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The Dirichlet Problem for a Certain Degenerate Parabolic Equation, II

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Abstract : We make an extension of the result obtained in our preceding paper [1] for the degenerate parabolic equation $u_t=[x(1-x)]^\alpha u_{xx}(x,t)$ $(\alpha=1)$, studying the case of $0<\alpha<1$.

1. Introduction

In the preceding paper [1], for the case of $\alpha=1$, we have considered the Dirichlet problem for the degenerate parabolic equation

$$(1.1)^1 \qquad u_t(x,\ t)=w(x)^\alpha u_{xx}(x,\ t) \quad (0<x<1\ ,\ t>0),$$

$$(1.1)^2 \qquad u(x,\ 0)=u_0(x) \quad (u_0(0)=u_0(1)=0),$$

$$(1.1)^3 \qquad u(0,\ t)=u(1,\ t)=0 \quad (t\geq 0),$$

$$(1.2) \qquad [w(x)=x(1-x)],$$

demonstrating the unique existence of the time-global solution of (1.1) under some conditions on $u_0(x)$. Here, we shall study the same problem for $\alpha\in(0,1)$. The notations are the same as in [1].

2. Preliminaries

We make some preparations for constructing the solution of (1.1) for $\alpha\in(0,1)$. Firstly, we use the conventional method of separation of variables as in [1]. If $U(x,\ t)=X(x)T(t)$ satisfies $(1.1)^1$-$(1.1)^3$, then we have

$$(2.1) \qquad T'(t)X(x)=w(x)^\alpha X''(x)T(t),$$

i.e.,

$$(2.2) \qquad \frac{T'(t)}{T(t)}=w(x)^\alpha\frac{X''(x)}{X(x)}=-\lambda\ (=\text{const.}),$$

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whence

(2.3)
$$\begin{cases} X''(x) = -\lambda w(x)^{-\alpha} X(x) & (0<x<1), \\ X(0)=X(1)=0, \end{cases}$$
$$T(t) = Ce^{-\lambda t} \quad (C, \text{const.}),$$

which shows that λ and $X(x)$ are an eigenvalue and an eigenfunction corresponding to λ for the eigenvalue problem (2.3), respectively. As easily seen (cf. [1, Afterword]), the ordinary differential equation (2.3) is equivalent to the following integral one:

(2.4)
$$\begin{cases} X(x) = \lambda \int_0^1 G(x,\xi) \dfrac{1}{w(\xi)^\alpha} X(\xi) d\xi = \lambda G \circ \dfrac{X}{w^\alpha}(x) \quad (X(x) \in C^0 \ ([0,1])), \\ G(x,\xi) = \begin{cases} (1-x)\xi & (x \geq \xi) \\ (1-\xi)x & (x \leq \xi), \end{cases} \end{cases}$$

where we note that the following inequality holds,

(2.4)′
$$|X(x)| = |\lambda| \cdot \left| \int_0^1 G(x,\xi) w(\xi)^{-\alpha} X(\xi) d\xi \right| \leq C_0 |\lambda| w(x) |X|_I^{(0)},$$
$$(I=[0,1];\ C_0, \text{const.}),$$

which is derived in the same way as in [1]. Now, apart from our subject, for a while we discuss the eigenvalue problem (2.4). Defining $G_\alpha(x,\xi)$ by

(2.5)
$$\begin{cases} G_\alpha(x,\xi) = [w(x)w(\xi)]^{-\frac{\alpha}{2}} G(x,\xi) = \begin{cases} \dfrac{(1-x)^{1-\frac{\alpha}{2}} \xi^{1-\frac{\alpha}{2}}}{x^{\frac{\alpha}{2}} (1-\xi)^{\frac{\alpha}{2}}} & (1 \geq x \geq \xi \geq 0) \\ \dfrac{(1-\xi)^{1-\frac{\alpha}{2}} x^{1-\frac{\alpha}{2}}}{\xi^{\frac{\alpha}{2}} (1-x)^{\frac{\alpha}{2}}} & (1 \geq \xi \geq x \geq 0) \end{cases} \\ ((x,\xi) \neq (0,0),(1,1)) \\ G_\alpha(0,0) = G_\alpha(1,1) = 0, \end{cases}$$

where we note that

(2.5)′
$$\begin{cases} G_\alpha(x,\xi) = G_\alpha(\xi,x),\ G(x,x) = (1-x)^{1-\alpha} x^{1-\alpha},\ 0 \leq G_\alpha(x,\xi) \leq 1, \\ G_\alpha(x,\xi) \text{ is continuous on } [0,1] \times [0,1]. \end{cases}$$

