) = \tilde{\Lambda}_\infty^{(1)}(A, \tilde{\omega})$ for $A \in \mathfrak{T}$ and $\tilde{\omega} \in \tilde{\Omega}$, we have a process $\Theta = \{\Theta(A); A \in \mathfrak{T}\}$ defined on $(\tilde{\Omega}, \tilde{\mathfrak{F}}, \tilde{\mathbf{P}}^{(j)})$ for each $j = 1, 2$ and characterized by

(4.13) $\qquad\qquad \Theta = {}^d[v_j, M^{(j)}]$ with respect to $\tilde{\mathbf{P}}^{(j)}$ $(j = 1, 2)$.

This implies the equalities $\mathbf{P}_{A_j} = [\tilde{\mathbf{P}}^{(j)}]_\Theta$ $(j = 1, 2)$, where $[\tilde{\mathbf{P}}^{(j)}]_\Theta$ stands for the image of $\tilde{\mathbf{P}}^{(j)}$ induced by the map $\Theta: \tilde{\Omega} \ni \tilde{\omega} \longrightarrow \Theta(\cdot, \tilde{\omega}) \in \mathbf{R}^{\mathfrak{T}}$. Thus we obtain the desired relation $\mathbf{P}_{A_1} \sim \mathbf{P}_{A_2}$ from $\tilde{\mathbf{P}}^{(1)} \sim \tilde{\mathbf{P}}^{(2)}$.

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