

***Analysis of solar neutrino problem by means of  
Nötzold and Nakagawa's approach including the  
interference term  
- Hyperbolic-tangent profile for electron density  
in the sun and exact solution -***

M. KANEYAMA and M. BIYAJIMA

Department of Physics, Faculty of Science, Shinshu University, Matsumoto 390-8621, Japan  
(Received November 19, 2003)

**Abstract**

Using an exact solution with the hyperbolic-tangent profile for the electron density in the sun, which is developed by Nötzold and later modified by Nakagawa, we have analyzed the solar neutrino problem. An interference term in their approach is correctly taken into account. Combining the hyperbolic-tangent profile with the BP2000, we obtain a phenomenological fitting in the analytic form. Combining recent observed results for survival probability  $P(\nu_e \rightarrow \nu_e)$  by the SNO, SK, SAGE, Gallex, GNO and Homestake Collaborations, we obtain a large mixing angle (LMA) whose figure looks like a shoulder.

## 1 Introduction

One of the most interesting subjects in elementary particle physics is the solar neutrino problem. As a possible explanation for it, the MSW mechanism seems the best solution [1, 2, 3]. Very recently the SNO Collaboration has reported that the survival probability of  $\nu_e$  from the sun is  $P(\nu_e \rightarrow \nu_e) = 0.348 \pm 0.03$ , and the large mixing angle (LMA),  $\Delta m^2 = m_2^2 - m_1^2 = 0.35 \times 10^{-4} (\text{eV}^2)$  and  $\theta = 32^\circ$ , is favor for the explanation of the global measurements [4, 5, 6] ( $m_1, m_2$  are masses of the neutrino mass eigenstate  $\nu_1, \nu_2$ , respectively. Hereafter  $m_2 > m_1$  is assumed.  $\theta$  is the mixing angle defined by  $\nu_e = \cos\theta\nu_1 + \sin\theta\nu_2$  and  $\nu_\mu = -\sin\theta\nu_1 + \cos\theta\nu_2$ ).

On the other hand in 1988 Nötzold proposed an exact solution for the solar neutrino oscillation, using the hyperbolic-tangent profile for the density of electrons defined by  $N_e$  in the sun [7]:

$$N_e = N_0 \left[ 1 - \tanh \frac{r}{l} \right], \quad (1)$$

where  $N_0 = V / (R_s \sqrt{2} G_F)$ ,  $R_s$  and  $G_F$  denote the radius of the sun  $R_s = 6.96 \times 10^6$  km and the Fermi coupling constant  $G_F = 8.917 \times 10^{-8}$  (GeV  $\cdot$  fm<sup>3</sup>). Moreover  $V$ ,  $r$  and  $l$  denote the magnitude of the density at the center of the sun,  $r = R/R_s$  and  $l = l/R_s$ , respectively. Nötzold has assumed  $r \rightarrow r + r_0 = r - \infty$ , a priori (See Fig. 1(a)).

Later, Nakagawa has proposed a modified expression for  $N_e$  as

$$N_e = N_0 \left[ 1 - \tanh \frac{r + r_0}{l} \right], \quad (2)$$

where  $r_0$  is an adjusted parameter which reproduces the  $N_e$  in better way than Eq. (1). Nakagawa has stressed that the result by Nötzold, the magnitude of  $\Delta m^2/E_\nu$ , is reduced about 26% in the figure of  $\Delta m^2/E_\nu$  vs.  $\sin^2 \theta$ , as Eq. (2) is used for the electron density in the sun [8]. His result with the standard solar model (SSM) in 1988 [9] is shown in Fig. 1(b) ( $V = 1.718 \times 10^4$ ,  $l = 0.155$ ,  $r_0 = -0.105$  for SSM in 1988).

In this report we would like to consider the BP2000 [10] with Eq. (2) and the solar neutrino problem based on Nötzold-Nakagawa's approach including the interference term. Our phenomenological fitting is shown in Fig. 2 ( $V = 1.740 \times 10^4$ ,  $l = 0.148$ ,  $r_0 = -0.115$  for BP2000).

In the second section, their theoretical formulas a la Nakagawa are explained. The explicit interference term is shown. In the third section, we estimate the allowed regions ( $\Delta m^2$  and mixing angle) for survival probabilities observed by the SNO, SK, SAGE, GNO, Gallex and Homestake Collaborations [4, 5, 6, 11, 12, 13, 14, 15]. In the final section, concluding remarks are presented.

