Orbifold family unification in $SO(2N)$ gauge theory

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We study the possibility of family unification on the basis of $SO(2N)$ gauge theory on the five-dimensional space-time, $M^4 \times S^1/Z_2$. Several $SO(10)$, $SU(4) \times SU(2)_L \times SU(2)_R$, or $SU(5)$ multiplets come from a single bulk multiplet of $SO(2N)$ after the orbifold breaking. Other multiplets including brane fields are necessary to compose three families of quarks and leptons.

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I. INTRODUCTION

The *family unification* or *flavor unification* based on a large symmetry group can provide a possible solution for the origin of the family replication [1–4]. However, we encounter difficulty in the unification on the four-dimensional Minkowski space, because of extra fields such as "mirror particles" existing in the higher-dimensional representation. The mirror particles are particles with opposite quantum numbers under the standard model (SM) gauge group. If the idea of family or flavor unification is to be realized in nature, extra particles must disappear from the low-energy spectrum around the weak scale. Several interesting mechanisms have been proposed to get rid of the unwelcomed particles. One is to adopt the "survival hypothesis," which is the assumption that if a symmetry is broken down to a smaller symmetry at a scale M_{SB}, then any fermion mass terms invariant under the smaller group induce fermion masses of order $O(M_{\mathrm{SB}})$ [2,5]. Georgi investigated whether an anomaly free set of no-repeated representations in $SU(N)$ models can lead to families based on the survival hypothesis, and found that three families are derived from $[11, 4] + [11, 8] + [11, 9] + [11, 10]$ in the $SU(11)$ model in four dimensions [2]. Another possibility is to confine extra particles at a high-energy scale by some strong interaction [6].

If we move from four dimensions to higher dimensions, there is a possibility to reduce substances including mirror particles using the symmetry reduction concerning extra dimensions, as originally discussed in superstring theory [7,8]. Hence it is meaningful to reexamine the idea of family or flavor unification using grand unified theories (GUTs) on a higer-dimensional space-time.[1] We refer to the family unification using orbifolds for extra dimensions as the orbifold family unification. There are several preceding studies on the orbifold family unification. The complete family unification has been suggested in E_8 GUT on $M^4 \times T^2/Z_3$ [11]. The model in which three families come from a combination of a bulk gauge multiplet and a few brane fields in $SO(10)$ GUT on $M^4 \times T^2/Z_3$ has been examined [12]. The gauge, Higgs, and matter unification has been proposed in $SU(8)$ GUT on $M^4 \times T^2/Z_6$ [13] and $M^4 \times T^2/Z_3$ [14] and $SO(16)$ on $M^4 \times T^2/Z_6$ [15]. The orbifold family unification has been studied in $SU(N)$ on $M^4 \times S^1/Z_2$ [16].

In this paper, we study the possibility of orbifold family unification on the basis of $SO(2N)$ gauge theory on $M^4 \times S^1/Z_2$ using the method in Ref. [16].[2] We investigate whether or not three families are derived from a single bulk multiplet of $SO(2N)$ for several orbifold symmetry breaking patterns.

The contents of this paper are as follows. In Sec. II, we review and provide general arguments on the orbifold breaking on S^1/Z_2. In Sec. III, we investigate unification of quarks and leptons in $SO(2N)$ gauge theory on $M^4 \times S^1/Z_2$. Section IV is devoted to conclusions and discussions. We discuss the gauge equivalence of boundary conditions (BCs) in Appendix A and the symmetry breaking of $SO(2N + 1)$ in Appendix B.

II. S^1/Z_2 ORBIFOLD BREAKING

In this section, we study the orbifold symmetry breaking mechanism in $SO(2N)$ gauge theory on $M^4 \times S^1/Z_2$, where M^4 is the four-dimensional Minkowski space.

A. Boundary conditions and symmetry reduction on S^1/Z_2

First we review the symmetry reduction mechanism on S^1/Z_2 briefly [20]. Let x (or x^μ, $\mu = 0, \ldots, 3$) and y (or x^5) be coordinates of M^4 and S^1/Z_2, respectively. The S^1/Z_2 is obtained by dividing the circle S^1 (with the identification $y \sim y + 2\pi R$) by the Z_2 transformation $y \to -y$ so that the point y is identified with $-y$. Here, R is the radius of S^1. Then the S^1/Z_2 is regarded as an interval with length πR. Both end points $y = 0$ and πR are fixed points under the Z_2 transformation. For the operations:

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[1] Five-dimensional supersymmetric GUTs on $M^4 \times S^1/Z_2$ possess the attractive feature that the triplet-doublet splitting of Higgs multiplets is elegantly realized [9,10].

[2] $SO(10)$ GUTs on $M^4 \times T^2/Z_2$ [17] and $M^4 \times S^1/Z_2$ [18] and $SO(12)$ GUT on $M^4 \times S^1/Z_2$ [19] have been constructed and their phenomenological implications have been studied.

$$s_0\colon y \to -y, \qquad s_1\colon y \to 2\pi R - y, \qquad t\colon y \to y + 2\pi R, \tag{1}$$

the following relations hold:

$$s_0^2 = s_1^2 = I, \qquad t = s_1 s_0, \tag{2}$$

where I is the identity operation. The operation s_1 is the reflection at the end point $y = \pi R$ and the S^1/Z_2 can be defined using s_0 and s_1.

