Clifford Theory for Association Schemes

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Abstract

Clifford theory of finite groups is generalized to association schemes. It shows a relation between irreducible complex characters of a scheme and a strongly normal closed subset of the scheme. The restriction of an irreducible character of a scheme to a strongly normal closed subset coniatns conjugate characters with same multiplicities. Moreover some strong relations are obtained.

1 Introduction

Let K be an algebraically closed field. Let G be a finite group, N a normal subgroup of G. The usual Clifford theory for finite groups shows that

(CF1) the restriction of an irreducible KG-module to KN is a direct sum of G-conjugates of an irreducible KN-module L with the same multiplicities,

(CF2) there exists a natural bijection between the set of irreducible KG-modules over L and the set of KT-modules over L, where T is the stabilizer of L in G,

(CF3) and there exists a natural bijection between the set of irreducible KT-modules over L and the set of irreducible modules of a generalized group algebra of T/N.

In this article, we will generalize them to association schemes. But we only consider module over the complex number field $\mathbb{C}$. The arguments also hold for modules over an arbitrary algebraically closed field of characteristic zero. To do this, the assumption "normal" is too weak. There exists an example of a scheme with a normal closed subset such that (CF1) does not hold for it (see [3]). So we consider "strongly normal" closed subsets. When a scheme is commutative, the author has already shown the results in [3].

Let (X, S) be an association scheme, T a strongly normal closed subset of S. Then the quotient $S/\!\!/T$ can be regarded as a finite group. So we use the theory of group-graded algebras by Dade [1]. Then we can define $S/\!\!/T$-conjugates of $\mathbb{C}T$-modules and prove generalizations of (CF1), (CF2), and (CF3) for association schemes. Dade's theory is very essential in our arguments, but we need only a very spacial case of his theory. So we restrict our attention to the spacial case and give easier proofs to the theory in section 2. In section 3, we apply Dade's theory to association schemes, and in section 4, we state our main theorems. The statement (CF1) will be generalized in Theorem 4.1, (CF2) in Theorem 4.2, and (CF3) in Theorem 4.3. In section 5, we consider the multiplicities of irreducible complex characters of association schemes in the standard characters. Finally, in section 6, we will give an application on a combinatorial property of schemes.

2 Group-graded algebras and their modules

In this section, we state some results in theory of group-graded algebras and their modules by Dade [1]. It is not so easy to understand all of his theory. So we restrict our attention to a spacial case which is needed later, and give proofs to the results.

Let K be a field, G a finite group, and A a finite dimensional K-algebra with the identity element. We say that A is a *G-graded algebra* if A has a decomposition

$$A = \bigoplus_{g \in G} A_g$$

of K-subspaces such that

$$A_g A_h \subset A_{gh}$$

for any $g, h \in G$. Obviously A_1 is a subalgebra of A and A_g is a both left and right A_1-submodule of A.

Let $A = \bigoplus_{g\in G} A_g$ be a G-graded algebra. A right A-module M is said to be a *G-graded A-module* if M has a decomposition

$$M = \bigoplus_{g\in G} M_g$$

of K-subspaces such that

$$M_g A_h \subset M_{gh}$$

for $g, h \in G$. We call M_g the *g-component* of M.

In this paper, we only consider finite dimensional modules over K.

Let A be a G-graded algebra, and M a G-graded A-module. For $g \in G$, we define the *conjugate* M^g of M as follows. Let $M^g = M$ as an A-module and put $(M^g)_h = M_{gh}$. Then M^g is again a G-graded A-module by

$$(M^g)_h A_k = M_{gh} A_k \subset M_{ghk} = (M^g)_{hk}$$

for $g, h, k \in G$. We say that M and M' are *G-conjugate* if there exists $g \in G$ such that $M' \cong M^g$.

Let $M = \bigoplus_{g\in G} M_g$ and $N = \bigoplus_{g\in G} N_g$ be G-graded A-modules. An A-homomorphism $f : M \to N$ is said to be a *G-graded A-homomorphism* if $f(M_g) \subset N_g$ for all $g \in G$.

Let $M = \bigoplus_{g\in G} M_g$ be a G-graded A-module. For a subset H of G, we say that M is *H-null* if $M_h = 0$ for all $h \in H$. The sum of all H-null G-graded A-submodules of M is also H-null. So there exists the unique maximal H-null G-graded A-submodule of M. We call it the *H-null socle* of M and write it $S_H(M)$.

Now we consider the induction of an A_1-module to A. Let L be a right A_1-module, and consider the decomposition

$$L \otimes_{A_1} A = \bigoplus_{g\in G} L \otimes A_g.$$

We call $L \otimes_{A_1} A$ the *induction* of L to A. Then $L \otimes_{A_1} A$ becomes a graded A-module with $(L \otimes_{A_1} A)_g = L \otimes A_g$. We can see that $L \otimes A_g$ is an A_1-submodule of $L \otimes_{A_1} A$ for any $g \in G$ and the decomposition is a direct sum decomposition of an A_1-module $L \otimes_{A_1} A$. Especially, $L \otimes A_1$ is isomorphic to L as an A_1-module. Here $L \otimes A_g$ is considered as a subset of $L \otimes_{A_1} A$. Since A_g has an (A_1, A_1)-bimodule structure, we can consider a right A_1-module $L \otimes_{A_1} A_g$. Then $L \otimes A_g \cong L \otimes_{A_1} A_g$ as right A_1-modules.