## 2 Theoretical formulas with the hyperbolic-tangent for $N_e$

To explain theoretical formulas first derived by Nötzold and later modified by Nakagawa, we briefly describe their framework. The coupled Schrödinger equations for the solar neutrino oscillation are given as

$$i \frac{dv_e}{dt} = \left( \frac{m_1^2 \cos^2 \theta + m_2^2 \sin^2 \theta}{2E} + \sqrt{2} G_F N_e \right) v_e + \frac{\Delta m^2}{2E} \sin \theta \cos \theta v_\mu, \quad (3)$$

$$i \frac{dv_\mu}{dt} = \frac{\Delta m^2}{2E} \sin \theta \cos \theta v_e + \frac{m_1^2 \sin^2 \theta + m_2^2 \cos^2 \theta}{2E} v_\mu. \quad (4)$$

Introducing the following notations,  $ct = R_s r$ ,  $M_0^2 = R_s (m_2^2 + m_1^2) / (4E)$ ,  $R_s \Delta m^2 / (2E) = x / B \equiv x_0$  with  $B \equiv E(\text{MeV}) / \Delta m^2 (\text{eV}^2)$  and  $x = 1.764 \times 10^9$ , we use the following abbreviations

$$G = \frac{x_0 \cos 2\theta}{2}, \quad (5)$$

$$H = \frac{x_0 \sin 2\theta}{2}. \quad (6)$$

Moreover, exchanging the variable from  $v_e$  to  $v(r)$ ,

$$v_e(r) = v(r) \exp\left\{-i \int_0^r \left[M_0^2 + \frac{1}{2} VU(r)\right] dr\right\}, \quad (7)$$

we have the following differential equation as

$$\frac{d^2 v}{dr^2} + \left[i \frac{1}{2} VU(r)' + \left\{\frac{1}{2} VU(r) - G\right\}^2 + H^2\right] v = 0, \quad (8)$$

where  $U(r) = 1 - \tanh[(r + r_0)/l]$ . Changing the variable  $r$  to  $y$  and assuming the factorized form for  $v(r)$ ,

$$y = \frac{1}{1 + \exp\left(-2 \frac{r + r_0}{l}\right)}, \quad (9)$$

$$U(r) = 2(1 - y), \quad (10)$$

$$v(r) = (1 - y)^a y^\nu f(y), \quad (11)$$

we obtain the following hyper-geometric equation

$$y(1 - y) \frac{d^2 f}{dy^2} + \left\{c - (a + b + 1)y\right\} \frac{df}{dy} - abf = 0, \quad (12)$$

where

$$a = \mu + \nu + \lambda, \quad (13)$$

$$b = \mu + \nu + 1 - \lambda, \quad (14)$$

$$c = 2\nu + 1, \quad (15)$$

$$\mu = i \frac{l}{2} \sqrt{G^2 + H^2}, \quad (16)$$

$$\nu = i \frac{l}{2} \sqrt{(V - G)^2 + H^2}, \quad (17)$$

$$\lambda = i \frac{l}{2} V. \quad (18)$$

The general solution of Eq. (12) is expressed by two independent hyper-geometrical functions.  $F(a, b; c; y)$ - is expressed by replacing  $\nu \rightarrow -\nu$  in a, b, c of  $F(a, b; c; y)$ .

$$f(y) = C_1 F(a, b; c; y)_+ + C_2 y^{-2\nu} F(a, b; c; y)_-, \quad (19)$$

where  $C_1$  and  $C_2$  are the integral constants. Using an ordinary procedure with the initial condition,  $\nu_e(y=y_0)=1$  and  $\nu_\mu(y=y_0)=0$ , we obtain the following formula for the survival probability

$$\begin{aligned} \langle P(\nu_e \rightarrow \nu_e) \rangle &= P_1 \cos^2 \theta + (1 - P_1) \sin^2 \theta \\ &- \sqrt{P_c(1 - P_c)} \cos 2\theta_m \sin 2\theta \cos(2.54 \frac{\Delta m^2}{E} L + \delta). \end{aligned} \quad (20)$$

where

$$P_1 = P_c \sin^2 \theta_m + (1 - P_c) \cos^2 \theta_m, \quad (21)$$

$$P_c = \frac{\cosh(\pi l V) - \cosh[\pi l(\Delta p - \Delta q)]}{\cosh[\pi l(\Delta p + \Delta q)] - \cosh[\pi l(\Delta p - \Delta q)]}, \quad (22)$$

$$\cos 2\theta_m = \frac{G - V(1 - y_0)}{\sqrt{[G - V(1 - y_0)]^2 + H^2}}, \quad (23)$$