Although the point y is identified with the points $-y$ and $2\pi R - y$ on S^1/Z_2, a field does not necessarily take an identical value at these points. We require that the Lagrangian density should be single valued. Then the following BCs of the field $\Phi(x, y)$ are allowed:

$$\begin{aligned} \Phi(x, -y) &= T_\Phi[s_0]\Phi(x, y), \\ \Phi(x, 2\pi R - y) &= T_\Phi[s_1]\Phi(x, y), \\ \Phi(x, y + 2\pi R) &= T_\Phi[t]\Phi(x, y), \end{aligned} \tag{3}$$

where $T_\Phi[s_0]$, $T_\Phi[s_1]$, and $T_\Phi[t]$ represent appropriate representation matrices for s_0, s_1, and t operations, respectively. The $T_\Phi[*]$ belong to the group elements of transformations which keep the action integral invariant and satisfy the counterparts of (2):

$$T_\Phi[s_0]^2 = T_\Phi[s_1]^2 = I, \qquad T_\Phi[t] = T_\Phi[s_0]T_\Phi[s_1], \tag{4}$$

where I stands for the unit matrix. For the eigenstates of $T_\Phi[s_0]$ and $T_\Phi[s_1]$, the eigenvalues are interpreted as the Z_2 parity for the fifth coordinate flip and take $+1$ or -1 by definition. Then the eigenvalues of $T_\Phi[t]$ also take $+1$ or -1. As the assignment of Z_2 parity determines BCs of each multiplet on S^1/Z_2, we use "Z_2 parity" as a parallel expression of "BCs on S^1/Z_2" in the remainder of the paper.

Let $\phi^{(\mathcal{P}_0,\mathcal{P}_1;\mathcal{U})}(x, y)$ be a component in a multiplet $\Phi(x, y)$ and have definite eigenvalues $(\mathcal{P}_0, \mathcal{P}_1; \mathcal{U})$ for s_0, s_1, and t operations. The Fourier expansion of $\phi^{(\mathcal{P}_0,\mathcal{P}_1;\mathcal{U})}(x, y)$ is given by

$$\phi^{(++;+)}(x, y) = \frac{1}{\sqrt{\pi R}}\phi_0(x) + \sqrt{\frac{2}{\pi R}}\sum_{n=1}^{\infty}\phi_n(x)\cos\frac{ny}{R}, \tag{5}$$

$$\phi^{(--;+)}(x, y) = \sqrt{\frac{2}{\pi R}}\sum_{n=1}^{\infty}\phi_n(x)\sin\frac{ny}{R}, \tag{6}$$

$$\phi^{(+-;-)}(x, y) = \sqrt{\frac{2}{\pi R}}\sum_{n=1}^{\infty}\phi_n(x)\cos\frac{(n-\frac{1}{2})y}{R}, \tag{7}$$

$$\phi^{(-+;-)}(x, y) = \sqrt{\frac{2}{\pi R}}\sum_{n=1}^{\infty}\phi_n(x)\sin\frac{(n-\frac{1}{2})y}{R}, \tag{8}$$

where $\pm$ indicates the eigenvalues ± 1.

In the above expansions (5)–(8), the coefficients $\phi_0(x)$ and $\phi_n(x)$ $(n = 1, 2, \dots)$ are four-dimensional fields, which are called zero mode and Kaluza-Klein (KK) modes, respectively. The KK modes $\phi_n(x)$ acquire the mass n/R for $(\mathcal{P}_0, \mathcal{P}_1; \mathcal{U}) = (\pm 1, \pm 1; +1)$, and $(n - \frac{1}{2})/R$ for $(\mathcal{P}_0, \mathcal{P}_1; \mathcal{U}) = (\pm 1, \mp 1; -1)$ upon compactification. Unless all components of the nonsinglet field have a common Z_2 parity, a symmetry reduction occurs upon compactification because $\phi_0(x)$ are absent in fields with an odd parity. This kind of symmetry breaking is called "orbifold breaking" [21].

Our four-dimensional world is assumed to be a Minkowski space at one of the fixed points, on the basis of the "brane world scenario." There exist two kinds of four-dimensional fields in our low-energy theory. One is the brane field which lives only at the boundary, and the other is the zero mode stemming from the bulk field. The massive modes $\phi_n(x)$ do not appear in our low-energy world because they have heavy masses of $O(1/R)$, with the same magnitude as the unification scale. Chiral anomalies may arise at the boundaries with the advent of chiral fermions. Those anomalies must be canceled in the four-dimensional effective theory by the contribution of brane chiral fermions and/or counterterms such as the Chern-Simons term [22,23].

B. Orbifold symmetry breaking of $SO(2N)$

The $SO(2N)$ is the orthogonal group whose determinant is 1 and number of elements are $N(2N-1)$. The representation matrices of $SO(2N)$ are expressed as $e^{i\theta^\alpha T^\alpha}$, where θ^α is a real parameter and T^α are elements of the Lie algebra $so(2N)$. The generators T^α $(\alpha = 1, \dots, N(2N-1))$ are pure imaginary antisymmetric matrices, i.e., $(T^\alpha)^t = -T^\alpha$ and $(T^\alpha)^* = -T^\alpha$. The generators for vector representation **2N** are written by the direct product of the 2×2 matrix and the $N \times N$ matrix:

$$\begin{gathered} \sigma_2 \otimes S_N,\ \left(\frac{N(N+1)}{2} - 1\right); \qquad \sigma_0 \otimes A_N,\ \left(\frac{N(N-1)}{2}\right); \\ \sigma_2 \otimes I_N,\ (1); \qquad \sigma_1 \otimes A_N,\ \left(\frac{N(N-1)}{2}\right); \\ \sigma_3 \otimes A_N,\ \left(\frac{N(N-1)}{2}\right), \end{gathered} \tag{9}$$

where σ_i $(i = 1, 2, 3)$ are Pauli matrices, σ_0 is the 2×2 unit matrix, S_N, A_N, and I_N stand for $N \times N$ symmetric matrices (whose components are real), $N \times N$ antisymmetric matrices (whose components are pure imaginary), and the $N \times N$ unit matrix and the numbers in the parentheses represent the numbers of elements. The elements of subalgebra $su(N)$ are $\sigma_2 \otimes S_N$ and $\sigma_0 \otimes A_N$.