We transform (2.4) into an integral equation having a symmetric and continuous kernel $G_\alpha(x,\xi)$ as follows:

(2.6)
$$\tilde{X}(x) = \lambda \int_0^1 G_\alpha(x,\xi) \tilde{X}(\xi) d\xi \,(= \lambda G_\alpha \circ \tilde{X}(x)), \quad (\tilde{X} = w^{\frac{\alpha}{2}} X),$$

(2.6)′
$$[\text{N.B.}: \tilde{X}(x) \in C^0([0,1]) (\text{cf.} (2.4)'), \tilde{X}(0) = \tilde{X}(1) = 0].$$

In the next place, we state the following 7 lemmas.

Lemma 2.1. *The functions of the set* $\{G_\alpha \circ f(x) | f \in C^0([0,1]), \|f\|_{L^2} \leq 1\}$ *are uniformly bounded and equicontinuous. Thus,* G_α *as an operator from* $C^0([0,1])$ *into itself is* *completely continuous (cf.[7]).*

Proof. (i) The uniform boundedness of the above-mentioned functions is obvious. (ii) The equicontinuity of those functions is easily to be demonstrated, if we note that

$$(2.7)\qquad \begin{aligned} |G_\alpha\circ f(x)-G_\alpha\circ f(x')| &\le \int_0^1 |G_\alpha(x,\xi)-G_\alpha(x',\xi)|\cdot|f(\xi)|d\xi \\ &\le \left[\int_0^1 |G_\alpha(x,\xi)-G_\alpha(x',\xi)|^2 d\xi\right]^{\frac{1}{2}} \|f\|_{L^2} \quad (0\le x\le x'\le 1) \\ &\le \left[\int_0^x |\cdots|^2 d\xi+\int_x^{x'} |\cdots|^2 d\xi+\int_{x'}^1 |\cdots|^2 d\xi\right]^{\frac{1}{2}} \\ &= [I_1+I_2+I_3]^{\frac{1}{2}} \end{aligned}$$

and that

$$(2.7)'\qquad 0\le (x')^\gamma - x^\gamma \le |x'-x|^\gamma \quad (0\le x\le x'\le 1, 0\le \gamma\le 1).$$

(Q.E.D.).

Lemma 2.2. *The eigenvalues λ's in the eigenvalue problem (2.4) (or, what is the same, (2.3)) are positive. Moreover, if $X(x)$ is an eigenfunction corresponding to λ, being normalized with $w(x)^{-\alpha}$, then we have,*

$$(2.8)\qquad |X(x)|\le\sqrt{\lambda}.$$

Proof. By virtue of the estimate $(2.4)'$ of $|X(x)|$, $X'(0)$ and $X'(1)$ exist, being continuous there. Thus, from (2.3) we obtain

$$(2.8)'\qquad 0<\int_0^1 X'(x)^2dx=\lambda\int_0^1 w(x)^{-\alpha}X(x)^2dx,$$

[N.B.: $X(x)$ is non-trival so that $X'(x)\not\equiv 0$.]

Moreover, if $X(x)$ is normalized with weight $w(x)^{-\alpha}$, then, by $(2.8)'$,

$$(2.8)''\qquad |X(x)|\le\int_0^1 |X'(x)|dx=\left(\int_0^1 X'(x)^2dx|\right)^{\frac{1}{2}}=\lambda^{\frac{1}{2}}.$$

(Q.E.D.).

Thus, by lemmas 2.1 and 2.2, the eigenvalues λ's are positive and countable.