Proposition 2.1. *Suppose L is a simple A_1-module. Let M be a proper G-graded A-submodule of $L \otimes_{A_1} A$. Then M is 1-null. So the 1-null socle $S_1(L \otimes_{A_1} A)$ is the unique maximal G-graded A-submodule of $L \otimes_{A_1} A$.*

Proof. First, we note that $L \otimes_{A_1} A$ is not 1-null since $(L \otimes A)_1 \cong L$. Let M be a G-graded A-submodule of $L \otimes_{A_1} A$. Suppose M is not 1-null. Then $M_1 \neq 0$. Since $L \cong L \otimes_{A_1} A_1$ is a simple A_1-module, M_1 contains $L \otimes_{A_1} A_1$. But $(L \otimes_{A_1} A_1)A = L \otimes_{A_1} A$. So $M = L \otimes_{A_1} A$. □

For a simple A_1-module L, define

$$L \bar{\otimes} A = (L \otimes_{A_1} A)/S_1(L \otimes_{A_1} A).$$

Then Proposition 2.1 shows that $L \bar{\otimes} A$ is a simple G-graded A-module.

Proposition 2.2. *Let M be a simple G-graded A-module. Then M_g is a simple A_1-module or 0 for every $g \in G$. Moreover if M is not 1-null, then $M_1 \bar{\otimes} A \cong M$.*

Proof. Let M be a simple G-graded A-module and suppose $M_g \neq 0$. Since the conjugate M^g is also a simple G-graded A-module, it is enough to show that M_1 is simple if M is not 1-null. Suppose M is not 1-null and let L be a simple A_1-submodule of M_1. Then $LA = \sum_{g \in G} LA_g$ and $LA_g \subset M_g$ for any $g \in G$. So $LA = \bigoplus_{g \in G} LA_g$ and LA is a G-graded A-submodule of M. Since M is a simple G-graded A-module, $LA = M$ holds. Then $L = (LA)_1 = M_1$. So M_1 is a simple A_1-module.

Let M be a simple G-graded A-module and suppose M is not 1-null. Then $M = M_1 A$. There exists a G-graded A-epimorphism $f : M_1 \otimes_{A_1} A \to M_1 A$ such that $f(m \otimes a) = ma$. But the 1-null socle of $M_1 \otimes_{A_1} A$ is the unique maximal G-graded A-submodule. So we have $M_1 \bar{\otimes} A \cong M$. □

Proposition 2.2 shows that every non 1-null simple G-graded A-module is of the form $L \bar{\otimes} A$ for some simple A_1-module L. So we have a bijection between the set of simple A_1-modules and the set of non 1-null simple G-graded A-modules.

Let $M = \bigoplus_{g \in G} M_g$ be a simple G-graded A-module. Then M_g is a simple A_1-module or 0 for any $g \in G$. Put

$$\mathrm{Supp}(M) = \{g \in G \mid M_g \neq 0\}$$

and call this the *support* of M. Note that $\mathrm{Supp}(M)$ is not necessary a subgroup of G. For $g, h \in \mathrm{Supp}(M)$, we say that simple A_1-modules M_g and M_h are G-*conjugate*. In this case, $M_g \bar{\otimes} A$ and $M_h \bar{\otimes} A$ are G-conjugate G-graded A-modules. So being G-conjugate is an equivalence relation on the set of isomorphism classes of simple A_1-modules.

For a simple A_1-module L, define the *stabilizer*

$$G\{L\} = \{g \in G \mid (L\bar{\otimes}A)_g \cong L\}$$

of L in G. Then $G\{L\}$ is a subgroup of G and the support $\mathrm{Supp}(L\bar{\otimes}A)$ is a union of some left $G\{L\}$-cosets.

3 Adjacency algebras of association schemes

We fix some notations for association schemes. In this paper an *association scheme* or a *scheme* means a finite scheme in [5].