As a warm-up, we consider the breakdown of $SO(2N)$ by the Z_2 projection with the following type of $2N \times 2N$ matrix:

$$P = \sigma_0 \otimes I_{m,n} \quad \text{or} \quad \sigma_2 \otimes I_{m,n}, \tag{10}$$

where $I_{m,n}$ is defined by

$$I_{m,n} \equiv \mathrm{diag}(\underbrace{+1, \ldots, +1}_{m}, \underbrace{-1, \ldots, -1}_{n(=N-m)}). \tag{11}$$

(1) $P = \sigma_0 \otimes I_{m,n}$

The generators for unbroken symmetry commute with P, i.e., $[P, T^a] = 0$, and they are given by

$$\begin{aligned} &\sigma_2 \otimes S_m; \quad \sigma_0 \otimes A_m; \quad \sigma_2 \otimes I_m; \quad \sigma_1 \otimes A_m; \\ &\sigma_3 \otimes A_m; \quad \sigma_2 \otimes S_n; \quad \sigma_0 \otimes A_n; \\ &\sigma_2 \otimes I_n; \quad \sigma_1 \otimes A_n; \quad \sigma_3 \otimes A_n, \end{aligned} \tag{12}$$

where S_m (S_n), A_m (A_n), and I_m (I_n) stand for $m \times m$ ($n \times n$) symmetric submatrices, $m \times m$ ($n \times n$) antisymmetric submatrices, and the $m \times m$ ($n \times n$) unit submatrix. Hence the unbroken symmetry is $SO(2m) \times SO(2n)$.

(2) $P = \sigma_2 \otimes I_{m,n}$

The generators which commute with P are given by

$$\begin{aligned} &\sigma_2 \otimes S_m; \quad \sigma_0 \otimes A_m; \quad \sigma_2 \otimes I_m; \quad \sigma_2 \otimes S_n; \\ &\sigma_0 \otimes A_n; \quad \sigma_2 \otimes I_n; \quad \sigma_1 \otimes A_{m,n}; \quad \sigma_3 \otimes A_{m,n}, \end{aligned} \tag{13}$$

where $A_{m,n}$ are antisymmetric matrices composed by off-diagonal $m \times n$ and $n \times m$ submatrices and commute with $I_{m,n}$. Hence the unbroken symmetry is $SU(N) \times U(1)$.

We study the combination of Z_2 projections with $\sigma_0 \otimes I_{m,n}$ and $\sigma_2 \otimes I_{m,n}$. The generators which simultaneously commute with $\sigma_0 \otimes I_{m,n}$ and $\sigma_2 \otimes I_{m,n}$ are given by

$$\begin{aligned} &\sigma_2 \otimes S_m; \quad \sigma_0 \otimes A_m; \quad \sigma_2 \otimes I_m; \\ &\sigma_2 \otimes S_n; \quad \sigma_0 \otimes A_n; \quad \sigma_2 \otimes I_n. \end{aligned} \tag{14}$$

The unbroken symmetry is $SU(m) \times SU(n) \times U(1)^2$. The same intersections can be obtained with the combination of $\sigma_0 \otimes I_{m,n}$ and $\sigma_2 \otimes I_N$ or that of $\sigma_2 \otimes I_{m,n}$ and $\sigma_2 \otimes I_N$.

We study the BCs in $SO(2N)$ gauge theory on $M^4 \times S^1/Z_2$. The BCs on S^1/Z_2 are specified by the $2N \times 2N$ matrices (P_0, P_1, U), where $P_0^2 = P_1^2 = I$ and $U = P_0 P_1$. For (P_0, P_1, U), we use the following type of matrices:

$$P^{(0)} \equiv \sigma_0 \otimes \tilde{I} \quad \text{or} \quad P^{(2)} \equiv \sigma_2 \otimes \tilde{I}', \tag{15}$$

where $\tilde{I}$ and $\tilde{I}'$ are $N \times N$ diagonal matrices whose diagonal components take $+1$ or -1. In this case, the relations $P_0 P_1 = P_1 P_0 = U$ and $U^2 = I$ hold and the symmetry breaking patterns are classified into the following two types.

(Type I) All matrices belong to the $P^{(0)}$ type. By the arrangement of the rows and columns, (P_0, P_1, U) are written by

$$(P_0, P_1, U) = (\sigma_0 \otimes \tilde{I}_1, \sigma_0 \otimes \tilde{I}_2, \sigma_0 \otimes \tilde{I}_3), \tag{16}$$

where $\tilde{I}_1$, $\tilde{I}_2$, and $\tilde{I}_3 (= \tilde{I}_1 \tilde{I}_2)$ are defined by

$$\begin{aligned} \tilde{I}_1 &\equiv \mathrm{diag}(\overbrace{+1, \ldots, +1, +1, \ldots, +1, -1, \ldots, -1, -1, \ldots, -1}^{N}), \\ \tilde{I}_2 &\equiv \mathrm{diag}(+1, \ldots, +1, -1, \ldots, -1, +1, \ldots, +1, -1, \ldots, -1), \\ \tilde{I}_3 &\equiv \mathrm{diag}(\underbrace{+1, \ldots, +1}_{p}, \underbrace{-1, \ldots, -1}_{q}, \underbrace{-1, \ldots, -1}_{r}, \underbrace{+1, \ldots, +1}_{s(=N-p-q-r)}), \end{aligned} \tag{17}$$

where $p, q, r, s \geq 0$, and $N = p + q + r + s$. We denote the above BC (16) as $[p, q; r, s]^{\mathrm{I}}$. The symmetry of $[p, q; r, s]^{\mathrm{I}}$ becomes

$$SO(2N) \to SO(2p) \times SO(2q) \times SO(2r) \times SO(2s), \tag{18}$$

where $SO(0)$ means nothing.

(Type II) Two of P_0, P_1, and U belong to the $P^{(2)}$ type and the remaining one is a $P^{(0)}$ type, and they are classified into the three subtypes:

$$(P_0, P_1, U) = (\sigma_0 \otimes \tilde{I}_1, \sigma_2 \otimes \tilde{I}_2, \sigma_2 \otimes \tilde{I}_3) \quad (\text{type IIa}), \tag{19}$$

$$= (\sigma_2 \otimes \tilde{I}_1, \sigma_0 \otimes \tilde{I}_2, \sigma_2 \otimes \tilde{I}_3) \quad (\text{type IIb}), \tag{20}$$

$$= (\sigma_2 \otimes \tilde{I}_1, \sigma_2 \otimes \tilde{I}_2, \sigma_0 \otimes \tilde{I}_3) \quad (\text{type IIc}), \tag{21}$$

