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The 2-components of the 31-stem homotopy groups of the 9 and 10-spheres

Dedicated to the memory of Professor Minoru Nakaoka

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Abstract

Group structures of the 2-primary components of the 31-stem homotopy groups of spheres were studied by Oda in 1979. There are, however, two incompletely determined groups. In this paper, our investigation with Toda's composition method gives structures of them.

1 Introduction.

We denote by π^n_k the direct sum of the torsion-free part and the 2-primary component of the k-th homotopy group $\pi_k(S^n)$ of the n-dimensional sphere S^n. The group π^n_k was studied by Toda [14] with his composition method, and several authors [7, 5, 6, 9] followed the method. In particular, Oda [9] investigated the 2-primary components of k-stem homotopy groups for $25 \leq k \leq 31$. There are, however, two incompletely determined groups in 31-stem: π^9_{40} and π^{10}_{41}.

We denote by $\{\chi_1, \ldots, \chi_n\}$ a group generated by elements $\chi_1, \ldots, \chi_n$. If the group is isomorphic to a group G, it is denoted by $G\{\chi_1, \ldots, \chi_n\}$. For the group $\mathbb{Z}$ of integers, we set $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$, and let $(\mathbb{Z}_n)^k$ be the direct sum $\mathbb{Z}_n \oplus \cdots \oplus \mathbb{Z}_n$ of k-copies of $\mathbb{Z}_n$. Let $E\colon \pi^n_k \to \pi^{n+1}_{k+1}$ be the suspension homomorphism, and let $P\colon \pi^{2n+1}_{k+2} \to \pi^n_k$ be the P-homomorphism which is denoted by Δ in [14]. We use generators of homotopy groups of spheres in the same symbol as defined in [14, 7, 5, 6, 9].

Oda [9, Theorem 3(c)] showed

$$\begin{aligned}(1.1)\qquad \pi^9_{40} = {} & \mathbb{Z}_2\{\sigma_9\delta_{16}\} \oplus \mathbb{Z}_2\{\sigma_9\bar{\mu}_{16}\sigma_{33}\} \oplus \mathbb{Z}_2\{\sigma_9\bar{\sigma}'_{16}\} \oplus \mathbb{Z}_2\{\delta_9\sigma_{33}\} \oplus \\ & \mathbb{Z}_{16}\{\alpha^{IV}_3\} \oplus G_1\{\bar{\nu}_9\nu_{17}\bar{\kappa}_{20}\}\end{aligned}$$

and

(1.2) $$\pi^{10}_{41} = \mathbb{Z}_8\{P\sigma^*_{21}\} \oplus \mathbb{Z}_{16}\{E\alpha^{IV}_3\} \oplus G_2\{\kappa^*_{10}, \delta_{10}\sigma_{34}\},$$

where G_1 is $\mathbb{Z}_2$ or 0, and G_2 is $(\mathbb{Z}_2)^2$ or $\mathbb{Z}_4$. We mention that equations

(1.3) $$P(\nu_{19}\bar{\kappa}_{22}) = \bar{\nu}_9\nu_{17}\bar{\kappa}_{20} \quad \text{and} \quad P(\nu_{21}\bar{\sigma}_{24}) = \delta_{10}\sigma_{34}$$

are obtained in [9, III-(10.3), Proposition 4.7(6)]. We denote by $[\chi_1, \chi_2]$ the Whitehead product of χ_1 and χ_2, and by $\iota_n \in \pi^n_n$ the homotopy class of the identity map on S^n. Since $PE^2\chi = \pm[\iota_n, \iota_n]\circ\chi$ for an element $\chi \in \pi^{2n-1}_k$ (see [14, Proposition 2.5]), we have $PE^{n+2}\chi = \pm[\iota_n, \iota_n]E^n\chi = \pm[\iota_n, E\chi]$ for an element $\chi \in \pi^{n-1}_k$. Then images of elements by P are frequently written as Whitehead products. For example, the equations of (1.3) are identical with

(1.4) $$[\iota_9, \nu_9\bar{\kappa}_{12}] = \bar{\nu}_9\nu_{17}\bar{\kappa}_{20} \quad \text{and} \quad [\iota_{10}, \nu_{10}\bar{\sigma}_{13}] = \delta_{10}\sigma_{34}.$$

Signs are unnecessary because $2\bar{\nu}_9\nu_{17}\bar{\kappa}_{20} = 0$ and $2\delta_{10}\sigma_{34} = 0$.

The purpose of this note is to determine the groups (1.1) and (1.2). Our results are stated as follows.

Theorem 1.1. (i) $[\iota_9, \nu_9\bar{\kappa}_{12}] = \bar{\nu}_9\nu_{17}\bar{\kappa}_{20} = 0$ *and*

$$\pi^9_{40} = \mathbb{Z}_2\{\sigma_9\delta_{16}\} \oplus \mathbb{Z}_2\{\sigma_9\bar{\mu}_{16}\sigma_{33}\} \oplus \mathbb{Z}_2\{\sigma_9\bar{\sigma}'_{16}\} \oplus \mathbb{Z}_2\{\delta_9\sigma_{33}\} \oplus \mathbb{Z}_{16}\{\alpha^{IV}_3\}.$$

(ii) $[\iota_{10}, \nu_{10}\bar{\sigma}_{13}] = \delta_{10}\sigma_{34} = 2\kappa^*_{10}$ *and*

$$\pi^{10}_{41} = \mathbb{Z}_8\{P\sigma^*_{21}\} \oplus \mathbb{Z}_{16}\{E\alpha^{IV}_3\} \oplus \mathbb{Z}_4\{\kappa^*_{10}\}.$$

To show Theorem 1.1, we use Toda's composition method which requires generators of homotopy groups. In Section 2, we describe some relations of generators. Section 3 gives details of our investigation concerning π^9_{40}. The structure of π^{10}_{41} is obtained in Section 4.

At the end of this section, we recall [14, Proposition 4.2]: there exists an exact sequence

$$\cdots \to \pi^{2n+1}_{k+2} \xrightarrow{P} \pi^n_k \xrightarrow{E} \pi^{n+1}_{k+1} \xrightarrow{H} \pi^{2n+1}_{k+1} \xrightarrow{P} \pi^n_{k-1} \xrightarrow{E} \pi^{n+1}_k \to \cdots,$$

called the EHP sequence, where H be the Hopf homomorphism.

2 Recollection of some relations.

We use notations in [14] and properties of Toda brackets freely. We know

(2.1) $2\eta_2 = [\iota_2, \iota_2]$, [14, Proposition 5.1];

(2.2) $2\nu_4 - E\nu' = \pm[\iota_4, \iota_4]$, [14, (5.8)];

(2.3) $\eta_3\nu_4 = \nu'\eta_6$, $\eta_4\nu_5 = (E\nu')\eta_7 = [\iota_4, \eta_4]$, [14, (5.9), (5.11)];

(2.4) $\nu_5\eta_8 = [\iota_5, \iota_5]$, [14, (5.10)];

(2.5) $2\sigma_8 - E\sigma' = \pm[\iota_8, \iota_8]$, [14, (5.16)];

(2.6) $\eta_9\sigma_{10} + \sigma_9\eta_{16} = [\iota_9, \iota_9]$, [14, Lemma 6.4, (7.1)];

(2.7) $\eta_7\sigma_8 = \bar{\nu}_7 + \varepsilon_7 + \sigma'\eta_{14}$, $\eta_9\sigma_{10} = \bar{\nu}_9 + \varepsilon_9$, [14, Lemma 6.4, (7.4)];

(2.8) $\eta_3\varepsilon_4 = \varepsilon_3\eta_{11}$, [14, (7.5)];

(2.9) $\varepsilon_4\nu_{12} = [\iota_4, \iota_4]\bar{\nu}_7$, [14, (7.13)];

(2.10) $\nu_6\bar{\nu}_9 = \nu_6\varepsilon_9 = 2\bar{\nu}_6\nu_{14} = [\iota_6, \nu_6^2]$, [14, (7.17), (7.18)];

(2.11) $2\sigma_{15}^2 = [\iota_{15}, \iota_{15}]$, [14, (10.10)];

(2.12) $\varepsilon_3\sigma_{11} = 0$, $\sigma_{11}\varepsilon_{18} = 0$, $\bar{\nu}_6\sigma_{14} = 0$, [14, Lemma 10.7];

(2.13) $\eta_6\kappa_7 = \bar{\varepsilon}_6$, $\kappa_9\eta_{23} = \bar{\varepsilon}_9$, [14, (10.23)];

(2.14) $\nu_5\sigma_8\nu_{15}^2 = \eta_5\bar{\varepsilon}_6$, $\varepsilon_3^2 = \varepsilon_3\bar{\nu}_{11} = \eta_3\bar{\varepsilon}_4 = \bar{\varepsilon}_3\eta_{18}$, [14, Lemma 12.10].

By (2.7) and (2.12), we have

(2.15) $$\eta_9\sigma_{10}^2 = (\bar{\nu}_9 + \varepsilon_9)\sigma_{17} = \bar{\nu}_9\sigma_{17} + \varepsilon_9\sigma_{17} = 0.$$

We recall [14, (7.19)]:

(2.16) $\sigma'\nu_{14} = x\nu_7\sigma_{10}$ for some odd integer x.