Let (X, S) be a scheme, K be a field. The *valency* of $s \in S$ is denoted by n_s. For a subset T of S, we also use the notation $n_T = \sum_{t \in T} n_t$. Especially, $n_S = |X|$. For $s \in S$, let σ_s denote the *adjacency matrix* of s. The adjacency matrix will be considered as a matrix over a suitable field. The *adjacency algebra* $\bigoplus_{s \in S} K\sigma_s$ of (X, S) over K will be denoted by KS. Mainly, we will consider the adjacency algebra over the complex number field $\mathbb{C}$. The *structure constant* will be denoted by p_{st}^u, namely $\sigma_s\sigma_t = \sum_{u \in S} p_{st}^u \sigma_u$ in $\mathbb{C}S$. Since the adjacency algebra $\mathbb{C}S$ is closed by the transpose conjugate, $\mathbb{C}S$ is a semisimple algebra. Let $\mathrm{Irr}(S)$ denote the set of all irreducible characters of $\mathbb{C}S$. For $\chi \in \mathrm{Irr}(S)$, the central primitive idempotent corresponding to χ will be denoted by e_χ. Naturally, we can regard the vector space $\mathbb{C}X$ as a right $\mathbb{C}S$-module. Let γ_S be the character of the $\mathbb{C}S$-module $\mathbb{C}X$. We call $\mathbb{C}X$ and γ_S the *standard module* and the *standard character* of (X, S), respectively. The multiplicity of $\chi \in \mathrm{Irr}(S)$ in γ_S will be called the *multiplicity* of χ and denoted by m_χ.

For (strongly normal) closed subsets, quotient (factor) schemes, and so on, see the Zieschang's book [5].

3.1 The Casimir element

Let (X, S) be a scheme. We consider the adjacency algebra $\mathbb{C}S$ of (X, S). Define a map $\zeta : \mathbb{C}S \to \mathbb{C}S$ by

$$\zeta(a) = \sum_{s \in S} \frac{1}{n_s} \sigma_{s^*} a \sigma_s.$$

We call ζ the *Casimir operator* of $\mathbb{C}S$.

Lemma 3.1 ([5, Theorem 4.2.1])**.** *For any $a \in \mathbb{C}S$, $\zeta(a)$ is in the center $Z(\mathbb{C}S)$ of $\mathbb{C}S$.*

The element $v = \zeta(1) = \sum_{s \in S} {n_s}^{-1} \sigma_{s^*} \sigma_s$ is called the *Casimir element* of $\mathbb{C}S$. By Lemma 3.1 v is in the center of $\mathbb{C}S$. We remark that the Casimir element is in $\mathbb{C}\mathbf{O}^\theta(S)$, where $\mathbf{O}^\theta(S)$ is the thin residue of S (see [5, §2.3]).

Proposition 3.2. *The Casimir element of $\mathbb{C}S$ is invertible in $\mathbb{C}S$.*

Proof. Put $v = \sum_{s \in S} {n_s}^{-1} \sigma_{s^*} \sigma_s$. We write the regular character of $\mathbb{C}S$ by reg. Then $\mathrm{reg}(\sigma_s) = \sum_{\chi \in \mathrm{Irr}(S)} \chi(1)\chi(\sigma_s) = \sum_{t \in S} p_{ts}^t$. So we have

$$\begin{aligned} v &= \sum_{s \in S} \frac{1}{n_s} \sum_{t \in S} p_{s^* s}^t \sigma_t = \sum_{s \in S} \sum_{t \in S} \frac{1}{n_t} p_{st^*}^s \sigma_t \\ &= \sum_{t \in S} \frac{1}{n_t} \mathrm{reg}(\sigma_{t^*}) \sigma_t = \sum_{t \in S} \sum_{\chi \in \mathrm{Irr}(S)} \frac{1}{n_t} \chi(1)\chi(\sigma_{t^*})\sigma_t \\ &= \sum_{\chi \in \mathrm{Irr}(S)} \frac{n_S \chi(1)}{m_\chi} e_\chi. \end{aligned}$$

and

$$v^{-1} = \sum_{\chi \in \mathrm{Irr}(S)} \frac{m_\chi}{n_S \chi(1)} e_\chi$$

since $1 = \sum_{\chi \in \mathrm{Irr}(S)} e_\chi$ is an orthogonal idempotent decomposition. □

3.2 Graded modules and simple modules

Let K be a field. Let (X, S) be a scheme and T a strongly normal closed subset of S. Then $S/\!\!/T$ is thin and we can regard it as a finite group. Then

$$KS = \bigoplus_{s^T \in S/\!\!/T} K(TsT)$$

is an $S/\!\!/T$-graded K-algebra, where $K(TsT) = \bigoplus_{u \in TsT} K\sigma_u$. Obviously $(KS)_{1^T} = KT$. We can apply Dade's theory for KS, but we restrict our attention to the case $K = \mathbb{C}$. We will show that every $S/\!\!/T$-graded $\mathbb{C}S$-module is semisimple.

Proposition 3.3. *Let M and N be $S/\!\!/T$-graded $\mathbb{C}S$-modules and $f : M \to N$ a $\mathbb{C}$-linear map such that $f(M_{s^T}) \subset N_{s^T}$ for any $s^T \in S/\!\!/T$. Define $\tilde{f} : M \to N$ by*

$$\tilde{f}(m) = \sum_{s \in S} \frac{1}{n_s} f(m\sigma_s)\sigma_{s^*}.$$

Then $\tilde{f}$ is an $S/\!\!/T$-graded $\mathbb{C}S$-homomorphism.

