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A classification of some S^3-bundles

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We denote by $\iota_n \in \pi_n(S^n)$ the homotopy class of the identity map of S^n. Let E_n be the S^3-bundle over S^7 induced from the canonical S^3-bundle $Sp(2)$ by $n\iota_7$. Let $E_{n,k}$ be the S^3-bundle over E_n induced from E_k by the projection $p_n : E_n \longrightarrow S^7$. Then we have a commutative diagram:

$$\begin{array}{ccccc} S^3 & & S^3 & & S^3 \\ \vdots & & \vdots & & \vdots \\ E_{n,k} & \longrightarrow & E_k & \longrightarrow & Sp(2) \\ \downarrow & & \downarrow p_k & & \downarrow \\ E_n & \xrightarrow{p_n} & S^7 & \xrightarrow{k\iota_7} & S^7. \end{array}$$

In [2] we encounter an obstruction element in $\pi_9(S^3)$ which detects the triviality of the bundle $E_{n,k}$. The purpose of this note is to show that $\pi_9(S^3)$ really classifies the S^3-bundles over E_n for some integer n.

As is well known ([2]), we have the following cell structure:

$$E_n = (S^3 \cup_{n\omega} e^7) \cup_{\gamma} e^{10},$$

where ω is the Blakers-Massey element generating $\pi_6(S^3) \cong \mathbf{Z}_{12}$ and γ is the attaching map of the top cell of E_n.

We set $Q_n = S^3 \cup_{n\omega} e^7$ and denote by $j : (Q_n, *) \longrightarrow (Q_n, S^3)$ the inclusion. Let χ be a generator of $\pi_7(Q_n, S^3) \cong \mathbf{Z}$. Then, by (5.1) of [3], we have

$$j_* \gamma = [\chi, \iota_3],$$

where $[\chi, \iota_3]$ is the relative Whitehead product of χ and ι_3.

We consider the following exact sequence induced from the cofibration $S^9 \xrightarrow{\gamma} Q_n \xrightarrow{i_n} E_n$:

$$\begin{array}{ccccccc} [E_n, BS^3] & \xleftarrow{q_n^*} & \pi_{10}(BS^3) & \xleftarrow{(\Sigma\gamma)^*} & [\Sigma Q_n, BS^3] & \xleftarrow{(\Sigma i_n)^*} & [\Sigma E_n, BS^3] \\ & & \| & & \| & & \| \\ & & \pi_9(S^3) & \xleftarrow{\gamma^*} & [Q_n, S^3] & \xleftarrow{i_n^*} & [E_n, S^3], \end{array}$$

where BS^3 is the classifying space and $q_n : E_n \longrightarrow S^{10}$ is a map pinching Q_n to one point.

If we can show that $i_n^* : [E_n, S^3] \longrightarrow [Q_n, S^3]$ is surjective, then the set $[E_n, BS^3]$

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is not trivial since $\pi_{10}(BS^3) \cong \pi_9(S^3) \cong \mathbf{Z}_3$. So our task is to examine the map i_n^*: $[E_n, S^3] \longrightarrow [Q_n, S^3]$.

Let $\eta_2 \in \pi_3(S^2)$ be the Hopf map and $\eta_n = \Sigma^{n-2}\eta_2$ for $n \geq 2$. We denote by (a,b) the greatest common divisor of two integers a and b. Set $c = \frac{12}{(12,n)}$. Then we have the following.

Lemma 1. i) *The set* $[Q_n, S^3]$ *consists of the element* $\omega\eta_6 g$ *and an extension* $\overline{cm\iota_3}$ *of* $cm\iota_3$ *for any integer* m, *where* $g: Q_n \longrightarrow S^7$ *is a map pinching* S^3 *to one point.*

ii) $[Q_n, BS^3] \cong \mathbf{Z}_{(12,n)}$.

Proof. In the exact sequence induced from the cofibration $S^6 \xrightarrow{n\omega} S^3 \xrightarrow{i} Q_n$, we have

$$\begin{array}{ccccccccc} \pi_6(S^3) & \xleftarrow{(n\omega)^*} & \pi_3(S^3) & \xleftarrow{i^*} & [Q_n, S^3] & \xleftarrow{g^*} & \pi_7(S^3) & \xleftarrow{(\Sigma n\omega)^*} & \pi_4(S^3) \\ \| & & \| & & & & \| & & \| \\ \mathbf{Z}_{12}\{\omega\} & & \mathbf{Z}\{\iota_3\} & & & & \mathbf{Z}_2\{\omega\eta_6\} & & \mathbf{Z}_2\{\eta_3\} \end{array}$$

and

$$\begin{array}{ccccccc} \pi_3(BS^3) & \xleftarrow{i^*} & [Q_n, BS^3] & \xleftarrow{g^*} & \pi_7(BS^3) & \xleftarrow{(\Sigma n\omega)^*} & \pi_4(BS^3) \\ \| & & & & \| & & \| \\ 0 & & & & \mathbf{Z}_{12} & & \mathbf{Z}. \end{array}$$

