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21-st and 22-nd homotopy groups of the n-th rotation group

To the memory of Professor Tatsuji Kudo

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Abstract

We denote by R_n the n-th rotatoin group and by $\pi_k(R_n)$ the k-th homotopy group of R_n. We determine the group structures of the homotopy groups $\pi_k(Rn)$ for $k=21$ and 22 by use of the fibration $R_{n+1}\xrightarrow{Rn}S^n$.

Introduction and statements of results

This paper is a sequel to [12]. According to [17] and [19], the group structures of $\pi_k(R_n)$ for $k\leq 22$ and $n\leq 9$ are known. The group structures of $\pi_k(R_n)$ for $k\leq 20$ are obtained by [2], [10], [11], [12], [14], [25], [26] and [27]. We denote by $\pi_k(X:2)$ the direct sums of a free abelian subgroup and the 2-primary components such that the index $[\pi_k(X):\pi_k(X:2)]$ is odd. The purpose of the present note is to determime

$\pi_k(R_n:2)$ for $k=21$ and 22. We write $\pi_k(R_n:2)=R^n_k$ and $\pi_k(S^n:2)=\pi^n_k$ [28].

Our method is the composition methods [28]. We freely use the generators and group structures of the homotopy groups $\pi_{n+k}(S^n)$ for $k\leq 22$ [16], [18], [28]. The main tool is the following exact sequence induced from the fibration $R_{n+1}\xrightarrow{Rn}S^n$:

$$(k)_n \qquad \pi_{k+1}(S^n)\xrightarrow{\Delta}\pi_k(R_n)\xrightarrow{i_*}\pi_k(R_{n+1})\xrightarrow{p_*}\pi_k(S^n)\xrightarrow{\Delta}\pi_{k-1}(R_n),$$

where $i=i_{n+1}:R_n\to R_{n+1}$ is the inclusion map, $p=p_{n+1}:R_{n+1}\to S^n$ is the projection and Δ is the connecting map. We set $i_{k,n}:R_k\to R_n$ for $k\leq n-1$ as $i_n\circ\cdots\circ i_{k+1}$.

The group structures of $\pi_{21}(R_{17})$ and $\pi_k(R_n)$ for $k=21$, 22 and $18\leq n\leq 22$ are obtained by [13]. We recall from [1] the splitting for $k\leq 2n-3$ and $n\geq 13$:

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$$\pi_k(R_n)\cong\pi_k(R_\infty)\oplus\pi_{k+1}(V_{2n,n}),$$

where $V_{m,r}=R_m/R_{m-r}$ for $m\geq r$ is the Stiefel manifold. By use of this splitting, by [3] and the group structures of $\pi_{k+1}(V_{2n,n})$ [8], we obtain the group structures of $\pi_k(R_n)$ for $k=21$, 22 and $n\geq 13$. Therefore our main task is to determine the group structures of $\pi_k(R_n)$ for $10\leq n\leq 12$.

Denote by $\#\alpha$ the order of α. For an element $\alpha\in\pi_k(S^n)$, we denote by $[\alpha]\in\pi_k(R_{n+1})$ an element satisfying $p_{n+1*}[\alpha]=\alpha$. Note that $[\alpha]$ is only determined modulo $i_{n+1*}(\pi_k(R_n))$. We set

$$[\alpha]_m=i_{n,m*}[\alpha]\in\pi_k(R_m)\ (n\leq m-1\,;\ \alpha\in\pi_k(S^n)\,;\ \Delta\alpha=0).$$

About the element $\kappa_7\in\pi^7_{21}$, we adopt the renamed one by [18]. We state our result.

Theorem 0.1 (i) $\pi_{21}(R_3:2)=\mathbb{Z}_4\{[\eta_2]\mu'\sigma_{14}\}\oplus\mathbb{Z}_2\{[\eta_2]\nu'\bar{\varepsilon}_6\}\oplus\mathbb{Z}_2\{[\eta_2]\eta_3\bar{\mu}_4\}$;

$\pi_{21}(R_4:2)=\mathbb{Z}_4\{[\eta_2]_4\mu'\sigma_{14}\}\oplus\mathbb{Z}_2\{[\eta_2]_4\nu'\bar{\varepsilon}_6\}\oplus\mathbb{Z}_2\{[\eta_2]_4\eta_3\bar{\mu}_4\}\oplus\mathbb{Z}_4\{[\iota_3]\mu'\sigma_{14}\}$
$\oplus\mathbb{Z}_2\{[\iota_3]\nu'\bar{\varepsilon}_6\}\oplus\mathbb{Z}_2\{[\iota_3]\eta_3\bar{\mu}_4\}$;

$\pi_{21}(R_5:2)=\mathbb{Z}_{32}\{[\nu_4\sigma'\sigma_{14}]\}\oplus\mathbb{Z}_2\{[\iota_3]_5\eta_3\bar{\mu}_4\}$;

$\pi_{21}(R_6:2)=\mathbb{Z}_{16}\{[\nu_4\sigma'\sigma_{14}]_6\}\oplus\mathbb{Z}_2\{[\eta_5^2]\kappa_7\}$.

(ii) $\pi_{21}(R_7:2)=\mathbb{Z}_8\{[\nu_4\sigma'\sigma_{14}]_7\}\oplus\mathbb{Z}_4\{[\eta_6]\kappa_7\}$;

$\pi_{21}(R_8:2)=\mathbb{Z}_8\{[\nu_4\sigma'\sigma_{14}]_8\}\oplus\mathbb{Z}_4\{[\eta_6]_8\kappa_7\}\oplus\mathbb{Z}_8\{[\iota_7]\sigma'\sigma_{14}\}\oplus\mathbb{Z}_4\{[\iota_7]\kappa_7\}$;

$\pi_{21}(R_9:2)=\mathbb{Z}_4\{[\eta_7]_9\kappa_7\}$.

(iii) $\pi_{21}(R_{10}:2)=\mathbb{Z}_2\{[\iota_7]_{10}\kappa_7\}$;

$\pi_{21}(R_{11}:2)=\mathbb{Z}_2\{[\iota_7]_{11}\kappa_7\}\oplus\mathbb{Z}_2\{[\eta_{10}^2]\mu_{12}\}$;

$\pi_{21}(R_{12}:2)=\mathbb{Z}_2\{[\iota_7]_{12}\kappa_7\}\oplus\mathbb{Z}_2\{[\eta_{11}]\mu_{12}\}\oplus\mathbb{Z}_2\{[\eta_{10}^2]_{12}\mu_{12}\}$;

$\pi_{21}(R_{13}:2)=\mathbb{Z}_2\{[\eta_{11}]_{13}\mu_{12}\}\oplus\mathbb{Z}_4\{[\nu_{12}^2]\nu_{18}\}$, *where* $2[\nu_{12}^2]\nu_{18}=[\iota_7]_{13}\kappa_7$;

$\pi_{21}(R_{14}:2)=\mathbb{Z}_4\{[\nu_{12}^2]_{14}\nu_{18}\}$;

(iv) $\pi_{21}(R_{15}:2)=\mathbb{Z}_2\{[\nu_{12}^2]_{15}\nu_{18}\}$;

$\pi_{21}(R_{16}:2)=\mathbb{Z}_2\{[\nu_{12}^2]_{16}\nu_{18}\}\oplus\mathbb{Z}_2\{[\nu_{15}]\nu_{18}\}$;

$\pi_{21}(R_n:2)=\mathbb{Z}_2\{[\nu_{15}]_n\nu_{18}\}$ *for* $n=17$ *and* 18 ;

$\pi_{21}(R_{19}:2)=\mathbb{Z}_2\{[\nu_{15}]_{19}\nu_{18}\}\oplus\mathbb{Z}_2\{[\eta_{18}^2]\eta_{20}\}$;

$\pi_{21}(R_{20}:2)=\mathbb{Z}_2\{[\eta_{18}^2]_{20}\eta_{20}\}\oplus\mathbb{Z}_2\{[\eta_{19}]\eta_{20}\}$;

$\pi_{21}(R_{21}:2)=\mathbb{Z}_2\{[\eta_{19}]_{21}\eta_{20}\}$;

$\pi_{21}(R_{22}:2)=\mathbb{Z}\{\Delta\iota_{22}\}$

$\pi_{21}(R_n:2)=0$ *for* $n\geq 23$.

Theorem 0.2 (i) $\pi_{22}(R_3:2)=\mathbb{Z}_4\{[\eta_2]\bar{\mu}'\}\oplus\mathbb{Z}_2\{[\eta_2]\nu'\mu_6\sigma_{15}\}$;

$\pi_{22}(R_4:2)=\mathbb{Z}_4\{[\eta_2]_4\bar{\mu}'\}\oplus\mathbb{Z}_2\{[\eta_2]_4\nu'\mu_6\sigma_{15}\}\oplus\mathbb{Z}_4\{[\iota_3]\bar{\mu}'\}\oplus\mathbb{Z}_2\{[\iota_3]\nu'\mu_6\sigma_{15}\}$;

$\pi_{22}(R_5:2)=\mathbb{Z}_{32}\{[\nu_4\rho'']\}\oplus\mathbb{Z}_2\{[\nu_4\sigma'\bar{\nu}_{14}]\}\oplus\mathbb{Z}_2\{[\nu_4\sigma'\varepsilon_{14}]\}$;

$\pi_{22}(R_6:2)=\mathbb{Z}_{16}\{[\nu_4\rho'']_6\}\oplus\mathbb{Z}_4\{[\nu_5]\kappa_8\}\oplus\mathbb{Z}_2\{[\nu_4\sigma'\bar{\nu}_{14}]_6\}\oplus\mathbb{Z}_2\{[\nu_4\sigma'\varepsilon_{14}]_6\}$.

(ii) $\pi_{22}(R_7:2)=\mathbb{Z}_8\{[\nu_4\rho'']_7\}\oplus\mathbb{Z}_2\{[\nu_5]_7\kappa_8\}\oplus\mathbb{Z}_2\{[\nu_4\sigma'\bar{\nu}_{14}]_7\}\oplus\mathbb{Z}_2\{[\nu_4\sigma'\varepsilon_{14}]_7\}\oplus\mathbb{Z}_8\{[\zeta'+\mu_6\sigma_{15}]\}$
$\oplus\mathbb{Z}_2\{[\eta_6]\bar{\varepsilon}_7\}$;

$\pi_{22}(R_8 : 2) = \mathbb{Z}_8\{[\nu_4\rho'']_8\} \oplus \mathbb{Z}_2\{[\nu_5]_8\kappa_8\} \oplus \mathbb{Z}_2\{[\nu_4\sigma'\bar{\nu}_{14}]_8\} \oplus \mathbb{Z}_2\{[\nu_4\sigma'\varepsilon_{14}]_8\} \oplus \mathbb{Z}_8\{[\zeta' + \mu_6\sigma_{15}]_8\}$
$\quad \oplus \mathbb{Z}_2\{[\eta_6]_8\bar{\varepsilon}_7\} \oplus \mathbb{Z}_8\{[\iota_7]\rho''\} \oplus \mathbb{Z}_2\{[\iota_7]\sigma'\bar{\nu}_{14}\} \oplus \mathbb{Z}_2\{[\iota_7]\sigma'\varepsilon_{14}\} \oplus \mathbb{Z}_2\{[\iota_7]\bar{\varepsilon}_7\}$;
$\pi_{22}(R_9 : 2) = \mathbb{Z}_{16}\{[8\sigma_8]\sigma_{15}\} \oplus \mathbb{Z}_8\{[\iota_7]_9\rho''\} \oplus \mathbb{Z}_2\{[\nu_5]_9\kappa_8\} \oplus \mathbb{Z}_2\{[\iota_7]_9\bar{\varepsilon}_7\}$,
where $2([8\sigma_8]\sigma_{15}) \equiv a[\zeta' + \mu_6\sigma_{15}]_9 + b[\iota_7]_9\rho'' \bmod [\nu_5]_9\kappa_8$ *for* a *and* b *odd.*

(iii) $\pi_{22}(R_{10} : 2) = \mathbb{Z}_{16}\{[8\sigma_8]_{10}\sigma_{15}\} \oplus \mathbb{Z}_8\{[\iota_7]_{10}\rho''\} \oplus \mathbb{Z}_2\{[\iota_7]_{10}\bar{\varepsilon}_7\}$;
$\pi_{22}(R_{11} : 2) = \mathbb{Z}_{16}\{[8\sigma_8]_{11}\sigma_{15}\} \oplus \mathbb{Z}_{32}\{[[\iota_{10}, \nu_{10}]]\}$;
$\pi_{22}(R_{12} : 2) = \mathbb{Z}_{32}\{[[\iota_{10}, \nu_{10}]]_{12}\} \oplus \mathbb{Z}_{32}\{[\zeta_{11}]\} \oplus \mathbb{Z}_4\{[8\sigma_8]_{12}\sigma_{15} + 2v[\zeta_{11}] + 2w[[\iota_{10}, \nu_{10}]]_{12}\}$
for v *odd and* w *an integer.*

(iv) $\pi_{22}(R_{13} : 2) = \mathbb{Z}_{16}\{[\eta_{12}^3]\sigma_{15}\}$, *where* $2[\eta_{12}^3]\sigma_{15} = [8\sigma_8]_{13}\sigma_{15}$, $[\zeta_{11}]_{13} \equiv [\eta_{12}^3]\sigma_{15} \bmod 2[\eta_{12}^3]\sigma_{15}$;
$\pi_{22}(R_{14} : 2) = \mathbb{Z}_{16}\{[\eta_{13}^2\sigma_{15}]\} \oplus \mathbb{Z}_2\{[\nu_{13}^2]\nu_{19}\}$, *where* $2[\eta_{13}^2\sigma_{15}] \equiv [\eta_{12}^3]_{14}\sigma_{15} \bmod 2[\eta_{12}^3]_{14}\sigma_{15}$;
$\pi_{22}(R_{15} : 2) = \mathbb{Z}_{16}\{[\eta_{14}\sigma_{15}]\} \oplus \mathbb{Z}_2\{[\nu_{13}^2]_{15}\nu_{19}\}$, *where* $2[\eta_{14}\sigma_{15}] = [\eta_{13}^2\sigma_{15}]_{15}$;
$\pi_{22}(R_{16} : 2) = \mathbb{Z}_{16}\{[\eta_{14}\sigma_{15}]_{16}\} \oplus \mathbb{Z}_2\{[\nu_{13}^2]_{16}\nu_{19}\} \oplus \mathbb{Z}_{16}\{[\sigma_{15}]\}$;
$\pi_{22}(R_{17} : 2) = \mathbb{Z}_2\{[\nu_{13}^2]_{17}\nu_{19}\} \oplus \mathbb{Z}_{16}\{[\sigma_{15}]_{17}\}$;
$\pi_{22}(R_n : 2) = \mathbb{Z}_{16}\{[\sigma_{15}]_n\}$ *for* $n = 18$ *and* 19 ;
$\pi_{22}(R_{20} : 2) = \mathbb{Z}_4\{[2\nu_{19}] - 2[\sigma_{15}]_{20}\} \oplus \mathbb{Z}_{16}\{[\sigma_{15}]_{20}\}$;
$\pi_{22}(R_{21} : 2) = \mathbb{Z}_8\{[\sigma_{15}]_{21}\}$;
$\pi_{22}(R_{22} : 2) = \mathbb{Z}_4\{[\sigma_{15}]_{22}\}$;
$\pi_{22}(R_{23} : 2) = \mathbb{Z}_2\{[\sigma_{15}]_{23}\}$;
$\pi_{22}(R_n : 2) = 0$ *for* $n \geq 24$.

The group structure $\pi_{22}(R_9 : 2)$ corrects that of $\pi_{22}(Spin(9) : 2)$ [17].

1 Recollection of fundamental facts

First of all we recall a formula [13, Lemma 1],

(1.1) $$\Delta(\alpha \circ \Sigma\beta) = \Delta\alpha \circ \beta.$$

Suppose given elements $\alpha \in \pi_k(S^n)$ and $\beta \in \pi_m(S^k)$ so that $\Delta\alpha = 0$. Since $\Delta(\alpha \circ \beta) = \Delta(p_{n+1*}([\alpha] \circ \beta))$, we obtain

(1.2) $$\Delta(\alpha \circ \beta) = 0 \ (\Delta\alpha = 0).$$

From the relation $p_{n*}\Delta\iota_n = (1 + (-1)^n)\iota_{n-1}$, we obtain

(1.3) $p_{n+1*}\Delta(\Sigma\alpha) = 0$ (n : even) ; $p_{n+1*}\Delta(\Sigma\alpha) = 2\iota_n \circ \alpha$ (n : odd) for $\alpha \in \pi_k(S^n)$.

Let $J : \pi_k(R_n) \to \pi_{k+n}(S^n)$ be the J-homomorphism. We have a formula

$$J(\alpha \circ \beta) = J(\alpha) \circ \Sigma^n \beta.$$

We consider the commutative diagram up to sign [29] :

(1.4) $$\begin{array}{ccccccc} R_k^{n-1} & \xrightarrow{i_*} & R_k^n & \xrightarrow{p_*} & \pi_k^{n-1} & \xrightarrow{\Delta} & R_{k-1}^{n-1} \\ \downarrow J & & \downarrow J & & \downarrow \Sigma^n & & \downarrow J \\ \pi_{k+n-1}^{n-1} & \xrightarrow{\Sigma} & \pi_{k+n}^n & \xrightarrow{H} & \pi_{k+n}^{2n-1} & \xrightarrow{P} & \pi_{k+n-2}^{n-1}. \end{array}$$

where the upper sequence is $(k)_n$ and the lower is the EHP sequence. The following formula [20, Theorem 5.2] about Toda brackets is useful:

$$\Delta\{\alpha, \Sigma\beta, \Sigma\gamma\}_1 \subset \{\Delta\alpha, \beta, \gamma\}. \tag{1.5}$$

We need the following [19, Theorem 2.1].

Lemma 1.1 (Mimura-Toda) *Let (X, p, B) be a fibre space with a fibre F. Assume that $\alpha \in \pi_{i+1}(B)$, $\beta \in \pi_j(S^i)$ and $\gamma \in \pi_k(S^j)$ satisfy the conditions $(\Delta\alpha)\circ\beta = 0$ and $\beta\circ\gamma = 0$. For an arbitrary element δ of $\{\Delta\alpha, \beta, \gamma\} \subset \pi_{k+1}(F)$, there exists an element $\varepsilon \in \pi_{j+1}(X)$ such that*

$$p_*\varepsilon = \alpha\circ\Sigma\beta \text{ and } i_*\delta = \varepsilon\circ\Sigma\gamma.$$

Let $Spin(n)$, $U(n)$ and $Sp(n)$ be the the spinor, unitary and symplectic groups, respectively. As it is well known [17], $Sp(2) = Spin(5)$. There exists a natural epimorphism $f: Sp(2) \to R_5$ which induces a Hopf map $h: S^7 = Sp(2)/Sp(1) \to R_5/R_4 = S^4$ [7]. This shows the following.