Proof. First we show that $\tilde{f}$ is a $\mathbb{C}S$-homomorphism. For $t \in S$, we have

$$\begin{aligned} \tilde{f}(m\sigma_t) &= \sum_{s \in S} \frac{1}{n_s} f(m\sigma_t\sigma_s)\sigma_{s^*} = \sum_{s \in S}\sum_{u \in S} \frac{1}{n_s} p_{ts}^u f(m\sigma_u)\sigma_{s^*} \\ &= \sum_{s \in S}\sum_{u \in S} \frac{1}{n_u} p_{u^*t}^{s^*} f(m\sigma_u)\sigma_{s^*} = \sum_{u \in S} \frac{1}{n_u} f(m\sigma_u)\sigma_{u^*}\sigma_t \\ &= \tilde{f}(m)\sigma_t. \end{aligned}$$

So $\tilde{f}$ is a $\mathbb{C}S$-homomorphism.

Now we show that $\tilde{f}$ is an $S/\!\!/T$-graded $\mathbb{C}S$-homomorphism. Suppose $m \in M_{u^T}$. Then

$$\begin{aligned} f(m\sigma_{s^*})\sigma_s &\in f(M_{u^T}(\mathbb{C}S)_{(s^*)^T})(\mathbb{C}S)_{s^T} \subset f(M_{u^T(s^*)^T})(\mathbb{C}S)_{s^T} \\ &\subset N_{u^T(s^*)^T}(\mathbb{C}S)_{s^T} \subset N_{u^T}. \end{aligned}$$

So $\tilde{f}(m) \in N_{u^T}$. This means that $\tilde{f}$ is $S/\!\!/T$-graded. □

Proposition 3.4. *Let M be an $S/\!\!/T$-graded $\mathbb{C}S$-module and N an $S/\!\!/T$-graded $\mathbb{C}S$-submodule of M. Then there exists an $S/\!\!/T$-graded $\mathbb{C}S$-submodule L such that $M = N \oplus L$.*

Proof. For every $s^T \in S/\!\!/T$, there exists a $\mathbb{C}$-subspace L_{s^T} of M_{s^T} such that $M_{s^T} = N_{s^T} \oplus L_{s^T}$ as a $\mathbb{C}$-space. Put $L = \bigoplus_{s^T \in S/\!\!/T} L_{s^T}$. Then $M = N \oplus L$ as a $\mathbb{C}$-space. Let $f : M \to N$ be the projection with respect to this direct sum. Then $f(M_{s^T}) \subset N_{s^T}$ for any $s^T \in S/\!\!/T$. Define $p : M \to M$ by

$$p(m) = \tilde{f}(m)v^{-1},$$

where $\tilde{f}$ is defined in Proposition 3.3 and $v = \sum_{s \in S} {n_s}^{-1} \sigma_s \sigma_{s^*}$ is the Casimir element of $\mathbb{C}S$. Note that v is invertible in $\mathbb{C}S$ by Proposition 3.2. Then p is a $\mathbb{C}S$-homomorphism by Proposition 3.3 and that v is in the center of $\mathbb{C}S$. Note that $v \in (\mathbb{C}S)_{1^T} = \mathbb{C}T$ since $v \in \mathbb{C}\mathbf{O}^\theta(S)$ and $T \supset \mathbf{O}^\theta(S)$. Also $v^{-1} \in \mathbb{C}T$. Then clearly $p(M) \subset N$ and if $n \in N$, then

$$\begin{aligned} p(n) &= \tilde{f}(n)v^{-1} = \sum_{s \in S} \frac{1}{n_s} f(n\sigma_s)\sigma_{s^*} v^{-1} \\ &= \sum_{s \in S} \frac{1}{n_s} n\sigma_s\sigma_{s^*} v^{-1} = nvv^{-1} = n. \end{aligned}$$

So $p(M) = N$. It is easy to see that $M = N \oplus (1-p)(M)$ as a $\mathbb{C}S$-module. Now it is enough to show that $p(M_{u^T}) \subset M_{u^T}$ for any $u^T \in S/\!\!/T$. Suppose $m \in M_{u^T}$. Then $m\sigma_s \in M_{u^T s^T}$, $f(m\sigma_s) \in N_{u^T s^T}$, and $f(m\sigma_s)\sigma_{s^*} \in N_{u^T}$. So $f(m\sigma_s)\sigma_{s^*} v^{-1} \in N_{u^T}$. Now $p(m) \in N_{u^T} \subset M_{u^T}$ and the proof is completed. □

Theorem 3.5. *For a simple $\mathbb{C}T$-module L, $L \otimes_{\mathbb{C}T} \mathbb{C}S$ is a simple $S/\!\!/T$-graded $\mathbb{C}S$-module. Especially $L \bar{\otimes} \mathbb{C}S \cong L \otimes_{\mathbb{C}T} \mathbb{C}S$.*

Proof. If $L \otimes_{\mathbb{C}T} \mathbb{C}S$ is not a simple $S/\!\!/T$-graded $\mathbb{C}S$-module, then it is a direct sum of some $S/\!\!/T$-graded $\mathbb{C}S$-submodules by Proposition 3.4. But Proposition 2.1 says that the 1^T-null socle is the unique maximal $S/\!\!/T$-graded $\mathbb{C}S$-submodule of $L \otimes_{\mathbb{C}T} \mathbb{C}S$. This is a contradiction. □

Combining Proposition 2.2 and Theorem 3.5, we have the following fact.