$$\Delta p = \frac{R_s}{4E} \Delta m^2, \quad (24)$$

$$\Delta q = \sqrt{V^2 - 2V\Delta p \cos 2\theta + \Delta p^2}, \quad (25)$$

$$y_0 = \frac{1}{1 + \exp(-2 \frac{r_0}{l})}. \quad (26)$$

In Refs. [7] [8] the third term of the right hand side, the interference term, in Eq. (20) is assumed to be zero because of the oscillation. We have obtained the following formula, after integration,

$$\begin{aligned} &- \frac{R_s}{10.16 \times 10^3 \Delta L \Delta p} \sqrt{P_c(1 - P_c)} \cos 2\theta_m \sin 2\theta \\ &\times \sin\left(\frac{5.08 \times 10^3}{R_s} \Delta L \Delta p\right) \cos\left[\frac{5.08 \times 10^3}{R_s} (L_2 + L_1) \Delta p\right], \end{aligned} \quad (27)$$

where  $L_1$  and  $L_2$  are the minimum distance from the sun to earth and the maximum one, respectively, and

$$\Delta L = L_2 - L_1. \quad (28)$$

### 3 Analysis of solar neutrino problem with empirical values by means of Eq. (20)

The data on the solar neutrino, the survival probability  $P(\nu_e \rightarrow \nu_e)$ , reported by the SNO, Gallex, GNO, SAGE, Homestake Collaborations are shown in Table 1.

Using empirical values in Table 1 and Eq. (20) with  $\langle E_\nu \rangle = 8\text{MeV}$  and  $\langle E_\nu \rangle =$

Table 1: Data on the solar neutrino [6, 13, 14, 15].  $\langle E_\nu \rangle$  is the average neutrino energy.

Exp	$\langle E_\nu \rangle$	$P(\nu_e \rightarrow \nu_e)$
SNO	$\sim 8$ MeV	$0.348 \pm 0.029$
SAGE	$\sim 0.8$ MeV	$0.54 \pm 0.06$
Gallex+GNO	$\sim 0.8$ MeV	$0.56 \pm 0.07$
Homestake	$\sim 8$ MeV	$0.34 \pm 0.03$

0.8MeV, we obtain two allowed regions (AR) which are shown in Fig. 3(a). The small mixing angle (SMA) region is located in  $(\tan^2\theta, \Delta m^2)=(0.8 \times 10^{-3}, 10^{-5})$  and the large mixing angle (LMA) region is located in  $(0.5, 10^{-4} \sim 4 \times 10^{-5})$ . Combining the global measurement (including the day-night effect by the SNO Collaboration and by the SK Collaboration) and our result, we obtain Fig. 3(b). The shoulder-like region nearly coincides with the large mixing angle (LMA) with C.L.99.73%. (See Ref. [4])

As seen in Fig. 3, our results based on Eq. (2) and Eq. (20) is partially consistent with the LMA solutions.

#### 4 Concluding remarks

1. As seen in Fig. 2, Eq. (2) is useful to reproduce the electron density based on BP2000.
2. Using Eq. (20), we have examined the AR expressed by  $(\Delta m^2, \tan^2 \theta)$ . In the present calculation we have added the third term Eq. (27) to Nötzold and Nakagawa's formula. As compared with result of Refs. [7][8], we have additional AR: The contribution is seen in the most right curve and the bottom region with oscillations near  $(\Delta m^2, \tan^2 \theta)=(0.2 \sim 5, 1 \times 10^{-14} \sim 1 \times 10^{-13})$ . The main reason is attributed to Eq. (27). Moreover, we can compare our results with the exponential profile, for example, see Fig. 2 of Ref. [16]. A similar allowed region ( $0.1 \lesssim \tan^2 \theta \lesssim 1.0$ ) is observed.
3. From the empirical results in Table 1 and Eq. (20), we have obtained two AR's, the LMA and SMA. Combining the AR's reported by the SNO, the SK Collaborations and our Fig. 3(a), we have obtained Fig. 3(b). This is fairly well consistent with the LMA reported by the SNO Collaboration.
4. Moreover, we have to add the following fact: For the figure of  $(\sin^2\theta, \Delta m^2/E_\nu)$  with the BP2000, the result of calculation a la Ref. [7], the magnitude of  $\Delta m^2/E_\nu$ , is about 20% larger than that of present calculation using values reported by the SNO Collaboration [4].