Lemma 1.2 *There exists a natural map between the exact sequences induced from fiberings $Sp(2) \xrightarrow{Sp(1)} S^7$ and $R_5 \xrightarrow{R_4} S^4$:*

$$\begin{array}{ccccccc}
\pi_n(S^3) & \xrightarrow{i_*} & \pi_n(Sp(2)) & \xrightarrow{p_*} & \pi_n(S^7) & \xrightarrow{\Delta_{\mathbb{H}}} & \pi_{n-1}(S^3) \\
\downarrow [\iota_3]_* & & \downarrow f_* & & \downarrow h_* & & \downarrow [\iota_3]_* \\
\pi_n(R_4) & \xrightarrow{i_*} & \pi_n(R_5) & \xrightarrow{p_*} & \pi_n(S^4) & \xrightarrow{\Delta_{\mathbb{R}}} & \pi_{n-1}(R_4).
\end{array}$$

Let $\omega_n(\mathbb{C}) \in \pi_{2n}(U(n))$ be the characteristic class and $r: U(n) \to R_{2n}$ be the canonical inclusion. Then we know that $r_*\omega_n(\mathbb{C})$ is a generator of $\pi_{2n}(R_{2n})$ and $[\iota_{2n+1}, \iota_{2n+1}] = \Sigma\tau_n$, where $\tau_n = J(r_*\omega_n(\mathbb{C}))$ [9]. So, we can take $\Delta\iota_{2n+1} = i_{2n+1*}r_*\omega_n(\mathbb{C})$. As $p_{2n*}r_*\omega_n(\mathbb{C}) = (n+1)\eta_{2n-1}$, we can set $[\eta_{4n-1}] = r_*\omega_{2n}(\mathbb{C})$ and obtain

$$\Delta\iota_{4n+1} = [\eta_{4n-1}]_{4n+1}. \tag{1.6}$$

Let $\alpha \in \pi_k^{4n+2}$ be an element satisfying $2\iota_{4n+2}\circ\alpha = 0$. By (1.1) and the fact that $R_{4n+2}^{4n+2} \cong \mathbb{Z}_4$, we obtain

$$\Delta(\eta_{4n+2}\Sigma\alpha) = 0 \ (2\iota_{4n+2}\circ\alpha = 0) \tag{1.7}$$

and an element $[\eta_{4n+2}\Sigma\alpha] \in R_{k+1}^{4n+3}$. Since $2R_{4n+4}^{4n+3} = 0$ [13], we have $\#[\eta_{4n+2}^2] = 2$. By [13], for $n \geq 2$,

$$R_{4n+4}^{4n+3} \cong \begin{cases} \mathbb{Z}_2, & n \text{ even}; \\ (\mathbb{Z}_2)^2, & n \text{ odd}, \end{cases}$$

$$R_{4n+4}^{4n+4} \cong \begin{cases} (\mathbb{Z}_2)^2, & n \text{ even}; \\ (\mathbb{Z}_2)^3, & n \text{ odd}, \end{cases}$$

$$R_{4n+4}^{4n+5} \cong \begin{cases} \mathbb{Z}_2, & n \text{ even}; \\ (\mathbb{Z}_2)^2, & n \text{ odd} \end{cases}$$

and

$$R^{4n+4}_{4n+4}=\mathbb{Z}_2\{[\eta_{4n+3}]\}\oplus i_{4n+4*}R^{4n+3}_{4n+4}.$$

So, by use of the exact sequences $(4n+3)_{4n+4}$ and $(4n+4)_{4n+4}$:

$$R^{4n+4}_{4n+5}\xrightarrow{\Delta}R^{4n+4}_{4n+4}\xrightarrow{i_*}R^{4n+5}_{4n+4}\xrightarrow{p_*}\pi^{4n+4}_{4n+4},$$

we obtain

(1.8) $$\Delta\eta_{4n+4}=[\eta^2_{4n+2}]_{4n+4} \text{ for } n\geq 2.$$

Notice that $\Delta\eta_8=[\eta_6]_8\eta_7$, because

(1.9) $$\Delta\iota_8=2[\iota_7]-[\eta_6]_8.$$

By use of the exact sequences $(8n+k)_{8n+2}$ for $0\leq k\leq 2$ and [13], we obtain

(1.10) $$\Delta\iota_{8n+3}=[\nu_{8n-1}]_{8n+3} \text{ for } n\geq 2.$$

Notice that $\Delta\iota_{11}=[\iota_7]_{11}\nu_7$ [10].

We recall from [15] the following.

Theorem 1.3 (Mahowald) *If* $n\neq 1$, 2 or 4, *then* $\#\Delta[\iota_{2n}, \iota_{2n}]=a_n(2n-1)!/8$, *where* $a_n=1$ *if* n *is even and* 2 *if* n *is odd.*

Let P^n be the real n-dimensional projective space and j_n: $\mathrm{P}^{n-1}\to R_n$ be the canonical inclusion. For the J-map $J:[\mathrm{P}^{n-1}, R_n]\to[\Sigma^n\mathrm{P}^{n-1}, S^n]$, we set $g_n=J(j_n)$. We show the following.

Lemma 1.4 *Let* $p=p_{2n}$. *Then,* $p_*\Delta[\iota_{2n}, \iota_{2n}]=[\iota_{2n-1}, \eta_{2n-1}]$. *In particular* $p_*\Delta[\iota_{2n}, \iota_{2n}]=0$ *for* n *even and* $p_*\Delta[\iota_{2n}, \iota_{2n}]=[\iota_{2n-1}, \eta_{2n-1}]\neq 0$ *for* n *odd.*

Proof. By use of [26, Lemma 1] we obtain $\Sigma^{2n+3}p_*\Delta[\iota_{2n}, \iota_{2n}]=0$. By the Freudenthal suspension theorem, Σ^{2n+1}: $\pi^{2n+1}_{4n}\to\pi^{4n+2}_{6n+1}$ is an isomorphism, and so $\Sigma^2p_*\Delta[\iota_{2n}, \iota_{2n}]=0$. By the EHP sequence

$$\pi^{4n+1}_{4n+1}\xrightarrow{P}\pi^{2n}_{4n-1}\xrightarrow{\Sigma}\pi^{2n+1}_{4n},$$

we have

$$\Sigma p_*\Delta[\iota_{2n}, \iota_{2n}]\in P\pi^{4n+1}_{4n+1}\cap\Sigma\pi^{2n-1}_{4n-2}.$$

As $P\pi^{4n+1}_{4n+1}\cap\Sigma\pi^{2n-1}_{4n-2}=0$ by the Serre theorem, we obtain $\Sigma p_*\Delta[\iota_{2n}, \iota_{2n}]=0$. So, by the EHP sequense

$$\pi^{4n-1}_{4n}\xrightarrow{P}\pi^{2n-1}_{4n-2}\xrightarrow{\Sigma}\pi^{2n}_{4n-1},$$

we obtain $p_*\Delta[\iota_{2n}, \iota_{2n}]\in P\pi^{4n-1}_{4n}=\{[\iota_{2n-1}, \eta_{2n-1}]\}$. By [6], $[\iota_{2n-1}, \eta_{2n-1}]\neq 0$ for n odd and $=0$ for n even. Hence,

$$p_*\Delta[\iota_{2n}, \iota_{2n}]=[\iota_{2n-1}, \eta_{2n-1}]=0 \ (n: \text{even}).$$

Let $\gamma_n: S^n \to \mathrm{P}^n$ be the covering map and $p_n: \mathrm{P}^n \to S^n$ be the pinch map. Since $\Delta\iota_n = j_n\gamma_{n-1}$, we have $g_n\Sigma^n\gamma_{n-1}=\pm[\iota_n, \iota_n]$ by (1.4). We recall from [21] that

$$\pm g_n \in \{\Sigma g_{n-1}, \Sigma^n\gamma_{n-2}, \Sigma^{n-1}p_{n-1}\}_1.$$

By (1.5), we have

$$\begin{aligned}\pm\Delta g_{2n} &\in \{\Delta(\Sigma g_{2n-1}), \Sigma^{2n-1}\gamma_{2n-2}, \Sigma^{2n-2}p_{2n-1}\}\\ &=\{\Delta\iota_{2n}\circ g_{2n-1}, \Sigma^{2n-1}\gamma_{2n-2}, \Sigma^{2n-2}p_{2n-1}\}\\ &\subset\{\Delta\iota_{2n}, [\iota_{2n-1}, \iota_{2n-1}], \Sigma^{2n-2}p_{2n-1}\}.\end{aligned}$$

So, by (1.3), we obtain

$$\begin{aligned}\pm p_*\Delta[\iota_{2n}, \iota_{2n}] &= p_*\Delta g_{2n}\circ\Sigma^{2n-1}\gamma_{2n-1}\\ &\in p_*\{\Delta\iota_{2n}, [\iota_{2n-1}, \iota_{2n-1}], \Sigma^{2n-2}p_{2n-1}\}\circ\Sigma^{2n-1}\gamma_{2n-1}\\ &\subset\{2\iota_{2n-1}, [\iota_{2n-1}, \iota_{2n-1}], 2\iota_{4n-3}\}.\end{aligned}$$

We recall that $[\iota_{2n-1}, \iota_{2n-1}]=\Sigma\tau_{n-1}$. By the fact that $\pi_{2n-2}(R_{2n-2})\cong(\mathbb{Z}_2)^2$ or $(\mathbb{Z}_2)^3$ for n odd [13] and that $\Sigma\pi^{2n-2}_{4n-4}=\pi^{2n-1}_{4n-3}$, we obtain

$$\{2\iota_{2n-1}, [\iota_{2n-1}, \iota_{2n-1}], 2\iota_{4n-3}\}=\{2\iota_{2n-1}, \Sigma\tau_{n-1}, 2\iota_{4n-3}\}_1.$$

So, by [28, Corollary 3.7], we obtain

$$\{2\iota_{2n-1}, [\iota_{2n-1}, \iota_{2n-1}], 2\iota_{4n-3}\}\ni(\Sigma\tau_{n-1})\eta_{4n-3}=[\iota_{2n-1}, \eta_{2n-1}].$$

Hence we have

$$p_*\Delta[\iota_{2n}, \iota_{2n}]\equiv[\iota_{2n-1}, \eta_{2n-1}] \bmod 2\pi^{2n-1}_{4n-2}.$$

Since $\pi^5_{10}\cong\mathbb{Z}_2$, we have $p_*\Delta[\iota_6, \iota_6]=[\iota_5, \eta_5]$. As it is well known, $[\iota_{2n-1}, \eta_{2n-1}]\circ\eta_{4n-2}=[\iota_{2n-1}, \eta^2_{2n-1}]\neq 0$ for n odd and $n\geq 5$. So $[\iota_{2n-1}, \eta_{2n-1}]\notin 2\pi^{2n-1}_{4n-2}$ in this case. This implies $p_*\Delta[\iota_{2n}, \iota_{2n}]=[\iota_{2n-1}, \eta_{2n-1}]$ for n odd and $n\geq 3$. This completes the proof. □

2 Elements in the J-image

Suppose that $\alpha\in\pi^{n-1}_k$ satisfies the relations $\Delta\alpha=0$ and $\pm\Sigma^n\alpha=H(\beta)$ for an element $\beta\in\pi^n_{k+n}$. Then, by (1.4), we have $H(J[\alpha])=\pm\Sigma^n\alpha=H(\beta)$. So, by the EHP sequence, we obtain

(2.1) $$J[\alpha]\equiv\beta \bmod \Sigma\pi^{n-1}_{k+n-1}.$$

By (2.1), we obtain $J[\iota_3]\equiv\nu_4 \bmod \Sigma\nu'$ and $J[\iota_7]\equiv\sigma_8 \bmod \Sigma\sigma'$. By [28, Proposition 4.4], $\pi^n_i=\Sigma\pi^{n-1}_{i-1}\oplus J[\iota_n]_*\pi^{2n-1}_i$ for $n=3$ or 7. From this fact, and to avoid messiness of changing generators of π^n_i, we take as

(2.2) $$J[\iota_3]=\nu_4 \text{ and } J[\iota_7]=\sigma_8.$$

We know $\Delta\nu_5=0$ [10, Table 3]. By the same argument as in [12, Lemma 1.3], we can take $[\zeta_5]\in\{[\nu_5], 8\iota_8, \Sigma\sigma'\}_1$. That is,

(2.3) $$\Delta\nu_5=0,\ \Delta\zeta_5=0 \text{ and } [\zeta_5]\in\{[\nu_5], 8\iota_8, \Sigma\sigma'\}_1.$$

Hereafter we use the following convention: Our notation $J[\alpha]=\beta$ stands for $J[\alpha]=x\beta$ for x odd.

By [28, Lemma 12.10] and [24, Proposition 2.8.(2)],

(2.4) $$\varepsilon_6^2=\varepsilon_6\bar{\nu}_{14}=\eta_6\bar{\varepsilon}_7 \text{ and } \bar{\nu}_6^2=\bar{\nu}_6\varepsilon_{14}=0.$$

Now we obtain the following result overlapping with [4].

Lemma 2.1 (i) $J[\eta_2]=\nu'$; $J[4\nu_4]=\sigma'''$; $J[\eta_5^2]=\sigma''$; $J[\eta_6]=\sigma'$.
(ii) $J[\nu_5]=\bar{\nu}_6+\varepsilon_6$; $J[\nu_4^2]=\nu_5\sigma_8$.
(iii) $J[\sigma''']=[\iota_6, \sigma'']=4P(\sigma_{13})$; $J[\eta_5\varepsilon_6]=-\sigma''\sigma_{13}+\bar{\nu}_6\nu^2{}_{14}$; $J[2[\iota_6, \iota_6]]=\zeta_7$; $J[\bar{\nu}_6+\varepsilon_6]=\sigma'\sigma_{14}$.
(iv) $J[\eta_{10}^2]=\theta'$; $J[\eta_{11}]\equiv\theta \bmod \Sigma\theta'$; $J[8\sigma_8]\equiv\rho' \bmod \sigma_9\bar{\nu}_{16}, \sigma_9\varepsilon_{16}$; $J[\eta_{12}^3]=\rho_{13}$.
(v) $J[\zeta_5]\equiv\zeta'+\mu_6\sigma_{15} \bmod \eta_6\bar{\varepsilon}_7$; $J[\zeta_5]_9=\mu_9\sigma_{18}$; $J([\zeta_5]_7+[\eta_6]\eta_7\varepsilon_8+b[\nu_5]_7\varepsilon_8)=\mu_7\sigma_{16}$;
$J([\zeta_5]_7\eta_{16}+b[\nu_5]_7\varepsilon_8\eta_{16})=\mu_7\sigma_{16}\eta_{23}=\eta_7\mu_8\sigma_{17}$ *for* $b=0$ *or* 1.

Proof. By the fact that $HJ[\eta_2]=\eta_5=H(\nu')$ and (2.1), we have $J[\eta_2]\equiv\nu' \bmod 2\nu'$. This yields the first of (i). The rest of (i), (ii) and (iii) are obtained from [10] and [11].

The first two relations of (iv) are obtained by [12]. The fourth of (iv) is obtained by [10]. By [10] and [28], $J[8\sigma_8]\equiv\rho' \bmod \Sigma\pi_{23}^8=\{2\rho', \sigma_9\bar{\nu}_{16}, \sigma_9\varepsilon_{16}, \bar{\varepsilon}_9\}$. In the stable range, $\bar{\varepsilon}$ is not in the J-image, so we have the third of (iv).

By the fact that $\{\varepsilon_3, 2\iota_{11}, 8\sigma_{11}\}=\mu_3\sigma_{12}$ and $\{\bar{\nu}_6, 8\iota_{14}, 2\sigma_{14}\}\ni\zeta' \bmod 2\zeta'$ [22, Proposition I.3.2.(1)], we see that

$$\{\bar{\nu}_6+\varepsilon_6, 8\iota_{14}, 2\sigma_{14}\}\subset\{\bar{\nu}_6, 8\iota_{14}, 2\sigma_{14}\}+\{\varepsilon_6, 8\iota_{14}, 2\sigma_{14}\}\ni\zeta'+\mu_6\sigma_{15}$$
$$\bmod \{2\zeta'\}+\{\bar{\nu}_6, \varepsilon_6\}\circ\pi_{22}^{14}+\pi_{15}^6\circ 2\sigma_{15}.$$

Since $\pi_{15}^6\cong(\mathbb{Z}_2)^3$ [28], we have $\pi_{15}^6\circ 2\sigma_{15}=0$. By (2.4), $\{\bar{\nu}_6, \varepsilon_6\}\circ\pi_{22}^{14}=\{\eta_6\bar{\varepsilon}_7\}$ and

(2.5) $$\zeta'+\mu_6\sigma_{15}\in\{\bar{\nu}_6+\varepsilon_6, 8\iota_{14}, 2\sigma_{14}\}_1 \bmod 2\zeta', \eta_6\bar{\varepsilon}_7.$$

Then, by (2.3) and the fact that $J[\nu_5]=\bar{\nu}_6+\varepsilon_6$, we obtain

$$J[\zeta_5]\in\{\bar{\nu}_6+\varepsilon_6, 8\iota_{14}, 2\sigma_{14}\}\ni\zeta'+\mu_6\sigma_{15} \bmod 2\zeta', \eta_6\bar{\varepsilon}_7.$$

This leads to the relation $J[\zeta_5]\equiv\zeta'+\mu_6\sigma_{15} \bmod 2\zeta', \eta_6\bar{\varepsilon}_7$. Since $J([\nu_4^2]_6\nu_{10}^2)=\nu_6\sigma_9\nu_{16}^2=\eta_6\bar{\varepsilon}_7$ [28, Lemma 12. 10], we obtain the first of (v). The rest of (v) is obviously obtained. This completes the proof. □

Lemma 2.2 (i) $J[\eta_{14}^2]\equiv\eta^{*\prime} \bmod \sigma_{15}\mu_{22}$; $J[\eta_{15}]\equiv\eta_{16}^*+\omega_{16} \bmod \Sigma\eta^{*\prime}, \sigma_{16}\mu_{23}$.
(ii) $J[[\iota_9, \eta_9]]=\sigma_{10}\zeta_{17}$; $J[\eta_9\varepsilon_{10}]=\lambda''$; $J[\varepsilon_{10}]=\lambda'+2\xi'$; $J[\nu_{12}^2]=\lambda+2\xi_{13}$;

$J[\nu_{15}] \equiv \nu_{16}^* + \xi_{16} \bmod 4\Sigma^3\lambda$.

(iii) $J([\sigma'''] \sigma_{12}) = 8\bar{\sigma}_6$; $J[32[\iota_{10}, \iota_{10}]] = \bar{\zeta}_{11}$.

(iv) $J[\varepsilon_{11}] \equiv \omega' \bmod \theta\sigma_{24}, (\Sigma\xi')\eta_{30}, (\Sigma\lambda')\eta_{30}$; $J[\nu_{13}^2] = \omega_{14}\nu_{30}$; $J[\rho^{IV}] = 8P(\rho_{13})$.

(v) $J[\zeta_9] = \beta'$; $J[\eta_{10}\mu_{11}] \equiv \beta''' \bmod \Sigma\beta'$; $J[\mu_{11}] \equiv \beta''' \bmod \Sigma\beta'', \Sigma^2\beta'$; $J[\eta_{18}^2] = \bar{\beta}$; $J[\eta_{19}] = \bar{\bar{\beta}} \bmod [\iota_{20}, \eta_{20}]$.

Proof. First, we recall from [24, Lemma 2.10] that

$$\Sigma\eta^{*\prime} \equiv [\iota_{16}, \eta_{16}] \bmod \sigma_{16}\mu_{23}. \tag{2.6}$$

The first of (i) is obtained by the fact that $H(J[\eta_{14}^2]) = H(\eta^{*\prime})$, $\pi_{30}^{14} = \mathbb{Z}_8\{\omega_{14}\} \oplus \mathbb{Z}_2\{\sigma_{14}\mu_{21}\}$ and (1.8). The second of (i) is obtained by (2.1), (1.6), (2.6) and [28, Proposition 12.20.(ii)]. These correct the assertions of [10, pp. 30–31].

For the proof of the rest of (i) and (ii), we need the fact that the stable J-image is 0 in $\pi_{18}^S(S^0 : 2)$ and the result [28, Corollary 12.25]:

$$[\iota_{19}, \iota_{19}] = \nu_{19}^* + \xi_{19}. \tag{2.7}$$

The first of (ii) is obtained by [11, p. 343]. The second of (ii) follows from the fact that $\Sigma\pi_{27}^9 = \{\sigma_{10}\zeta_{17}, \eta_{10}\bar{\mu}_{11}\}$, $\Sigma^\infty\lambda = 2\nu^*$ and [23]:

$$H(\lambda'') = \eta_{19}\varepsilon_{20} ; \ 2\lambda'' \equiv \sigma_{10}\zeta_{17} \bmod 2\sigma_{10}\zeta_{17}.$$

The third of (ii) is obtained by the fact that $\xi = -\nu^*$ (2.7) and [23]:

$$H(\lambda') = \varepsilon_{21}.$$

The fourth of (ii) is obtained by the result $H(\lambda) = \nu_{25}^2$. The last of (ii) is obtained by the fact that $HJ[\nu_{15}] = H(c\nu_{16}^*)$ for c odd [28, Lemma 12.14], $\Sigma\pi_{33}^{15} = \{\xi_{16}, \Sigma^3\lambda, \eta_{16}\bar{\mu}_{17}\}$, (1.10), (2.1) and (2.7).