Theorem 3.6. *For any simple $\mathbb{C}T$-module L and $s \in S$, $L \otimes \mathbb{C}(TsT)$ is a simple $\mathbb{C}T$-module or 0.*

For any simple $\mathbb{C}T$-module L, the set of $S/\!\!/T$-conjugates is $\{L \otimes \mathbb{C}(TsT) \mid s \in S,\ L \otimes \mathbb{C}(TsT) \neq 0\}$. We remark that there exist examples such that L and L' are $S/\!\!/T$-conjugate simple $\mathbb{C}T$-modules but their dimensions are different. In general, characters of $S/\!\!/T$-conjugate simple $\mathbb{C}T$-modules have the same multiplicities (see section 5).

4 Clifford Theory

First, we state some notations. Let K be a field, A a finite dimensional K-algebra, B an subalgebra of A. For a right B-module L, the *induction*

$L \otimes_B A$ of L to A will be denoted by $L{\uparrow}^A$. For a right A-module M, we write $M{\downarrow}_B$ if M is considered as a B-module and we call $M{\downarrow}_B$ the *restriction* of M to B. The Frobenius reciprocity holds, namely

$$\mathrm{Hom}_A(L{\uparrow}^A, M) \cong \mathrm{Hom}_B(L, M{\downarrow}_B)$$

as K-spaces (for example, see [4, Theorem I.11.3]). Let $\mathrm{IRR}(A)$ denote a complete set of representatives for the isomorphism classes of irreducible (simple) A-modules. Suppose both A and B are semisimple. For a simple B-module L, define

$$\mathrm{IRR}(A \mid L) = \{M \in \mathrm{IRR}(A) \mid \mathrm{Hom}_A(L{\uparrow}^A, M) \neq 0\}.$$

For any $M \in \mathrm{IRR}(A)$, there exists $L \in \mathrm{IRR}(B)$ such that $M \in \mathrm{IRR}(A \mid L)$.

For a scheme (X, S), its closed subset T, a right $\mathbb{C}T$-module L, and a right $\mathbb{C}S$-module M, we write $L{\uparrow}^S$ and $M{\downarrow}_T$ instead of $L{\uparrow}^{\mathbb{C}S}$ and $M{\downarrow}_{\mathbb{C}T}$, respectively.

In the rest of this section, we fix a scheme (X, S) and its strongly normal closed subset T.

Let $M \in \mathrm{IRR}(\mathbb{C}S)$. Then $M \in \mathrm{IRR}(\mathbb{C}S \mid L)$ for some $L \in \mathrm{IRR}(\mathbb{C}T)$. Since M is a direct summand of $L{\uparrow}^S$, any simple submodule of $M{\downarrow}_T$ is an $S/\!\!/T$-conjugate of L. If L and L' are $S/\!\!/T$-conjugate, then $L{\uparrow}^S \cong L'{\uparrow}^S$ as $\mathbb{C}S$-modules. So

$$\begin{aligned} & \dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}T}(L, M{\downarrow}_T) = \dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}S}(L{\uparrow}^S, M) \\ = \ & \dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}S}(L'{\uparrow}^S, M) = \dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}T}(L', M{\downarrow}_T). \end{aligned}$$

This shows the following theorem.

Theorem 4.1. *Let $M \in \mathrm{IRR}(\mathbb{C}S)$. There exists $L \in \mathrm{IRR}(\mathbb{C}T)$ such that $M \in \mathrm{IRR}(\mathbb{C}S \mid L)$. Then there exists a positive integer e such that*

$$M{\downarrow}_T \cong e\left(\bigoplus_{L'} L'\right),$$

where L' runs over all $S/\!\!/T$-conjugates of L.

Remark. Theorem 4.1 is true for over not only $\mathbb{C}$ but also an arbitrary coefficient field K, by [1, Theorem 12.4 and 12.10]. See also [2, Theorem 2].