## 5 Acknowledgments

The authors would like to thank their colleagues for their assistance, in particular T. Mizoguchi, and useful communication with T. Kaneko and H. Nunokawa.

## References

- [ 1 ] L. Wolfenstein, *Phys. Rev.* **D17**, 2369 (1978).
- [ 2 ] S. P. Mikheev and A. Yu. Smirnov, *Yad. Fiz.* **42**, 1441 (1985).
- [ 3 ] H. A. Bethe, *Phys. Rev. Lett.* **56**, 1305 (1986).
- [ 4 ] SNO Collaboration, (2002), [nucl-ex/0204009](#).
- [ 5 ] SNO Collaboration, (2002), [nucl-ex/0204008](#).
- [ 6 ] SNO Collaboration, *Phys. Rev. Lett.* **87**, 071301 (2001).
- [ 7 ] D. Nötzold, *Phys. Rev.* **D36**, 1625 (1987); See also J. N. Bahcall et al., *Rev. Mod. Phys.* **54**, 767 (1982).
- [ 8 ] M. Nakagawa, *Phys. Rev.* **D41**, 685 (1990): Eq. (2.11) has a misprint. The corrected one is our Eq. (8).
- [ 9 ] J. N. Bahcall, R. K. Ulrich, *Rev. Mod. Phys.* **60**, 297 (1988).
- [10] J. N. Bahcall, H. M. Pinsonneault and S. Basu, *Astrophys. J* **555**, 990 (2001).
- [11] Super-Kamiokande Collaboration, *Phys. Rev. Lett.* **86**, 5651 (2001).
- [12] Super-Kamiokande Collaboration, *Phys. Rev. Lett.* **86**, 5656 (2001).
- [13] J. N. Abdurashitov et al., (2002), [astro-ph/0204245](#).
- [14] C. Cattadori, Update of solar neutrino interaction rate measurement from GNO at LNGS, TAUP 2001, LNGS, (September 2001).
- [15] B. T. Cleveland et al, *Astrophys. J* **496**, 505 (1998).
- [16] A. de Gouvea, A. Friedland, and H. Murayama, *Phys. Lett.* **B 490**, 125 (2000).

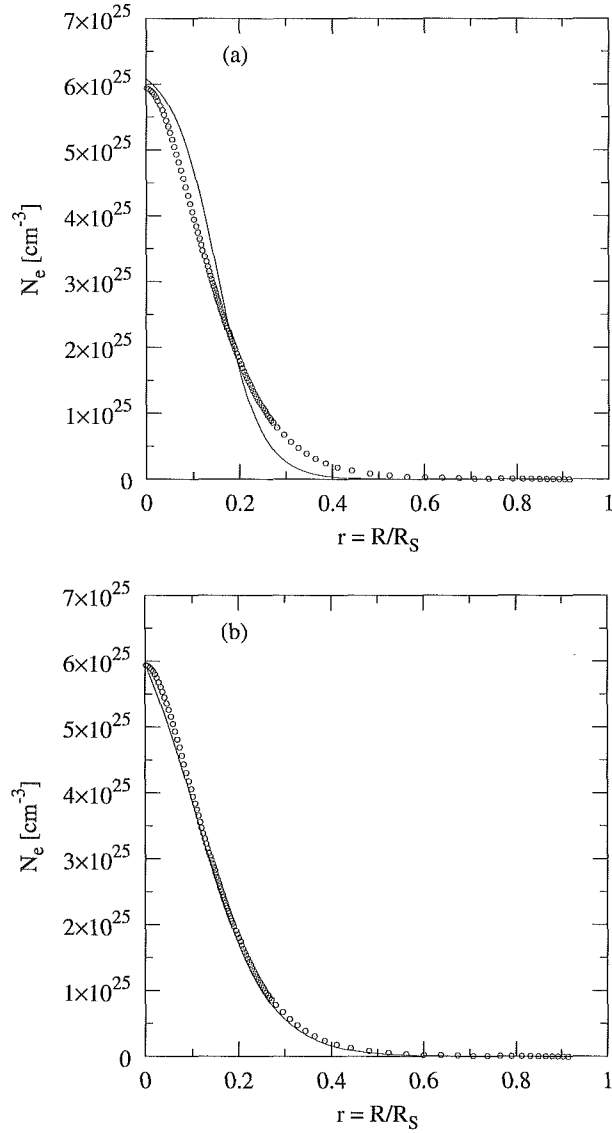


Figure 1: (a) The electron number density from SSM1988 in Ref. [9], which is fitted by Eq. (1). See Fig. 2 of Ref. [7]. (b)  $N_e$  is fitted by Eq. (2). See Ref. [8].