Since $J[\sigma'''] = 4P(\sigma_{13})$ by Lemma 2.1.(iii), $J([\sigma''']\sigma_{12}) = 4P(\sigma_{13}^2)$. By the fact that $2\bar{\sigma}_6 \equiv xP(\sigma_{13}^2) \bmod \Sigma\pi_{24}^5 = \{\bar{\zeta}_6, 16\bar{\sigma}_6\}$ for x odd [28, p. 151], $\#\bar{\zeta}_n = 8$ for $n \geq 5$ and $\#\bar{\sigma}_7 = 2$, we obtain $4\bar{\sigma}_6 = 2xP(\sigma_{13}^2)$. This leads to the first of (iii). The second of (iii) is a direct consequence from the fact that

$$R_{19}^{11} \cong \mathbb{Z}\{[32[\iota_{10}, \iota_{10}]]\} \oplus (\mathbb{Z}_2)^2, \ R_{19}^n \cong \{[32[\iota_{10}, \iota_{10}]]_n\} (n \geq 22)$$

and that $\bar{\zeta}$ is in the stable J-image.

We recall the relations $H(\omega') \equiv \varepsilon_{23} \bmod \eta_{23}\sigma_{24}$ [28, Lemma 12.21], $\theta'\sigma_{23} = \xi'\eta_{29}$, $\theta\sigma_{24} \equiv \xi_{12}\eta_{30} \bmod (\Sigma\xi')\eta_{30}, (\Sigma\lambda')\eta_{30}$ [24, Proposition 2.20] and $\omega' \in \{[\iota_{12}, \iota_{12}], \nu_{23}^2, \eta_{29}\} \bmod \xi_{12}\eta_{30}, (\Sigma\xi')\eta_{30}, (\Sigma\lambda')\eta_{30}$ [24, Lemma 2.21]. Since $\Delta(\nu_{12}^2) = 0$ [11], the Toda bracket $\{\Delta\iota_{12}, \nu_{11}^2, \eta_{17}\} \subset R_{19}^{12}$ is well defined and

$$J\{\Delta\iota_{12}, \nu_{11}^2, \eta_{17}\} \subset \{[\iota_{12}, \iota_{12}], \nu_{23}^2, \eta_{29}\}.$$

So, for a representative α of $\{\Delta\iota_{12}, \nu_{11}^2, \eta_{17}\}$,

$$J\alpha \equiv \omega' \bmod \xi_{12}\eta_{30},\ (\Sigma\xi')\eta_{30},\ (\Sigma\lambda')\eta_{30}.$$

By Lemma 2.1.(iv) and Lemma 2.2.(ii), we see that $J([\eta_{11}]\sigma_{12}) \equiv \theta\sigma_{24} \bmod (\Sigma\theta')\sigma_{24}$, $J([\eta_{10}^2]\sigma_{12}) = \theta'\sigma_{23} = \xi'\eta_{29}$ and $J([\varepsilon_{10}]_{12}\eta_{18}) = (\Sigma\lambda')\eta_{30}$. By (1.3),

$$p_{12*}\{\Delta\iota_{12},\ \nu_{11}^2,\ \eta_{17}\} \subset \{2\iota_{11},\ \nu_{11}^2,\ \eta_{17}\} \ni \varepsilon_{11} \bmod \eta_{11}\sigma_{12},$$

so α is taken as $[\varepsilon_{11}] \bmod [\eta_{11}]\sigma_{12}$. This leads to the first of (iv).

By [12], we have $2[\nu_{13}^2] = [\varepsilon_{11}]_{14}$. So we obtain

$$2J[\nu_{13}^2] = J[\varepsilon_{11}]_{14} \equiv \Sigma^2\omega' \bmod \xi_{14}\eta_{32},\ (\Sigma^3\xi')\eta_{32},\ (\Sigma^3\lambda')\eta_{32}.$$

By [28, (12.27)], $\Sigma^2\omega' = 2\omega_{14}\nu_{30}$. Since $2\xi_{14} = \Sigma^3\xi'$ and $2\Sigma\lambda = \Sigma^3\lambda'$ [28, Lemma 12.19], $(\Sigma^3\xi')\eta_{32} = (\Sigma^3\lambda')\eta_{32} = 0$. By the fact that $P(\sigma_{27}) = \xi_{13}\eta_{31}$ [28, (12.26)], we have $\xi_{14}\eta_{32} = 0$ and $2J[\nu_{13}^2] = 2\omega_{14}\nu_{30}$. Since $\bar{\sigma}$ is not in the stable J-image, we obtain the relation $J[\nu_{13}^2] \equiv \omega_{14}\nu_{30} \bmod 4\bar{\xi}_{14}$. Hence, by the fact that $\Delta\nu_{17} = [\nu_{13}^2]_{17}$ [12] and $\omega\nu = 0 \in \pi_{19}^S(S^0)$, we have the second of (iv).

By [12], $[\rho^{IV}] \in \{[\sigma'''],\ 4\iota_{12},\ 4\sigma_{12}\}$. So, by the fact that $J[\sigma'''] = 4P(\sigma_{13})$,

$$J[\rho^{IV}] \in \{4P(\sigma_{13}),\ 4\iota_{18},\ 4\sigma_{18}\} \supset P\{4\sigma_{13},\ 4\iota_{20},\ 4\sigma_{20}\}.$$

By [28, Lemma 10.9] and [24, Lemma 2.9], we have $8\rho_{13} = \Sigma^7\rho'''$. So, by the definition of ρ''' [28, p. 103], $8\rho_{13} \in \{4\sigma_{13},\ 4\iota_{20},\ 4\sigma_{20}\} \bmod 4\sigma_{13}\circ\pi_{28}^{20} + \pi_{21}^{13}\circ 4\sigma_{21} = 0$. This leads to the last of (iv). By [12], we have (v). This completes the proof. □

3 Determination of $\pi_k(R_n : 2)$ $(k=21,\ 22\ ;\ n \leq 8)$

First of all, from Lemma 2.1.(i), [10] and [25], we obtain

Lemma 3.1 $R_7^5 = \mathbb{Z}\{[4\nu_4]\}$, $R_7^6 = \mathbb{Z}\{[\eta_5^2]\}$ *and* $R_7^7 = \mathbb{Z}\{[\eta_6]\}$, *where* $2[\eta_5^2] = [4\nu_4]_6$ *and* $2[\eta_6] = [\eta_5^2]_7$.

Hereafter we shall use the fact [3] that the stable J-image is trivial in $\pi_k^S(S^0)$ for $k=21,22$.

As it is well known, $\pi_k(R_3) \cong \pi_k(S^3)$ and $\pi_k(R_4) \cong \pi_k(R_3) \oplus \pi_k(S^3)$. We know $\pi_{21}^3 = \mathbb{Z}_4\{\mu'\sigma_{14}\} \oplus \mathbb{Z}_2\{\nu'\bar{\varepsilon}_6\} \oplus \mathbb{Z}_2\{\eta_3\bar{\mu}_4\}$ and $\pi_{22}^3 = \mathbb{Z}_4\{\bar{\mu}'\} \oplus \mathbb{Z}_2\{\nu'\mu_6\sigma_{15}\}$, where $2\mu' = \eta_3^2\mu_5$ [28, (7.7)] and $2\bar{\mu}' = \eta_3^2\bar{\mu}_5$ [28, Lemma 12.4]. This yields the groups R_n^3 and R_n^4 for $n=21$ and 22:

$$R_{21}^4 = \mathbb{Z}_4\{[\eta_2]_4\mu'\sigma_{14}\} \oplus \mathbb{Z}_2\{[\eta_2]_4\nu'\bar{\varepsilon}_6\} \oplus \mathbb{Z}_2\{[\eta_2]_4\eta_3\bar{\mu}_4\}$$
$$\oplus\, \mathbb{Z}_4\{[\iota_3]\mu'\sigma_{14}\} \oplus \mathbb{Z}_2\{[\iota_3]\nu'\bar{\varepsilon}_6\} \oplus \mathbb{Z}_2\{[\iota_3]\eta_3\bar{\mu}_4\}\ ;$$
$$R_{22}^4 = \mathbb{Z}_4\{[\eta_2]_4\bar{\mu}'\} \oplus \mathbb{Z}_2\{[\eta_2]_4\nu'\mu_6\sigma_{15}\} \oplus \mathbb{Z}_4\{[\iota_3]\bar{\mu}'\} \oplus \mathbb{Z}_2\{[\iota_3]\nu'\mu_6\sigma_{15}\}.$$

By [19, Theorem 5.1], we have the following:

$\pi_{14}(Sp(2) : 2) = \mathbb{Z}_{16}\{[2\sigma']\}$, where $4[2\sigma'] = \pm i_*\mu'$;
$\pi_{15}(Sp(2) : 2) = \mathbb{Z}_2\{[\sigma'\eta_{14}]\}$; $\pi_{16}(Sp(2) : 2) = \mathbb{Z}_2\{[\sigma'\eta_{14}]\eta_{15}\} \oplus \mathbb{Z}_2\{[\nu_7]\nu_{10}^2\}$;
$\pi_{21}(Sp(2) : 2) = \mathbb{Z}_{32}\{[\sigma'\sigma_{14}]\} \oplus \mathbb{Z}_2\{i_*\eta_3\bar{\mu}_4\}$, where $8[\sigma'\sigma_{14}] = \pm i_*\mu'\sigma_{14}$;
$\pi_{22}(Sp(2) : 2) = \mathbb{Z}_{32}\{[\rho'']\} \oplus \mathbb{Z}_2\{[\sigma'\bar{\nu}_{14}]\} \oplus \mathbb{Z}_2\{[\sigma'\varepsilon_{14}]\}$, where $8[\rho''] = \pm i_*\bar{\mu}'$.

By [24, Propositions 2.2.(1), 2.13.(6), 2.17.(7)],

$$(\Sigma\nu')\sigma'=2\Sigma\varepsilon',\ \varepsilon'\sigma_{13}=2\bar{\varepsilon}' \text{ and } (\Sigma\nu')\rho''=0.$$

So, by setting $h\equiv\nu_4 \bmod \Sigma\nu'$ in Lemma 1.2, we obtain the following.

Lemma 3.2 (i) $R^5_{14}=\mathbb{Z}_{16}\{[2\nu_4\sigma']\}$ *and* $4[2\nu_4\sigma']=\pm[\iota_3]_5\mu'$.

(ii) $R^5_{15}=\mathbb{Z}_2\{[\nu_4\sigma'\eta_{14}]\}$; $R^5_{16}=\mathbb{Z}_2\{[\nu_4\sigma'\eta_{14}]\eta_{15}\}\oplus\mathbb{Z}_2\{[\nu_4^2]\nu_{10}^2\}$.

(iii) $R^5_{21}=\mathbb{Z}_{32}\{[\nu_4\sigma'\eta_{14}]\}\oplus\mathbb{Z}_2\{[\iota_3]_5\eta_3\bar{\mu}_4\}$ *and* $8[\nu_4\sigma'\sigma_{14}]=4[2\nu_4\sigma']\sigma_{14}$.

(iv) $R^5_{22}=\mathbb{Z}_{32}\{[\nu_4\rho'']\}\oplus\mathbb{Z}_2\{[\nu_4\sigma'\bar{\nu}_{14}]\}\oplus\mathbb{Z}_2\{[\nu_4\sigma'\varepsilon_{14}]\}$ *and* $8[\nu_4\rho'']=\pm[\iota_3]_5\bar{\mu}'$.

By [10, Table 3],

$$\Delta\iota_5=[\iota_3]_5\eta_3 \tag{3.1}$$

We consider the exact sequence $(21)_5$:

$$\pi^5_{22}\xrightarrow{\Delta}R^5_{21}\xrightarrow{i_*}R^6_{21}\xrightarrow{p_*}\pi^5_{21}\xrightarrow{\Delta}R^5_{20}.$$

We recall that $\eta_6\kappa_7=\bar{\varepsilon}_6$ [28, (10.23)], $\pi^5_{21}=\mathbb{Z}_2\{\mu_5\sigma_{14}\}\oplus\mathbb{Z}_2\{\eta_5^2\kappa_7\}$ and $\pi^5_{22}=\mathbb{Z}_4\{\nu_5\kappa_8\}\oplus\mathbb{Z}_2\{\bar{\mu}_5\}\oplus\mathbb{Z}_2\{\eta_5\mu_6\sigma_{15}\}$. By Lemma 3.1, $\mathrm{Ker}\Delta=\mathbb{Z}_2\{\eta_5^2\kappa_7\}$ for Δ on the right. By (2.3) and (1.2), $\Delta(\nu_5\kappa_8)=0$. By (3.1) and (1.1), $\Delta\bar{\mu}_5=[\iota_3]_5\eta_3\bar{\mu}_4$ and $\Delta(\eta_5\mu_6\sigma_{15})=[\iota_3]_5\eta_3^2\mu_5\sigma_{14}=2[\iota_3]_5\mu'\sigma_{14}=16[\nu_4\sigma'\sigma_{14}]$. Since $2\kappa_7=\bar{\nu}_7\nu_{15}^2$ [19], we have $2[\eta_5^2]\kappa_7=[\eta_5^2]\bar{\nu}_7\nu_{15}^2$. By [10], $R^5_{15}=\mathbb{Z}_2\{[\nu_4\sigma'\eta_{14}]\}$. So, by the relations $\eta_5^2\bar{\nu}_7=\eta_5\nu_6^3=0$ [28, (7.3), (5.9)], $[\eta_5^2]\bar{\nu}_7=a[\nu_4\sigma'\eta_{14}]_6$ for $a\in\{0,1\}$. By the fact that $[\nu_4\sigma'\eta_{14}]_6=[\eta_5\varepsilon_6]\eta_{14}+4[\nu_5]\sigma_8$ [11, Lemma 1.2.(iii)], we have $[\eta_5^2]\bar{\nu}_7\nu_{15}^2=a[\nu_4\sigma'\eta_{14}]_6\nu_{15}^2=0$. Hence

$$R^6_{21}=\mathbb{Z}_{16}\{[\nu_4\sigma'\sigma_{14}]_6\}\oplus\mathbb{Z}_2\{[\eta_5^2]\kappa_7\}.$$

In the exact sequence $(22)_5$:

$$\pi^5_{23}\xrightarrow{\Delta}R^5_{22}\xrightarrow{i_*}R^6_{22}\xrightarrow{p_*}\pi^5_{22}\xrightarrow{\Delta}R^5_{21},$$

we know that $\mathrm{Ker}\{\Delta:\pi^5_{22}\to R^5_{21}\}=\mathbb{Z}_4\{\nu_5\kappa_8\}$ and $\pi^5_{23}=\mathbb{Z}_8\{\zeta_5\sigma_{16}\}\oplus\mathbb{Z}_2\{\nu_5\eta_8\kappa_9\}\oplus\mathbb{Z}_2\{\eta_5\bar{\mu}_6\}$. By (2.3) and (1.2), $\Delta(\nu_5\eta_8\kappa_9)=\Delta(\zeta_5\sigma_{16})=0$. By (3.1), we see that

$$\Delta(\eta_5\mu_6)=2[\iota_3]_5\mu'=8[2\nu_4\sigma']\ ;\ \Delta(\eta_5\bar{\mu}_6)=2[\iota_3]_5\bar{\mu}'=16[\nu_4\rho''].\tag{3.2}$$

We recall from [28] that $2\nu_5\kappa_8\neq0$ and $4\kappa_8=0$. This yields the group

$$R^6_{22}=\mathbb{Z}_{16}\{[\nu_4\rho'']_6\}\oplus\mathbb{Z}_4\{[\nu_5]\kappa_8\}\oplus\mathbb{Z}_2\{[\nu_4\sigma'\bar{\nu}_{14}]_6\}\oplus\mathbb{Z}_2\{[\nu_4\sigma'\varepsilon_{14}]_6\}.$$

By [17, Proposition 9.1], $\pi_n(Spin(7):2)\cong\pi_n(G_2:2)\oplus\pi_n(S^7:2)$. We know $\pi^7_{21}=\mathbb{Z}_8\{\sigma'\sigma_{14}\}\oplus\mathbb{Z}_4\{\kappa_7\}$ and $\pi^7_{22}=\mathbb{Z}_8\{\rho''\}\oplus\mathbb{Z}_2\{\sigma'\bar{\nu}_{14}\}\oplus\mathbb{Z}_2\{\sigma'\varepsilon_{14}\}\oplus\mathbb{Z}_2\{\bar{\varepsilon}_7\}$. By [17, Theorem 6.1], $\pi_{21}(G_2:2)=0$ and $\pi_{22}(G_2:2)=\mathbb{Z}_8\{\langle\zeta'+\mu_6\sigma_{15}\rangle\}\oplus\mathbb{Z}_2\{\langle\eta_6^2\kappa_8\rangle\}$. Here the notation $\langle\alpha\rangle$ is an element such that $p_*\langle\alpha\rangle=\alpha$ for the projection $p:G_2\to G_2/SU(3)=S^6$. So we have $R^7_{21}\cong\mathbb{Z}_8\oplus\mathbb{Z}_4$ and $R^7_{22}\cong(\mathbb{Z}_8)^2\oplus(\mathbb{Z}_2)^4$ [17]. By (2.5) and the group structure of $R^7_{15}\cong(\mathbb{Z}_2)^4$ [17], we take $[\zeta'+\mu_6\sigma_{15}]\in R^7_{22}$ as follows:

(3.3) $[\zeta'+\mu_6\sigma_{15}]\in\{[\bar{\nu}_6+\varepsilon_6], 8\iota_{14}, 2\sigma_{14}\}$ mod $2[\zeta'+\mu_6\sigma_{15}], [\bar{\nu}_6+\varepsilon_6]\bar{\nu}_{14}, [\bar{\nu}_6+\varepsilon_6]\varepsilon_{14}$.