By use of the first exact sequence, we have that there exists an extension $\overline{cm\iota_3}$ for each $m \in \mathbf{Z}$ since

$$(n\omega)^*(cm\iota_3) = (cm\iota_3) \circ n\omega = \frac{mn}{(12,n)} 12\omega = 0.$$

On the other hand, by Lemma 5.7 of [5], we have

$$(\Sigma n\omega)^*\eta_3 = n(\eta_3 \circ \Sigma\omega) = 3n(\eta_3 \circ \Sigma\omega) = n\Sigma(\eta_2 \circ \nu') = 0.$$

Thus g^* is injective and $\omega\eta_6 g$ is a non-zero element of $[Q_n, S^3]$. This proves i).

In the second exact sequence, as $(\Sigma n\omega)^*: \mathbf{Z} \longrightarrow \mathbf{Z}_{12}$ maps 1 to n and g^* is surjective, we have $\mathrm{Ker}(g^*) = \mathrm{Im}(\Sigma n\omega)^* \cong n\mathbf{Z}_{12}$ and $[Q_n, BS^3] \cong \frac{\mathbf{Z}_{12}}{n\mathbf{Z}_{12}} \cong \mathbf{Z}_{(12,n)}$. This proves ii).

Let $h: S^7 \longrightarrow S^4$ be the Hopf map. Then we know the following ([1], [5]):

$$[\iota_4, \iota_4] = 2h \pm \Sigma\omega \text{ and } \pi_{10}(S^4) = \mathbf{Z}_{24}\{h \circ \Sigma^3 h\} \oplus \mathbf{Z}_3\{\Sigma(\omega \circ \Sigma^3\omega)\}.$$

We show

Lemma 2. $\Sigma\gamma = (\Sigma i)_*(a(h \circ \Sigma^3 h) + b\Sigma(\omega \circ \Sigma^3\omega))$ *for some integers* a *and* b.

Proof. We consider an anti-commutative diagram ([4]):

$$\begin{array}{ccccc} & & \pi_9(Q_n) & \xrightarrow{j_*} & \pi_9(Q_n, S^3) \\ & & \downarrow \Sigma & & \downarrow \Sigma' \\ \pi_{10}(S^4) & \xrightarrow{(\Sigma i)_*} & \pi_{10}(\Sigma Q_n) & \xrightarrow{j'_*} & \pi_{10}(\Sigma Q_n, S^4), \end{array}$$

where Σ' stands for the relative suspension and the lower sequence is exact. Since $j'_*(\Sigma\gamma) = -\Sigma'(j_*\gamma) = -\Sigma'[\chi, \iota_3] = 0$ by (2.30) of [4], there exists an element $\delta \in \pi_{10}(S^4)$ satisfying $\Sigma\gamma = (\Sigma i)_*\delta$. This completes the proof.

Remark. According to Lemma 2.32 of [4], δ is represented by the Hopf construction of a mapping of type $(\iota_3, n\omega)$. So we have

$$\Sigma\gamma = \pm 2n(\Sigma i)_*(h\circ\Sigma^3 h).$$

Now we shall prove the following.

Theorem. *$\gamma^*: [Q_n, S^3]\longrightarrow \pi_9(S^3)$ is trivial and $i_n^*: [E_n, S^3]\longrightarrow [Q_n, S^3]$ is surjective if* $n\not\equiv 0 \bmod 3$.

Proof. We consider the commutative diagram:

$$\begin{array}{ccc} \pi_9(S^3) & \xleftarrow{\gamma^*} & [Q_n, S^3] \\ \downarrow\Sigma & & \downarrow\Sigma \\ \pi_{10}(S^4) & \xleftarrow{(\Sigma\gamma)^*} & [\Sigma Q_n, S^4]. \end{array}$$

By [5], we have $\gamma^*(\omega\eta_6 g)=\omega\eta_6 g\gamma\in\omega\eta_6\pi_9(S^7)=\{\omega\eta_6\circ(\eta_7\circ\eta_8)\}=\{\omega\eta_6^3\}=\{6(\omega\circ\Sigma^3\omega)\}=0$.