Then (2.5) gives $2\sigma_9\nu_{16} = x\nu_9\sigma_{12}$. Since $8\sigma_9\nu_{16} = 8\nu_9\sigma_{12} = 0$ (see [14, Theorem 7.3]), we have

(2.17) $$\nu_9\sigma_{12} = \pm 2\sigma_9\nu_{16}$$

and hence, by [14, (7.21)],

(2.18) $$\nu_{10}\sigma_{13} = 2\sigma_{10}\nu_{17} = [\iota_{10}, \eta_{10}], \quad \sigma_{11}\nu_{18} = [\iota_{11}, \iota_{11}].$$

Similarly, we obtain

$$\zeta_9\sigma_{20} = \pm 2\sigma_9\zeta_{16}$$

because $\sigma'\zeta_{14} = x\zeta_7\sigma_{18}$ for some odd integer x (see [14, Lemma 12.12]) and $8\sigma_9\zeta_{16} = 8\zeta_9\sigma_{20} = 0$ (see [14, Theorem 12.8]). The relation and [14, (12.25)] imply

$$\zeta_{10}\sigma_{21} = 2\sigma_{10}\zeta_{17} = [\iota_{10}, \mu_{10}]. \tag{2.19}$$

By [12, Propositions (2.2)(5), (2.2)(6)] and [14, (7.25)], we have

$$\eta_5\zeta_6 = 0, \quad \zeta_6\eta_{17} = \nu_6\mu_9 = 8[\iota_6, \iota_6]\sigma_{11}. \tag{2.20}$$

By (2.18) and (2.4), we have

$$\eta_{11}[\iota_{12}, \iota_{12}] = [\eta_{11}, \eta_{11}] = [\iota_{11}, \iota_{11}]\eta_{21}^2 = \sigma_{11}\nu_{18}\eta_{21}^2 = 0. \tag{2.21}$$

Toda brackets $\{\eta_n, \nu_{n+1}, \sigma_{n+4}\}$ for $n = 10, 11$ are well defined by (2.3) and (2.18). By the fact that $\{\eta_{10}, \nu_{11}, \sigma_{14}\} \subset \pi_{22}^{10} = \{P(\nu_{21})\}$ (see [14, Theorem 7.6]), we have $E\{\eta_{10}, \nu_{11}, \sigma_{14}\} = 0$. Hence, by [14, (1.15), Proposition 1.3], we have

$$0 = -E\{\eta_{10}, \nu_{11}, \sigma_{14}\} \in \{\eta_{11}, \nu_{12}, \sigma_{15}\} \bmod \pi_{16}^{11} \circ \sigma_{16} + \eta_{11} \circ \pi_{23}^{12}.$$

By the fact that $\pi_{16}^{11} = 0$ and $\pi_{23}^{12} = \{[\iota_{12}, \iota_{12}], \zeta_{12}\}$ (see [14, Proposition 5.4, Theorem 7,4]) and relations (2.21) and (2.20), we have $\pi_{16}^{11} \circ \sigma_{16} + \eta_{11} \circ \pi_{23}^{12} = \{\eta_{11}[\iota_{12}, \iota_{12}], \eta_{11}\zeta_{12}\} = 0$. Hence, we obtain

$$\{\eta_{11}, \nu_{12}, \sigma_{15}\} = 0. \tag{2.22}$$

By [14, Lemma 5.12 and Remark],

$$\{\eta_n, \nu_{n+1}, \eta_{n+4}\} = \nu_n^2 \text{ for } n \geq 5, \tag{2.23}$$

and hence, by $\bar{\sigma}_6 \in \{\nu_6, \varepsilon_9 + \bar{\nu}_9, \sigma_{17}\}_1$ (see [14, p. 138]), (2.7), (2.16) and (2.18),

$$\begin{aligned}\eta_5\bar{\sigma}_6 &\in \eta_5 \circ \{\nu_6, \varepsilon_9 + \bar{\nu}_9, \sigma_{17}\} = \eta_5 \circ \{\nu_6, \eta_9\sigma_{10}, \sigma_{17}\} \\ &= -\{\eta_5, \nu_6, \eta_9\sigma_{10}\} \circ \sigma_{18} \supset -\{\eta_5, \nu_6, \eta_9\} \circ \sigma_{11}^2 = -\nu_5^2\sigma_{11}^2 \\ &= -x\nu_5(E\sigma')\nu_{15}\sigma_{18} = 0 \bmod \eta_5\nu_6 \circ \pi_{25}^9 + \eta_5 \circ \pi_{18}^6 \circ \sigma_{18}\end{aligned}$$

for some odd integer x. Here, $\eta_5\nu_6 = 0$ by (2.3); $\eta_5 \circ \pi_{18}^6 = \{\eta_5[\iota_6, \iota_6]\sigma_{11}\}$ by [14, Theorem 7.6]. By (2.2), (2.4) and by relations $\eta_3^3 = 2\nu'$ and $\nu'\nu_6 = 0$ (see [14, (5.3), Proposition 5.11]), we have $\eta_5[\iota_6, \iota_6] = [\eta_5, \eta_5] = [\iota_5, \iota_5]\eta_9^2 =$

$\nu_5\eta_8^3 = \nu_5(2E^5\nu') = 2\nu_5(E^5\nu') = (E^2\nu')(2\nu_8) = 2E^2(\nu'\nu_6) = 0$. This implies $\eta_5 \circ \pi_{18}^6 = 0$. Thus

$$\eta_5\bar{\sigma}_6 = 0. \tag{2.24}$$

We know [7, Lemma 16.1, p. 50]:

$$\pi_{30}^{10} = \mathbb{Z}_8\{\bar{\kappa}_{10}\} \oplus \mathbb{Z}_8\{\beta'\}, \tag{2.25}$$

where

$$E\beta' = \theta'\varepsilon_{23}, \quad H\beta' = \zeta_{19} \quad \text{and} \quad 2\beta' = \pm[\iota_{10}, \zeta_{10}]. \tag{2.26}$$

By [12, Proposition (2.6)(2)], $\nu_8\theta' = (E\sigma')\varepsilon_{15}$ or $(E\sigma')\bar{\nu}_{15}$. Then $E(\nu_7\beta') = \nu_8\theta'\varepsilon_{23}$ equals $(E\sigma')\varepsilon_{15}^2$ or $(E\sigma')\bar{\nu}_{15}\varepsilon_{23}$. Here,

$$\varepsilon_9^2 = \eta_9\bar{\varepsilon}_{10} = \nu_9\sigma_{12}\nu_{19}^2 = 0 \tag{2.27}$$

by (2.14) and (2.18). Furthermore, $\bar{\nu}_6\varepsilon_{14} = 0$ by [12, Proposition (2.8)(2)]. These imply $E(\nu_7\beta') = 0$ and

$$\nu_7\beta' = 0 \tag{2.28}$$

because $E\colon \pi_{30}^7 \to \pi_{31}^8$ is a monomorphism. On the other hand, $E(\beta'\sigma_{30}) = \theta'\varepsilon_{23}\sigma_{31} = 0$ by (2.12) and $H(\beta'\sigma_{30}) = \zeta_{19}\sigma_{30} = 0$ by (2.19). Then $\beta'\sigma_{30} \in \ker E = \mathbb{Z}_4\{P\nu_{21}^*\}$ (see [9, Theorem 2(a)]) and $\beta'\sigma_{30} \in \ker H$. Since $HP\nu_{21}^* = \pm 2\nu_{19}^*$ (see [14, Proposition 2.7]) and ν_{19}^* is of order 8 (see [14, Theorem 12.22]), we have $\ker E \cap \ker H = 0$, that is,

$$\beta'\sigma_{30} = 0. \tag{2.29}$$

We show the following lemma.

Lemma 2.1. $\lambda'\eta_{29}\sigma_{30} = \xi'\eta_{29}\sigma_{30} = 0$.

Proof. First, we show the relation $\lambda'\eta_{29}\sigma_{30} = 0$. Since $H(\lambda'\sigma_{29}) = \varepsilon_{21}\sigma_{29} = 0$ from [11, Proposition 4(3)] and [14, Lemma 10.7], we have $\lambda'\sigma_{29} \in E\pi_{35}^{10}$. Moreover, by the fact that $E\pi_{35}^{10} = \{\sigma_{11}\xi_{18}, \sigma_{11}\nu_{18}^*, \mu_{3,11}, \eta_{11}\bar{\mu}_{12}\sigma_{29}\}$ and $\pi_{25}^s = \{\mu_3, \eta\bar{\mu}\sigma\}$ (see [6, Theorem 1(a)]) and the relation $E^2(\lambda'\sigma_{29}) = 2\lambda\sigma_{31} = 0$ (see [9, III-Proposition 2.2(2)]), we have $\lambda'\sigma_{29} \in \{\sigma_{11}\xi_{18}, \sigma_{11}\nu_{18}^*\}$. Thus, by relations $\xi_{13}\eta_{31} = P(\sigma_{27})$ (see [14, p. 166]) and $\nu^*_{16}\eta_{34} = 0$ (see [12, Proposition 2.20(3)]), we obtain

$$\lambda'\eta_{29}\sigma_{30} = \lambda'\sigma_{29}\eta_{36} \in \{\sigma_{11}\xi_{18}, \sigma_{11}\nu^*_{18}\} \circ \eta_{36} = 0.$$

Next, we show the relation $\xi'\eta_{29}\sigma_{30} = 0$. By relations $\xi'\sigma_{29} \equiv \sigma_{11}\nu^*_{18} \mod 2\sigma_{11}\nu^*_{18}$ (see [9, II-Proposition 2.1(6)]) and $\nu^*_{16}\eta_{34} = 0$, we obtain

$$\xi'\eta_{29}\sigma_{30} = \xi'\sigma_{29}\eta_{36} \equiv \sigma_{11}\nu^*_{18}\eta_{36} = 0 \mod 2\sigma_{11}\nu^*_{18}\eta_{36} = 0.$$

This completes the proof. □

We also show the following lemma.