We consider the exact sequence $(21)_6$:

$$\pi^6_{22}\xrightarrow{\Delta}R^6_{21}\xrightarrow{i_*}R^7_{21}\xrightarrow{p_*}\pi^6_{21}\xrightarrow{\Delta}R^6_{20}.$$

We know $\pi^6_{21}=\mathbb{Z}_4\{\rho'''\}\oplus\mathbb{Z}_2\{\eta_6\kappa_7\}$ and $\pi^6_{22}=\mathbb{Z}_8\{\zeta'\}\oplus\mathbb{Z}_2\{\mu_6\sigma_{15}\}\oplus\mathbb{Z}_2\{\eta_6\bar{\varepsilon}_7\}$. By [12], $\Delta\rho'''=[\rho^{IV}]$. By (1.2), $\Delta(\eta_6\kappa_7)=0$ and $\Delta(\eta_6\bar{\varepsilon}_7)=0$. In the exact sequence $(14)_5$:

$$\pi^5_{15}\xrightarrow{\Delta}R^5_{14}\xrightarrow{i_*}R^6_{14}\xrightarrow{p_*}\pi^5_{14}.$$

$\pi^5_{15}=\mathbb{Z}_8\{\nu_5\sigma_8\}\oplus\mathbb{Z}_2\{\eta_5\mu_6\}$ [28, Theorem 7.3], $R^5_{14}=\mathbb{Z}_{16}\{[2\nu_4\sigma']\}$ and $R^6_{14}=\mathbb{Z}_{16}\{\eta_5\varepsilon_6\}\oplus\mathbb{Z}_2\{[\nu_5]\nu_8^2\}$ [10, Table 2]. By (1.2), $\Delta(\nu_5\sigma_8)=0$. So, by the fact that $\Delta(\eta_5\mu_6)=8[2\nu_4\sigma']$ (3.2), we obtain

(3.4) $$2[\eta_5\varepsilon_6]=y[2\nu_4\sigma']_6 \ (y: odd).$$

In the the exact sequence $(14)_6$:

$$\pi^6_{15}\xrightarrow{\Delta}R^6_{14}\xrightarrow{i_*}R^7_{14}\xrightarrow{p_*}\pi^6_{14},$$

$\pi^6_{15}=\mathbb{Z}_2\{\mu_6\}\oplus\mathbb{Z}_2\{\eta_6\bar{\nu}_7\}\oplus\mathbb{Z}_2\{\eta_6\varepsilon_7\}$ [28, Theorem 7.2, (7.3)] and $R^7_{14}=\mathbb{Z}_8\{[\eta_5\varepsilon_6]_7\}\oplus\mathbb{Z}_2\{[\nu_5]_7\nu_8^2\}\oplus\mathbb{Z}_8\{[\bar{\nu}_6+\varepsilon_6]\}$ [10, Table 2]. By the fact that $\mu_6\in\{\eta_6, 2\iota_7, 2\Sigma\sigma''\}_1+\{\nu_6^3\}$, $\Delta\eta_6=0$, $\Delta(\nu_6^3)=2[\nu_5]\nu_8^2=0$ and by (1.5),

$$\Delta\mu_6\in\Delta\{\eta_6, 2\iota_7, 2\Sigma\sigma''\}_1\subset\{0, 2\iota_6, 2\sigma''\}=R^6_7\circ 4\sigma'.$$

So, by the fact that $R^6_7=\mathbb{Z}\{[\eta_5^2]\}$ (Lemma 3.1), we obtain $R^6_7\circ 4\sigma'=\{2[4\nu_4]_6\sigma'\}$. Since $([4\nu_4]\sigma'-2[2\nu_4\sigma'])\in i_{5*}R^4_{14}=\mathbb{Z}_4\{4[2\nu_4\sigma']\}$ [10, Table 2], we have $[4\nu_4]\sigma'=2b[2\nu_4\sigma']$ for b odd. Hence, by (3.4) and the group R^7_{14}, we obtain

(3.5) $$\Delta\mu_6=8[\eta_5\varepsilon_6].$$

From the fact that $[2\nu_4\sigma']\sigma_{14}-2[\nu_4\sigma'\sigma_{14}]\in i_{5*}R^4_{21}\cong\mathbb{Z}_4\oplus\mathbb{Z}_2$, we have $4[2\nu_4\sigma']\sigma_{14}=8[\nu_4\sigma'\sigma_{14}]$. Hence, by (3.4), (3.5) and the fact that $\Delta\zeta'=\Delta(\mu_6\sigma_{15})$, we obtain $\Delta(\mu_6\sigma_{15})=\Delta\mu_6\circ\sigma_{14}=4[2\nu_4\sigma']_6\sigma_{14}=8[\nu_4\sigma'\sigma_{14}]_6$ and $\Delta\zeta'=8[\nu_4\sigma'\sigma_{14}]_6$. By Lemma 3.1, we get a relation $2[\eta_6]\kappa_7=[\eta_5^2]_7\kappa_7$ and conclude that

$$R^7_{21}=\mathbb{Z}_8\{[\nu_4\sigma'\sigma_{14}]_7\}\oplus\mathbb{Z}_4\{[\eta_6]\kappa_7\}.$$

We recall [10, Table 3]

(3.6) $$\Delta\nu_6=2[\nu_5].$$

Next we show the following.

Lemma 3.3 $\Delta\bar{\mu}_6=8[\nu_4\rho'']_6$ and $i_{4,7*}R^4_{22}=0$.

Proof. $\bar{\mu}_6$ and $\bar{\mu}'$are taken as follows [28, pp. 136–7]:

$$\bar{\mu}_6 \in \{\mu_6, 4\iota_{15}, 4\sigma_{15}\}_1 \text{ and } \bar{\mu}' \in \{\mu', 4\iota_{14}, 4\sigma_{14}\}_1.$$

So, by the fact that $\Delta\mu_6 = 4[2\nu_4\sigma']_6 = [\iota_3]_6\mu'$ ((3.5), (3.4), Lemma 3.2.(i)) and by (1.5),

$$\begin{aligned}\Delta\bar{\mu}_6 &\in \{\Delta\mu_6, 4\iota_{14}, 4\sigma_{14}\}\\ &= \{[\iota_3]_6\mu', 4\iota_{14}, 4\sigma_{14}\}\\ &\supset [\iota_3]_6\circ\{\mu', 4\iota_{14}, 4\sigma_{14}\}\\ &\ni [\iota_3]_6\bar{\mu}' \bmod [\iota_3]_6\mu'\circ\pi_{22}^{14} + R_{15}^6\circ 4\sigma_{15}.\end{aligned}$$

Since $[\iota_3]_6\bar{\mu}' = 4[2\nu_4\sigma']_6$ and $\pi_{22}^{14} \cong (\mathbb{Z}_2)^2$, we have $[\iota_3]_6\mu'\circ\pi_{22}^{14} = 0$. By [10], $R_{15}^6 = \mathbb{Z}_8\{[\nu_5]\sigma_8\} \oplus \mathbb{Z}_2\{[\nu_4\sigma'\eta_{14}]_6\}$. So we obtain $R_{15}^6\circ 4\sigma_{15} = \{4[\nu_5]\sigma_8^2\}$. By the fact that $\Delta\nu_6 = 2[\nu_5]$ (3.6) and $2\nu_6\sigma_9^2 = 0$ [28, Theorem 12.7], we have $4[\nu_5]\sigma_8^2 = \Delta(2\nu_6\sigma_9^2) = 0$. This leads to the first half.

We recall

$$R_{22}^4 = \mathbb{Z}_4\{[\eta_2]_4\bar{\mu}'\} \oplus \mathbb{Z}_2\{[\eta_2]_4\nu'\mu_6\sigma_{15}\} \oplus \mathbb{Z}_4\{[\iota_3]\bar{\mu}'\} \oplus \mathbb{Z}_2\{[\iota_3]\nu'\mu_6\sigma_{15}\}.$$

By (3.2), $\Delta(\eta_5\bar{\mu}_6) = 2[\iota_3]_5\bar{\mu}'$. Then, by the fact that $[\eta_2]_5 = 2[\iota_3]_5$, we have $[\eta_2]_6\bar{\mu}' = 2[\iota_3]_6\bar{\mu}' = 0$. By Lemma 3.2.(iv) and the first half, $\pm[\iota_3]_6\bar{\mu}' = 8[\nu_4\rho'']_6 = \Delta\bar{\mu}_6$. So we obtain $[\iota_3]_7\bar{\mu}' = 0$. Since $[\eta_2]_5\nu', [\iota_3]_5\nu' \in R_6^5 = 0$, we have $[\eta_2]_5\nu'\mu_6\sigma_{15} = [\iota_3]_5\nu'\mu_6\sigma_{15} = 0 \in R_{22}^5$. This leads to the second half and completes the proof. □

By [24, Proposition 2.6.(4)] and [18, (15.3)′], $\kappa_7\eta_{21} = \bar{\varepsilon}_7 + \sigma'\bar{\nu}_{14}$. So we have $H(\kappa_7)\circ\eta_{21} = H(\sigma')\circ\bar{\nu}_{14} = \eta_{13}\bar{\nu}_{14} = \bar{\nu}_{13}\eta_{21}$, and hence

(3.7) $$H(\kappa_7) = \bar{\nu}_{13} \text{ and } \kappa_7\eta_{21} = \bar{\varepsilon}_7 + \sigma'\bar{\nu}_{14}.$$

In the exact sequence $(22)_6$:

$$\pi_{23}^6 \xrightarrow{\Delta} R_{22}^6 \xrightarrow{i_*} R_{22}^7 \xrightarrow{p_*} \pi_{22}^6 \xrightarrow{\Delta} R_{21}^6,$$

we know $\mathrm{Ker}\{\Delta : \pi_{22}^6 \to R_{21}^6\} = \{\zeta' + \mu_6\sigma_{15}, \eta_6\bar{\varepsilon}_7\} \cong \mathbb{Z}_8 \oplus \mathbb{Z}_2$ and $\pi_{23}^6 = \{P(\Sigma\theta), \nu_6\kappa_9, \bar{\mu}_6, \eta_6\mu_7\sigma_{16}\} \cong (\mathbb{Z}_2)^4$. We have $\Delta(\nu_6\kappa_9) = 2[\nu_5]\kappa_8$, $\Delta(\eta_6\mu_7\sigma_{16}) = 0$ and $\Delta(\bar{\mu}_6) = 8[\nu_4\rho'']_6$ (Lemma 3.3).

By Lemma 3.3, Lemma 3.5 and the group structure of $\pi_{22}(Spin(7):2) \cong (\mathbb{Z}_8)^2 \oplus (\mathbb{Z}_2)^4$ [17], we obtain

$$R_{22}^7 = \mathbb{Z}_8\{[\zeta' + \mu_6\sigma_{15}]\} \oplus \mathbb{Z}_2\{[\eta_6]\bar{\varepsilon}_7\} \oplus \mathbb{Z}_8\{[\nu_4\rho'']_7\} \oplus \mathbb{Z}_2\{[\nu_5]_7\kappa_8\} \oplus \mathbb{Z}_2\{[\nu_4\sigma'\bar{\nu}_{14}]_7\} \oplus \mathbb{Z}_2\{[\nu_4\sigma'\varepsilon_{14}]_7\}.$$

We need

Lemma 3.4 $[\nu_4\sigma'\sigma_{14}]\eta_{21} \equiv [\nu_4\sigma'\eta_{14}]\sigma_{15} \equiv [\nu_4\sigma'\bar{\nu}_{14}] + [\nu_4\sigma'\varepsilon_{14}] \bmod 16[\nu_4\rho'']$.

Proof. Since $p_{5*}([\nu_4\sigma'\sigma_{14}]\eta_{21} + [\nu_4\sigma'\bar{\nu}_{14}] + [\nu_4\sigma'\varepsilon_{14}]) = 0$, we obtain $[\nu_4\sigma'\sigma_{14}]\eta_{21} + [\nu_4\sigma'\bar{\nu}_{14}] + [\nu_4\sigma'\varepsilon_{14}] \in i_{5*}R_{22}^4 = \mathbb{Z}_4\{8[\nu_4\rho'']\}$ and

$$[\nu_4\sigma'\sigma_{14}]\eta_{21} \equiv [\nu_4\sigma'\bar{\nu}_{14}] + [\nu_4\sigma'\varepsilon_{14}] \bmod 16[\nu_4\rho''].$$

By the parallel argument,

$$[\nu_4\sigma'\eta_{14}]\sigma_{15}\equiv[\nu_4\sigma'\sigma_{14}]\eta_{21} \bmod 16[\nu_4\rho''].$$

This completes the proof. □

Although the following is obtained by a similar calculation to the last case in the proof of [17, Proposition 6.3], we give a proof.

Lemma 3.5 $\Delta(P(\Sigma\theta))=2[\nu_5]\kappa_8$.

Proof. By the fact that $\pi^6_{22}=\{\zeta', \mu_6\sigma_{15}, \eta_6\bar{\varepsilon}_7\}$ and (3.5), we have $\Delta\pi^6_{22}=\{8[\eta_5\varepsilon_6]\sigma_{15}\}$. So, by the proof of [28, Lemma 12.11], we can take $\Delta(P(\Sigma\theta))\in\{\Delta(P(\sigma_{13})), \nu_{17}, \eta_{20}\}$. By the relation $\Delta[\iota_6, \iota_6]=[\nu_5]\eta^2_8+4[\nu^2_4]_6$ [11, Lemma 1.1.(iii)], we obtain

$$\begin{aligned}\Delta(P(\Sigma\theta))&\in\{[\nu_5]\eta^2_8\sigma_{10}, \nu_{17}, \eta_{20}\}+\{4[\nu^2_4]_6\sigma_{10}, \nu_{17}, \eta_{20}\}\\ &\supset[\nu_5]\circ\{\eta^2_8\sigma_{10}, \nu_{17}, \eta_{20}\}+[\nu^2_4]_6\circ\{4\sigma_{10}, \nu_{17}, \eta_{20}\}.\end{aligned}$$

We have $\{\eta^2_8\sigma_{10}, \nu_{17}, \eta_{20}\}=\{(\bar{\nu}_8+\varepsilon_8)\eta_{16}, \nu_{17}, \eta_{20}\}\ni(\bar{\nu}_8+\varepsilon_8)\nu^2_{16}=2\kappa_8 \bmod \pi^8_{21}\circ\eta_{21}=0$ and $\{4\sigma_{10}, \nu_{17}, \eta_{20}\}\subset\pi^{10}_{22}=\{[\iota_{10}, \nu_{10}]\}$. This implies

$$[\nu_5]\circ\{\eta^2_8\sigma_{10}, \nu_{17}, \eta_{20}\}+[\nu^2_4]_6\circ\{4\sigma_{10}, \nu_{17}, \eta_{20}\}=\{2[\nu_5]\kappa_8\}.$$

We have to examine the indeterminacy $R^6_{21}\circ\eta_{21}=\{[\nu_4\sigma'\sigma_{14}]_6\eta_{21}, [\eta^2_5]\kappa_7\eta_{21}\}$. By (3.7), $[\eta^2_5]\kappa_7\eta_{21}=[\eta^2_5](\eta_7\kappa_{21}+\sigma'\bar{\nu}_{14})$. Since $[\eta^2_5]\eta_7=4[\nu_5]$, we obtain $[\eta^2_5]\eta_7\kappa_{21}=4[\nu_5]\kappa_8=0$. On the other hand, by the fact that $2[2\nu_4\sigma']=\pm[4\nu_4]\sigma'$ and $2[\eta^2_5]\sigma'=\pm2[4\nu_4]\sigma'$, we have $[\eta^2_5]\sigma'=\pm[2\nu_4\sigma']_6$. Since $[2\nu_4\sigma']\eta_{14}\in i_{5*}R^4_{15}=0$ (Lemma 3.2.(ii)), we obtain $[2\nu_4\sigma']\bar{\nu}_{14}=[2\nu_4\sigma']\varepsilon_{14}$ and

$$[2\nu_4\sigma']\varepsilon_{14}\in\{[2\nu_4\sigma'], \eta_{14}, 2\iota_{15}\}\circ\nu^2_{16} \bmod 0.$$

Since $\{[2\nu_4\sigma'], \eta_{14}, 2\iota_{15}\}\subset R^5_{16}=\{[\nu_4\sigma'\eta_{14}]\eta_{15}, [\nu^2_4]\nu^2_{10}\}$ (Lemma 3.2.(ii)), we get that $[2\nu_4\sigma']\bar{\nu}_{14}=0$. Hence, $[\eta^2{}_5]\sigma'\bar{\nu}_{14}=0$, and thus, we see that $[\eta^2_5]\kappa_7\eta_{21}=0$.

Finally, by Lemma 3.4 and the group structure of R^7_{22}, we have $[\nu_4\sigma'\sigma_{14}]_7\eta_{21}=[\nu_4\sigma'\bar{\nu}_{14}]_7+[\nu_4\sigma'\varepsilon_{14}]_7\neq0$. This completes the proof. □

Obviously we see that

$$R^8_{21}=\mathbb{Z}_8\{[\nu_4\sigma'\sigma_{14}]_8\}\oplus\mathbb{Z}_4\{[\eta_6]_8\kappa_7\}\oplus\mathbb{Z}_8\{[\iota_7]\sigma'\sigma_{14}\}\oplus\mathbb{Z}_4\{[\iota_7]\kappa_7\}$$

and

$$\begin{aligned}R^8_{22}=&\mathbb{Z}_8\{[\zeta'+\mu_6\sigma_{15}]_8\}\oplus\mathbb{Z}_2\{[\eta_6]_8\bar{\varepsilon}_7\}\oplus\mathbb{Z}_8\{[\nu_4\rho'']_8\}\oplus\mathbb{Z}_2\{[\nu_5]_8\kappa_8\}\oplus\mathbb{Z}_2\{[\nu_4\sigma'\bar{\nu}_{14}]_8\}\\ &\oplus\mathbb{Z}_2\{[\nu_4\sigma'\varepsilon_{14}]_8\}\oplus\mathbb{Z}_8\{[\iota_7]\rho''\}\oplus\mathbb{Z}_2\{[\iota_7]\bar{\varepsilon}_7\}\oplus\mathbb{Z}_2\{[\iota_7]\sigma'\bar{\nu}_{14}\}\oplus\mathbb{Z}_2\{[\iota_7]\sigma'\varepsilon_{14}\}.\end{aligned}$$

By [11, Lemma 1.1.(v)],

(3.8) $$[\eta_6]\sigma'=4[\bar{\nu}_6+\varepsilon_6]+[\nu_5]_7\nu^2_8+[\eta_5\varepsilon_6]_7.$$

By [28, Lemma 10.7],

(3.9) $$\varepsilon_3\sigma_{11}=0 \text{ and } \bar{\nu}_6\sigma_{14}=0.$$

Finally we show the following.

Lemma 3.6 (i) $[\eta_5\varepsilon_6]\sigma_{14}=a[\nu_4\sigma'\sigma_{14}]_6$, $[\bar{\nu}_6+\varepsilon_6]\sigma_{14}=b[\nu_4\sigma'\sigma_{14}]_7+2c[\eta_6]\kappa_7$ *for* a *odd,* $b\in\mathbb{Z}$, $c\in\{0, 1\}$ *and* $[\eta_6]\sigma'\sigma_{14}=(a+4b)[\nu_4\sigma'\sigma_{14}]_7$.

(ii) $[\nu_4\sigma'\sigma_{14}]_6\eta_{21}=[\nu_4\sigma'\eta_{14}]_6\sigma_{15}=[\nu_4\sigma'\bar{\nu}_{14}]_6+[\nu_4\sigma'\varepsilon_{14}]_6$.

(iii) $[\eta_5\varepsilon_6]_7\varepsilon_{14}=[\nu_4\sigma'\varepsilon_{14}]_7+d([\nu_4\sigma'\bar{\nu}_{14}]_7+[\nu_4\sigma'\varepsilon_{14}]_7)$;

$[\eta_5\varepsilon_6]_7\bar{\nu}_{14}=[\nu_4\sigma'\bar{\nu}_{14}]_7+d([\nu_4\sigma'\bar{\nu}_{14}]_7+[\nu_4\sigma'\varepsilon_{14}]_7)$ *for* $d\in\{0, 1\}$.

(iv) $[\bar{\nu}_6+\varepsilon_6]\bar{\nu}_{14}\equiv[\bar{\nu}_6+\varepsilon_6]\varepsilon_{14}\equiv[\nu_5]_7\kappa_8+[\eta_6]\bar{\varepsilon}_7 \bmod 4[\nu_4\rho'']_7, [\nu_4\sigma'\bar{\nu}_{14}]_7, [\nu_4\sigma'\varepsilon_{14}]_7$.

Proof. By (3.9), $[\eta_5\varepsilon_6]\sigma_{14}\in i_{6*}R^5_{21}=\mathbb{Z}_{16}\{[\nu_4\sigma'\sigma_{14}]_6\}$. By (3.4) and Lemma 3.2.(iii),

$$8[\eta_5\varepsilon_6]\sigma_{14}=4[2\nu_4\sigma']_6\sigma_{14}=8[\nu_4\sigma'\sigma_{14}]_6.$$

This implies the first relation of (i). By (3.9), we have $[\bar{\nu}_6+\varepsilon_6]\sigma_{14}\in i_{7*}R^6_{21}=\mathbb{Z}_8\{[\nu_4\sigma'\sigma_{14}]_7\}\oplus\mathbb{Z}_2\{2[\eta_6]\kappa_7\}$. This implies the second of (i). So, by (3.8) and the fact that $\nu_{11}\sigma_{14}=0$ [28, (7.20)], $[\eta_6]\sigma'\sigma_{14}=(4[\bar{\nu}_6+\varepsilon_6]+[\nu_5]_7\nu_8^2+[\eta_5\varepsilon_6]_7)\sigma_{14}=4[\bar{\nu}_6+\varepsilon_6]\sigma_{14}+[\eta_5\varepsilon_6]_7\sigma_{14}=(a+4d)[\nu_4\sigma'\sigma_{14}]_7$. This leads to the third relation of (i).