where $\tilde{I}_1$, $\tilde{I}_2$, and $\tilde{I}_3$ are defined by (17). We denote the above BCs (19)–(21) as $[p, q; r, s]^{\mathrm{IIa}}$, $[p, q; r, s]^{\mathrm{IIb}}$, and $[p, q; r, s]^{\mathrm{IIc}}$, respectively. The symmetries of $[p, q; r, s]^{\mathrm{IIa}}$, $[p, q; r, s]^{\mathrm{IIb}}$, and $[p, q; r, s]^{\mathrm{IIc}}$ become

$$SO(2N) \to SU(p+q) \times SU(r+s) \times U(1)^{2-k} \quad (\text{type IIa}), \tag{22}$$

$$SO(2N) \to SU(p+r) \times SU(q+s) \times U(1)^{2-k} \quad (\text{type IIb}), \tag{23}$$

$$SO(2N) \to SU(p+s) \times SU(q+r) \times U(1)^{2-k} \quad (\text{type IIc}), \tag{24}$$

where k is a sum of the number of $SU(0)$ and $SU(1)$, $SU(0)$ means nothing, and $SU(1)$ unconventionally stands for $U(1)$. Because type IIa, type IIb, and type IIc are interchanged among them by the interchange of P_0, P_1, and U and the same results for the numbers of each species are obtained, we use type IIa as the representative of type II. If two BCs are transformed into each other by a global $SO(2N)$ transformation and/or a gauge transformation, they are equivalent. The $[p, q; r, s]^{\mathrm{IIa}}$ is transformed into $[p + \ell_1, q - \ell_1, r + \ell_2, s - \ell_2]^{\mathrm{IIa}}$ using the global $SO(2N)$ symmetry which changes σ_2 into $-\sigma_2$ partially. Here, ℓ_1 and ℓ_2 are arbitrary integers which satisfy $p + \ell_1$, $q - \ell_1$, $r + \ell_2$, and $s - \ell_2 \geq 0$. Hence we use $[m, 0, n, 0]^{\mathrm{IIa}}$ ($N = m + n$) in place of $[p, q; r, s]^{\mathrm{IIa}}$ with $m = p + q$ and $n = r + s$. In Appendix A, we discuss the gauge invariance of

BCs and the equivalence relations for the sake of completeness.

Strictly speaking, we must find the minimum of the effective potential for the Wilson line phases in order to know physical gauge symmetry [24]. It requires a model-dependent analysis because the effective potential depends on the particle contents and their BCs. In the following discussion, we suppose that the BC belongs to the same equivalence class of $(P_0^{\rm sym}, P_1^{\rm sym}, U^{\rm sym})$ defined by Eq. (A6).

C. Z_2 parity assignment

We study the Z_2 parity assignment for gauge fields and matter fermions for two types.

(Type I) The BCs of gauge fields, $A_M(x,y) = A_M^\alpha(x,y)T^\alpha$, are given by

$$\begin{aligned} s_0\colon A_\mu(x,-y) &= P_0 A_\mu(x,y) P_0^{-1}, \\ A_y(x,-y) &= -P_0 A_y(x,y) P_0^{-1}, \\ s_1\colon A_\mu(x, 2\pi R - y) &= P_1 A_\mu(x,y) P_1^{-1}, \\ A_y(x, 2\pi R - y) &= -P_1 A_y(x,y) P_1^{-1}, \\ t\colon A_M(x, y + 2\pi R) &= U A_M(x,y) U^{-1}, \end{aligned} \tag{25}$$

where $M = 0, \ldots, 3, 5$. Using the relation $\mathrm{tr}(T^\alpha T^\beta) = \delta^{\alpha\beta}/2$, the BCs for four-dimensional components of gauge bosons, $A_\mu(x,y) = A_\mu^\alpha(x,y)T^\alpha$, are rewritten as

$$\begin{aligned} A_\mu^\alpha(x,-y) &= 2\,\mathrm{tr}(T^\alpha P_0 T^\beta P_0^{-1}) A_\mu^\beta(x,y), \\ A_\mu^\alpha(x, 2\pi R - y) &= 2\,\mathrm{tr}(T^\alpha P_1 T^\beta P_1^{-1}) A_\mu^\beta(x,y), \\ A_\mu^\alpha(x, y + 2\pi R) &= 2\,\mathrm{tr}(T^\alpha U T^\beta U^{-1}) A_\mu^\beta(x,y). \end{aligned} \tag{26}$$

Under the BC $[p,q,r,s]^{\rm I}$, A_μ^α is decomposed into a sum of multiplets of the subgroup $SO(2N) \to SO(2p) \times SO(2q) \times SO(2r) \times SO(2s)$ $(N = p+q+r+s)$ as

$$\begin{aligned} \mathbf{N(2N-1)} = {} & \mathbf{(p(2p-1), 1, 1, 1)}^{++;+} \\ & + \mathbf{(1, q(2q-1), 1, 1)}^{++;+} \\ & + \mathbf{(1, 1, r(2r-1), 1)}^{++;+} \\ & + \mathbf{(1, 1, 1, s(2s-1))}^{++;+} \\ & + \mathbf{(2p, 2q, 1, 1)}^{+-;-} + \mathbf{(2p, 1, 2r, 1)}^{-+;-} \\ & + \mathbf{(2p, 1, 1, 2s)}^{--;+} + \mathbf{(1, 2q, 2r, 1)}^{--;+} \\ & + \mathbf{(1, 2q, 1, 2s)}^{-+;-} + \mathbf{(1, 1, 2r, 2s)}^{+-;-}, \end{aligned} \tag{27}$$

where Z_2 parities are obtained using Eq. (26), and $\mathbf{p(2p-1)}$ and $\mathbf{2p}$ represent the components of A_μ^α with adjoint and vector representation of $SO(2p)$, respectively. The index $+$ or $-$ stands for Z_2 parity $+1$ or -1. The A_y^α have the opposite Z_2 parities $\mathcal{P}_0$ and $\mathcal{P}_1$ to those of A_μ^α.