Assume that there exists an integer m such that $\gamma^*\overline{cm\iota_3}=\omega\circ\Sigma^3\omega$. Then, by use of Lemmas 1 and 2, we have

$$\begin{aligned}\Sigma(\omega\circ\Sigma^3\omega) &= \Sigma(\gamma^*\overline{cm\iota_3}) = \Sigma\overline{cm\iota_3}\circ\Sigma i\circ(a(h\circ\Sigma^3 h)+b\Sigma(\omega\circ\Sigma^3\omega))\\ &=(cm\iota_4)\circ(a(h\circ\Sigma^3 h)+b\Sigma(\omega\circ\Sigma^3\omega))\\ &=cm\iota_4\circ a(h\circ\Sigma^3 h)+cm\iota_4\circ b\Sigma(\omega\circ\Sigma^3\omega)\\ &=a((cm\iota_4\circ h)\circ\Sigma^3 h)+bcm\Sigma(\omega\circ\Sigma^3\omega)\\ &=a\left(\left(cmh+\frac{cm(cm-1)}{2}[\iota_4,\iota_4]H(h)\right)\circ\Sigma^3 h\right)+bcm\Sigma(\omega\circ\Sigma^3\omega).\end{aligned}$$

Here H is the Hopf invariant and we have used the Hilton formula. Hence we have

$$\begin{aligned}(1-bcm)\Sigma(\omega\circ\Sigma^3\omega) &= a\left(\left(cmh+\frac{cm(cm-1)}{2}(2h\pm\Sigma\omega)\right)\circ\Sigma^3 h\right)\\ &=a\left(\left((cm)^2 h\pm\frac{cm(cm-1)}{2}\Sigma\omega\right)\circ\Sigma^3 h\right)\\ &=a\left(((cm)^2 h)\circ\Sigma^3 h\pm\frac{cm(cm-1)}{2}(\Sigma\omega\circ\Sigma^3 h)\right)\\ &=a(cm)^2(h\circ\Sigma^3 h)\pm\frac{acm(cm-1)}{2}\Sigma(\omega\circ\Sigma^2 h).\end{aligned}$$

Since $\Sigma(\omega\circ\Sigma^2 h)$ and $\Sigma(\omega\circ\Sigma^3\omega)$ are elements of the 3-primary component of $\pi_{10}(S^4)\cong \mathbf{Z}_{24}\oplus\mathbf{Z}_3$ ([1], [5]), we have $\Sigma(\omega\circ\Sigma^2 h)=-2\Sigma\omega\circ\Sigma^3 h=-\Sigma\omega\circ 2\Sigma^3 h=\pm\Sigma(\omega\circ\Sigma^3\omega)=\mp 2\Sigma(\omega\circ\Sigma^3\omega)$.

Thus we have

$$(1-bcm)\Sigma(\omega\circ\Sigma^3\omega)=a(cm)^2(h\circ\Sigma^3 h)-acm(cm-1)\Sigma(\omega\circ\Sigma^3\omega)$$

and

$$a(cm)^2(h\circ\Sigma^3 h)-(1-bcm+acm(cm-1))\Sigma(\omega\circ\Sigma^3\omega)=0.$$

This implies that

$$a(cm)^2\equiv 0 \bmod 24 \tag{1}$$

and

$$cm(b-a(cm-1))\equiv 1 \bmod 3 \tag{2}$$

By hypothesis $n \not\equiv 0 \bmod 3$, we have $(12, n) = 1, 2$ or 4, and so $c = 12, 6$ or 3 which contradicts (2). Hence we conclude that γ^* is trivial. This completes the proof.

Finally we have

Corollary. *If* $n \not\equiv 0 \bmod 3$, *then* $[E_n, BS^3] \neq 0$ *and if* $(12, n) = 1$, *then* $[E_n, BS^3] \cong \pi_9(S^3)$.

Proof. Making use of the exact sequence induced from the cofibration $S^9 \xrightarrow{\gamma} Q_n \xrightarrow{i_n} E_n$ and by Lemma 1. ii), we have

$$\begin{array}{ccccccc} [Q_n, BS^3] & \xleftarrow{i_n^*} & [E_n, BS^3] & \xleftarrow{q_n^*} & \pi_{10}(BS^3) & \xleftarrow{(\Sigma\gamma)^*} & [\Sigma Q_n, BS^3] \\ \| & & & & \| & & \| \\ \mathbf{Z}_{(12,n)} & & & & \pi_9(S^3) & \xleftarrow{\gamma^*} & [Q_n, S^3]. \end{array}$$

By Theorem, γ^* is trivial if $n \not\equiv 0 \bmod 3$, and so q_n^* is injective. This implies the first half. If $(12, n) = 1$, we have $[Q_n, BS^3] = 0$. This implies that q_n^* is surjective, so the second half follows.

Remark. Corollary tells us that there exist only 3 S^3-bundles over $E_1 = Sp(2), E_5, E_7$ and E_{11} up to isomorphism of bundles.

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