Lemma 2.2. $\bar{\mu}_3\kappa_{20} = 0$.

Proof. Since $\bar{\mu}_3 \in \{\mu_3, 2\iota_{12}, 8\sigma_{12}\}_1$ (see [14, p. 136]), we have

$$\bar{\mu}_3\kappa_{20} \in \{\mu_3, 2\iota_{12}, 8\sigma_{12}\}_1 \circ \kappa_{20} = \mu_3 \circ E\{2\iota_{11}, 8\sigma_{11}, \kappa_{18}\} \subset \mu_3 \circ E\pi^{11}_{33}.$$

Then $\pi^{11}_{33} = \{\sigma_{11}\rho_{18}, \varepsilon_{11}\kappa_{19}, \nu_{11}\bar{\sigma}_{14}\}$ (see [5, Theorem B]) gives

$$\bar{\mu}_3\kappa_{20} \in \{\mu_3\sigma_{12}\rho_{19}, \mu_3\varepsilon_{12}\kappa_{20}, \mu_3\nu_{12}\bar{\sigma}_{15}\}.$$

By [9, III-Proposition 2.6(1)], the element $\mu_3\sigma_{12}\rho_{19}$ is equal to 0. Since relations $\eta_3\mu_4 = \mu_3\eta_{12}$, $\mu_3\varepsilon_{12} \equiv \eta_3\mu_4\sigma_{13} \mod 2\bar{\varepsilon}'$ (see [12, Propositions (2.2)(2), (2.13)(7)]), and κ_{10} is of order 2 (see [14, Theorem 10.3]), we have

$$\mu_3\varepsilon_{12}\kappa_{20} \equiv \eta_3\mu_4\sigma_{13}\kappa_{20} = \mu_3\eta_{12}\sigma_{13}\kappa_{20} \mod (2\bar{\varepsilon}')\kappa_{20} = \bar{\varepsilon}'(2\kappa_{20}) = 0.$$

Then $\mu_3\varepsilon_{12}\kappa_{20} = \mu_3\eta_{12}\nu_{13}E^3\lambda = 0$, as $\sigma_{10}\kappa_{17} = \nu_{10}\lambda$ (see [9, I-Proposition 3.1(1)]) and by (2.3). By relations $\mu_3\nu_{12} = \nu'\eta_6\varepsilon_7$ (see [12, Proposition (2.2)(4)]), (2.8) and (2.24), we have $\mu_3\nu_{12}\bar{\sigma}_{15} = \nu'\eta_6\varepsilon_7\bar{\sigma}_{15} = \nu'\varepsilon_6\eta_{14}\bar{\sigma}_{15} = 0$. Thus $\bar{\mu}_3\kappa_{20} = 0$. □

3 Proof of Theorem 1.1(i).

We recall the definition and a property of $\bar{\bar{\sigma}}'_6 \in \pi^6_{30}$ from [6, (3.8)].

(3.1) $$\bar{\bar{\sigma}}'_6 \in \{\nu_6, \eta_9, \bar{\sigma}_{10}\}_3, \quad 2\bar{\bar{\sigma}}'_6 = 0.$$

We shall show the following theorem.

Theorem 3.1. $\sigma'\delta_{14} \equiv \bar{\nu}_7\nu_{15}\bar{\kappa}_{18} \bmod \sigma'\bar{\bar{\sigma}}'_{14}, \sigma'\bar{\mu}_{14}\sigma_{31}$.

By the equation (2.5), $E^2(\sigma'\delta_{14})$, $E^2(\sigma'\bar{\bar{\sigma}}'_{14})$ and $E^2(\sigma'\bar{\mu}_{14}\sigma_{31})$ are equal to $2\sigma_9\delta_{16}$, $2\sigma_9\bar{\bar{\sigma}}'_{16}$ and $2\sigma_9\bar{\mu}_{16}\sigma_{33}$, respectively. By (1.1) and (3.1), these elements are 0. Then Theorem 3.1 gives $\bar{\nu}_9\nu_{17}\bar{\kappa}_{20} = 0$: it is suffice to show Theorem 3.1 to obtain Theorem 1.1(i).

Toda brackets $\{\nu_6, \eta_9, \zeta_{10}\}_1$, $\{\eta_{10}, \zeta_{11}, \sigma_{22}\}$ and $\{\zeta_{11}, \sigma_{22}, 2\sigma_{29}\}$ are well defined: $\nu_6\eta_9 = 0$ by (2.4); $\eta_5\zeta_6 = 0$ by (2.20); $\zeta_{11}\sigma_{22} = 0$ by (2.19); and $2\sigma_{16}^2 = 0$ by (2.11).

Lemma 3.2. $\langle \nu, \eta, \zeta \rangle = 0$.

Proof. By relations (2.23) and $\nu\zeta = 0$ (see [14, Theorem 14.1.ii)]), we obtain $\eta \circ \langle \nu, \eta, \zeta \rangle = \langle \eta, \nu, \eta \rangle \circ \zeta = \nu^2\zeta = 0$. Hence, we obtain the equation $\langle \nu, \eta, \zeta \rangle = 0$ since $\eta_* : \pi_{16}^s \to \pi_{17}^s$ is a monomorphism from the fact that $\pi_{16}^s = \{\eta^*, \eta\rho\}$ and $\pi_{17}^s = \{\eta\eta^*, \eta^2\rho, \nu\kappa, \bar{\mu}\}$ (see [14, p. 189]). □

Lemma 3.3. $\{\nu_6, \eta_9, \zeta_{10}\} \ni \zeta' \bmod \eta_6\bar{\varepsilon}_7, 2\zeta'$ *and* $\{\nu_7, \eta_{10}, \zeta_{11}\} \ni \sigma'\eta_{14}\varepsilon_{15} \bmod \eta_7\bar{\varepsilon}_8$.

Proof. By [14, Proposition 5.9, Theorem 7.7], we have $\pi_{11}^6 = \{P\iota_{13}\}$, $\pi_{12}^7 = 0$, $\pi_{n+13}^n = \{\sigma_n\nu_{n+7}^2\}$ for $n = 9, 10$ and $E\pi_{21}^8 = \pi_{22}^9$. The last equation implies that the Toda bracket $\{\nu_6, \eta_9, \zeta_{10}\}$ is equal to $\{\nu_6, \eta_9, \zeta_{10}\}_1$. The indeterminacy of $\{\nu_6, \eta_9, \zeta_{10}\}$ is $\nu_6 \circ \pi_{22}^9 + \pi_{11}^6 \circ \zeta_{11} = \{\nu_6\sigma_9\nu_{16}^2, P\zeta_{13}\}$. Here, $\nu_6\sigma_9\nu_{16}^2 = \eta_6\bar{\varepsilon}_7$ by (2.14) and $P\zeta_{13} = \pm 2\zeta'$ by [14, (12.4)]. The indeterminacy of $\{\nu_7, \eta_{10}, \zeta_{11}\}$ is obtained in the same way. By (2.4), we have $P\iota_{11} = \nu_5\eta_8$ and hence the kernel of $P\colon \pi_{11}^{11} \to \pi_9^5 = \mathbb{Z}_2\{\nu_5\eta_8\}$ (see [14, Proposition 5.8]) is generated by $2\iota_{11}$. Then an equation

$$H\{\nu_6, \eta_9, \zeta_{10}\}_1 = -P^{-1}(\nu_5\eta_8) \circ \zeta_{11} = \{a\zeta_{11} \mid a \text{ is odd}\}$$

is obtained by the use of [14, Proposition 2.6]. Since $\pi_{22}^{11} = \mathbb{Z}_8\{\zeta_{11}\}$, $\pi_{22}^6 = \mathbb{Z}_8\{\zeta'\} \oplus \mathbb{Z}_2\{\mu_6\sigma_{15}\} \oplus \mathbb{Z}_2\{\eta_6\bar{\varepsilon}_7\}$, and $H\zeta' \equiv \zeta_{11} \bmod 2\zeta_{11}$ (see [14, Theorems 7.4, 12.6, Lemma 12.1]), we have

$$H^{-1}\{a\zeta_{11} \mid a \text{ is odd}\} = \{b\zeta' + c\mu_6\sigma_{15} + d\eta_6\bar{\varepsilon}_7 \mid b = 1,3,5,7,\ c = 0,1,\ d = 0,1\}.$$

This yields

$$\{\nu_6, \eta_9, \zeta_{10}\} \ni \zeta' + c\mu_6\sigma_{15}$$

for $c = 0$ or 1, because the indeterminacy of $\{\nu_6, \eta_9, \zeta_{10}\}$ is $\{\eta_6\bar{\varepsilon}_7, 2\zeta'\}$ as above. Therefore, relations $E\zeta' = \sigma'\eta_{14}\varepsilon_{15}$ and $2\varepsilon_{15} = 0$ (see [14, (12.4), Theorem 7.1]) give

$$\{\nu_7, \eta_{10}, \zeta_{11}\} \supset -E\{\nu_6, \eta_9, \zeta_{10}\} \ni -E\zeta' - c\mu_7\sigma_{16} = \sigma'\eta_{14}\varepsilon_{15} + c\mu_7\sigma_{16}.$$

By (2.5), we have $E^2(\sigma'\eta_{14}\varepsilon_{15}) = \sigma_9\eta_{16} \circ 2\varepsilon_{17} = 0$ and hence $\langle \nu, \eta, \zeta \rangle \ni c\mu\sigma$. Since $\mu\sigma = \sigma\mu \neq 0$ (see [14, p. 156, Theorem 12.16]), Lemma 3.2 leads to $c = 0$. □

Lemma 3.4. $\nu_7 \circ \{\eta_{10}, \zeta_{11}, \sigma_{22}\} = 0$ *and* $\{\eta_{10}, \zeta_{11}, \sigma_{22}\} \circ \sigma_{30} = 0$.