We require the Z_2 parity invariance for the interaction between the gauge fields and a matter fermion ψ:

$$\mathcal{P}_0(\bar\psi \gamma^M A_M^\alpha T^\alpha \psi) = \mathcal{P}_1(\bar\psi \gamma^M A_M^\alpha T^\alpha \psi) = +1. \tag{28}$$

The invariance under the shift $y \to y + 2\pi R$, i.e., $\mathcal{U}(\bar\psi \gamma^M A_M^\alpha T^\alpha \psi) = +1$, is automatically satisfied from $\mathcal{P}_0 \mathcal{P}_1 \mathcal{U} = +1$.

There are two inequivalent spinor representations $\mathbf{2}_1^{N-1}$ and $\mathbf{2}_2^{N-1}$ in $SO(2N)$. For $N = 4\ell + 1$ and $4\ell + 3$ $(\ell \in \{\mathbb{N}, 0\})$, $\mathbf{2}_a^{N-1}$ are complex representations and they are conjugate to each other, i.e., $\bar{\mathbf{2}}_1^{N-1} = \mathbf{2}_2^{N-1}$ and $\bar{\mathbf{2}}_2^{N-1} = \mathbf{2}_1^{N-1}$. For $N = 4\ell$ $(\ell \in \mathbb{N})$, $\mathbf{2}_a^{N-1}$ $(a = 1, 2)$ are real representations and self-conjugate, i.e., $\bar{\mathbf{2}}_a^{N-1} = \mathbf{2}_a^{N-1}$. For $4\ell + 2$ $(\ell \in \{\mathbb{N}, 0\})$, $\mathbf{2}_a^{N-1}$ are pseudoreal representations and self-conjugate. If the matter fermion forms the spinor representations $\mathbf{2}_a^{N-1}$ or the vector representation $\mathbf{2N}$, the following relations hold:

$$\begin{aligned} & \mathcal{P}_0(\bar{\mathbf{2}}_a^{N-1} \times \mathbf{N(2N-1)} \times \mathbf{2}_a^{N-1}) \\ & \quad = \mathcal{P}_1(\bar{\mathbf{2}}_a^{N-1} \times \mathbf{N(2N-1)} \times \mathbf{2}_a^{N-1}) = +1, \end{aligned} \tag{29}$$

$$\begin{aligned} & \mathcal{P}_0(\mathbf{2N} \times \mathbf{N(2N-1)} \times \mathbf{2N}) \\ & \quad = \mathcal{P}_1(\mathbf{2N} \times \mathbf{N(2N-1)} \times \mathbf{2N}) = +1. \end{aligned} \tag{30}$$

By the Z_2 projection with P_0, $SO(2N)$ is broken down to $SO(2(p+q)) \times SO(2(r+s))$ and $\mathbf{2}_a^{N-1}$ and $\mathbf{2N}$ are decomposed into

$$\begin{aligned} \mathbf{2}_1^{N-1} &= (\mathbf{2}_1^{p+q-1}, \mathbf{2}_1^{r+s-1}) + (\mathbf{2}_2^{p+q-1}, \mathbf{2}_2^{r+s-1}), \\ \mathbf{2}_2^{N-1} &= (\mathbf{2}_1^{p+q-1}, \mathbf{2}_2^{r+s-1}) + (\mathbf{2}_2^{p+q-1}, \mathbf{2}_1^{r+s-1}), \\ \mathbf{2N} &= \mathbf{(2(p+q), 1)} + \mathbf{(1, 2(r+s))}. \end{aligned} \tag{31}$$

Using (27), (29), and (30), we find that each multiplet has a definite $\mathcal{P}_0$ as

$$\begin{aligned} \mathcal{P}_0((\mathbf{2}_1^{p+q-1}, \mathbf{2}_1^{r+s-1})) &= +\eta_1^0, \\ \mathcal{P}_0((\mathbf{2}_2^{p+q-1}, \mathbf{2}_2^{r+s-1})) &= -\eta_1^0, \\ \mathcal{P}_0((\mathbf{2}_1^{p+q-1}, \mathbf{2}_2^{r+s-1})) &= +\eta_2^0, \\ \mathcal{P}_0((\mathbf{2}_2^{p+q-1}, \mathbf{2}_1^{r+s-1})) &= -\eta_2^0, \\ \mathcal{P}_0(\mathbf{(2(p+q), 1)}) &= +\eta_{\rm v}^0, \\ \mathcal{P}_0(\mathbf{(1, 2(r+s))}) &= -\eta_{\rm v}^0, \end{aligned} \tag{32}$$

where η_1^0, η_2^0, and $\eta_{\rm v}^0$ are intrinsic Z_2 parities. In the same way, $SO(2N)$ is broken down to $SO(2(p+r)) \times SO(2(q+s))$ by P_1, and $\mathbf{2}_a^{N-1}$ and $\mathbf{2N}$ are decomposed into

$$\begin{aligned} \mathbf{2}_1^{N-1} &= (\mathbf{2}_1^{p+r-1}, \mathbf{2}_1^{q+s-1}) + (\mathbf{2}_2^{p+r-1}, \mathbf{2}_2^{q+s-1}), \\ \mathbf{2}_2^{N-1} &= (\mathbf{2}_1^{p+r-1}, \mathbf{2}_2^{q+s-1}) + (\mathbf{2}_2^{p+r-1}, \mathbf{2}_1^{q+s-1}), \\ \mathbf{2N} &= \mathbf{(2(p+r), 1)} + \mathbf{(1, 2(q+s))}. \end{aligned} \tag{33}$$

Each multiplet has a definite $\mathcal{P}_1$ as

$$\mathcal{P}_1((\mathbf{2}_1^{p+r-1}, \mathbf{2}_1^{q+s-1})) = +\eta_1^1, \quad \mathcal{P}_1((\mathbf{2}_2^{p+r-1}, \mathbf{2}_2^{q+s-1})) = -\eta_1^1, \quad \mathcal{P}_1((\mathbf{2}_1^{p+r-1}, \mathbf{2}_2^{q+s-1})) = +\eta_2^1,$$
$$\mathcal{P}_1((\mathbf{2}_2^{p+r-1}, \mathbf{2}_1^{q+s-1})) = -\eta_2^1, \quad \mathcal{P}_1((\mathbf{2(p+r)}, \mathbf{1})) = +\eta_{\mathrm{v}}^1, \quad \mathcal{P}_1((\mathbf{1}, \mathbf{2(q+s)})) = -\eta_{\mathrm{v}}^1, \tag{34}$$

where η_1^1, η_2^1, and η_{v}^1 are intrinsic Z_2 parities. The same argument holds for U.