Fix a simple $\mathbb{C}T$-module L. Put $U/\!\!/T$ the stabilizer of L in $S/\!\!/T$. Then

$$\bigoplus_{s^T\in S/\!\!/T} L\otimes\mathbb{C}(TsT) = L\otimes_{\mathbb{C}T}\mathbb{C}S \supset L\otimes_{\mathbb{C}T}\mathbb{C}U = \bigoplus_{u^T\in U/\!\!/T} L\otimes\mathbb{C}(TuT)$$

and, by Theorem 3.6,

$$\bigoplus_{u^T\in U/\!\!/T} L\otimes\mathbb{C}(TuT) \cong n_{U/\!\!/T}L$$

as a $\mathbb{C}T$-module. So $\dim_{\mathbb{C}}\mathrm{Hom}_{\mathbb{C}U}(L\!\uparrow^U, L\!\uparrow^U) = \dim_{\mathbb{C}}\mathrm{Hom}_{\mathbb{C}T}(L, L\!\uparrow^U\!\downarrow_T) = n_{U/\!\!/T}$. On the other hand, by the Frobenius reciprocity, we have

$$\dim_{\mathbb{C}}\mathrm{Hom}_{\mathbb{C}S}(L\!\uparrow^S, L\!\uparrow^S) = \dim_{\mathbb{C}}\mathrm{Hom}_{\mathbb{C}T}(L, L\!\uparrow^S\!\downarrow_T) = n_{U/\!\!/T}.$$

So $\dim_{\mathbb{C}}\mathrm{Hom}_{\mathbb{C}S}(L\!\uparrow^S, L\!\uparrow^S) = \dim_{\mathbb{C}}\mathrm{Hom}_{\mathbb{C}U}(L\!\uparrow^U, L\!\uparrow^U)$. Let $L\!\uparrow^U\cong\bigoplus_i m_iM_i$ be the irreducible decomposition of $L\!\uparrow^U$, with the property that $M_i\cong M_j$ if and only if $i=j$. Then

$$\begin{aligned}\dim_{\mathbb{C}}\mathrm{Hom}_{\mathbb{C}U}(L\!\uparrow^U, L\!\uparrow^U) &= \dim_{\mathbb{C}}\mathrm{Hom}_{\mathbb{C}U}(\bigoplus_i m_iM_i,\ \bigoplus_i m_iM_i)\\ &\le \dim_{\mathbb{C}}\mathrm{Hom}_{\mathbb{C}S}(\bigoplus_i m_iM_i\!\uparrow^S,\ \bigoplus_i m_iM_i\!\uparrow^S)\\ &= \dim_{\mathbb{C}}\mathrm{Hom}_{\mathbb{C}S}(L\!\uparrow^S, L\!\uparrow^S).\end{aligned}$$

This means that $\dim_{\mathbb{C}}\mathrm{Hom}_{\mathbb{C}S}(M_i\!\uparrow^S,\ M_i\!\uparrow^S) = 1$ and $M_i\!\uparrow^S$ is a simple $\mathbb{C}S$-module for every i. Also $M_i\!\uparrow^S\cong M_j\!\uparrow^S$ if and only if $i=j$. Obviously $M_i\in\mathrm{IRR}(\mathbb{C}U\mid L)$ and $M_i\!\uparrow^S\in\mathrm{IRR}(\mathbb{C}S\mid L)$.

Conversely, let $N\in\mathrm{IRR}(\mathbb{C}S\mid L)$. Then N is a direct summand of $L\!\uparrow^S$. So there exists some M_i such that N is a direct summand of $M_i\!\uparrow^S$. Since $M_i\!\uparrow^S$ is simple, such M_i is uniquely determined. This shows the following theorem.

Theorem 4.2. *Fix a simple $\mathbb{C}T$-module L. Put $U/\!\!/T$ the stabilizer of L in $S/\!\!/T$. Then there exists a bijection $\tau:\mathrm{IRR}(\mathbb{C}U\mid L)\to\mathrm{IRR}(\mathbb{C}S\mid L)$ such that $\tau(M)=M\!\uparrow^S$ and $\tau^{-1}(N)$ is the unique direct summand of $N\!\downarrow_U$ contained in* $\mathrm{IRR}(\mathbb{C}U\mid L)$.

Next, we consider $\mathrm{IRR}(\mathbb{C}U\mid L)$. Since $L\!\uparrow^U\cong\bigoplus_i m_iM_i$, we have

$$m_i = \dim_{\mathbb{C}}\mathrm{Hom}_{\mathbb{C}U}(L\!\uparrow^U, M_i) = \dim_{\mathbb{C}}\mathrm{Hom}_{\mathbb{C}T}(L, M_i\!\downarrow_T)$$

and

$$\mathrm{End}_{\mathbb{C}U}(L\!\uparrow^U) \cong \bigoplus_i \mathrm{Mat}_{m_i}(\mathbb{C}).$$

We have that $\dim_{\mathbb{C}} \mathrm{End}_{\mathbb{C}U}(L\!\uparrow^U) = \sum_i {m_i}^2 = n_{U/\!\!/T}$. We consider another description of the structure of $\mathrm{End}_{\mathbb{C}U}(L\!\uparrow^U)$.