Proof. By Lemma 3.3, we have

$$\nu_7 \circ \{\eta_{10}, \zeta_{11}, \sigma_{22}\} = -(\{\nu_7, \eta_{10}, \zeta_{11}\} \circ \sigma_{23}) \subset \{\sigma'\eta_{14}\varepsilon_{15}, \eta_7\bar{\varepsilon}_8\} \circ \sigma_{23}.$$

Then (2.12) and

$$\bar{\varepsilon}_3\sigma_{18} = 0, \quad [6, (2.4)], \tag{3.2}$$

give the first half. By (2.25), $\{\eta_{10}, \zeta_{11}, \sigma_{22}\} \subset \mathbb{Z}_8\{\bar{\kappa}_{10}\} \oplus \mathbb{Z}_8\{\beta'\}$. We have $\nu_7\bar{\kappa}_{10}$ is of order 8 by [6, Theorem 1.1(a)], and $\nu_7\beta' = 0$ by (2.28). Then the first half shows $\{\eta_{10}, \zeta_{11}, \sigma_{22}\} \subset \mathbb{Z}_8\{\beta'\}$. Hence, by (2.29), we obtain the second half. □

By Lemma 3.4, we may consider the Jacobi identity:

$$\begin{aligned}\{\{\nu_7, \eta_{10}, \zeta_{11}\}, \sigma_{23}, 2\sigma_{30}\} + \{\nu_7, \{\eta_{10}, \zeta_{11}, \sigma_{22}\}, 2\sigma_{30}\}& \\ + \{\nu_7, \eta_{10}, \{\zeta_{11}, \sigma_{22}, 2\sigma_{29}\}\} \ni 0.&\end{aligned} \tag{3.3}$$

As a preparation, we show the next lemma, which is followed by observations of Toda brackets in (3.3).

Lemma 3.5. $\pi^7_{31} \circ 2\sigma_{31} = 0$ *and* $\nu_7 \circ \pi^{10}_{38} = 0$.

Proof. Since $\pi^7_{31} \cong (\mathbb{Z}_2)^7$ (see [6, Theorem 1.1(b)]), the first half of the lemma is obvious. The group π^{10}_{38} is obtained in [9, Theorem 2(b)]:

$$\pi^{10}_{38} = \mathbb{Z}_8\{F^{(1)}_1\} \oplus \mathbb{Z}_2\{\sigma^4_{10}\} \oplus \mathbb{Z}_2\{\sigma_{10}\nu^*_{17}\nu_{35}\} \oplus \mathbb{Z}_2\{\bar{\nu}_{10}\bar{\kappa}_{18}\} \oplus \mathbb{Z}_2\{\varepsilon_{10}\bar{\kappa}_{18}\},$$

where $F_1^{(1)} \in \{F_1, 8\iota_{30}, 2\sigma_{30}\}_1$ for $F_1 \in \{P\iota_{21}, \eta_{19}^2, \varepsilon_{21}\}_1$ (see [8, Definition 3.18]). Since $F_1 = x\beta'$ for some odd integer x (see [9, I-(2.1)]), (2.28) and the first half lead to

$$\nu_7 F_1^{(1)} \in \nu_7 \circ \{x\beta', 8\iota_{30}, 2\sigma_{30}\} = -\{\nu_7, x\beta', 8\iota_{30}\} \circ 2\sigma_{31} \subset \pi_{31}^7 \circ 2\sigma_{31} = 0.$$

By (2.16) and (2.18),

$$\nu_7\sigma_{10}^4 = \sigma'\nu_{14}\sigma_{17}^3 = 0.$$

Since $\nu_{16}^* \in \{\sigma_{16}, 2\sigma_{23}, \nu_{30}\}_1$ (see [14, p. 153]) and

$$\{\nu_{13}, \sigma_{16}, 2\sigma_{23}\} = \xi_{13} + y(\lambda + 2\xi_{13})$$

for some odd integer y (see [9, I-Proposition 3.4(8)]), relations (2.11), (2.18) and $\xi_{12}\nu_{30} = \sigma_{12}^3$ (see [9, II-Proposition 2.1(2)]) yield

$$\begin{aligned}\nu_{13}\nu_{16}^* &\in \nu_{13} \circ \{\sigma_{16}, 2\sigma_{23}, \nu_{30}\} \\ &= -\{\nu_{13}, \sigma_{16}, 2\sigma_{23}\} \circ \nu_{31} \\ &= (-\xi_{13} - y(\lambda + 2\xi_{13}))\nu_{31} = \sigma_{13}^3 - y\lambda\nu_{31},\end{aligned}$$

and hence, by (2.16), (2.18), $\sigma' E\lambda \equiv 0 \bmod 4E^2\phi''$ (see [9, II-(6.4)]) and $2\nu_{32}^2 = 0$ (see [14, Proposition 5.11]), we obtain

$$\begin{aligned}\nu_7\sigma_{10}\nu_{17}^*\nu_{35} = \sigma'\nu_{14}\nu_{17}^*\nu_{35} = \sigma'(\sigma_{14}^3 - y(E\lambda)\nu_{32})\nu_{35} = \sigma'(E\lambda)\nu_{32}^2 \equiv 0 \\ \bmod 4E^2\phi'' \circ \nu_{32}^2 = E^2\phi'' \circ 4\nu_{32}^2 = 0.\end{aligned}$$

By (2.10),

$$\nu_7\bar{\nu}_{10}\bar{\kappa}_{18} = \nu_7\varepsilon_{10}\bar{\kappa}_{18} = 0.$$

Therefore, every element which is composite of generators of π_{38}^{10} and ν_7 is 0. □

Lemma 3.6. $\{\nu_7, \chi, 2\sigma_{30}\} = 0$ *for* $\chi \in \{\eta_{10}, \zeta_{11}, \sigma_{22}\}$.

Proof. The indeterminacy of $\{\nu_7, \chi, 2\sigma_{30}\}$ is trivial by Lemma 3.5. We may take $\chi = x\beta'$ for some integer x from the proof of Lemma 3.4. By [9, Theorem 3(c)] and (2.29), we have

$$\{\nu_7, x\beta', 2\sigma_{30}\} = \{\nu_7, x\beta', \sigma_{30}\} \circ 2\iota_{38} \in 2\pi_{38}^7 = \{2\alpha_3'''\},$$

and hence $\{\nu_7, x\beta', 2\sigma_{30}\} = 2y\alpha'''_3$ for some integer y. By (2.26) and $E^2\theta' = 0$ (see [14, p. 80]), we have

$$2yE^5\alpha'''_3 = E^5\{\nu_7, x\beta', 2\sigma_{30}\} \in -\{\nu_{12}, 0, 2\sigma_{35}\} = \nu_{12} \circ \pi^{15}_{43} + \pi^{12}_{36} \circ 2\sigma_{36}.$$

The Toda bracket $-\{\nu_{12}, 0, 2\sigma_{35}\}$ vanishes because $\pi^{12}_{36} \cong (\mathbb{Z}_2)^4$ (see [6, Theorem 1.1(b)]), $\nu_{12} \circ \pi^{15}_{43} = \{\nu_{12}\varepsilon_{15}\bar{\kappa}_{23}\}$ (see [9, Theorem 2(b)]), and $\nu_{12}\varepsilon_{15} = 0$ by (2.10). On the other hand, $2yE^5\alpha'''_3 \equiv 4yE^3\alpha^{IV}_3 \bmod E^3\pi^9_{33} \circ \sigma_{36}$ (see [9, III-Proposition 3.4]). Here, the group $E^3\pi^9_{33} \circ \sigma_{36}$ has three generators $\delta_{12}\sigma_{36}$, $\bar{\mu}_{12}\sigma^2_{29}$ and $\bar{\sigma}'_{12}\sigma_{36}$ (see [6, pp. 34–35]), each of which is 0 by [9, III-Propositions 2.6(1), 2.6(5)]:

$$\bar{\mu}_3\sigma^2_{20} = 0, \quad \bar{\sigma}'_7\sigma_{31} = 0 \quad \text{and} \quad \delta_{11}\sigma_{35} = 0. \tag{3.4}$$

Therefore, we have $4yE^3\alpha^{IV}_3 = 0$. This yields $\{\nu_7, x\beta', 2\sigma_{30}\} = 2y\alpha'''_3 = 0$, since $E^3\alpha^{IV}_3$ and α'''_3 are of order 16 and 8 (see [9, Theorem 3(c)]), respectively. □

Lemma 3.7. $\{\chi, \sigma_{23}, 2\sigma_{30}\} = \sigma'\eta_{14}\phi_{15}$ *for* $\chi \in \{\nu_7, \eta_{10}, \zeta_{11}\}$.