Combining the Z_2 projections with P_0 and P_1, $SO(2N)$ is broken down to $SO(2p) \times SO(2q) \times SO(2r) \times SO(2s)$, and $\mathbf{2}_a^{N-1}$ and $\mathbf{2N}$ are decomposed into

$$\begin{aligned}
\mathbf{2}_1^{N-1} &= (\mathbf{2}_1^{p-1}, \mathbf{2}_1^{q-1}, \mathbf{2}_1^{r-1}, \mathbf{2}_1^{s-1}) + (\mathbf{2}_1^{p-1}, \mathbf{2}_1^{q-1}, \mathbf{2}_2^{r-1}, \mathbf{2}_2^{s-1}) + (\mathbf{2}_2^{p-1}, \mathbf{2}_2^{q-1}, \mathbf{2}_1^{r-1}, \mathbf{2}_1^{s-1}) + (\mathbf{2}_2^{p-1}, \mathbf{2}_2^{q-1}, \mathbf{2}_2^{r-1}, \mathbf{2}_2^{s-1}) \\
&\quad + (\mathbf{2}_1^{p-1}, \mathbf{2}_2^{q-1}, \mathbf{2}_1^{r-1}, \mathbf{2}_2^{s-1}) + (\mathbf{2}_1^{p-1}, \mathbf{2}_2^{q-1}, \mathbf{2}_2^{r-1}, \mathbf{2}_1^{s-1}) + (\mathbf{2}_2^{p-1}, \mathbf{2}_1^{q-1}, \mathbf{2}_1^{r-1}, \mathbf{2}_2^{s-1}) + (\mathbf{2}_2^{p-1}, \mathbf{2}_1^{q-1}, \mathbf{2}_2^{r-1}, \mathbf{2}_1^{s-1}), \\
\mathbf{2}_2^{N-1} &= (\mathbf{2}_1^{p-1}, \mathbf{2}_1^{q-1}, \mathbf{2}_1^{r-1}, \mathbf{2}_2^{s-1}) + (\mathbf{2}_1^{p-1}, \mathbf{2}_1^{q-1}, \mathbf{2}_2^{r-1}, \mathbf{2}_1^{s-1}) + (\mathbf{2}_2^{p-1}, \mathbf{2}_2^{q-1}, \mathbf{2}_1^{r-1}, \mathbf{2}_2^{s-1}) + (\mathbf{2}_2^{p-1}, \mathbf{2}_2^{q-1}, \mathbf{2}_2^{r-1}, \mathbf{2}_1^{s-1}) \\
&\quad + (\mathbf{2}_1^{p-1}, \mathbf{2}_2^{q-1}, \mathbf{2}_1^{r-1}, \mathbf{2}_1^{s-1}) + (\mathbf{2}_1^{p-1}, \mathbf{2}_2^{q-1}, \mathbf{2}_2^{r-1}, \mathbf{2}_2^{s-1}) + (\mathbf{2}_2^{p-1}, \mathbf{2}_1^{q-1}, \mathbf{2}_1^{r-1}, \mathbf{2}_1^{s-1}) + (\mathbf{2}_2^{p-1}, \mathbf{2}_1^{q-1}, \mathbf{2}_2^{r-1}, \mathbf{2}_2^{s-1}), \\
\mathbf{2N} &= (\mathbf{2p}, \mathbf{1}, \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{2q}, \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}, \mathbf{2r}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{2s}).
\end{aligned} \tag{35}$$

The Z_2 parities of each multiplet are listed in Table I. The eigenvalues of $\mathcal{U}$ are determined from $\mathcal{P}_0 \mathcal{P}_1 \mathcal{U} = +1$.

In the case with $s = 0$, $\mathbf{2}_a^{N-1}$ and $\mathbf{2N}$ are decomposed into

$$\begin{aligned}
\mathbf{2}_1^{N-1} &= (\mathbf{2}_1^{p-1}, \mathbf{2}_1^{q-1}, \mathbf{2}_1^{r-1}) + (\mathbf{2}_2^{p-1}, \mathbf{2}_2^{q-1}, \mathbf{2}_1^{r-1}) \\
&\quad + (\mathbf{2}_1^{p-1}, \mathbf{2}_2^{q-1}, \mathbf{2}_2^{r-1}) + (\mathbf{2}_2^{p-1}, \mathbf{2}_1^{q-1}, \mathbf{2}_2^{r-1}), \\
\mathbf{2}_2^{N-1} &= (\mathbf{2}_1^{p-1}, \mathbf{2}_1^{q-1}, \mathbf{2}_2^{r-1}) + (\mathbf{2}_1^{p-1}, \mathbf{2}_2^{q-1}, \mathbf{2}_1^{r-1}) \\
&\quad + (\mathbf{2}_2^{p-1}, \mathbf{2}_1^{q-1}, \mathbf{2}_1^{r-1}) + (\mathbf{2}_2^{p-1}, \mathbf{2}_2^{q-1}, \mathbf{2}_2^{r-1}), \\
\mathbf{2N} &= (\mathbf{2p}, \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{2q}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}, \mathbf{2r}),
\end{aligned} \tag{36}$$

under $SO(2p) \times SO(2q) \times SO(2r)$. In the case with $r = s = 0$, $\mathbf{2}_a^{N-1}$ and $\mathbf{2N}$ are decomposed into

TABLE I. Z_2 parities of matter fermions in type I.