(ii) is a direct consequence of Lemma 3.4 and the group structure R^7_{22}.

Since $\varepsilon_{14}\in\{\eta_{14}, 2\iota_{15}, \nu_{15}^2\}$ and $R^5_{16}\circ\nu_{16}^2=\{[\nu_4\sigma'\eta_{14}]\eta_{15}, [\nu_4^2]\nu_{10}^2\}\circ\nu_{16}^2=0$, we see that

$$[\nu_4\sigma'\eta_{14}]\in\{[\nu_4\sigma'\eta_{14}], 2\iota_{15}, \nu_{15}^2\} \bmod \{[\nu_4\sigma'\eta_{14}]\sigma_{15}\}+i_{5*}R^4_{22}.$$

We obtain

$$2\{[\nu_4\sigma'\eta_{14}], 2\iota_{15}, \nu_{15}^2\}=[\nu_4\sigma'\eta_{14}]\circ\{2\iota_{15}, \nu_{15}^2, 2\iota_{21}\}=\{2[\nu_4\sigma'\eta_{14}]\sigma_{15}\}=0.$$

Hence we have

$$[\nu_4\sigma'\varepsilon_{14}]\in\{[\nu_4\sigma'\eta_{14}], 2\iota_{15}, \nu_{15}^2\} \bmod [\nu_4\sigma'\eta_{14}]\sigma_{15}, 16[\nu_4\rho'']$$

and, by (ii),

$$[\nu_4\sigma'\varepsilon_{14}]_7\in\{[\nu_4\sigma'\eta_{14}]_7, 2\iota_{15}, \nu_{15}^2\} \bmod \{[\nu_4\sigma'\bar{\nu}_{14}]_7+[\nu_4\sigma'\varepsilon_{14}]_7\}+R^7_{16}\circ\nu_{16}^2.$$

By [12], $R^7_{16}\circ\nu_{16}\subset R^7_{19}=\mathbb{Z}_2\{[\nu_5]_7\bar{\nu}_8\nu_{16}\}$. So, by the fact that $2\kappa_8=\bar{\nu}_8\nu_{16}^2$ and $2[\nu_5]_7\kappa_8=0$, we have $R^7_{16}\circ\nu_{16}^2=0$. By [11, Lemma 1.2.(iii)], $[\eta_5\varepsilon_6]_7\eta_{14}=[\nu_4\sigma'\eta_{14}]_7$. Thus we obtain

$$[\nu_4\sigma'\varepsilon_{14}]_7\in\{[\nu_4\sigma'\eta_{14}]_7, 2\iota_{15}, \nu_{15}^2\}\supset[\eta_5\varepsilon_6]_7\circ\{\eta_{14}, 2\iota_{15}, \nu_{15}^2\}\ni[\eta_5\varepsilon_6]_7\varepsilon_{14} \bmod [\nu_4\sigma'\bar{\nu}_{14}]_7+[\nu_4\sigma'\varepsilon_{14}]_7.$$

This leads to the first half of (iii). We have

$$[\eta_5\varepsilon_6](\bar{\nu}_{14}+\varepsilon_{14})=[\eta_5\varepsilon_6]\sigma_{14}\eta_{21}=[\nu_4\sigma'\sigma_{14}]_6\eta_{21}=[\nu_4\sigma'\eta_{14}]_6\sigma_{15}=[\nu_4\sigma'\bar{\nu}_{14}]_6+[\nu_4\sigma'\varepsilon_{14}]_6.$$

This and the first half of (iii) lead to the second half of (iii).

By the fact that $(\bar{\nu}_6+\varepsilon_6)\bar{\nu}_{14}=(\bar{\nu}_6+\varepsilon_6)\varepsilon_{14}=\eta_6\bar{\varepsilon}_7$ (2.4), we obtain $[\bar{\nu}_6+\varepsilon_6]\bar{\nu}_{14}-[\eta_6]\bar{\varepsilon}_7\in i_{7*}R^6_{22}=\mathbb{Z}_8\{[\nu_4\rho'']_7\}\oplus\mathbb{Z}_2\{[\nu_5]_7\kappa_8\}\oplus\mathbb{Z}_2\{[\nu_4\sigma'\bar{\nu}_{14}]_7\}\oplus\mathbb{Z}_2\{[\nu_4\sigma'\varepsilon_{14}]_7\}$. Since $\pi^5_{27}=\{\varepsilon_5\kappa_{13}, \nu_5\zeta_8,$

$\nu_5\bar{\sigma}_8\} \cong (\mathbb{Z}_2)^3$, we have $J[\nu_4\rho''] \in \{\nu_5\zeta_8\}$. By Lemma 3.6.(iii) and [11, Lemma 1.1.(ii)], $J[\nu_4\sigma'\bar{\nu}_{14}]_7 = J[\nu_4\sigma'\varepsilon_{14}]_7 = 0$. By [16, Lemma 8.1],

$$J([\eta_6]\bar{\varepsilon}_7) = \sigma'\eta_{14}\kappa_{15} = \eta_7\sigma_8\kappa_{15} + (\bar{\nu}_7 + \varepsilon_7)\kappa_{15} = (\bar{\nu}_7 + \varepsilon_7)\kappa_{15} = J([\nu_5]_7\kappa_8)$$

Hence, by the relation $J([\bar{\nu}_6 + \varepsilon_6]\bar{\nu}_{14}) = \sigma'\sigma_{14}\bar{\nu}_{21} = 0$ (Lemma 2.1.(iii)), we obtain

$$[\bar{\nu}_6 + \varepsilon_6]\bar{\nu}_{14} \equiv [\nu_5]_7\kappa_8 + [\eta_6]\bar{\varepsilon}_7 \bmod 4[\nu_4\rho'']_7,\ [\nu_4\sigma'\bar{\nu}_{14}]_7,\ [\nu_4\sigma'\varepsilon_{14}]_7.$$

The same argument is used for $[\bar{\nu}_6 + \varepsilon_6]\varepsilon_{14}$. This leads to (iv) and completes the proof. □

4 Determination of $\pi_{21}(R_n : 2)$ $(n \geq 9)$

We know that $\Delta\sigma_9 = [\nu_5]_9\sigma_8 + [\iota_7]_9(\bar{\nu}_7 + \varepsilon_7 + \sigma'\eta_{14})$ [11, Lemma 1.2.(v)]. By [11, Lemmas 1.3 and 2.3], $[\nu_5]_9\sigma_8 = [\iota_7]_9\sigma'\eta_{14}$. So we obtain

(4.1) $$\Delta\sigma_9 = [\iota_7]_9(\bar{\nu}_7 + \varepsilon_7)$$

By [11, Lemma 1.2.(ii)] and its proof,

(4.2) $$\Delta\sigma_8 = [\iota_7]\sigma' + x[\bar{\nu}_6 + \varepsilon_6]_8 + y([\eta_5\varepsilon_6]_8 + [\nu_5]_8\nu_8^2) \ (x : \text{odd};\ y \in \mathbb{Z}).$$

In the exact sequence $(21)_8$:

$$\pi_{22}^8 \xrightarrow{\Delta} R_{21}^8 \xrightarrow{i_*} R_{21}^9 \xrightarrow{p_*} \pi_{21}^8 \xrightarrow{\Delta} R_{20}^8,$$

$\Delta : \pi_{21}^8 \to R_{20}^8$ is a monomorphism and $\pi_{22}^8 = \mathbb{Z}_{16}\{\sigma_8^2\} \oplus \mathbb{Z}_8\{(\Sigma\sigma')\sigma_{15}\} \oplus \mathbb{Z}_4\{\kappa_8\}$. By (1.9) and Lemma 3.6.(i),

$$\Delta\kappa_8 = 2[\iota_7]\kappa_7 - [\eta_6]_8\kappa_7$$

and

$$\Delta((\Sigma\sigma')\sigma_{15}) = 2[\iota_7]\sigma'\sigma_{14} - [\eta_6]_8\sigma'\sigma_{14} = 2[\iota_7]\sigma'\sigma_{14} - (a + 4b)[\nu_4\sigma'\sigma_{14}]_8 \ (a : \text{odd}).$$

By (4.2) and Lemma 3.6.(i),

$$\Delta(\sigma_8^2) = [\iota_7]\sigma'\sigma_{14} + (xb + ya)[\nu_4\sigma'\sigma_{14}]_8 + 2xc[\eta_6]_8\kappa_7.$$

Hence we obtain

$$R_{21}^9 = \mathbb{Z}_4\{[\iota_7]_9\kappa_7\}.$$

By the fact that $2[\eta_6]_9\kappa_7 = 4[\iota_7]_9\kappa_7 = 0$ and Lemma 3.6.(i), $[\iota_7]_9\sigma'\sigma_{14} \equiv 0 \bmod 2[\iota_7]_9\sigma'\sigma_{14}$ and hence $[\iota_7]_9\sigma'\sigma_{14} = 0$. Thus, by Lemma 3.6.(i),

(4.3) $$[\iota_7]_9\sigma'\sigma_{14} = [\eta_5\varepsilon_6]_9\sigma_{14} = [\bar{\nu}_6 + \varepsilon_6]_9\sigma_{14} = [\nu_4\sigma'\sigma_{14}]_9 = [\eta_6]_9\sigma'\sigma_{14} = 0.$$

In the exact sequence $(21)_9$:

$$\pi^9_{22} \xrightarrow{\Delta} R^9_{21} \xrightarrow{i_*} R^{10}_{21} \xrightarrow{p_*} \pi^9_{21},$$

$\pi^9_{21}=0$ and $\pi^9_{22}=\mathbb{Z}_2\{\sigma_9\nu^2_{16}\}$. By (4.1), $\Delta(\sigma_9\nu^2_{16})=[\iota_7]_9(\bar{\nu}_7+\varepsilon_7)\nu^2_{15}=[\iota_7]_9\bar{\nu}_7\nu^2_{15}=2[\iota_7]_9\kappa_7$ and

$$R^{10}_{21}=\mathbb{Z}_2\{[\iota_7]_{10}\kappa_7\}.$$

In the exact sequence $(21)_{10}$:

$$\pi^{10}_{22} \xrightarrow{\Delta} R^{10}_{21} \xrightarrow{i_*} R^{11}_{21} \xrightarrow{p_*} \pi^{10}_{21} \xrightarrow{\Delta} R^{10}_{20},$$

$\pi^{10}_{21}=\mathbb{Z}_8\{\zeta_{10}\}$ and $\pi^{10}_{22}=\mathbb{Z}_4\{[\iota_{10},\nu_{10}]\}$. We know [12]

$$\mathrm{Ker}\{\Delta: \pi^{10}_{21} \to R^{10}_{20}\}=\mathbb{Z}_2\{\eta^2_{10}\mu_{12}\}.$$

Since $J\Delta[\iota_{10},\nu_{10}]=[\iota_{10},[\iota_{10},\nu_{10}]]=0$ and $J([\iota_7]_{10}\kappa_7)=\sigma_{10}\kappa_7\neq 0$ [16], we obtain

$$\Delta[\iota_{10},\nu_{10}]=0 \tag{4.4}$$

and

$$R^{11}_{21}=\mathbb{Z}_2\{[\iota_7]_{11}\kappa_7\}\oplus\mathbb{Z}_2\{[\eta^2_{10}]\mu_{12}\}.$$

In the exact sequence $(21)_{11}$:

$$\pi^{11}_{22} \xrightarrow{\Delta} R^{11}_{21} \xrightarrow{i_*} R^{12}_{21} \xrightarrow{p_*} \pi^{11}_{21} \xrightarrow{\Delta} R^{11}_{20},$$

$\pi^{11}_{21}=\mathbb{Z}_2\{\sigma_{11}\nu_{18}\}\oplus\mathbb{Z}_2\{\eta_{11}\mu_{12}\}$ and $\pi^{11}_{22}=\mathbb{Z}_8\{\zeta_{11}\}$. We know $\mathrm{Ker}\{\Delta: \pi^{11}_{21}\to R^{11}_{20}\}=\mathbb{Z}_2\{\eta_{11}\mu_{12}\}$. By the fact that $\Delta\iota_{11}=[\iota_7]_{11}\nu_7$ [10] and $\nu_6\zeta_9=2\sigma''\sigma_{13}$ [28, (10.7)], we have $\Delta\zeta_{11}=[\iota_7]_{11}\nu_7\zeta_{10}=2[\iota_7]_{11}\Sigma(\sigma''\sigma_{13})\in 2\pi_{21}(R_{11})=0$. This implies

$$R^{12}_{21}=\mathbb{Z}_2\{[\iota_7]_{12}\kappa_7\}\oplus\mathbb{Z}_2\{[\eta^2_{10}]_{12}\mu_{12}\}\oplus\mathbb{Z}_2\{[\eta_{11}]\mu_{12}\}.$$

In the exact sequence $(21)_{12}$:

$$\pi^{12}_{22} \xrightarrow{\Delta} R^{12}_{21} \xrightarrow{i_*} R^{13}_{21} \xrightarrow{p_*} \pi^{12}_{21} \xrightarrow{\Delta} R^{12}_{20},$$

$\pi^{12}_{21}=\mathbb{Z}_2\{\mu_{12}\}\oplus\mathbb{Z}_2\{\eta_{12}\varepsilon_{13}\}\oplus\mathbb{Z}_2\{\nu^3_{12}\}$ and $\pi^{12}_{22}=\mathbb{Z}_2\{\eta_{12}\mu_{13}\}$. By (1.8), $\Delta(\eta_{12}\mu_{13})=[\eta^2_{10}]_{12}\mu_{12}$. We know $\Delta\mu_{12}\equiv[\eta_{10}\mu_{11}]_{12} \bmod [\zeta_9]_{12}$ [12, Lemma 3.9.(2)]. By [12, Lemma 3.9.(1)] and (1.8),

$$\Delta(\eta_{12}\varepsilon_{13})=[\eta^2_{10}]_{12}\varepsilon_{12}=[\zeta_9]_{12}. \tag{4.5}$$

We consider an element $[\nu^2_{12}]\nu_{18}\in R^{13}_{21}$. By the fact that $J[\nu^2_{12}]=\lambda+2\zeta_{13}$ (Lemma 2.2.(ii)) and $\zeta_{13}\nu_{31}=\sigma^3_{13}$ [22, Proposition II.2.1.(2)], we obtain $J([\nu^2_{12}]\nu_{18})=\lambda\nu_{31}$. Hence, by the fact that $J([\eta_{11}]_{13}\mu_{12})=(\Sigma\theta)\mu_{25}\neq 0$ [16, Theorem A] and $2\lambda\nu_{31}=\sigma_{13}\kappa_{20}$ [16, Proposition 7.2], we obtain

$$R^{13}_{21}=\mathbb{Z}_4\{[\nu^2_{12}]\nu_{18}\}\oplus\mathbb{Z}_2\{[\eta_{11}]_{13}\mu_{12}\}, \text{ where } 2[\nu^2_{12}]\nu_{18}=[\iota_7]_{13}\kappa_7.$$

Now consider the exact sequence $(21)_{13}$:

$$\pi^{13}_{22} \xrightarrow{\Delta} R^{13}_{21} \xrightarrow{i_*} R^{14}_{21} \xrightarrow{p_*} \pi^{13}_{21} \xrightarrow{\Delta} R^{13}_{20}.$$

$\Delta: \pi^{13}_{21} \to R^{13}_{20}$ is a monomorphism [12]. Since $\Delta\iota_{13}=[\eta_{11}]_{13}$ (1.6), $\Delta\mu_{13}=[\eta_{11}]_{13}\mu_{12}$ and $\Delta(\eta^2_{13}\sigma_{15})=[\eta_{11}]_{13}\eta^2_{12}\sigma_{14}$. By the fact that $[\iota_{13},\eta^2_{13}]=(\Sigma\theta)\eta^2_{25}=8\sigma^2_{13}$ [28, (7.30), (10.10)], $R^{13}_{14}=\mathbb{Z}_8\{[\bar{\nu}_6+\varepsilon_6]_{13}\}$[10, Table 2] and by Lemma 2.1.(iii); (iv), we have $[\eta_{11}]_{13}\eta^2_{12}=4[\bar{\nu}_6+\varepsilon_6]_{13}$. Hence $[\eta_{11}]_{13}\eta^2_{12}\sigma_{14}=4[\bar{\nu}_6+\varepsilon_6]_{13}\sigma_{14}=0$ by (4.3). We know $\Delta(\nu^2_{13})=0$ in determining R^{14}_{19} [12]. So, by (1.2), $\Delta(\nu^3_{13})=0$. This implies $R^{14}_{21}=\mathbb{Z}_4\{[\nu^2_{12}]_{14}\nu_{18}\}$.

Next, in the exact sequence

$$\pi^{14}_{22} \xrightarrow{\Delta} R^{14}_{21} \xrightarrow{i_*} R^{15}_{21} \xrightarrow{p_*} \pi^{14}_{21} \xrightarrow{\Delta} R^{14}_{20},$$

$\Delta: \pi^{14}_{21} \to R^{14}_{20}$ is an isomorphism [12]. Since $P(\bar{\nu}_{29})=P(\varepsilon_{29})=2(\Sigma\lambda)\nu_{32}$ [16], we see that $\Delta\bar{\nu}_{14}=\Delta\varepsilon_{14}=2[\nu^2_{12}]_{14}\nu_{18}$ and $R^{15}_{21}=\mathbb{Z}_2\{[\nu^2_{12}]_{15}\nu_{18}\}$.

In the exact sequence

$$\pi^{15}_{22} \xrightarrow{\Delta} R^{15}_{21} \xrightarrow{i_*} R^{16}_{21} \xrightarrow{p_*} \pi^{15}_{21} \xrightarrow{\Delta} R^{15}_{20},$$

as $\Delta\iota_{15}=[\bar{\nu}_6+\varepsilon_6]_{15}$ and $[\bar{\nu}_6+\varepsilon_6]_{15}\nu_{14}=0$ ([10], [11]), there exists an element $[\nu_{15}]\nu_{18}\in R^{16}_{21}$. By (4.3), $\Delta\sigma_{15}=[\bar{\nu}_6+\varepsilon_6]_{15}\sigma_{14}=0$. By Lemma 2.2.(ii); (iii); (iv), the fact that $\nu^*_{16}\eta_{34}=\zeta_{16}\eta_{34}=(\Sigma^3\lambda')\eta_{34}=0$ [24, Proposition 2.20] and $i_{16*}R^{15}_{19}=\mathbb{Z}\{[32[\iota_{10},\iota_{10}]]_{16}\}\oplus\mathbb{Z}_2\{[\nu^2_{13}]_{16}\}$ [12], we see that $[\nu_{15}]\eta_{18}=0$. So, by the fact that $R^{16}_{20}=0$ [12],

$$2[\nu_{15}]\nu_{18}\in[\nu_{15}]\circ\{\eta_{18},\ 2\iota_{19},\ \eta_{19}\}=-\{[\nu_{15}],\ \eta_{18},\ 2\iota_{19}\}\circ\eta_{20}\subset R^{16}_{20}\circ\eta_{20}=0.$$

This implies that

$$R^{16}_{21}=\mathbb{Z}_2\{[\nu_{15}]\nu_{18}\}\oplus\mathbb{Z}_2\{[\nu^2_{12}]_{16}\nu_{18}\}.$$

By the fact that $\pm P(\nu_{33})=2\nu^*_{16}-\Sigma^3\lambda$ [28, Lemma 12.18], we have a relation $\Delta\nu_{16}=2[\nu_{15}]-[\nu^2_{12}]_{16}$, and so $\Delta(\nu^2_{16})=[\nu^2_{12}]_{16}\nu_{18}$. So, by the exact sequence $(21)_{16}$, we have the group $R^{17}_{21}=\mathbb{Z}_2\{[\nu_{15}]_{17}\nu_{18}\}$. Obviously we get that $R^{18}_{21}=\mathbb{Z}_2\{[\nu_{15}]_{18}\nu_{18}\}$.