Proof. By Lemma 3.3, $\chi = \sigma'\eta_{14}\varepsilon_{15} + x\eta_7\bar{\varepsilon}_8$ for some integer x. By [10, Theorem 2(2)], we have $\sigma'\eta_{14}\varepsilon_{15} = \sigma'\eta^2_{14}\sigma_{16} + \eta_7\bar{\varepsilon}_8$. Then $\chi = \sigma'\eta^2_{14}\sigma_{16} + (x+1)\eta_7\bar{\varepsilon}_8$ and

$$\{\chi, \sigma_{23}, 2\sigma_{30}\} \subset \{\sigma'\eta^2_{14}\sigma_{16}, \sigma_{23}, 2\sigma_{30}\} + \{(x+1)\eta_7\bar{\varepsilon}_8, \sigma_{23}, 2\sigma_{30}\}.$$

By (2.14) and (2.18),

$$\{\eta_7\bar{\varepsilon}_8, \sigma_{23}, 2\sigma_{30}\} = \{\nu_7\sigma_{10}\nu^2_{17}, \sigma_{23}, 2\sigma_{30}\} \supset \nu_7 \circ \{\sigma_{10}\nu^2_{17}, \sigma_{23}, 2\sigma_{30}\} \bmod \nu_7\sigma_{10}\nu^2_{17} \circ \pi^{23}_{38} + \pi^7_{31} \circ 2\sigma_{31}.$$

By Lemma 3.5, we have $\nu_7 \circ \{\sigma_{10}\nu^2_{17}, \sigma_{23}, 2\sigma_{30}\} \subset \nu_7 \circ \pi^{10}_{38} = 0$ and

$$\nu_7\sigma_{10}\nu^2_{17} \circ \pi^{23}_{38} + \pi^7_{31} \circ 2\sigma_{31} \subset \nu_7 \circ \pi^{10}_{38} + \pi^7_{31} \circ 2\sigma_{31} = 0.$$

These give $\{(x+1)\eta_7\bar{\varepsilon}_8, \sigma_{23}, 2\sigma_{30}\} = 0$. On the other hand, we have

$$\{\sigma'\eta^2_{14}\sigma_{16}, \sigma_{23}, 2\sigma_{30}\} \supset \sigma'\eta_{14} \circ \{\eta_{15}\sigma_{16}, \sigma_{23}, 2\sigma_{30}\} \ni \sigma'\eta_{14}\phi_{15} \bmod \sigma'\eta^2_{14}\sigma_{16} \circ \pi^{23}_{38} + \pi^7_{31} \circ 2\sigma_{31}$$

since $\{\eta_9\sigma_{10}, \sigma_{17}, 2\sigma_{24}\} \ni \phi_9$ (see [9, I-Proposition 3.4(5)]). Lemma 3.5, (2.14) and the above equation $\sigma'\eta_{14}\varepsilon_{15} = \sigma'\eta_{14}^2\sigma_{16} + \eta_7\bar{\varepsilon}_8$ yield

$$\sigma'\eta_{14}^2\sigma_{16} \circ \pi_{38}^{23} + \pi_{31}^7 \circ 2\sigma_{31} = (\sigma'\eta_{14}\varepsilon_{15} - \nu_7\sigma_{10}\nu_{17}^2) \circ \pi_{38}^{23} = \sigma'\eta_{14}\varepsilon_{15} \circ \pi_{38}^{23}.$$

Then the indeterminacy is trivial because $\pi_{38}^{23} = \{\rho_{23}, \bar{\varepsilon}_{23}\}$ (see [14, Theorem 10.10]), $\varepsilon_5\rho_{13} = 0$ (see [9, I-Proposition 3.1(4)]), and $\eta_3\varepsilon_4\bar{\varepsilon}_{12} = \varepsilon_3\eta_{11}\bar{\varepsilon}_{12} = 0$ by (2.8) and (2.27). Therefore, we have $\{\sigma'\eta_{14}^2\sigma_{16}, \sigma_{23}, 2\sigma_{30}\} = \sigma'\eta_{14}\phi_{15}$ and this completes the proof. □

Lemmas 3.6 and 3.7 simplify (3.3) as follows.

$$(3.5) \qquad \sigma'\eta_{14}\phi_{15} \in \{\nu_7, \eta_{10}, \{\zeta_{11}, \sigma_{22}, 2\sigma_{29}\}\}.$$

Next, we investigate the Toda bracket $\{\nu_7, \eta_{10}, \{\zeta_{11}, \sigma_{22}, 2\sigma_{29}\}\}$. The following lemma extends [2, Lemma 3.5].

Lemma 3.8. $\{\zeta_{11}, \sigma_{22}, 2\sigma_{29}\} = \nu_{11}^2\bar{\kappa}_{17} \pm 2\tau''' + x\sigma_{11}\bar{\sigma}_{18}$ *for* $x = 0$ *or* 1.

Proof. We consider an inclusion

$$\{\zeta_{11}, 2\sigma_{22}, \sigma_{29}\} \supset \{\zeta_{11}, \sigma_{22}, 2\sigma_{29}\} \bmod \zeta_{11} \circ \pi_{37}^{22} + \pi_{30}^{11} \circ \sigma_{30}.$$

The groups $\pi_{37}^{22} = \{\rho_{22}, \bar{\varepsilon}_{22}\}$ and $\pi_{30}^{11} = \{\lambda'\eta_{29}, \xi'\eta_{29}, \bar{\sigma}_{11}, \bar{\zeta}_{11}\}$ are obtained in [14, Theorems 10.10, 12.23]. Since $\zeta_{11}\rho_{22} \equiv 0 \bmod 8\pi_{37}^{11}$ (see [9, I-Proposition 3.5(7)]) and $\pi_{37}^{11} \cong \mathbb{Z}_8 \oplus (\mathbb{Z}_2)^4$ (see [9, Theorem 1(b)]), we have $\zeta_{11}\rho_{22} = 0$. We also have $\zeta_{11}\bar{\varepsilon}_{22} = \zeta_{11}\eta_{22}\kappa_{23} = 0$ by (2.13) and (2.20). Thus $\zeta_{11} \circ \pi_{37}^{22} = 0$. Moreover, we obtain $\pi_{30}^{11} \circ \sigma_{30} = 0$: $\lambda'\eta_{29}\sigma_{30} = \xi'\eta_{29}\sigma_{30} = 0$ by Lemma 2.1; $\bar{\sigma}_{10}\sigma_{29} = 0$ by [9, I-Proposition 6.4(6)]; and $\bar{\zeta}_{11}\sigma_{30} = 0$ by [9, I-(8.14)]. This implies that $\{\zeta_{11}, \sigma_{22}, 2\sigma_{29}\}$ equals $\{\zeta_{11}, 2\sigma_{22}, \sigma_{29}\}$ and consists of one element. Therefore, we examine the Toda bracket $\{\zeta_{11}, 2\sigma_{22}, \sigma_{29}\}$ below. By [9, the proof of I-(8.22)], we have $H\{\zeta_{10}, 2\sigma_{21}, \sigma_{28}\}_1 = H\tau''$ and hence

$$\{\zeta_{10}, 2\sigma_{21}, \sigma_{28}\}_1 \subset \tau'' + \ker H = \tau'' + \{2\tau'', \sigma_{10}\bar{\sigma}_{17}, \bar{\kappa}_{10}\nu_{30}^2, \nu_{10}^2\bar{\kappa}_{16}, \eta_{10}\mu_{3,11}\}$$

by [9, I-(8.12), (8.13)]. Moreover, by a relation $\bar{\kappa}_{11}\nu_{31}^2 = \nu_{11}^2\bar{\kappa}_{17}$ (see [9, I-Proposition 6.4(7)]), we obtain

$$\begin{aligned}\{\zeta_{11}, 2\sigma_{22}, \sigma_{29}\} &= -E\{\zeta_{10}, 2\sigma_{21}, \sigma_{28}\}_1 \\ &\in -E\tau'' + \{2E\tau'', \sigma_{11}\bar{\sigma}_{18}, \nu_{11}^2\bar{\kappa}_{17}, \eta_{11}\mu_{3,12}\}.\end{aligned}$$

By [9, I-(8.18), Theorem 1(b)], $2\tau''' = -E\tau''$, τ''' is of order 8, and $\sigma_{11}\bar{\sigma}_{18}$, $\nu_{11}^2\bar{\kappa}_{17}$, $\eta_{11}\mu_{3,12}$ are of order 2. Then $\{\zeta_{11}, 2\sigma_{22}, \sigma_{29}\}$ is written as

$$\{\zeta_{11}, 2\sigma_{22}, \sigma_{29}\} = \pm 2\tau''' + a\sigma_{11}\bar{\sigma}_{18} + b\nu_{11}^2\bar{\kappa}_{17} + c\eta_{11}\mu_{3,12}$$

with coefficients a, b, c, each of which is 0 or 1. Since $\sigma_{14}\bar{\sigma}_{21} = 0$ and $E^8(2\tau''') = -E^9\tau'' = 4E^7\tau^{IV} = 0$ (see [9, I-Proposition 6.4(10), (8.22), (8.24)]), we have

$$\langle\zeta, 2\sigma, \sigma\rangle = b\nu^2\bar{\kappa} + c\eta\mu_{3,*} \in \pi_{26}^s.$$

Here, $\pi_{26}^s = \mathbb{Z}_2\{\nu^2\bar{\kappa}\} \oplus \mathbb{Z}_2\{\eta\mu_{3,*}\}$ by [9, Theorem 1(b)] and $\langle\zeta, 2\sigma, \sigma\rangle = \langle\sigma, 2\sigma, \zeta\rangle = \nu^2\bar{\kappa}$ by [2, Theorem 1]. Thus $b = 1$ and $c = 0$: the proof is completed. □

Lemma 3.9. $\{\nu_7, \eta_{10}, \chi\} \equiv \bar{\nu}_7\nu_{15}\bar{\kappa}_{18} \bmod \sigma'\bar{\bar{\sigma}}'_{14}$ *for* $\chi \in \{\zeta_{11}, \sigma_{22}, 2\sigma_{29}\}$.