Representation		$\mathcal{P}_0$	$\mathcal{P}_1$	$\mathcal{U}$
$\mathbf{2}_1^{N-1}$	$(\mathbf{2}_1^{p-1}, \mathbf{2}_1^{q-1}, \mathbf{2}_1^{r-1}, \mathbf{2}_1^{s-1})$	$+\eta_1^0$	$+\eta_1^1$	$+\eta_1^0\eta_1^1$
	$(\mathbf{2}_1^{p-1}, \mathbf{2}_1^{q-1}, \mathbf{2}_2^{r-1}, \mathbf{2}_2^{s-1})$	$+\eta_1^0$	$-\eta_1^1$	$-\eta_1^0\eta_1^1$
	$(\mathbf{2}_2^{p-1}, \mathbf{2}_2^{q-1}, \mathbf{2}_1^{r-1}, \mathbf{2}_1^{s-1})$	$+\eta_1^0$	$-\eta_1^1$	$-\eta_1^0\eta_1^1$
	$(\mathbf{2}_2^{p-1}, \mathbf{2}_2^{q-1}, \mathbf{2}_2^{r-1}, \mathbf{2}_2^{s-1})$	$+\eta_1^0$	$+\eta_1^1$	$+\eta_1^0\eta_1^1$
	$(\mathbf{2}_1^{p-1}, \mathbf{2}_2^{q-1}, \mathbf{2}_1^{r-1}, \mathbf{2}_2^{s-1})$	$-\eta_1^0$	$+\eta_1^1$	$-\eta_1^0\eta_1^1$
	$(\mathbf{2}_1^{p-1}, \mathbf{2}_2^{q-1}, \mathbf{2}_2^{r-1}, \mathbf{2}_1^{s-1})$	$-\eta_1^0$	$-\eta_1^1$	$+\eta_1^0\eta_1^1$
	$(\mathbf{2}_2^{p-1}, \mathbf{2}_1^{q-1}, \mathbf{2}_1^{r-1}, \mathbf{2}_2^{s-1})$	$-\eta_1^0$	$-\eta_1^1$	$+\eta_1^0\eta_1^1$
	$(\mathbf{2}_2^{p-1}, \mathbf{2}_1^{q-1}, \mathbf{2}_2^{r-1}, \mathbf{2}_1^{s-1})$	$-\eta_1^0$	$+\eta_1^1$	$-\eta_1^0\eta_1^1$
$\mathbf{2}_2^{N-1}$	$(\mathbf{2}_1^{p-1}, \mathbf{2}_1^{q-1}, \mathbf{2}_1^{r-1}, \mathbf{2}_2^{s-1})$	$+\eta_2^0$	$+\eta_2^1$	$+\eta_2^0\eta_2^1$
	$(\mathbf{2}_1^{p-1}, \mathbf{2}_1^{q-1}, \mathbf{2}_2^{r-1}, \mathbf{2}_1^{s-1})$	$+\eta_2^0$	$-\eta_2^1$	$-\eta_2^0\eta_2^1$
	$(\mathbf{2}_2^{p-1}, \mathbf{2}_2^{q-1}, \mathbf{2}_1^{r-1}, \mathbf{2}_2^{s-1})$	$+\eta_2^0$	$-\eta_2^1$	$-\eta_2^0\eta_2^1$
	$(\mathbf{2}_2^{p-1}, \mathbf{2}_2^{q-1}, \mathbf{2}_2^{r-1}, \mathbf{2}_1^{s-1})$	$+\eta_2^0$	$+\eta_2^1$	$+\eta_2^0\eta_2^1$
	$(\mathbf{2}_1^{p-1}, \mathbf{2}_2^{q-1}, \mathbf{2}_1^{r-1}, \mathbf{2}_1^{s-1})$	$-\eta_2^0$	$+\eta_2^1$	$-\eta_2^0\eta_2^1$
	$(\mathbf{2}_1^{p-1}, \mathbf{2}_2^{q-1}, \mathbf{2}_2^{r-1}, \mathbf{2}_2^{s-1})$	$-\eta_2^0$	$-\eta_2^1$	$+\eta_2^0\eta_2^1$
	$(\mathbf{2}_2^{p-1}, \mathbf{2}_1^{q-1}, \mathbf{2}_1^{r-1}, \mathbf{2}_1^{s-1})$	$-\eta_2^0$	$-\eta_2^1$	$+\eta_2^0\eta_2^1$
	$(\mathbf{2}_2^{p-1}, \mathbf{2}_1^{q-1}, \mathbf{2}_2^{r-1}, \mathbf{2}_2^{s-1})$	$-\eta_2^0$	$+\eta_2^1$	$-\eta_2^0\eta_2^1$
$\mathbf{2N}$	$(\mathbf{2p}, \mathbf{1}, \mathbf{1}, \mathbf{1})$	$+\eta_{\mathrm{v}}^0$	$+\eta_{\mathrm{v}}^1$	$+\eta_{\mathrm{v}}^0\eta_{\mathrm{v}}^1$
	$(\mathbf{1}, \mathbf{2q}, \mathbf{1}, \mathbf{1})$	$+\eta_{\mathrm{v}}^0$	$-\eta_{\mathrm{v}}^1$	$-\eta_{\mathrm{v}}^0\eta_{\mathrm{v}}^1$
	$(\mathbf{1}, \mathbf{1}, \mathbf{2r}, \mathbf{1})$	$-\eta_{\mathrm{v}}^0$	$+\eta_{\mathrm{v}}^1$	$-\eta_{\mathrm{v}}^0\eta_{\mathrm{v}}^1$
	$(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{2s})$	$-\eta_{\mathrm{v}}^0$	$-\eta_{\mathrm{v}}^1$	$+\eta_{\mathrm{v}}^0\eta_{\mathrm{v}}^1$

$$\begin{aligned}
\mathbf{2}_1^{N-1} &= (\mathbf{2}_1^{p-1}, \mathbf{2}_1^{q-1}) + (\mathbf{2}_2^{p-1}, \mathbf{2}_2^{q-1}), \\
\mathbf{2}_2^{N-1} &= (\mathbf{2}_1^{p-1}, \mathbf{2}_2^{q-1}) + (\mathbf{2}_2^{p-1}, \mathbf{2}_1^{q-1}), \\
\mathbf{2N} &= (\mathbf{2p}, \mathbf{1}) + (\mathbf{1}, \mathbf{2q}),
\end{aligned} \tag{37}$$

under $SO(2p) \times SO(2q)$. The Z_2 parities of each multiplet are understood from those for the corresponding representations in Table I.