The sequence $(21)_{18}$ induces the group $R^{19}_{21}=\mathbb{Z}_2\{[\eta^2_{18}]\eta_{20}\}\oplus\mathbb{Z}_2\{[\nu_{15}]_{19}\nu_{18}\}$.

By Lemma 2.2.(ii) and the fact that $P(\nu_{39})=(\nu^*_{19}+\zeta_{19})\nu_{37}$ [28, Corollary 12.25], $2\pi^{19}_{40}=0$ [16, Theorem A], we obtain $J([\nu_{15}]_{19}\nu_{18})=(\nu^*_{19}+\zeta_{19})\nu_{37}$. Hence, by the relation $\bar{\beta}\eta_{39}\neq 0$ [16, Theorem A], we get $\Delta\nu_{19}=[\nu_{15}]_{19}\nu_{18}$. Hence the sequence $(21)_{20}$ implies

$$R^{20}_{21}=\mathbb{Z}_2\{[\eta_{19}]\eta_{20}]\}\oplus\mathbb{Z}_2\{[\eta^2_{18}]_{20}\eta_{20}\}.$$

By (1.8) and (1.6), $\Delta\eta_{20}=[\eta^2_{18}]_{20}$ and $\Delta\iota_{21}=[\eta_{19}]_{21}$. Thus we obtain the following:

$$R^{21}_{21}=\mathbb{Z}_2\{[\eta_{19}]_{21}\eta_{20}\},\ R^{22}_{21}=\mathbb{Z}\{\Delta\iota_{22}\}\ \text{and}\ R^{23}_{21}=0.$$

This completes the proof of Theorem 0.1.

Finally, we give a remark. By the fact that $\theta'\mu_{23}\neq 0$ [16, Theorem A], $\zeta'\nu_{29}=0$, $\lambda'\nu_{29}=\sigma_{11}\kappa_{18}$ [22, Proposition I.3.1.(1)] and that $J: R^{13}_{18}\to\pi^{13}_{31}$ is a monomorphism, we obtain

$$[\varepsilon_{10}]\nu_{18}=[\iota_7]_{11}\kappa_7$$

and

$$2[\nu_{12}^2]=[\varepsilon_{10}]_{13}.$$

On [11, p.343], the second and third authors claim to have proved the equality $2[\nu_{12}^2]=4([\varepsilon_{10}]_{13}-2[2\sigma_{11}]_{13})$. This is wrong and should be corrected as above.

5 Determination of $\pi_{22}(R_n:2)$ $(n=9,10)$

We have $HJ[\nu_4\sigma'\bar{\nu}_{14}]=HJ[\nu_4\sigma'\varepsilon_{14}]=0$ and

$$HJ[\nu_4\rho'']=2\nu_9\Sigma\rho'\equiv 0 \bmod 4\zeta_9\sigma_{20}=8\sigma_9\zeta_{16}=0.$$

We recall that $\Sigma\pi_{26}^4=\pi_{27}^5=\mathbb{Z}_2\{\nu_5\bar{\zeta}_8\}\oplus\mathbb{Z}_2\{\varepsilon_5\kappa_{13}\}\oplus\mathbb{Z}_2\{\nu_5\bar{\sigma}_8\}$ [16]. Therefore we see that $J[\nu_4\sigma'\bar{\nu}_{14}]$, $J[\nu_4\sigma'\varepsilon_{14}]$, $J[\nu_4\rho'']\in\{\nu_5\bar{\zeta}_8\}$. We determine the J-image of $[\zeta'+\mu_6\sigma_{15}]\in R_{22}^7$. By the fact that $\Sigma\zeta'=\sigma'\eta_{14}\varepsilon_{15}$ [28, (12.4)], $\mu_{13}\sigma_{22}=\eta_{13}\rho_{14}=H(\sigma'\rho_{14})$ [28, Proposition 12.20, Lemma 5.14] and by (2.1), we obtain $J[\zeta'+\mu_6\sigma_{15}]\equiv\sigma'\rho_{14} \bmod \Sigma\pi_{28}^6$. By [16], $\Sigma\pi_{28}^6=\mathbb{Z}_4\{2\sigma'\rho_{14}\}\oplus\mathbb{Z}_2\{\bar{\nu}_7\kappa_{15}\}\oplus\mathbb{Z}_2\{\varepsilon_7\kappa_{15}\}\oplus\mathbb{Z}_2\{\nu_7\bar{\sigma}_{10}\}$. So we get that

$$J[\zeta'+\mu_6\sigma_{15}]\equiv c\sigma'\rho_{14} \bmod (\bar{\nu}_7+\varepsilon_7)\kappa_{15}\ (c: odd). \tag{5.1}$$

Next we show the following.

Lemma 5.1 $[\eta_6]\rho''\equiv 4[\zeta'+\mu_6\sigma_{15}]+d[\nu_4\rho'']_7 \bmod [\nu_4\sigma'\bar{\nu}_{14}]_7, [\nu_4\sigma'\varepsilon_{14}]_7$ *for d odd.*

Proof. By the definition of ρ'' [28, p.103], $4\rho''=4\iota_7\circ\rho''\in\{4\sigma', 8\iota_{14}, 2\sigma_{14}\}$ and $4\nu_4\rho''\in\{4\nu_4\sigma', 8\iota_{14}, 2\sigma_{14}\}$. By the fact that $\eta_6\rho''=4\zeta'$ [24, Proposition 2.8.(3)], $[\eta_6]\rho''-4[\zeta'+\mu_6\sigma_{15}]\in i_{7*}R_{22}^6=\mathbb{Z}_8\{[\nu_4\rho'']_7\}\oplus\mathbb{Z}_2\{[\nu_5]_7\kappa_8\}\oplus\mathbb{Z}_2\{[\nu_4\sigma'\bar{\nu}_{14}]_7\}\oplus\mathbb{Z}_2\{[\nu_4\sigma'\varepsilon_{14}]_7\}$. By (3.8) and (3.4), we have $4[\eta_6]\sigma'=4[\eta_5\varepsilon_6]_7=2[2\nu_4\sigma']_7$. Hence, by the fact that $R_{15}^7\cong(\mathbb{Z}_2)^4$ [10] and $\pi_{22}^{14}\cong(\mathbb{Z}_2)^2$, we have

$$4[\eta_6]\rho''\in\{4[\eta_6]\sigma', 8\iota_{14}, 2\sigma_{14}\}=\{2[2\nu_4\sigma']_7, 8\iota_{14}, 2\sigma_{14}\} \bmod 2[2\nu_4\sigma']_7\circ\pi_{22}^{14}+R_{15}^7\circ 2\sigma_{15}=0.$$

That is,

$$4[\eta_6]\rho''=\{2[2\nu_4\sigma']_7, 8\iota_{14}, 2\sigma_{14}\}.$$

We consider $\{2[2\nu_4\sigma'], 8\iota_{14}, 2\sigma_{14}\}$. Since $R_{15}^5\cong\mathbb{Z}_2$ [10, Proposition 2.1], the indeterminacy is $2[2\nu_4\sigma']\circ\pi_{22}^{14}+R_{15}^7\circ 2\sigma_{15}=0$, and so this bracket consists of a single element. We see that

$$p_{5*}\{2[2\nu_4\sigma'], 8\iota_{14}, 2\sigma_{14}\}\subset\{4\nu_4\sigma', 8\iota_{14}, 2\sigma_{14}\}\ni 4\nu_4\rho'',$$

where the indeterminacy of $\{4\nu_4\sigma', 8\iota_{14}, 2\sigma_{14}\}$ is

$$4\nu_4\sigma'\circ\pi_{22}^{14}+\pi_{15}^4\circ 2\sigma_{15}=\{2(\Sigma\mu')\sigma_{15}\}.$$

As

$$\Delta(2(\Sigma\mu')\sigma_{15})=\Delta\iota_4\circ 2\mu'\sigma_{14}=2[\eta_2]_4\mu'\sigma_{14}\neq 0$$

by Theorem 0.1.(i), we obtain $p_{5*}\{2[2\nu_4\sigma'],\ 8\iota_{14},\ 2\sigma_{14}\}=4\nu_4\rho''$ and

$$\{2[2\nu_4\sigma'],\ 8\iota_{14},\ 2\sigma_{14}\}\equiv 4[\nu_4\rho'']\ \mathrm{mod}\ i_{5*}R^4_{22}.$$

By Lemma 3.3, we obtain $i_{4,7*}\{2[2\nu_4\sigma'],\ 8\iota_{14},\ 2\sigma_{14}\}=4[\nu_4\rho'']_7$ and $4[\nu_4\rho'']_7=4[\eta_6]\rho''=\{2[2\nu_4\sigma']_7,\ 8\iota_{14},\ 2\sigma_{14}\}$. Notice that $J([\eta_6]\rho'')=4\sigma'\rho_{14}$ and $4J[\zeta'+\mu_6\sigma_{15}]=4\sigma'\rho_{14}$ (5.1) and $J([\nu_5]_7\kappa_8)=(\bar{\nu}_7+\varepsilon_7)\kappa_{15}\neq 0$ [16]. This leads to the assertion and completes the proof. □

We need the following.

Lemma 5.2 (i) $\Delta(\sigma_8\bar{\nu}_{15})=[\iota_7]\sigma'\bar{\nu}_{14}+[\bar{\nu}_6+\varepsilon_6]_8\bar{\nu}_{14}\ \mathrm{mod}\ [\eta_5\varepsilon_6]_8\bar{\nu}_{14}$;

$\Delta(\sigma_8\bar{\nu}_{15})\equiv[\iota_7]\sigma'\bar{\nu}_{14}+[\nu_5]_8\kappa_8+[\eta_6]_8\bar{\varepsilon}_7\ \mathrm{mod}\ 4[\nu_4\rho'']_8,\ [\nu_4\sigma'\bar{\nu}_{14}]_8,\ [\nu_4\sigma'\zeta_4]_8$.

(ii) $\Delta(\sigma_8\varepsilon_{15})\equiv[\iota_7]\sigma'\varepsilon_{14}+[\bar{\nu}_6+\varepsilon_6]_8\varepsilon_{14}\ \mathrm{mod}\ [\eta_5\varepsilon_6]_8\varepsilon_{14}$;

$\Delta(\sigma_8\varepsilon_{15})\equiv[\iota_7]\sigma'\varepsilon_{14}+[\nu_5]_8\kappa_8+[\eta_6]_8\bar{\varepsilon}_7\ \mathrm{mod}\ 4[\nu_4\rho'']_8,\ [\nu_4\sigma'\bar{\nu}_{14}]_8,\ [\nu_4\sigma'\varepsilon_{14}]_8$.

Proof. By (4.2) and the relation $\nu_7\bar{\nu}_{10}=0$ [28, p.70],

$$\Delta(\sigma_8\bar{\nu}_{15})=\Delta\sigma_8\circ\bar{\nu}_{14}=[\iota_7]\sigma'\bar{\nu}_{14}+x[\bar{\nu}_6+\varepsilon_6]_8\bar{\nu}_{14}+y[\eta_5\varepsilon_6]_8\bar{\nu}_{14}\ (x:\ odd,\ y\in\mathbb{Z}).$$

This implies the first half of (i). The second half of (i) follows the first half and Lemma 3.6.(iii); (iv). By the parallel argument to (i), we obtain (ii). □

Now we consider the exact sequence $(22)_8$:

$$\pi^8_{23}\xrightarrow{\Delta}R^8_{22}\xrightarrow{i_*}R^9_{22}\xrightarrow{p_*}\pi^8_{22}\xrightarrow{\Delta}R^8_{21}.$$

In determining R^9_{21}, we know $\mathrm{Ker}\{\Delta:\pi^8_{22}\to R^8_{21}\}=\mathbb{Z}_2\{8\sigma_8^2\}$. We know $\pi^8_{23}=\mathbb{Z}_2\{\sigma_8\bar{\nu}_{15}\}\oplus\mathbb{Z}_2\{\sigma_8\varepsilon_{15}\}\oplus\mathbb{Z}_8\{\Sigma\rho''\}\oplus\mathbb{Z}_2\{(\Sigma\sigma')\bar{\nu}_{15}\}\oplus\mathbb{Z}_2\{(\Sigma\sigma')\varepsilon_{15}\}\oplus\mathbb{Z}_2\{\bar{\varepsilon}_8\}$. By (1.9) and Lemma 5.1, $\Delta\bar{\varepsilon}_8=[\eta_6]_8\bar{\varepsilon}_7$ and

$$\Delta(\Sigma\rho'')\equiv 2[\iota_7]\rho''-4[\zeta'+\mu_6\sigma_{15}]_8-d[\nu_4\rho'']_8\ \mathrm{mod}\ [\nu_4\sigma'\bar{\nu}_{14}]_8,\ [\nu_4\sigma'\varepsilon_{14}]_8\ (d:\ odd).$$

By the fact that $[\eta_6]\sigma'\bar{\nu}_{14}=[\eta_5\varepsilon_6]_7\bar{\nu}_{14}$, $[\eta_6]\sigma'\varepsilon_{14}=[\eta_5\varepsilon_6]_7\varepsilon_{14}$ and Lemma 3.6.(iii), we have the following for $b\in\{0,1\}$:

$$\Delta((\Sigma\sigma')\bar{\nu}_{15})=[\nu_4\sigma'\bar{\nu}_{14}]_8+b([\nu_4\sigma'\bar{\nu}_{14}]_8+[\nu_4\sigma'\varepsilon_{14}]_8) \tag{5.2}$$

and

$$\Delta((\Sigma\sigma')\varepsilon_{15})=[\nu_4\sigma'\varepsilon_{14}]_8+b([\nu_4\sigma'\varepsilon_{14}]_8+[\nu_4\sigma'\varepsilon_{14}]_8). \tag{5.3}$$

Hence, by the group structure of R^8_{22} and Lemma 5.2, we get that

$$\mathrm{Coker}\{\Delta:\pi^8_{23}\to R^8_{22}\}=\mathbb{Z}_8\{[\zeta'+\mu_6\sigma_{15}]_8\}\oplus\mathbb{Z}_8\{[\iota_7]\rho''\}\oplus\mathbb{Z}_2\{[\nu_5]_8\kappa_7\}\oplus\mathbb{Z}_2\{[\iota_7]\bar{\varepsilon}_7\}.$$

We need the following.

Lemma 5.3 $\{[\nu_5]_7\nu_8^2,\ 8\iota_{14},\ 2\sigma_{14}\}=0$ and $\{[\eta_5\varepsilon_6]_7,\ 8\iota_{14},\ 2\sigma_{14}\}\ni[\eta_6]\rho''+4[\zeta'+\mu_6\sigma_{15}]_7$ mod

$[\nu_4\sigma'\bar{\nu}_{14}]_7$, $[\nu_4\sigma'\varepsilon_{14}]_7$.

Proof. Since $\nu_6\zeta_9=2\sigma''\sigma_{13}$ and $\{\nu_{11}, 8\iota_{14}, 2\sigma_{14}\}\ni\zeta_8$, we have

$$\{[\nu_5]_7\nu_8^2, 8\iota_{14}, 2\sigma_{14}\}\supset[\nu_5]_7\nu_8\circ\{\nu_{11}, 8\iota_{14}, 2\sigma_{14}\}\ni 4[\nu_5]_7(\Sigma\sigma')\sigma_{15} \bmod [\nu_5]_7\nu_8^2\circ\pi_{22}^{14}+R_{15}^7\circ 2\sigma_{15}.$$

We know $\nu_{11}\circ\pi_{22}^{14}=0$. By the fact that $R_{15}^7\cong(\mathbb{Z}_2)^4$ [17], we have $4[\nu_5]_7(\Sigma\sigma')\sigma_{15}=0$ and $R_{15}^7\circ 2\sigma_{15}=0$. This implies the first half.

By the fact that $[\eta_5\varepsilon_6]_7=[\nu_5]_7\nu_8^2+[\eta_6]\sigma'+4[\bar{\nu}_6+\varepsilon_6]$ (3.8), we obtain

$$\{[\eta_5\varepsilon_6]_7, 8\iota_{14}, 2\sigma_{14}\}\subset\{[\eta_6]\sigma', 8\iota_{14}, 2\sigma_{14}\}+\{4[\bar{\nu}_6+\varepsilon_6], 8\iota_{14}, 2\sigma_{14}\} \bmod [\eta_6]\sigma'\circ\pi_{22}^{14}.$$

By the definition of ρ'', we have

$$\{[\eta_6]\sigma', 8\iota_{14}, 2\sigma_{14}\}\supset[\eta_6]\{\sigma', 8\iota_{14}, 2\sigma_{14}\}\ni[\eta_6]\rho'' \bmod [\eta_6]\sigma'\circ\pi_{22}^{14}.$$

By (3.3),

$$\{4[\bar{\nu}_6+\varepsilon_6], 8\iota_{14}, 2\sigma_{14}\}=4[\zeta'+\mu_6\sigma_{15}].$$

We have

$$[\eta_6]\sigma'\circ\pi_{22}^{14}=([\nu_5]_7\nu_8^2+[\eta_5\varepsilon_6]_7+4[\bar{\nu}_6+\varepsilon_6])\circ\{\bar{\nu}_{14}, \varepsilon_{14}\}=[\eta_5\varepsilon_6]_7\circ\{\bar{\nu}_{14}, \varepsilon_{14}\}=\{[\nu_4\sigma'\bar{\nu}_{14}]_7, [\nu_4\sigma'\varepsilon_{14}]_7\}$$

by Lemma 3.6.(iii). This leads to the second half and completes the proof. □

Now we show the following.