Proof. By Lemma 3.8, χ has the form $\nu_{11}^2\bar{\kappa}_{17} \pm 2\tau''' + x\sigma_{11}\bar{\sigma}_{18}$ for $x = 0$ or 1. Then there is an inclusion

$$\{\nu_7, \eta_{10}, \chi\} \subset \{\nu_7, \eta_{10}, \nu_{11}^2\bar{\kappa}_{17}\} + \{\nu_7, \eta_{10}, 2\tau'''\} + \{\nu_7, \eta_{10}, x\sigma_{11}\bar{\sigma}_{18}\}.$$

Notice that $\nu_7 \circ \pi_{38}^{10} = 0$, $\pi_{12}^7 = 0$ and $\pi_{19}^7 = 0$ by Lemma 3.5 and [14, Proposition 5.9, Theorem 7.6]. Since

$$\bar{\nu}_6 \in \{\nu_6, \eta_9, \nu_{10}\}, \quad [14, \text{p. } 53], \tag{3.6}$$

we have

$$\bar{\nu}_7\nu_{15}\bar{\kappa}_{18} \in \{\nu_7, \eta_{10}, \nu_{11}\} \circ \nu_{15}\bar{\kappa}_{18} \subset \{\nu_7, \eta_{10}, \nu_{11}^2\bar{\kappa}_{17}\} \bmod \nu_7 \circ \pi_{38}^{10} + \pi_{12}^7 \circ \nu_{12}^2\bar{\kappa}_{18} = 0.$$

Since $E\colon \pi_{37}^{11} \to \pi_{38}^{12}$ is a monomorphism (see [9, I-(8.17)]), a relation $2\iota_{11} \circ \tau''' = 2\tau'''$ holds, and hence

$$\{\nu_7, \eta_{10}, 2\tau'''\} \supset \{\nu_7, \eta_{10}, 2\iota_{11}\} \circ E\tau''' \subset \pi_{12}^7 \circ E\tau''' = 0 \bmod \nu_7 \circ \pi_{38}^{10} + \pi_{12}^7 \circ 2E\tau''' = 0.$$

By (2.6), (2.16), (2.24) and (3.1),

$$\begin{aligned}\{\nu_7, \eta_{10}, \sigma_{11}\bar{\sigma}_{18}\} &\subset \{\nu_7, \eta_{10}\sigma_{11}, \bar{\sigma}_{18}\} \supset \{\nu_7\sigma_{10}, \eta_{17}, \bar{\sigma}_{18}\} \\ &\supset y\sigma' \circ \{\nu_{14}, \eta_{17}, \bar{\sigma}_{18}\} \\ &\ni \sigma'\bar{\bar{\sigma}}'_{14} \bmod \nu_7 \circ \pi_{38}^{10} + \pi_{19}^7 \circ \bar{\sigma}_{19} = 0\end{aligned}$$

for some odd integer y. These results imply $\{\nu_7, \eta_{10}, \nu_{11}^2\bar{\kappa}_{17}\} = \bar{\nu}_7\nu_{15}\bar{\kappa}_{18}$, $\{\nu_7, \eta_{10}, 2\tau'''\} = 0$ and $\{\nu_7, \eta_{10}, \sigma_{11}\bar{\sigma}_{18}\} = \sigma'\bar{\bar{\sigma}}'_{14}$. □

From Lemma 3.9 and (3.5), we have

$$\sigma'\eta_{14}\phi_{15} \equiv \bar{\nu}_7\nu_{15}\bar{\kappa}_{18} \bmod \sigma'\bar{\bar{\sigma}}'_{14}.$$

Since $\eta_4\phi_5 \equiv \delta_4 \bmod \bar{\mu}_4\sigma_{21}, (E\varepsilon')\kappa_{14}$ (see [9, I-Proposition 3.5(9)]),

$$\sigma'\delta_{14} \equiv \bar{\nu}_7\nu_{15}\bar{\kappa}_{18} \bmod \sigma'\bar{\bar{\sigma}}'_{14}, \sigma'\bar{\mu}_{14}\sigma_{31}, \sigma'(E^{11}\varepsilon')\kappa_{24}.$$

Here, $E^{11}\varepsilon' = \pm 2\nu_{14}\sigma_{17} = 0$ by [14, (7.10)] and (2.18). This completes the proof of Theorem 3.1.

4 Proof of Theorem 1.1(ii).

According to [9, the proof of III-Proposition 3.3(2)],

(4.1) $$\{\kappa_{10} + 8x\sigma_{10}^2, 2\iota_{24}, \eta_{24}\} \ni 0, \quad \{2\iota_{23}, \eta_{23}, \sigma_{24}^2\} = 0$$

and κ_{10}^* is a representative of a tertiary composition

$$\{\kappa_{10} + 8x\sigma_{10}^2, 2\iota_{24}, \eta_{24}, \sigma_{25}^2\}_1,$$

where x is an integer satisfying that $\{\kappa_{11}, 2\iota_{25}, \eta_{25}\} = x\sigma_{11}\mu_{18}$. It holds that [9, III-(10.5)]:

$$2\kappa_{10}^* \in \{\bar{\varepsilon}_{10}, \eta_{25}, \sigma_{26}^2\} \bmod P(\nu_{21}\bar{\sigma}_{24}) \quad \text{and} \quad 2\kappa_{10}^* = x\delta_{10}\sigma_{34} \text{ for } x = 0 \text{ or } 1,$$

where $\delta_{10}\sigma_{34} = P(\nu_{21}\bar{\sigma}_{24}) = [\iota_{10}, \nu_{10}\bar{\sigma}_{13}]$ by (1.3) and (1.4).

We denote by $M^n = S^{n-1} \cup_{2\iota_{n-1}} e^n$ the $\mathbb{Z}_2$-Moore space, $i_n\colon S^{n-1} \to M^n$ the inclusion map, and $p_n\colon M^n \to S^n$ the collapsing map. Let $\tilde{\eta}_n \in \pi_{n+2}(M^{n+1})$ be a coextension of η_n for $n \geq 3$. Notice that $\tilde{\eta}_n \in \pi_{n+2}(M^{n+1}) \cong \mathbb{Z}_4$ is a generator satisfying

(4.2) $$p_{n+1}\tilde{\eta}_n = \eta_{n+1}, \quad 2\tilde{\eta}_n = i_{n+1}\eta_n^2 \quad \text{and} \quad \tilde{\eta}_n \in \{i_{n+1}, 2\iota_n, \eta_n\}.$$

By (2.1) and (2.15), a Toda bracket $\{2\iota_{11}, \eta_{11}, \sigma_{12}^2\} \subset \pi_{27}^{11}$ is well defined. Since $\pi_{27}^{11} = \mathbb{Z}_2\{\sigma_{11}\mu_{18}\}$, $\pi_{39}^{23} = \mathbb{Z}_2\{\omega_{23}\} \oplus \mathbb{Z}_2\{\sigma_{23}\mu_{30}\}$ (see [14, Theorem 12.16]) and $E^{12}\colon \pi_{27}^{11} \to \pi_{39}^{23}$ is a monomorphism, (4.1) implies $\{2\iota_{11}, \eta_{11}, \sigma_{12}^2\} = 0$. Then (4.2) gives

$$\tilde{\eta}_{11}\sigma_{13}^2 \in \{i_{12}, 2\iota_{11}, \eta_{11}\} \circ \sigma_{13}^2 = -(i_{12} \circ \{2\iota_{11}, \eta_{11}, \sigma_{12}^2\}) = 0.$$

By (4.1), there exist an extension $\overline{\kappa_{10} + 8x\sigma_{10}^2} \in [M^{25}, S^{10}]$ of $\kappa_{10} + 8x\sigma_{10}^2$ and a coextension $\tilde{\eta}'_{24} \in \pi_{26}(M^{25})$ of η_{24} such that $\overline{\kappa_{10} + 8x\sigma_{10}^2} \circ \tilde{\eta}'_{24} = 0$. Since $\tilde{\eta}'_{24} \in \mathbb{Z}_4\{\tilde{\eta}_{24}\}$, we have $\tilde{\eta}'_{24} = \tilde{\eta}_{24}$ or $-\tilde{\eta}_{24}$. If $\tilde{\eta}'_{24} = -\tilde{\eta}_{24}$, then we obtain

$$\overline{\kappa_{10} + 8x\sigma_{10}^2} \circ \tilde{\eta}_{24} = -\left(\overline{\kappa_{10} + 8x\sigma_{10}^2} \circ (-\tilde{\eta}_{24})\right) = 0.$$

Thus, we may define a Toda bracket

$$\{\overline{\kappa_{10} + 8x\sigma_{10}^2}, \tilde{\eta}_{24}, \sigma_{26}^2\}_{13}.$$

Hereafter, κ_{10}^* is chosen as a representative of this Toda bracket.