Lemma 2.3. *The eigenvalues λ's are positive and countably infinite, i.e. $\{\lambda\}=\{\lambda_n\}_{n=0}^{\infty}$. Moreover, $\lambda_0\le\lambda_1\le\cdots\le\lambda_n\le\cdots$ $(\lambda_n\to\infty(n\to\infty))$.*

Proof. If $\{G_\alpha\circ f\,|\,f\in C^0([0,1])\}$ is of a finite demension n_0, then $G_\alpha\circ 1$, $G_\alpha\circ x$, $\cdots$, $G_\alpha\circ x^{n_0}$ are linearly dependent (which is easily to be shown). Thus, there exist constants c_0, c_1, $\cdots$, c_{n_0} such that

(2.9)
$$\begin{cases}(c_0, c_1, \cdots, c_{n_0}) \neq (0, 0, \cdots, 0), \\ 0=\sum_{j=0}^{n_0} c_j G_x \circ x^j = G_\alpha \circ \sum_{j=0}^{n_0} c_j x^j,\end{cases}$$

which implies that

(2.9)′
$$0=\frac{d^2}{dx^2}0=-w(x)^{-\frac{\alpha}{2}}\sum_{j=0}^{n_0} c_j x^j, \ c_j=0 \ (j=0,1,\cdots,n_0).$$

The above conclusion is contradictory to (2.9).

(Q.E.D.).

Lemma 2.4. *The system of the eigenfunctions* $\{\tilde{X}_n(x)\}_{n=0}^{\infty}$ *of the eigenvalue problem (2.6) is complete in* $C^0([0,1])$.

Proof. Let $\{\tilde{X}_n\}$ be normalized with weight 1, f belong to $C^0([0,1])$, and, for any n, $(\tilde{X}_n, f)=0$. Then,

(2.10)
$$(\tilde{X}_n, f)=\lambda_n(G_\alpha \circ \tilde{X}_n, f)=\lambda_n(\tilde{X}, G_\alpha \circ f)=0, \text{ i.e.,}$$

(2.10)′
$$(\tilde{X}_n, G_\alpha \circ f)=0 \quad (\text{for any } n).$$

$G_\alpha \circ f(x)$ is to be absolutely and uniformly expanded in the following way (Hilbert-Schmidt's theorem),

(2.11)
$$G_\alpha \circ f(x)=\sum_{j=0}^{\infty} a_j \tilde{X}_j(x) \quad (a_j=(G_\alpha \circ f, \tilde{X}_j)),$$

where, by (2.10)′,

(2.13)
$$0=(\tilde{X}_n, \sum_{j=0}^{\infty} a_j \tilde{X}_j)=\sum_{j=0}^{\infty} a_j(\tilde{X}_n, \tilde{X}_j)=a_n \quad (n=0,1,2,\cdots).$$

Thus, $G_\alpha \circ f(x)\equiv 0$, which implies that

(2.12)
$$0=\frac{d^2}{dx^2}0=-w(x)^{-\frac{\alpha}{2}}f(x), \quad \text{i.e., } f(x)\equiv 0.$$

(Q.E.D.).

Lemma 2.5. *Let* λ_j*'s be as in lemmas 2.2 and 2.4. Then it holds that*

(2.14)
$$\sum_{j=0}^{\infty} \lambda_j^{-1}=B(2-\alpha, 2-\alpha),$$

where $B(\cdot,\cdot)$ *is the Beta function.*

Proof. The kernel $G_\alpha(x, \xi)$ of the integral equation (2.6) is obviously a positive-definite and symmetric one, being continuous on $[0,1]\times[0,1]$. Hence, by Mercer's theorem we have, (2.15) $$G_\alpha(x,\xi)=\sum_{j=0}^{\infty}\lambda_j^{-1}\tilde{X}_j(x)\tilde{X}_j(\xi) \quad (X_j\text{'s are normalized}),$$

whence are obtained the equalities

(2.15)′ $$\int_0^1 G(x,x)dx=\int_0^1(1-x)^{1-\alpha}x^{1-\alpha}dx=B(2-\alpha,2-\alpha)$$
$$=\sum_{j=0}^{\infty}\lambda_j^{-1}\int_0^1\tilde{X}_j(x)^2dx=\sum_{j=0}^{\infty}\lambda^{-1}.$$

(Q.E.D.).