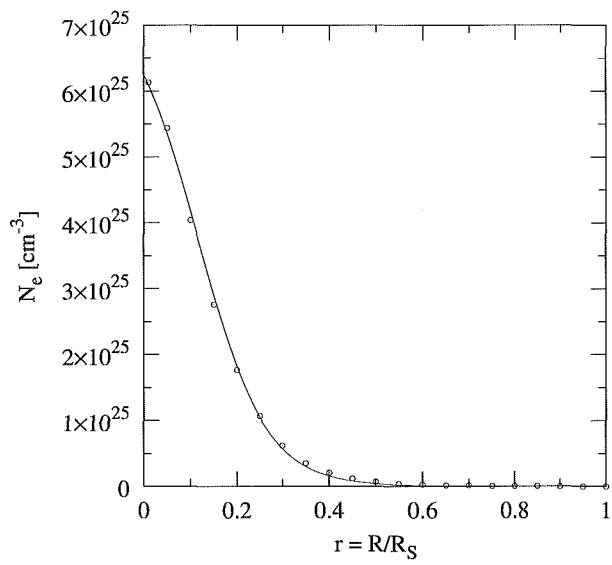


Figure 2 : The electron number density from BP2000, which is fitted by Eq. (2).



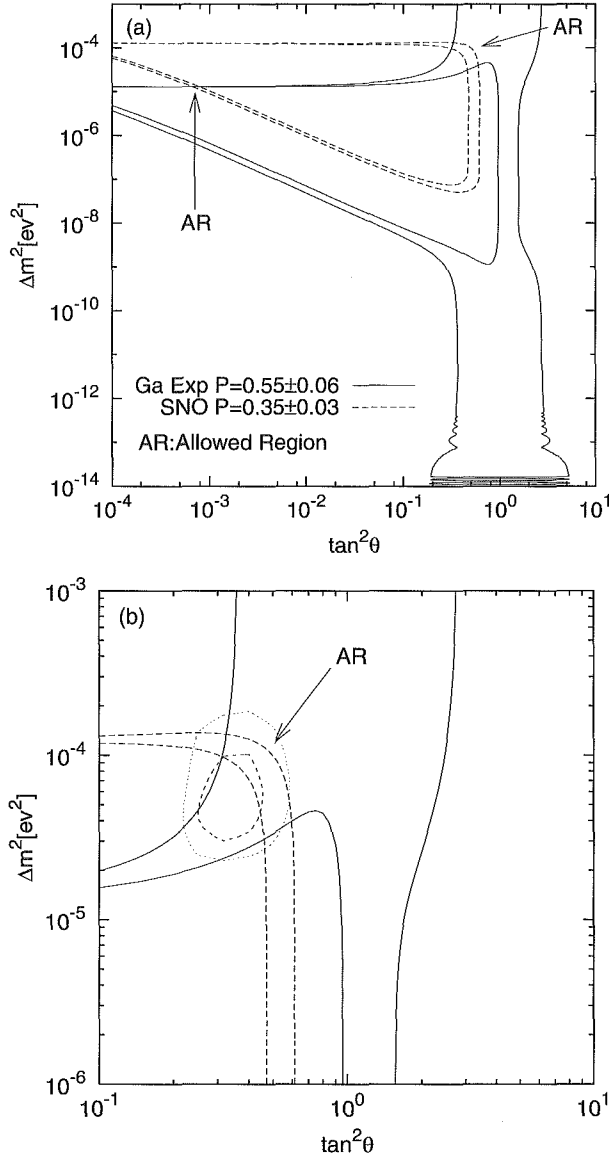


Figure 3 : (a)  $\Delta m^2$  vs.  $\tan^2 \theta$ . Long dashed and solid lines are obtained from empirical values by SNO Collaboration ( $P(\nu_e \rightarrow \nu_e) = 0.348 \pm 0.03$ ), and those by Ga experiment ( $P(\nu_e \rightarrow \nu_e) = 0.55 \pm 0.06$ ) and assuming  $\langle E_\nu \rangle = 8 \text{ MeV}$  and  $\langle E_\nu \rangle = 0.8 \text{ MeV}$ , respectively. (b) Enlarged figure of (a). The dashed circle with C.L. = 90% and dotted circle with C.L. = 99.73% are given in Ref. [4]. The SMA is excluded by observations by the SNO and SK Collaboration. AR is an abbreviation for the allowed region.