Lemma 4.1. $\kappa_{10}^* = \{\overline{\kappa_{10} + 8x\sigma_{10}^2}, \tilde{\eta}_{24}, \sigma_{26}^2\}_{13}$ *and* $2\kappa_{10}^* = E\{\bar{\varepsilon}_9, \eta_{24}, \sigma_{25}^2\}_{12}$.

Proof. Notice that $\{\bar{\varepsilon}_9, \eta_{24}, \sigma_{25}^2\}_{12}$ is well defined by (2.14), (2.15) and (2.27). The indeterminacy of $\{\overline{\kappa_{10} + 8x\sigma_{10}^2}, \tilde{\eta}_{24}, \sigma_{26}^2\}_{13}$ is

$$\overline{\kappa_{10} + 8x\sigma_{10}^2} \circ E^{13}\pi_{28}(M^{12}) + \pi_{27}^{10} \circ \sigma_{27}^2. \tag{4.3}$$

If the group is trivial, the first half is proved. Moreover, the triviality shows the second half since (4.2), (2.1) and (2.13) give

$$\begin{aligned}
2\kappa_{10}^* &\in \{\overline{\kappa_{10} + 8x\sigma_{10}^2}, \tilde{\eta}_{24}, \sigma_{26}^2\}_{13} \circ 2\iota_{41} \\
&= \{\overline{\kappa_{10} + 8x\sigma_{10}^2}, 2\tilde{\eta}_{24}, \sigma_{26}^2\}_{13} = \{\overline{\kappa_{10} + 8x\sigma_{10}^2}, i_{25}\eta_{24}^2, \sigma_{26}^2\}_{13} \\
&\supset \{\kappa_{10}\eta_{24}, \eta_{25}, \sigma_{26}^2\}_{13} = \{\bar{\varepsilon}_{10}, \eta_{25}, \sigma_{26}^2\}_{13} \\
&\supset E\{\bar{\varepsilon}_9, \eta_{24}, \sigma_{25}^2\}_{12} \bmod \overline{\kappa_{10} + 8x\sigma_{10}^2} \circ E^{13}\pi_{28}(M^{12}) + \pi_{27}^{10} \circ \sigma_{27}^2.
\end{aligned}$$

Therefore, we shall prove the group (4.3) to be 0.

By [14, Theorem 12.17], we have $\pi_{27}^{10} = \{\sigma_{10}\eta_{17}\mu_{18}, \nu_{10}\kappa_{13}, \bar{\mu}_{10}\}$. Then $\pi_{27}^{10} \circ \sigma_{27}^2 = 0$ because $\mu_3\sigma_{12}^2 = 0$ by [6, (2.9)],

$$\kappa_7\sigma_{21} = 0 \tag{4.4}$$

by [9, II-Proposition 2.1(2)], and $\bar{\mu}_3\sigma_{20}^2 = 0$ by (3.4).

Let $\varphi_n \in \pi_n(M^n, S^{n-1})$ be the characteristic map of the n-cell of M^n and let $j\colon (M^n, *) \to (M^n, S^{n-1})$ be the inclusion. By [14, Theorem 12.16], we know that $\pi_{n+16}(S^n) = \mathbb{Z}_2\{\sigma_n\mu_{n+7}\}$ for $n = 11, 12$. Then, by [3, (2.7)], there is a split exact sequence

$$0 \to Q\pi_{17}(S^{11}) \hookrightarrow \pi_{28}(M^{12}, S^{11}) \xrightarrow{p_{12*}} \pi_{28}(S^{12}) \to 0,$$

where $Q\colon \pi_{17}(S^{11}) \to \pi_{28}(M^{12}, S^{11})$ assigns to $x \in \pi_{17}(S^{11})$ the relative Whitehead product $[\varphi_{12}, x]$, and the kernel of Q is $(2\iota_{11*}\circ E^{-12}\circ H)\pi_{29}(S^{12}) \subset 2\pi_{17}(S^{11})$. Since $\pi_{17}(S^{11}) = \mathbb{Z}_2\{\nu_{11}^2\}$ (see [14, Proposition 5.11]), the kernel of Q is trivial, and hence $\pi_{28}(M^{12}, S^{11}) = \mathbb{Z}_2\{[\varphi_{12}, \nu_{11}^2]\} \oplus \mathbb{Z}_2\{\chi\}$ for an element $\chi \in p_{12*}^{-1}(\sigma_{12}\mu_{19})$. Notice that $\partial[\varphi_{12}, \nu_{11}^2] = [2\iota_{11}, \nu_{11}^2] = [\iota_{11}, 2\nu_{11}^2] = 0$ and $\partial\chi = 2\sigma_{11}\mu_{18} = 0$, where ∂ is the connecting homomorphism $\pi_{28}(M^{12}, S^{11}) \to \pi_{27}(S^{11})$. So, by making use of the homotopy exact sequence of a pair (M^{12}, S^{11}), we obtain

$$\pi_{28}(M^{12}) = \{\chi_1, \chi_2\} + i_{12*}\pi_{28}(S^{11}),$$

where $\chi_1 \in j_*^{-1}\chi \subset j_*^{-1}p_{12*}^{-1}(\sigma_{12}\mu_{19})$ and $\chi_2 \in j_*^{-1}[\varphi_{12}, \nu_{11}^2]$. We have

$$p_{12*}j_*(\tilde{\eta}_{11}\rho_{13}) = p_{12}\tilde{\eta}_{11}\rho_{13} = \eta_{12}\rho_{13} = \sigma_{12}\mu_{19}$$

because $\sigma_{12}\mu_{19} = \eta_{12}\rho_{13}$ (see [14, Proposition 12.20]) and by (4.2). Thus, we may take $\chi_1 = \tilde{\eta}_{11}\rho_{13}$. Since $j_*E\chi_2 = E'[\varphi_{12}, \nu_{11}^2] = 0$ for the relative suspension $E'\colon \pi_{28}(M^{12}, S^{11}) \to \pi_{29}(M^{13}, S^{12})$ (see [13]), $E\chi_2$ is contained in the kernel of $j_*\colon \pi_{29}(M^{13}) \to \pi_{29}(M^{13}, S^{12})$, that is, $E\chi_2 \in i_{13*}\pi_{29}(S^{12})$. Therefore, (4.3) is a subgroup of

$$\{\overline{\kappa_{10} + 8x\sigma_{10}^2} \circ \tilde{\eta}_{24}\rho_{26}\} + \overline{\kappa_{10} + 8x\sigma_{10}^2} \circ E^{12}(i_{13*}\pi_{29}(S^{12})).$$

From the definitions of $\overline{\kappa_{10} + 8x\sigma_{10}^2}$, we have $\overline{\kappa_{10} + 8x\sigma_{10}^2} \circ \tilde{\eta}_{24}\rho_{26} = 0$. By the fact that $E^{12}\pi_{29}(S^{12}) = \mathbb{Z}_2\{\varepsilon_{24}^*\} \oplus \mathbb{Z}_2\{\sigma_{24}\eta_{31}\mu_{32}\} \oplus \mathbb{Z}_2\{\nu_{24}\kappa_{27}\} \oplus \mathbb{Z}_2\{\bar{\mu}_{24}\}$ (see [14, Theorem 12.17]),

$$\begin{aligned}\overline{\kappa_{10} + 8x\sigma_{10}^2} \circ E^{12}(i_{13*}\pi_{29}(S^{12})) &= (\kappa_{10} + 8x\sigma_{10}^2) \circ E^{12}\pi_{29}(S^{12}) \\ &= \kappa_{10} \circ E^{12}\pi_{29}(S^{12}) \\ &= \kappa_{10} \circ \{\varepsilon_{24}^*, \sigma_{24}\eta_{31}\mu_{32}, \nu_{24}\kappa_{27}, \bar{\mu}_{24}\}.\end{aligned}$$

By equations (2.13), $\eta_{13}\omega_{14} = \varepsilon_{13}^*$ (see [12, Proposition (2.13)(1)]) and

$$\bar{\varepsilon}_3\omega_{18} = 0, \quad [9, \text{III-}(9.5)], \tag{4.5}$$

we have $\kappa_{10}\varepsilon_{24}^* = \kappa_{10}\eta_{24}\omega_{25} = \bar{\varepsilon}_{10}\omega_{25} = 0$. We also have $\kappa_{10}\sigma_{24}\eta_{31}\mu_{32} = 0$ by (4.4). A relation $\kappa_7\nu_{21} = \nu_7\kappa_{10}$ (see [12, Proposition (2.13)(2)]) and Lemma 3.5 give $\kappa_{10}\nu_{24}\kappa_{27} = \nu_{10}\kappa_{13}^2 \in E^3(\nu_7 \circ \pi_{38}^{10}) = 0$. By [14, Proposition 3.1] and Lemma 2.2, $\kappa_{10}\bar{\mu}_{24} = \bar{\mu}_{10}\kappa_{27} = 0$ is obtained. Therefore, (4.3) is trivial. □