Theorem 5.4 $R_{22}^9=\mathbb{Z}_{16}\{[8\sigma_8]\sigma_{15}\}\oplus\mathbb{Z}_8\{[\iota_7]_9\rho''\}\oplus\mathbb{Z}_2\{[\iota_7]_9\bar{\varepsilon}_7\}\oplus\mathbb{Z}_2\{[\nu_5]_9\kappa_8\}$, *where* $2([8\sigma_8]\sigma_{15})\equiv a[\zeta'+\mu_6\sigma_{15}]_9+b[\iota_7]_9\rho'' \bmod [\nu_5]_9\kappa_8$ *for a and b odd.*

Proof. We consider the bracket $\{\Delta\sigma_8, 8\iota_{14}, 2\sigma_{14}\}\subset R_{22}^8$. The indeterminacy of this bracket is $\Delta\sigma_8\circ\pi_{22}^{14}+R_{15}^8\circ 2\sigma_{15}=\{\Delta(\sigma_8\bar{\nu}_{15}), \Delta(\sigma_8\varepsilon_{15})\}$. By (4.2), we have

$$\begin{aligned}\{\Delta\sigma_8, 8\iota_{14}, 2\sigma_{14}\}\subset&\{[\iota_7]\sigma', 8\iota_{14}, 2\sigma_{14}\}+x\{[\bar{\nu}_6+\varepsilon_6]_8, 8\iota_{14}, 2\sigma_{14}\}\\&+y\{[\eta_5\varepsilon_6]_8, 8\iota_{14}, 2\sigma_{14}\}+y\{[\nu_5]_8\nu_8^2, 8\iota_{14}, 2\sigma_{14}\}.\end{aligned}$$

We see that

$$\{[\iota_7]\sigma', 8\iota_{14}, 2\sigma_{14}\}\supset[\iota_7]\circ\{\sigma', 8\iota_{14}, 2\sigma_{14}\}\ni[\iota_7]\rho'' \bmod [\iota_7]\sigma'\circ\pi_{22}^{14}$$

and

$$\{[\bar{\nu}_6+\varepsilon_6]_8, 8\iota_{14}, 2\sigma_{14}\}\ni[\zeta'+\mu_6\sigma_{15}]_8 \bmod [\bar{\nu}_6+\varepsilon_6]_8\bar{\nu}_{14}, [\bar{\nu}_6+\varepsilon_6]_8\varepsilon_{14}.$$

So, by Lemma 5.3, we obtain

$$\{\Delta\sigma_8, 8\iota_{14}, 2\sigma_{14}\}\ni[\iota_7]\rho''+(x+4y)[\zeta'+\mu_6\sigma_{15}]_8+y[\eta_6]_8\rho'' \bmod H,$$

where

$$H=\{[\iota_7]\sigma'\bar{\nu}_{14}, [\iota_7]\sigma'\varepsilon_{14}, [\nu_4\sigma'\bar{\nu}_{14}]_8, [\nu_4\sigma'\varepsilon_{14}]_8, [\eta_6]_8\bar{\varepsilon}_7, [\bar{\nu}_6+\varepsilon_6]_8\bar{\nu}_{14}, [\bar{\nu}_6+\varepsilon_6]_8\varepsilon_{14}\}.$$

By Lemma 5.1, $4[\nu_4\rho'']=4[\eta_6]\rho''$ and $4[\nu_4\rho'']_9=4[\eta_6]_9\rho''=8[\iota_7]_9\rho''=0$. We have $[\eta_6]_9\bar{\varepsilon}_7=2[\iota_7]_9\bar{\varepsilon}_7=0$. So, by Lemma 5.2, Lemma 3.6.(iv) and (5.2), (5.3), we get that

$$[\iota_7]_9\sigma'\bar{\nu}_{14}=[\iota_7]_9\sigma'\varepsilon_{14}=[\nu_5]_9\kappa_8.$$

Hence, by making use of Lemma 1.1 for the bracket $\{\Delta\sigma_8, 8\iota_{14}, 2\sigma_{14}\}$, we obtain

$$2([8\sigma_8]\sigma_{15})\equiv(x+4y)[\zeta'+\mu_6\sigma_{15}]_9+(1+2y)[\iota_7]_9\rho'' \bmod [\nu_5]_9\kappa_8,$$

and thus we get the group R^9_{22}. This completes the proof. □

This theorem corrects the group structure of $\pi_{22}(Spin(9):2)\cong(Z_8)^2\oplus(Z_2)^2$ in [17].

Next we show the following.

Lemma 5.5 *$R^{10}_{22}=Z_{16}\{[8\sigma_8]_{10}\sigma_{15}\}\oplus Z_8\{[\iota_7]_{10}\rho''\}\oplus Z_2\{[\iota_7]_{10}\bar{\varepsilon}_7\}$ and $[\nu_5]_{10}\kappa_8=[\iota_7]_{10}\bar{\varepsilon}_7$.*

Proof. In the exact sequence $(22)_9$:

$$\pi^9_{23}\xrightarrow{\Delta}R^9_{22}\xrightarrow{i_*}R^{10}_{22}\xrightarrow{p_*}\pi^9_{22}\xrightarrow{\Delta}R^9_{21},$$

$\Delta:\pi^9_{22}\to R^9_{21}$ is a monomorphism and $\pi^9_{23}=Z_{16}\{\sigma^2_9\}\oplus Z_4\{\kappa_9\}$. By (4.1) and (3.9), we have $\Delta(\sigma^2_9)=[\iota_7]_9(\bar{\nu}_7+\varepsilon_7)\circ\sigma_{15}=0$. We also have $\Delta\kappa_9=[\nu_5]_9\kappa_8+[\iota_7]_9\bar{\varepsilon}_7$. Hence we obtain the group R^{10}_{22}. This completes the proof. □

Finally we show the following.

Lemma 5.6 (i) *$[\eta_5\varepsilon_6]_9\bar{\nu}_{14}=[\eta_5\varepsilon_6]_9\varepsilon_{14}=[\nu_4\sigma'\bar{\nu}_{14}]_9=[\nu_4\sigma'\varepsilon_{14}]_9=0$.*

(ii) *$[\bar{\nu}_6+\varepsilon_6]_9\bar{\nu}_{14}=[\bar{\nu}_6+\varepsilon_6]_9\varepsilon_{14}=[\nu_5]_9\kappa_8=[\iota_7]_9\sigma'\bar{\nu}_{14}=[\iota_7]_9\sigma'\varepsilon_{14}$ and $[\iota_7]_9\sigma'\sigma_{14}\eta_{21}=0$.*

(iii) *$[\nu_5]_{10}\kappa_8=[\iota_7]_{10}\bar{\varepsilon}_7$ and $[\iota_7]_{10}\kappa_7\eta_{21}=0$.*

Proof. By [11, Lemma 1.2.(i)] and the relations $\nu_{11}\bar{\nu}_{14}=\nu_{11}\varepsilon_{14}=0$,

$$\Delta((\Sigma\sigma')\bar{\nu}_{15})=[\eta_5\varepsilon_6]_8\bar{\nu}_{14} \text{ and } \Delta((\Sigma\sigma')\varepsilon_{15})=[\eta_5\varepsilon_6]_8\varepsilon_{14}.$$

This, (5.2) and (5.3) imply (i).

By Lemma 3.6.(iv), we obtain

$$[\bar{\nu}_6+\varepsilon_6]_9\bar{\nu}_{14}\equiv[\bar{\nu}_6+\varepsilon_6]_9\varepsilon_{14}\equiv[\nu_5]_9\kappa_8 \bmod 4[\nu_4\rho'']_9.$$

By use of (4.3), $[\bar{\nu}_6+\varepsilon_6]_9\sigma_{14}\eta_{21}=0$, and so $[\bar{\nu}_6+\varepsilon_6]_9\bar{\nu}_{14}=[\bar{\nu}_6+\varepsilon_6]_9\varepsilon_{14}$. By Lemma 5.2.(i) and the first assertion, we obtain $[\bar{\nu}_6+\varepsilon_6]_9\bar{\nu}_{14}=[\iota_7]_9\sigma'\bar{\nu}_{14}$. By the parallel argument, we obtain $[\bar{\nu}_6+\varepsilon_6]_9\varepsilon_{14}=[\iota_7]_9\sigma'\varepsilon_{14}$. By Lemma 5.1, we see that $4[\nu_4\rho'']_7=4[\eta_6]\rho''$. So we obtain $4[\eta_6]_9\rho''=8[\iota_7]_9\rho''=0$. Thus (i) and Lemma 3.6.(iv) imply (ii).

We have $\Delta\kappa_9=([\nu_5]_9+[\iota_7]_9\eta_7)\kappa_8=[\nu_5]_9\kappa_8+[\iota_7]_9\bar{\varepsilon}_7$. This leads to the first half of (iii). By (3.7), we obtain $[\iota_7]\kappa_7\eta_{21}=[\iota_7](\bar{\varepsilon}_7+\sigma'\bar{\nu}_{14})$. So, by Lemma 3.6.(iv) and the first half of (ii), $[\iota_7]_{10}\kappa_7\eta_{21}=0$. Since $\sigma_{14}\eta_{21}=\bar{\nu}_{14}+\varepsilon_{14}$, we obtain $[\iota_7]_9\sigma'\sigma_{14}\eta_{21}=[\iota_7]_9\sigma'\bar{\nu}_{14}+[\iota_7]_9\sigma'\varepsilon_{14}=0$ by (iii). This leads to the second half of (iii) and completes the proof. □

6 Determination of $\pi_{22}(R_n:2)$ $(n\geq 11)$

First we recall from [18] that $\pi^{10}_{30}=Z_8\{\bar{\kappa}_{10}\}\oplus Z_8\{\beta'\}$, $\Sigma\beta'=\theta'\varepsilon_{23}$, $H(\beta')=\zeta_{19}$ and

$P(\zeta_{21})=\pm 2\beta'$. We show the following.

Lemma 6.1 (i) $\zeta_n\in\{\eta_n\varepsilon_{n+1}, \eta_{n+9}, 2\iota_{n+10}\}_{n-6}$ mod $2\zeta_n$ *for* $n\geq 10$ *except for* $n=12$; $\zeta_{12}\in\{\eta_{12}\varepsilon_{13}, \eta_{21}, 2\iota_{22}\}_6$ mod $2\zeta_{12}, 2[\iota_{12}, \iota_{12}]$.

(ii) $\bar{\varepsilon}_5=\{2\nu_5\sigma_8, \nu_{15}, \eta_{18}\}$.

(iii) $\sigma_{10}\rho_{17}\in\{\beta', \eta_{30}, 2\iota_{31}\}$ mod $2\sigma_{10}\rho_{17}$.

Proof. By the fact that $\eta_5^2\varepsilon_7=4(\nu_5\sigma_8)$ [28, (7.10)], $4(\nu_9\sigma_{12})=0$ [28, (7.20)] and $\eta_n\varepsilon_{n+1}=\varepsilon_n\eta_{n+8}$ for $n\geq 3$ [28, (7.5)], a Toda bracket $\{\eta_n\varepsilon_{n+1}, \eta_{n+9}, 2\iota_{n+10}\}_{n-6}$ is defined for $n\geq 9$. Since $\Sigma^\infty: \pi_{21}^{10}\to\pi_{11}^S(S^0:2)$ is an isomorphism and $\zeta\in\langle\eta\varepsilon, \eta, 2\iota\rangle$ mod 2ζ [28, Lemma 9.1], we obtain (i).

By [24, Lemma 2.14] and the fact that $\pi_{16}^4\circ\nu_{16}=0$, we have $\bar{\varepsilon}_4=\{\eta_4, 2\iota_5, \nu_5\sigma_8\nu_{15}\}=\{\eta_4, 2\nu_5\sigma_8, \nu_{15}\}$. By [28, Lemma 12.10], $\eta_3\bar{\varepsilon}_4=\bar{\varepsilon}_3\eta_{18}$. We have

$$\bar{\varepsilon}_4\eta_{19}\in\{\eta_4, 2\nu_5\sigma_8, \nu_{15}\}\circ\eta_{19}=\eta_4\circ\{2\nu_5\sigma_8, \nu_{15}, \eta_{18}\}.$$

The indeterminacy of $\{2\nu_5\sigma_8, \nu_{15}, \eta_{18}\}$ is 0, because $2\nu_5\sigma_8\circ\pi_{20}^{15}+\pi_{19}^5\circ\eta_{19}=\{\nu_5\zeta_8, \nu_5\bar{\nu}_8\nu_{16}\}\circ\eta_{19}=0$.

On the other hand, $\langle 2\nu\sigma, \nu, \eta\rangle=\pi_{14}^S(S^0:2)\circ\eta=\{\bar{\varepsilon}\}\cong\mathbb{Z}_2$. This and the fact that $\pi_{20}^5=\{\bar{\varepsilon}_5, \rho^{IV}\}\cong(\mathbb{Z}_2)^2$ lead to (ii).

Since $\Sigma^3: \pi_{31}^{10}\to\pi_{34}^{13}$ is a monomorphism [16] and $\Sigma^3\beta'=(\Sigma^2\theta')\varepsilon_{25}=0$, we obtain $\beta'\eta_{30}=0$. By Lemma 6.1.(i),

$$P(\zeta_{25})=P(\iota_{25})\circ\zeta_{23}\in\{[\iota_{12}, \eta_{12}\varepsilon_{13}], \eta_{32}, 2\iota_{33}\}.$$

Since $\varepsilon_{12}\kappa_{20}$ and $\nu_{12}\bar{\sigma}_{15}$ survive in the stable range, we get that $P(\zeta_{25})=2\sigma^{*'''}+(2n-1)\sigma_{12}\rho_{19}$ for $n\in\mathbb{Z}$ [16, p.320]. Hence, by the fact that $\Sigma^2\beta'=(\Sigma\theta')\varepsilon_{24}=P(\eta_{25}\varepsilon_{26})$ and $2\pi_{34}^{12}=\{2\sigma^{*'''}, 2\sigma_{12}\rho_{19}\}$ [16], we obtain

$$\begin{aligned}\Sigma^2\{\beta', \eta_{30}, 2\iota_{31}\} &\subset \{[\iota_{12}, \eta_{12}\varepsilon_{13}], \eta_{32}, 2\iota_{33}\}\\ &\ni P(\zeta_{25})=2\sigma^{*'''}+(2n-1)\sigma_{12}\rho_{19} \text{ mod } 2\pi_{34}^{12}=\{2\sigma^{*'''}, 2\sigma_{12}\rho_{19}\}.\end{aligned}$$

Since $\Sigma^2: \pi_{32}^{10}\to\pi_{34}^{12}$ is a monomorphism [16], $H(\sigma^{*'''})\equiv\zeta_{23}$ mod $2\zeta_{23}$ and $8\sigma^{*'''}\equiv 4\sigma_{12}\rho_{19}$ mod $8\sigma_{12}\rho_{19}$ [16, Lemma 6.2], we obtain $\sigma_{10}\rho_{17}\in\{\beta', \eta_{30}, 2\iota_{31}\}$. This leads to (iii) and completes the proof. □

Now we show the following.

Lemma 6.2 (i) $\Delta(\sigma_{10}\nu_{17}^2)=[\iota_7]_{10}\bar{\varepsilon}_7$.

(ii) $p_*\Delta\theta=\theta'$.

Proof. Since

$\sigma_{10}\nu_{17}^2=\sigma_{10}\circ\{\eta_{17}, \nu_{18}, \eta_{21}\}_1\subset\{\sigma_{10}\eta_{17}, \nu_{18}, \eta_{21}\}_1=\{\eta_{10}\sigma_{11}, \nu_{18}, \eta_{21}\}_1$

and $\Delta\eta_{10}=2[\iota_7]_{10}\nu_7$, we obtain

$$\Delta(\sigma_{10}\nu_{17}^2)\in\{2[\iota_7]_{10}\nu_7\sigma_{10}, \nu_{17}, \eta_{20}\}\supset[\iota_7]_{10}\{2\nu_7\sigma_{10}, \nu_{17}, \eta_{20}\}.$$

By Lemma 6.1.(ii), $\bar{\varepsilon}_7\in\{2\nu_7\sigma_{10}, \nu_{17}, \eta_{20}\}$ mod $\pi_{21}^7\circ\eta_{21}=\{\kappa_7\eta_{21}, \sigma'\sigma_{14}\eta_{21}\}$. By Lemma 5.6,

$\Delta(\sigma_{10}\nu_{17}^2)\equiv[\iota_7]_{10}\bar{\varepsilon}_7 \bmod [\iota_7]_{10}\circ\{\kappa_7\eta_{21},\ \sigma'\sigma_{14}\eta_{21}\}=0$. This leads to (i).

We recall that $\theta\in\{\sigma_{12},\ \nu_{19},\ \eta_{22}\}_1$ and $\theta'=\{\sigma_{11},\ 2\nu_{18},\ \eta_{21}\}$. By (1.3), we have

$$p_{12*}\Delta\theta\in p_{12*}\Delta\{\sigma_{12},\ \nu_{19},\ \eta_{22}\}_1\subset\{2\sigma_{11},\ \nu_{18},\ \eta_{21}\}\subset\{\sigma_{11},\ 2\nu_{18},\ \eta_{21}\}=\theta'.$$

This leads to (ii) and completes the proof. □

By (1.4) and (4.4), $H(J[[\iota_{10},\ \nu_{10}]])=0$ and so,

$$J[[\iota_{10},\ \nu_{10}]]\in\Sigma\pi_{32}^{10}=\pi_{33}^{11}=\mathbb{Z}_{16}\{\sigma_{11}\rho_{18}\}\oplus\mathbb{Z}_2\{\varepsilon_{11}\kappa_{19}\}\oplus\mathbb{Z}_2\{\nu_{11}\bar{\sigma}_{14}\}.$$

We can set

$$J[[\iota_{10},\ \nu_{10}]]=a'\sigma_{11}\rho_{18}\ (a'\in\mathbb{Z}).$$

By [22, Proposition II.2.3.(1)] and [16, (8.4)],

$$\rho'\rho_{24}\equiv b\sigma_9\rho_{16} \bmod [\iota_9,\ \kappa_9]\ (b:\text{odd}). \tag{6.1}$$

Hence, by the exact sequence $(22)_{10}$ and Lemmas 5.5, 6.2.(i), the following relation holds for integers r and s:

$$4[[\iota_{10},\ \nu_{10}]]=r[8\sigma_8]_{11}\sigma_{15}+s[\iota_7]_{11}\rho''.$$

Taking the J-image of this relation and using (6.1), we see that

$$4a\sigma_{11}\rho_{18}=r\Sigma^2(\rho'\sigma_{24})+4s\sigma_{11}\rho_{18}=(br+4s)\sigma_{11}\rho_{18}.$$

Hence, we get that $br\equiv4(a-s)$ (mod 16) and $r=4r'$ for some integer $r'$and

$$4([[\iota_{10},\ \nu_{10}]]+r'[8\sigma_8]_{11}\sigma_{15})=s[\iota_7]_{11}\rho''.$$

As $p_{12*}\Delta[\iota_{12},\ \iota_{12}]=0$ by Lemma 1.4, we have $\Delta[\iota_{12},\ \iota_{12}]\in i_{12*}R_{22}^{11}$. By Lemma 6.2.(ii), $\Delta\theta'=0$ and $i_{12*}: R_{22}^{11}\to R_{22}^{11}$ is a monomorphism. By Theorem 1.3, $\#\Delta[\iota_{12},\ \iota_{12}]=32$. Hence there exists an element of R_{22}^{11} with order 32. Thus s is odd. This determines the group R_{32}^{11}. By noticing the fact that $\#[\iota_{12},[\iota_{12},\ \iota_{12}]]=3$, we obtain the following.

Lemma 6.3 (i) $R_{22}^{11}=\mathbb{Z}_{16}\{[8\sigma_8]_{11}\sigma_{15}\}\oplus\mathbb{Z}_{32}\{[[\iota_{10},\ \nu_{10}]]\}$ *and* $4([[\iota_{10},\ \nu_{10}]]+r'[8\sigma_8]_{11}\sigma_{15})=s[\iota_7]_{11}\rho''$ *for* s *odd an integer* r'.

(ii) $\Delta[\iota_{12},\ \iota_{12}]=[[\iota_{10},\ \nu_{10}]]_{12}-a'[8\sigma_8]_{12}\sigma_{15}$.

We show the following.

Lemma 6.4 $[\eta_{10}^2]\eta_{12}\mu_{13}=8[8\sigma_8]_{11}\sigma_{15}$ *and* $4\Delta\zeta_{12}=[\eta_{10}^2]_{12}\eta_{12}\mu_{13}=8[8\sigma_8]_{12}\sigma_{15}$.