Consider the grading

$$L\!\uparrow^U = \bigoplus_{u^T \in U/\!\!/T} (L\!\uparrow^U)_{u^T} = \bigoplus_{u^T \in U/\!\!/T} L \otimes \mathbb{C}(TuT).$$

For $u^T \in U/\!\!/T$, put

$$E_{u^T} = \{\rho \in \mathrm{End}_{\mathbb{C}U}(L\!\uparrow^U) \mid \rho((L\!\uparrow^U)_{v^T}) \subset (L\!\uparrow^U)_{u^T v^T} \text{ for any } v^T \in S/\!\!/T\}.$$

Put $E = \sum_{u^T \in U/\!\!/T} E_{u^T}$. Then the sum is direct, the identity map is in E_{1^T}, and $E_{u^T} E_{v^T} \subset E_{u^T v^T}$. So E is a $U/\!\!/T$-graded $\mathbb{C}$-algebra.

For $u^T \in U/\!\!/T$, we define $\rho_{u^T} \in E_{u^T}$. Actually, this is a conjugation, but we explain its details. We know that $(L\!\uparrow^U)_{v^T} \cong L$ as $\mathbb{C}T$-modules for $v^T \in U/\!\!/T$. So we fix isomorphisms $\tau_{v^T} : (L\!\uparrow^U)_{v^T} \to L$ for all $v^T \in U/\!\!/T$. Note that τ_{v^T} is not unique. But, through τ_{v^T}, we identify $(L\!\uparrow^U)_{v^T}$ with L. For $\ell \in L\!\uparrow^U$, write $\ell = \sum_{v^T} \ell_{v^T}$, $\ell_{v^T} \in (L\!\uparrow^U)_{v^T}$. Define ρ_{u^T} by putting

$$(\rho_{u^T}(\ell))_{v^T} = \ell_{u^T v^T}.$$

Clearly ρ_{u^T} is a $\mathbb{C}$-linear isomorphism. For $w \in U$, we have

$$\begin{aligned}
(\rho_{u^T}(\ell\sigma_w))_{v^T} &= (\ell\sigma_w)_{u^T v^T}, \\
(\rho_{u^T}(\ell)\sigma_w)_{v^T} &= ((\rho_{u^T}(\ell))_{v^T (w^*)^T})\sigma_w = \ell_{u^T v^T (w^*)^T}\sigma_w = (\ell\sigma_w)_{u^T v^T}.
\end{aligned}$$

So ρ_{u^T} is a $\mathbb{C}U$-homomorphism. Now $\rho_{u^T} \in E_{u^T}$ by the definition and ρ_{u^T} is invertible in $\mathrm{End}_{\mathbb{C}U}(L\!\uparrow^U)$. Especially $\dim_{\mathbb{C}} E_{u^T} \geq 1$ for any $u^T \in U/\!\!/T$. Thus

$$n_{U/\!\!/T} = \dim_{\mathbb{C}} \mathrm{End}_{\mathbb{C}U}(L\!\uparrow^U) \geq \dim_{\mathbb{C}} E \geq n_{U/\!\!/T}.$$

This shows that $E = \mathrm{End}_{\mathbb{C}U}(L\!\uparrow^U)$. Now $E = \bigoplus_{u^T \in U/\!\!/T} \mathbb{C}\rho_{u^T}$ is a generalized group ring $\mathbb{C}^{(\alpha)}(U/\!\!/T)$ for some factor set α (see [4, §II.8]). Now we have the following theorem.

Theorem 4.3. *Fix a simple $\mathbb{C}T$-module L. Put $U/\!\!/T$ the stabilizer of L in $S/\!\!/T$. Then there exists a factor set α of $U/\!\!/T$ and a bijection μ : $\mathrm{IRR}(\mathbb{C}^{(\alpha)}(U/\!\!/T)) \to \mathrm{IRR}(\mathbb{C}U \mid L)$ such that $\dim_{\mathbb{C}} \mu(N) = (\dim_{\mathbb{C}} L)(\dim_{\mathbb{C}} N)$ for $N \in \mathrm{IRR}(\mathbb{C}^{(\alpha)}(U/\!\!/T))$.*

Suppose there exists a simple $\mathbb{C}^{(\alpha)}(U/\!\!/T)$-module of dimension one. Let Γ be the corresponding representation. Then

$$\Gamma(\rho_{u^T})\Gamma(\rho_{v^T}) = \Gamma(\rho_{u^T}\rho_{v^T}) = \alpha(u^T, v^T)\Gamma(\rho_{u^T v^T}).$$

This means that α is a coboundary and $\mathbb{C}^{(\alpha)}(U/\!\!/T) \cong \mathbb{C}(U/\!\!/T)$.

Corollary 4.4. *In Theorem 4.3, if there is $M \in \mathrm{IRR}(\mathbb{C}U \mid L)$ such that $M\downarrow_T \cong L$, then the factor set α can be chosen to be the trivial one.*

If the quotient $U/\!\!/T$ is a cyclic group, then the second cohomology group $H^2(U/\!\!/T, \mathbb{C}^\times)$ is trivial and L is extendible to U (see [4, Lemma III.5.4]).

5 Multiplicities of induced characters

In this section, we will give a formula on the multiplicities of induced characters, though it is not related to Clifford theory. This formula shows that multiplicities of conjugate characters are the same as we said in the end of section 3. In this section, we will use characters instead of modules, but they are essentially the same.