We know $\bar{\nu}_{17} = \{\nu_{17}, \eta_{20}, \nu_{21}\}$, $\pi^9_{22} = \mathbb{Z}_2\{\sigma_9\nu^2_{16}\}$ and $\sigma_9\nu^2_{16} \circ \nu_{22} = \sigma_9\nu^3_{16} \neq 0$ (see [14, Lemma 6.2, Theorems 7.7, 12.6]). A Toda bracket $\{\varepsilon_9, \nu_{17}, \eta_{20}\} \subset \pi^9_{22}$ is well defined by (2.9) and (2.4). Since $\{\varepsilon_9, \nu_{17}, \eta_{20}\} \circ \nu_{22} = -(\varepsilon_9 \circ \{\nu_{17}, \eta_{20}, \nu_{21}\}) = \varepsilon_9\bar{\nu}_{17}$ and

$$\varepsilon_9\bar{\nu}_{17} = 0, \tag{4.6}$$

by (2.18) and (2.14), we have

$$\{\varepsilon_9, \nu_{17}, \eta_{20}\} = 0. \tag{4.7}$$

Lemma 4.2. $\{\varepsilon_9, \bar{\nu}_{17}, \sigma^2_{25}\}_9 = 0$ *and* $\{\varepsilon_9, \varepsilon_{17}, \sigma^2_{25}\}_9 = \{\bar{\varepsilon}_9, \eta_{24}, \sigma^2_{25}\}_9$ *consists of one element.*

Proof. By (2.12), (2.27) and (4.6), $\{\varepsilon_9, \bar{\nu}_{17}, \sigma^2_{25}\}_9$ and $\{\varepsilon_9, \varepsilon_{17}, \sigma^2_{25}\}_9$ are well defined. Well-definedness of the remaining Toda bracket $\{\bar{\varepsilon}_9, \eta_{24}, \sigma^2_{25}\}_9$ is stated in the proof of Lemma 4.1. The indeterminacy of $\{\varepsilon_9, \bar{\nu}_{17}, \sigma^2_{25}\}_9$ is $\varepsilon_9 \circ E^9\pi^8_{31} + \pi^9_{26} \circ \sigma^2_{26}$, which is the same as that of $\{\varepsilon_9, \varepsilon_{17}, \sigma^2_{25}\}_9$. Here,

$$E^9\pi^8_{31} = \{2\bar{\rho}_{17}, \nu_{17}\bar{\kappa}_{20}, \phi_{17}\}$$

and

$$\pi^9_{26} = \{\sigma_9\eta_{16}\mu_{17}, \nu_9\kappa_{12}, \bar{\mu}_9, \eta_9\mu_{10}\sigma_{19}\}$$

by [6, pp. 24–27] and [14, Theorem 12.7], respectively. The group $\varepsilon_9 \circ E^9\pi^8_{31}$ is trivial: $\varepsilon_9 \circ 2\bar{\rho}_{17} = 2\varepsilon_9 \circ \bar{\rho}_{17} = 0$ as ε_9 is of order 2 (see [14, Theorem 7.1]); $\varepsilon_9\nu_{17} = 0$ by (2.9); and $\varepsilon_9\phi_{17} = 0$ by [9, III-Proposition 2.6(1)]. We also have $\pi^9_{26} \circ \sigma^2_{26} = 0$ in the same way that $\pi^{10}_{27} \circ \sigma^2_{27} = 0$ is proved in Lemma 4.1. Then the two Toda brackets consist of one element.

By (3.6), we may set $\bar{\nu}_6 = \chi_1\chi_2$, where χ_1 is an extension of ν_6 and χ_2 is a coextension of ν_{10} with respect to η_9. Let $\mathbb{C}\mathrm{P}^2$ be the complex plane, $i_{\mathbb{C}}\colon S^2 \to \mathbb{C}\mathrm{P}^2$ be the inclusion and $p_{\mathbb{C}}\colon \mathbb{C}\mathrm{P}^2 \to S^4$ be the collapsing map. By (4.7), we obtain

$$\varepsilon_9 E^{11}\chi_1 \in \varepsilon_9 \circ \{\nu_{17}, \eta_{20}, E^{17}p_{\mathbb{C}}\} = -(\{\varepsilon_9, \nu_{17}, \eta_{20}\} \circ E^{18}p_{\mathbb{C}}) = 0.$$

By the relation (2.22), we obtain

$$(E^2\chi_2)\sigma_{16} \in \{E^9 i_{\mathbb{C}}, \eta_{11}, \nu_{12}\} \circ \sigma_{16} = -(E^9 i_{\mathbb{C}} \circ \{\eta_{11}, \nu_{12}, \sigma_{15}\}) = 0.$$

So, we have

$$\{\varepsilon_9, \bar{\nu}_{17}, \sigma^2_{25}\}_9 = \{\varepsilon_9, E^{11}(\chi_1\chi_2), \sigma^2_{25}\}_9$$

$$= \{\varepsilon_9 E^{11}\chi_1, E^{11}\chi_2, \sigma_{25}^2\}_9 = \{0, E^{11}\chi_2, \sigma_{25}^2\}_9 = 0.$$

This leads to the first half.

Since $\varepsilon_5 = \{\nu_5^2, 2\iota_{11}, \eta_{11}\}$ (see [14, (7.6)]), we may set $\varepsilon_5 = \chi_3\tilde{\eta}_{11}$, where χ_3 is an extension of ν_5^2 with respect to $2\iota_{11}$. Since $\bar{\varepsilon}_5 = \{\varepsilon_5, \nu_{13}^2, 2\iota_{19}\}_1$ (see [9, III-Proposition 2.3(5)]), we have

$$\varepsilon_5 E^8\chi_3 \in \varepsilon_5 \circ E\{\nu_{12}^2, 2\iota_{18}, p_{18}\} = \{\varepsilon_5, \nu_{13}^2, 2\iota_{19}\}_1 \circ p_{20} = \bar{\varepsilon}_5 p_{20}.$$

Therefore,

$$\begin{aligned}\{\varepsilon_9, \varepsilon_{17}, \sigma_{25}^2\}_9 &= \{\varepsilon_9, E^{12}(\chi_3\tilde{\eta}_{11}), \sigma_{25}^2\}_9 = \{\varepsilon_9 E^{12}\chi_3, \tilde{\eta}_{23}, \sigma_{25}^2\}_9 \\ &= \{\bar{\varepsilon}_9 p_{24}, \tilde{\eta}_{23}, \sigma_{25}^2\}_9 \\ &\in \{\bar{\varepsilon}_9, \eta_{24}, \sigma_{25}^2\}_9 \bmod \bar{\varepsilon}_9 \circ E^9\pi_{31}^{15} + \pi_{26}^9 \circ \sigma_{26}^2.\end{aligned}$$

Here, $E^9\pi_{31}^{15} = \{\omega_{24}, \sigma_{24}\mu_{31}\}$ by [14, Theorem 12.16], $\bar{\varepsilon}_9\omega_{24} = 0$ by (4.5), and $\bar{\varepsilon}_9\sigma_{24}\mu_{31} = 0$ by (3.2). Then the indeterminacy is $\pi_{26}^9 \circ \sigma_{26}^2$, which equals 0 as above. This leads to the second half and completes the proof. □

By [6, (3.5)], δ_3 is an element in $\{\varepsilon_3, \varepsilon_{11} + \bar{\nu}_{11}, \sigma_{19}\}_1$. Since $E^3\pi_{24}^8 = E\pi_{26}^{10} = \{\sigma_{11}\mu_{18}\}$ (see [14, Theorems 12.6, 12.16]), the indeterminacy of the Toda bracket coincides with that of $\{\varepsilon_3, \varepsilon_{11} + \bar{\nu}_{11}, \sigma_{19}\}_3$. Then $\delta_3 \in \{\varepsilon_3, \varepsilon_{11} + \bar{\nu}_{11}, \sigma_{19}\}_3$, and hence, by Lemma 4.2,

$$\begin{aligned}\delta_9\sigma_{33} &\in \{\varepsilon_9, \varepsilon_{17} + \bar{\nu}_{17}, \sigma_{25}\}_9 \circ \sigma_{33} \subset \{\varepsilon_9, \varepsilon_{17} + \bar{\nu}_{17}, \sigma_{25}^2\}_9 \\ &= \{\varepsilon_9, \varepsilon_{17}, \sigma_{25}^2\}_9 + \{\varepsilon_9, \bar{\nu}_{17}, \sigma_{25}^2\}_9 = \{\bar{\varepsilon}_9, \eta_{24}, \sigma_{25}^2\}_9.\end{aligned}$$

This gives $\delta_9\sigma_{33} = \{\bar{\varepsilon}_9, \eta_{24}, \sigma_{25}^2\}_9$. Therefore, by Lemma 4.1, we have

$$2\kappa_{10}^* = E\{\bar{\varepsilon}_9, \eta_{24}, \sigma_{25}^2\}_{12} \subset E\{\bar{\varepsilon}_9, \eta_{24}, \sigma_{25}^2\}_9 = \delta_{10}\sigma_{34}.$$

This yields the assertion of Theorem 1.1(ii).

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