Lemma 2.6. $X_n'(x)=\dfrac{d}{dx}X_n(x)$ $(n=0,1,2,\cdots)$ *satisfy*

(2.16) $$|X_n'(x)|\le B(1-\alpha,1-\alpha)\lambda_n^{\frac{3}{2}}.$$

Proof. From (2.4) we have easily,

(2.16)′ $$X_n'(x)=\lambda_n[-\int_0^x\xi^{1-\alpha}(1-\xi)^{-\alpha}X_n(\xi)d\xi+\int_x^1(1-\xi)^{1-\alpha}\xi^{-\alpha}X_n(\xi)d\xi].$$

Thus, by (2.8)″, $|X_n'|$ is estimated as

(2.16)″ $$|X_n'(x)|\le\lambda_n\int_0^1(1-\xi)^{-\alpha}\xi^{-\alpha}d\xi\cdot|X_n|_I^{(0)}\le B(1-\alpha,1-\alpha)\lambda_n^{\frac{3}{2}}.$$

(Q.E.D.).

Lemma 2.7. $\dfrac{d^2}{dx^2}X_n(x)=X_n''(x)$ $(n=0,1,2,\cdots)$ *satisfy*

(2.17) $$|X_n''(x)|=\lambda_n w(x)^{-\alpha}|X(x)|\le C_0\lambda_n^{\frac{5}{2}}w(x)^{1-\alpha}\le\frac{C_0}{4}\lambda_n^{\frac{5}{2}}.$$

Proof. By (2.3) and (2.4)′, the relation (2.17) is obvious.

(Q.E.D.).

3. Main theorem

Let λ_n, $X_n(x)$ and $\tilde{X}_n(x)$ be as in the preceding sections 1 and 2. Now, we construct a formal solution $U(x,t)$ of (1.1) ($\alpha\in(0,1)$) such that

(3.1) $$\begin{cases} U(x,t)=\sum_{n=0}^{\infty}e^{-\lambda_n t}a_nX_n(x),\ U(x,0)=u_0(x)=\sum_{n=0}^{\infty}a_nX_n(x), \\ a_n=\int_0^1 u_0(x)w(x)^{-\alpha}X_n(x)dx \quad (n=0,1,2,\cdots), \end{cases}$$
$$\left[\int_0^1 w(x)^{-\alpha}X_n(x)^2dx=\int_0^1\tilde{X}_n(x)^2dx=1\right].$$

If $u_0(x)$ satisfies

$$\begin{cases} u_0(x)\in C^0([0,1])\cap C^2((0,1)) \quad (u_0(0)=u_0(1)=0), \\ w(x)^{\frac{\alpha}{2}}u_0''(x)\in C^0([0,1]), \end{cases} \tag{3.2}$$

then $u_0(x)$ is to be expanded abslutely and uniformly on $[0,1]$ (Hilbert-Schmidt's theorem) as below,

$$\begin{aligned} u_0(x) &= w(x)^{\frac{\alpha}{2}}\sum_{n=0}^{\infty}(G_\alpha\circ(-w^{\frac{\alpha}{2}}u_0''), \tilde{X}_n)\tilde{X}_n(x) = \sum_{n=0}^{\infty}(u_0, w^{-\alpha}X_n)X_n(x) \\ &= \sum_{n=0}^{\infty} a_nX_n(x), \end{aligned} \tag{3.3}$$

$$[\text{N.B.}: a_n=(G_\alpha\circ(-w^{\frac{\alpha}{2}}u_0''),\tilde{X}_n) = -\lambda_n^{-1}(w^{\frac{\alpha}{2}}u_0'',\tilde{X}_n) = -\lambda_n^{-1}(u_0'',X_n)\equiv\lambda_n^{-1}b_n].$$

As a final assertion we state:

Main Theorem. *Under the assumption (3.2) on u_0, there exists a unique solution $u(x,t)$, belonging to $C^0([0,1]\times[0,T])\cap C^{2,1}([0,1]\times(0,T])$ for an arbitrary $T\in(0,\infty)$. Moreover,*

$$\begin{cases} |u(x,t)|\le|u_0|_I^{(0)}, \quad t|u_x(x,t)|\le C_1B_0, \\ t^2|u_{xx}(x,t)|\le C_2B_0, \quad t|u_t(x,t)|\le C_3B_0 \end{cases} \tag{3.4}$$

where

$$\begin{cases} B_0=(\sum_{n=0}^{\infty}b_n^2)^{\frac{1}{2}}(\sum_{n=0}^{\infty}\lambda_n^{-1})^{\frac{1}{2}}=\|w^{\frac{\alpha}{2}}u_0''\|_{L^2}B(2-\alpha,2-\alpha)^{\frac{1}{2}}, \\ C_1,\ C_2 \text{ and } C_3 \text{ are positive constants.} \end{cases} \tag{3.4$'$}$$

Proof. The procedure of demonstrating the assertion is almost the same as in the proof of the main theorem of [1]. The inequalities in (3.4) are obtained by making use of the maximum principle, and lemmas 2.5, 2.6 and 2.7, and by noting that

$$\begin{cases} e^{-\lambda_n t}\lambda_n^{\beta}\le e^{-\beta}\beta^{\beta}t^{-\beta} \quad (\beta>0,\ t>0), \\ e^{-\lambda_n t}\lambda_n^{\beta}\le t^{-\beta}C_4(\beta)e^{-C_5(\beta)\lambda_n t}, \\ (C_4(\beta),\ C_5(\beta)>0). \end{cases} \tag{3.5}$$

(Q.E.D.).

Remark: Under another condition on u_0

$$\begin{cases} u_0\in C^0([0,1])\cap C^6((0,1)), \\ w^{\frac{\alpha}{2}}(w^{\alpha}u_0'')'')''\in C^0([0,1]), \\ w^{\alpha}u_0''(0)=w^{\alpha}u_0''(1)=w^{\alpha}(w^{\alpha}u_0'')''(0)=w^{\alpha}(w^{\alpha}u_0'')''(1)=0, \end{cases} \tag{3.6}$$

$u_x(x,t)$, $u_{xx}(x,t)$ and $u_t(x,t)$ are continuous on $[0,1]\times[0,\infty)$, which is easily to be demonstrated. We have only to see that, under the condition (3.6),

$$\begin{cases} a_n=-\lambda_n^{-1}(u_0'',X_n)=\lambda_n^{-1}(w^{\frac{-\alpha}{2}}G_\alpha\circ w^{\frac{\alpha}{2}}(w^{\alpha}u_0'')'', w^{\frac{\alpha}{2}}\tilde{X}_n) \\ =\lambda_n^{-2}((w^{\alpha}u_0'')'', X_n)=-\lambda_n^{-3}((w^{\alpha}u_0'')'', X_n). \end{cases} \tag{3.7}$$

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Corrigenda to the paper [1]

	Erratum	Correction
p.10, *l*.3	$\cdots \leq \int_I (X_n')^2 dx = \cdots$	$\cdots \leq \left(\int_I (X_n')^2 dx\right)^{\frac{1}{2}} = \cdots$
p.11, *ll*. 7～8	$\begin{cases} \cdots, \lvert u_x(x,t)\rvert \leq \\ \lvert u_{xx}(x,t)\rvert \leq \cdots, \lvert u_t(x,t)\rvert \leq \cdots \end{cases}$	$\begin{cases} \cdots, t\lvert u_x(x,t)\rvert \leq \cdots \\ t^2\lvert u_{xx}(x,t)\rvert \leq \cdots, t\lvert u_t(x,t)\rvert \leq \cdots \end{cases}$
p.12, *ll*. 13～14	$\begin{cases} \cdots \leq A_1, \cdots \leq A_2 \\ \cdots A_3(A_1, A_2, A_3, \cdots) \end{cases}$	$\begin{cases} \cdots \leq t^{-1}\tilde{A}_1, \cdots \leq t^{-1}\tilde{A}_2 \\ \cdots \leq t^{-2}\tilde{A}_3(\tilde{A}_1, \tilde{A}_2, \tilde{A}_3, \cdots) \end{cases}$