For a character η and an irreducible character χ of a scheme, let $\mathrm{m}(\chi \text{ in } \eta)$ denote the multiplicity of χ in η. Usually we consider the multiplicity m_χ only for an irreducible character χ. It is defined by $\mathrm{m}(\chi \text{ in } \gamma_S)$ for $\chi \in \mathrm{Irr}(S)$. Let η be an arbitrary character of S. Let we define the *multiplicity* m_η of η by

$$m_\eta = \sum_{\chi \in \mathrm{Irr}(S)} \mathrm{m}(\chi \text{ in } \eta)\, m_\chi.$$

If η is irreducible, then m_η is same as the original one. We have the following theorem.

Theorem 5.1. *Let (X, S) be a scheme and T a closed subset of S. Let φ be a character of T. Then $m_{\varphi\uparrow^S} = (n_S/n_T)m_\varphi$.*

Proof. Without loss of generality, we may assume that φ is irreducible. By the orthogonality relations [5, Theorem 4.1.5 (ii)], we have

$$\begin{aligned}
m_{\varphi\uparrow^S} &= \sum_{\chi\in\mathrm{Irr}(S)} \mathrm{m}(\chi \text{ in } \varphi\uparrow^S)\ m_\chi = \sum_{\chi\in\mathrm{Irr}(S)} \mathrm{m}(\varphi \text{ in } \chi\downarrow_T)\ m_\chi \\
&= \sum_{\chi\in\mathrm{Irr}(S)} m_\chi \frac{m_\varphi}{n_T\varphi(1)} \sum_{t\in T} \frac{1}{n_t}\varphi(\sigma_t)\chi(\sigma_{t^*}) \\
&= \frac{m_\varphi}{n_T\varphi(1)} \sum_{t\in T} \frac{1}{n_t}\varphi(\sigma_t) \sum_{\chi\in\mathrm{Irr}(S)} m_\chi\chi(\sigma_{t^*}) \\
&= \frac{m_\varphi}{n_T\varphi(1)} \sum_{t\in T} \frac{1}{n_t}\varphi(\sigma_t)\gamma_S(\sigma_{t^*}) = \frac{m_\varphi}{n_T\varphi(1)}\varphi(1)n_S = \frac{n_S}{n_T}m_\varphi.
\end{aligned}$$

This shows the assertion. □

Let T be a strongly normal closed subset of S, φ and ψ are $S/\!\!/T$-conjugate. Then we know that $\varphi\uparrow^S = \psi\uparrow^S$. So Theorem 5.1 shows that $m_\varphi = m_\psi$.

6 An application

In this final section, we will give an application of our arguments. This is an answer to a question by Mikhail Muzychuk.

Theorem 6.1. *Let (X, S) be a scheme, T a strongly normal closed subset of S. Then $|TsT| \le |T|$ for any $s \in S$.*

Proof. Let $s \in S$. We have

$$\mathbb{C}(TsT) \cong \mathbb{C}T \otimes_{\mathbb{C}T} \mathbb{C}(TsT) \cong \bigoplus_{L\in\mathrm{IRR}(\mathbb{C}T)} (\dim_{\mathbb{C}} L)L \otimes \mathbb{C}(TsT).$$

Note that, for $L,\ L' \in \mathrm{IRR}(\mathbb{C}T)$, $L \otimes \mathbb{C}(TsT) \cong L'$ if and only if $L' \otimes \mathbb{C}(Ts^*T) \cong L$. So $L \otimes \mathbb{C}(TsT) \cong L' \otimes \mathbb{C}(TsT)$ if and only if $L \cong L'$ for $L,\ L' \in \mathrm{IRR}(\mathbb{C}T)$ with the property $L \otimes \mathbb{C}(TsT) \ne 0$. We define a bijection $f : \mathrm{IRR}(\mathbb{C}T) \to \mathrm{IRR}(\mathbb{C}T)$ as follows. Put $f(L) = L \otimes \mathbb{C}(TsT)$ if $L \otimes \mathbb{C}(TsT) \ne 0$. Keeping f to be an injection, we define $f(L)$ arbitrarily for the other L. Then we have

$$\begin{aligned} |TsT| &= \dim_{\mathbb{C}} \mathbb{C}(TsT) = \sum_{L \in \mathrm{IRR}(\mathbb{C}T)} (\dim_{\mathbb{C}} L)(\dim_{\mathbb{C}} L \otimes \mathbb{C}(TsT)) \\ &\leq \sum_{L \in \mathrm{IRR}(\mathbb{C}T)} (\dim_{\mathbb{C}} L)(\dim_{\mathbb{C}} f(L)) \leq \sum_{L \in \mathrm{IRR}(\mathbb{C}T)} (\dim_{\mathbb{C}} L)^2 \\ &= \dim_{\mathbb{C}} \mathbb{C}T = |T| \end{aligned}$$

as desired. □

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