Proof. From the fact that $p_{11*}[\eta_{10}^2]\eta_{12}^2=0$ and $i_{11*}R_{14}^{10}=\mathbb{Z}_8\{[\bar{\nu}_6+\varepsilon_6]_{11}\}$, we can set $[\eta_{10}^2]\eta_{12}^2=4a[\bar{\nu}_6+\varepsilon_6]_{11}$ for $a\in\{0,1\}$. We know that $J[\bar{\nu}_6+\varepsilon_6]_{11}=2\sigma_{11}^2$ (Lemma 2.1.(iii)), $[\eta_{10}^2]_{13}=0$ by (1.8) and $\#\sigma_{13}^2=16$. So we obtain $a=0$ and $[\eta_{10}^2]\eta_{12}^2=0$. By this relation and by the fact that $\mu_{13}\in\{\eta_{13},\ 2\iota_{14},\ 8\sigma_{14}\}$, we see that

$$[\eta_{10}^2]\eta_{12}\mu_{13}\in[\eta_{10}^2]\eta_{12}\circ\{\eta_{13},\ 2\iota_{14},\ 8\sigma_{14}\}=\{[\eta_{10}^2]\eta_{12},\ \eta_{13},\ 2\iota_{14}\}\circ8\sigma_{15}.$$

By [10],

$$\{[\eta_{10}^2]\eta_{12},\ \eta_{13},\ 2\iota_{14}\}\subset R_{15}^{11}=\mathbb{Z}\{[8\sigma_8]_{11}\}\oplus\mathbb{Z}_2\{[\nu_5]_{11}\sigma_8\}.$$

So we have $[\eta_{10}^2]\eta_{12}\mu_{13}=8x[8\sigma_8]_{11}\sigma_{15}$ for $x\in\{0,1\}$. We obtain $J([\eta_{10}^2]\eta_{12}\mu_{13})=\theta'\eta_{23}\mu_{24}$ and $8J([8\sigma_8]_{11}\sigma_{15})=8(\Sigma^2\rho')\sigma_{26}=8\sigma_{11}\rho_{18}\neq 0$ by Lemma 2.1.(iv) and (6.1). By the fact that $\Sigma\theta'=P(\eta_{25})$ [28, p.80] and $\Sigma(\theta'\eta_{23}\mu_{24})=4P(\zeta_{25})=8\sigma_{12}\rho_{19}\neq 0$ [16, p.320]. This implies $x=1$, and hence the first relation follows.

By the fact that $4\zeta_{12}=\eta_{12}^2\mu_{14}$ and $\Delta\eta_{12}=[\eta_{10}^2]_{12}$ (1.8), we see that

$$4\Delta(\zeta_{12})=\Delta(\eta_{12}^2\mu_{14})=[\eta_{10}^2]_{12}\eta_{12}\mu_{13}.$$

Hence the first relation leads to the second one. This completes the proof. □

By (2.1) and the fact that $\Delta\zeta_{11}=0$, $H(\sigma^{*'''})\equiv\zeta_{23} \bmod 2\zeta_{23}$,

$$J[\zeta_{11}]\equiv y\sigma^{*'''} \bmod \sigma_{12}\rho_{19}\ (y:\text{odd}). \tag{6.2}$$

By (6.2) and the fact that $\#\sigma^{*'''}=32$ [16], $\#[\zeta_{11}]$ is a multiple of 32. By the exact sequence $(22)_{11}$, by Lemma 6.2 and Lemma 6.3, R_{22}^{12} is generates by $[8\sigma_8]_{12}\sigma_{15}$, $[[\iota_{10},\ \nu_{10}]]_{12}$ and $[\zeta_{11}]$.

We show the following.

Lemma 6.5 $\Delta\zeta_{12}\equiv u[8\sigma_8]_{12}\sigma_{15}+2[\zeta_{11}] \bmod 2[[\iota_{10},\ \nu_{10}]]_{12}$ *for u odd and* $\#[\zeta_{11}]=32$.

Proof. It suffices to show the relation. It follows from the relation $\zeta_9\eta_{20}=0$ [28, Theorem 7.6] that $[\zeta_9]\eta_{20}\in i_{10*}R_{21}^9$. So, by Theorem 0.1, we can set $[\zeta_9]\eta_{20}=x[\iota_7]_{10}\kappa_7$ for $x\in\{0,1\}$. Since $J[\zeta_9]=\beta'$ (Lemma 2.2.(v)), $\beta'\eta_{30}=0$ and $J([\iota_7]_{10}\kappa_7)=\sigma_{10}\kappa_{17}\neq 0$, we get $x=0$. So the Toda bracket $\{[\zeta_9],\ \eta_{20},\ 2\iota_{21}\}$ is well-defined. By (6.1) and Lemma 5.5, Lemma 6.1.(iii), we conclude that

$$[8\sigma_8]_{10}\sigma_{15}\in\{[\zeta_9],\ \eta_{20},\ 2\iota_{21}\} \bmod 2[8\sigma_8]_{10}\sigma_{15},\ [\iota_7]_{10}\rho'',\ [\iota_7]_{10}\bar{\varepsilon}_7.$$

By (4.5) and the fact that $\zeta_{12}\in\{\eta_{12}\varepsilon_{13},\ \eta_{21},\ 2\iota_{22}\}_1$ (Lemma 6.1.(i)), we see that

$$\Delta\zeta_{12}\in\{[\zeta_9]_{12},\ \eta_{20},\ 2\iota_{21}\}\supset i_{10,12*}\{[\zeta_9],\ \eta_{20},\ 2\iota_{21}\}.$$

Hence, by Lemmas 6.2 and 6.3, we can set

$$\Delta\zeta_{12}\equiv[8\sigma_8]_{12}\sigma_{15} \bmod 2R_{22}^{12}=\{2[8\sigma_8]_{12}\sigma_{15},\ 2[[\iota_{10},\ \nu_{10}]]_{12},\ 2[\zeta_{11}]\}.$$

Since $p_{12*}\Delta\zeta_{12}=2\zeta_{11}$ (1.3), we obtain the relation. This completes the proof. □

Now we show the following.

Proposotion 6.6 (i) $R_{22}^{12}=\mathbb{Z}_{32}\{[\zeta_{11}]\}\oplus\mathbb{Z}_{32}\{[[\iota_{10},\ \nu_{10}]]_{12}\}\oplus\mathbb{Z}_4\{[8\sigma_8]_{12}\sigma_{15}+2v[\zeta_{11}]+2w[[\iota_{10},\ \nu_{10}]]_{12}\}$ *for v odd and w an integer.*

(ii) $R_{22}^{13}=\mathbb{Z}_{16}\{[\eta_{12}^3]\sigma_{15}\}$, *where* $2[\eta_{12}^3]\sigma_{15}=[8\sigma_8]_{13}\sigma_{15}$ *and* $[\zeta_{11}]_{13}=y[\eta_{12}^3]\sigma_{15}$ *for y odd.*

Proof. We know the short exact sequence ($i=i_{12}$, $p=p_{12}$):

$$0\longrightarrow R_{22}^{11}\xrightarrow{i_*}R_{22}^{12}\xrightarrow{p_*}\pi_{22}^{11}\longrightarrow 0.$$

By Lemmas 6.4 and 6.5, we have the relation

$$8[8\sigma_8]_{12}\sigma_{15}\equiv 4u[8\sigma_8]_{12}\sigma_{15}+8[\zeta_{11}] \bmod 8[[\iota_{10},\ \nu_{10}]]_{12}.$$

This implies the relation for v odd and an integer w:

$$4([8\sigma_8]_{12}\sigma_{15}+2v[\zeta_{11}]+2w[[\iota_{10},\ \nu_{10}]]_{12})=0$$

and leads to (i).

In the exact sequence $(22)_{12}$:

$$\pi_{23}^{12}\xrightarrow{\Delta}R_{22}^{12}\xrightarrow{i_*}R_{22}^{13}\xrightarrow{p_*}\pi_{22}^{12}\xrightarrow{\Delta}R_{21}^{12},$$

$\pi_{23}^{12}=\mathbb{Z}\{[\iota_{12},\ \iota_{12}]\}\oplus\mathbb{Z}_8\{\zeta_{12}\}$ and that $\Delta:\ \pi_{22}^{12}\to R_{21}^{12}$ is a monomorphism. So, Lemmas 6.3.(i) and 6.5, we obtain

$$R_{22}^{13}=G\{[\zeta_{11}]_{13},\ [8\sigma_8]_{13}\sigma_{15}\},$$

where G is a group of order 16. Consider an element $[\eta_{12}^3]\sigma_{15}\in R_{22}^{13}$, where $J[\eta_{12}^3]\equiv\rho_{13}$ mod $2\rho_{13}$ (Lemma 2.1.(iv)). Then, by the relation $2[\eta_{12}^3]=[8\sigma_8]_{13}$ [10], we obtain the first relation of (ii):

$$2[\eta_{12}^3]\sigma_{15}=[8\sigma_8]_{13}\sigma_{15}.$$

Since $J([8\sigma_8]_{13}\sigma_{15})=2\rho_{13}\sigma_{28}$ is of order 8 [16], the order of $[\eta_{12}^3]\sigma_{15}$ is 16. This yields the group R_{22}^{13}. By the fact that $8\Sigma\sigma^{*\prime\prime\prime}=\pm 8\rho_{13}\sigma_{28}$ [16, (8.7)], $\Sigma^\infty\sigma^{*\prime\prime\prime}=\rho\sigma=0$ and (6.2), we obtain $\Sigma\sigma^{*\prime\prime\prime}=y\rho_{13}\sigma_{28}$ for y odd. This leads to the second relation of (ii) and completes the proof. □

Next we show the following.

Lemma 6.7 *$\Delta(\eta_{13}\mu_{14})=8[\eta_{12}^3]\sigma_{15}$ and $R_{22}^{14}=\mathbb{Z}_{16}\{[\eta_{13}^2\sigma_{15}]\}\oplus\mathbb{Z}_2\{[\nu_{13}^2]\nu_{19}\}$, where $2[\eta_{13}^2\sigma_{15}]=z[\eta_{12}^3]_{14}\sigma_{15}$ for z odd.*

Proof. In the exact sequence $(22)_{13}$, we know

$$\mathrm{Ker}\{\Delta:\ \pi_{22}^{13}\to R_{21}^{13}\}=\mathbb{Z}_2\{\eta_{13}^2\sigma_{15}\}\oplus\mathbb{Z}_2\{\nu_{13}^3\}.$$

So there exist elements $[\eta_{13}^2\sigma_{15}]$ and $[\nu_{13}^2]\nu_{19}\in R_{22}^{14}$. By the fact that $R_{20}^{14}=\mathbb{Z}_{16}\{\Delta\sigma_{14}\}$ [12], $J([\nu_{13}^2]\eta_{19})=0$ (Lemma 2.2.(iv)) and $8[\iota_{14},\ \sigma_{14}]\neq 0$ [18], we have $[\nu_{13}^2]\eta_{19}=0$. So, by the fact that $R_{21}^{14}=\mathbb{Z}_4\{[\nu_{12}^2]_{14}\nu_{18}\}$, we obtain

$$2[\nu_{13}^2]\nu_{19}\in[\nu_{13}^2]\circ\{\eta_{19},\ 2\iota_{20},\ \eta_{20}\}=-\{[\nu_{13}^2],\ \eta_{19},\ 2\iota_{20}\}\circ\eta_{21}\subset R_{21}^{14}\circ\eta_{21}=0.$$

Since $J\Delta(\eta_{13}\mu_{14})=[\iota_{13},\ \eta_{13}\mu_{14}]=8\rho_{13}\sigma_{28}\neq 0$ [16], we have the first relation. By (2.1) and the fact $\eta_{27}^2\sigma_{29}=H(\sigma^{*\prime\prime})$ [16, Lemma 6.2.(2)], we have

$$J[\eta_{13}^2\sigma_{15}]\equiv\sigma^{*\prime\prime} \bmod \Sigma\pi_{35}^{13}=\{\rho_{14}\sigma_{29},\ \varepsilon_{14}\kappa_{22},\ \nu_{14}\bar{\sigma}_{17}\}.$$

Since $2\sigma^{*\prime\prime}\equiv\rho_{14}\sigma_{29}$ mod $2\rho_{14}\sigma_{29}$ [16, Lemma 6.2.(2)], we obtain $2[\eta_{13}^2\sigma_{15}]=z[\eta_{12}^3]_{14}\sigma_{15}$ for z

odd. This leads to the second relation, yields the group structure of R^{14}_{22} and completes the proof. □

Next we show the following.

Lemma 6.8 $R^{15}_{22}=Z_{16}\{[\eta_{14}\sigma_{15}]\}\oplus Z_2\{[\nu^2_{13}]_{15}\nu_{19}\}$ *and* $2[\eta_{14}\sigma_{15}]=[\eta^2_{13}\sigma_{15}]_{15}$.

Proof. We know $\mathrm{Ker}\{\Delta:\pi^{14}_{22}\to R^{14}_{21}\}=Z_2\{\eta_{14}\sigma_{15}\}$ and $\Delta\bar{\nu}_{14}=\Delta\varepsilon_{14}=2[\nu^2_{12}]_{14}\nu_{18}$. By (1.7), $\Delta(\eta_{14}\varepsilon_{15})=\Delta(\eta^2_{14}\sigma_{16})=0$. We know $H(\sigma^{*\prime})=\eta_{29}\sigma_{30}$, $P(\mu_{29})=8\sigma^{*\prime\prime}\neq 0$ [16, pp. 321-2] and $H(\omega_{14}\nu_{30})=\nu^2_{27}$. This implies the relations $\Delta\mu_{14}=8[\eta^2_{13}\sigma_{15}]$, $J[\eta_{14}\sigma_{15}]\equiv x\sigma^{*\prime}$ mod $\omega_{15}\nu^2_{31}$ for x odd and $J([\nu^2_{13}]_{15}\nu_{19})=\omega_{15}\nu^2_{31}$. This yields the group R^{15}_{22} and completes the proof. □

Since $\Delta\sigma_{15}=[\bar{\nu}_6+\varepsilon_6]_{15}\sigma_{14}=0$, there exists an element $[\sigma_{15}]\in R^{16}_{22}$. Then we show the following.

Lemma 6.9 $R^{16}_{22}=Z_{16}\{[\eta_{14}\sigma_{15}]_{16}\}\oplus Z_2\{[\nu^2_{13}]_{16}\nu_{19}\}\oplus Z_{16}\{[\sigma_{15}]\}$.

Proof. By the fact that $\Delta\iota_{15}=[\bar{\nu}_6+\varepsilon_6]_{15}$ [10], by Lemma 5.6 and Lemma 6.2, we get that $\Delta\pi^{15}_{23}=0$ and $i_{16*}: R^{15}_{22}\to R^{16}_{22}$ is a monomorphism. We recall from [16] that $H(\sigma^*_{16})\equiv\sigma_{31}$ mod $2\sigma_{31}$,

$$\pi^{15}_{37}=Z_{16}\{\sigma^{*\prime}\}\oplus Z_2\{\omega_{15}\nu^2_{31}\}\oplus Z_2\{\varepsilon_{15}\kappa_{23}\}\oplus Z_2\{\nu_{15}\bar{\sigma}_{18}\},$$

$\Sigma:\pi^{15}_{37}\to\pi^{16}_{38}$ is a monomorphism and $\pi^{16}_{38}=Z_{16}\{\sigma^*_{16}\}\oplus\Sigma\pi^{15}_{37}$. This implies the relation $J[\sigma_{15}]\equiv\sigma^*_{16}$ mod $\Sigma\sigma^{*\prime}$, $\omega_{16}\nu^2_{32}$. We obtain

$$\Delta\sigma_{16}-2[\sigma_{15}]\in i_*R^{15}_{22}=\{[\eta_{14}\sigma_{15}]_{16},\ [\nu^2_{13}]_{16}\nu_{19}\}$$

and $J([\nu^2_{13}]_{16}\nu_{19})=\omega_{16}\nu^2_{32}$. Hence, by the fact that $P(\sigma_{33})\equiv 2\sigma^*_{16}-\Sigma\sigma^{*\prime}$ mod $\rho_{16}\sigma_{31}=4\Sigma\sigma^{*\prime}$ [16], we obtain

$$\Delta\sigma_{16}=2[\sigma_{15}]+a[\eta_{14}\sigma_{15}]_{16}+b[\nu^2_{13}]_{16}\nu_{19}\ (a:\text{odd};\ b\in\{0,1\}).$$

and $32[\sigma_{15}]=0$. So, by Lemma 6.8, we can set $16[\sigma_{15}]=8b[\eta_{14}\sigma_{15}]_{16}+c[\nu^2_{13}]_{16}\nu_{19}$ for $b,c\in\{0,1\}$. Since $8J[\eta_{14}\sigma_{15}]_{16}=8\Sigma\sigma^{*\prime}$ and $J([\nu^2_{13}]_{16}\nu_{19})=\omega_{16}\nu^2_{32}$ are independent in π^{16}_{38} [16], the relation $0=16J[\sigma_{15}]=8b\Sigma\sigma^{*\prime}+c\omega_{16}\nu^2_{32}$ implies $b=c=0$. Thus $\#[\sigma_{15}]=16$. This completes the proof. □

In determining R^{17}_{21}, we know that $\Delta:\pi^{16}_{22}\to R^{16}_{21}$ is a monomorphism. This implies the following.

Lemma 6.10 $R^{17}_{22}=Z_{16}\{[\sigma_{15}]_{17}\}\oplus Z_2\{[\nu^2_{13}]_{17}\nu_{19}\}$.

Since $\Delta\nu_{17}=[\nu^2_{13}]_{17}$ [12], $\Delta(\nu^2_{17})=[\nu^2_{13}]_{17}\nu_{19}$. This implies that $R^{18}_{22}=Z_{16}\{[\sigma_{15}]_{18}\}$. Obviously $R^{19}_{22}=Z_{16}\{[\sigma_{15}]_{19}\}$. $i_{20*}: R^{19}_{22}\to R^{20}_{22}$ is a monomorphism, because $\pi^{19}_{23}=0$. We know $\mathrm{Ker}\{\Delta:\pi^{19}_{22}\to R^{19}_{21}\}=Z_4\{2\nu_{19}\}$. We recall $\pi^{19}_{41}=Z_{16}\{\sigma^*_{19}\}\oplus Z_2\{\varepsilon_{19}\kappa_{27}\}\oplus Z_2\{\nu_{19}\bar{\sigma}_{22}\}$. By [16, Lemma 8.3], $8\sigma^*_{20}=4P(\nu_{41})$. So we get that $4\Delta\nu_{20}=8[\sigma_{15}]_{20}$. This implies that $R^{20}_{22}=Z_{16}\{[\sigma_{15}]_{20}\}\oplus Z_4\{\Delta\nu_{20}-2[\sigma_{15}]_{20}\}$.

The groups R^n_{22} for $n\geq 21$ are obtained from the relations:

$$P(\eta^2_{43})=4\sigma^*_{21},\ P(\eta_{45})=2\sigma^*_{22}\ \text{and}\ P(\iota_{47})=\sigma^*_{23}.$$

We note the relations: $\Delta(\eta_{21}^2)=4[\sigma_{15}]_{21}$, $\Delta\eta_{22}=2[\sigma_{15}]_{22}$ and $\Delta\iota_{23}=[\sigma_{15}]_{23}$.

Finally we state the group structure of the p-primary components $\pi_k(R_n:p)$ for $k=21$ and 22, where p is an odd prime. We know isomorphisms for all k ([2], [5]);

$$\pi_k(R_{2n+1}:p)\cong\pi_k(Sp(n):p);$$
$$\pi_k(R_{2n+2}:p)\cong\pi_k(R_{2n+1}:p)\oplus\pi_k(S^{2n+1}:p).$$

We know the homotopy groups $\pi_k(Sp(n))$ by [19] and [20] for $k\leq 22$. So, by combining the results of [16], [18] and [28], we obtain our result which is summarized in the following table.

	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$
$k=21$	3	$(3)^2$	0	0	3	$(3)^2$
$k=22$	$3\cdot 11$	$(3\cdot 11)^2$	$3\cdot 5\cdot 11$	$3\cdot 5\cdot 11$	$27\cdot 5\cdot 7\cdot 11$	$27\cdot 5\cdot 7\cdot 11+3\cdot 5$

	$n=9$	$n=10$	$n=11$	$n=12$
$k=21$	3	3	0	3
$k=22$	$81\cdot 25\cdot 7\cdot 11$	$81\cdot 25\cdot 7\cdot 11+3$	$81\cdot 25\cdot 7\cdot 11$	$81\cdot 25\cdot 7\cdot 11+9\cdot 7$

	$n=13$	$n=14$	$n=15$	$n=16$	$n=17$	$n=18$	$n=19$	$n=20$
$k=21$	0	0	0	0	0	0	0	0
$k=22$	0	0	0	$3\cdot 5$	0	0	0	3

$\pi_k(R_n:p)=0$ $(n\geq 21,\ k=21,22)$.

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