(Type II) Under the BC $[m, 0, n, 0]^{\mathrm{IIa}}$, A_μ^α is decomposed into a sum of multiplets of the subgroup $SO(2N) \to SU(m) \times SU(n) \times U(1)^{2-k}$ ($N = m + n$) as

$$\begin{aligned}
\mathbf{N(2N-1)} &= (\mathbf{m^2-1}, \mathbf{1})^{++;+} + (\mathbf{1}, \mathbf{n^2-1})^{++;+} \\
&\quad + (\mathbf{1}, \mathbf{1})^{++;+} + (\mathbf{1}, \mathbf{1})^{++;+} + (\mathbf{m}, \bar{\mathbf{n}})^{-+;-} \\
&\quad + (\bar{\mathbf{m}}, \mathbf{n})^{-+;-} + (\mathbf{m(m-1)/2}, \mathbf{1})^{+-;-} \\
&\quad + (\mathbf{1}, \mathbf{n(n-1)/2})^{+-;-} + (\overline{\mathbf{m(m-1)/2}}, \mathbf{1})^{+-;-} \\
&\quad + (\mathbf{1}, \overline{\mathbf{n(n-1)/2}})^{+-;-} + (\mathbf{m}, \mathbf{n})^{--;+} \\
&\quad + (\bar{\mathbf{m}}, \bar{\mathbf{n}})^{--;+},
\end{aligned} \tag{38}$$

where Z_2 parities are obtained using Eq. (26), $U(1)$ charges are omitted, and $\mathbf{m}^2 - \mathbf{1}(\mathbf{n}^2 - \mathbf{1})$, $\mathbf{m}(\mathbf{n})$, and $\mathbf{m(m-1)/2(n(n-1)/2)}$ represent the components of A_μ^α with adjoint, vector, and rank 2 antisymmetric representation of $SU(m)$ [$SU(n)$], respectively. The representations with the overline stand for the complex conjugate ones.

By the Z_2 projection with P_1, $SO(2N)$ is broken down to its subgroup including $SU(N)$ whose adjoint representation $\mathbf{N^2-1}$ is given by

$$\begin{aligned}
\mathbf{N^2-1} &= (\mathbf{m^2-1}, \mathbf{1})^{++;+} + (\mathbf{1}, \mathbf{n^2-1})^{++;+} \\
&\quad + (\mathbf{1}, \mathbf{1})^{++;+} + (\mathbf{m}, \bar{\mathbf{n}})^{-+;-} + (\bar{\mathbf{m}}, \mathbf{n})^{-+;-}.
\end{aligned} \tag{39}$$

In the same way, by U, $SO(2N)$ is broken down to its subgroup including $SU(N)$ whose adjoint representation $\mathbf{N^2-1}$ is given by

$$\mathbf{N^2-1} = (\mathbf{m^2-1},\mathbf{1})^{++;+} + (\mathbf{1},\mathbf{n^2-1})^{++;+} + (\mathbf{1},\mathbf{1})^{++;+} + (\mathbf{m},\mathbf{n})^{--;+} + (\bar{\mathbf{m}},\bar{\mathbf{n}})^{--;+}. \tag{40}$$

Under the exchange of P_1 and U, the adjoint representations (39) and (40) are exchanged. It corresponds to the relation between the Georgi-Glashow type $SU(5)$ [25] and the flipped type $SU(5)$ [26] in $SO(10)$ GUTs [27,28].

We study the Z_2 parity assignment for matter fermions. By the Z_2 projection with P_0, $SO(2N)$ is broken down to $SO(2m)\times SO(2n)$, and $\mathbf{2}_a^{N-1}$ and $\mathbf{2N}$ are decomposed into

$$\begin{aligned}\mathbf{2}_1^{N-1} &= (\mathbf{2}_1^{m-1},\mathbf{2}_1^{n-1}) + (\mathbf{2}_2^{m-1},\mathbf{2}_2^{n-1}),\\ \mathbf{2}_2^{N-1} &= (\mathbf{2}_1^{m-1},\mathbf{2}_2^{n-1}) + (\mathbf{2}_2^{m-1},\mathbf{2}_1^{n-1}),\\ \mathbf{2N} &= (\mathbf{2m},\mathbf{1}) + (\mathbf{1},\mathbf{2n}).\end{aligned} \tag{41}$$

Using (27), (29), and (30), we find that each multiplet has a definite $\mathcal{P}_0$ as

$$\begin{aligned}&\mathcal{P}_0((\mathbf{2}_1^{m-1},\mathbf{2}_1^{n-1})) = +\eta_1^0, \quad &&\mathcal{P}_0((\mathbf{2}_2^{m-1},\mathbf{2}_2^{n-1})) = -\eta_1^0,\\ &\mathcal{P}_0((\mathbf{2}_1^{m-1},\mathbf{2}_2^{n-1})) = +\eta_2^0, &&\mathcal{P}_0((\mathbf{2}_2^{m-1},\mathbf{2}_1^{n-1})) = -\eta_2^0,\\ &\mathcal{P}_0((\mathbf{2m},\mathbf{1})) = +\eta_{\rm v}^0, &&\mathcal{P}_0((\mathbf{1},\mathbf{2n})) = -\eta_{\rm v}^0.\end{aligned} \tag{42}$$

In the same way, $SO(2N)$ is broken down to $SU(N)\times U(1)$ by P_1, and $\mathbf{2}_a^{N-1}$ and $\mathbf{2N}$ are decomposed into

$$\mathbf{2}_1^{N-1} = \sum_{k=0}^{[N/2]}[N,2k], \qquad \mathbf{2}_2^{N-1} = \sum_{k=0}^{[(N-1)/2]}[N,2k+1],$$
$$\mathbf{2N} = \mathbf{N} + \bar{\mathbf{N}}, \tag{43}$$

where $[N/2]$ and $[(N-1)/2]$ represent Gauss's symbol, and $[N,2k](={}_NC_{2k})$ and $[N,2k+1](={}_NC_{2k+1})$ are the rank $2k+1$ totally antisymmetric representations of the $SU(N)$ gauge group and the $U(1)$ charge is omitted. Each multiplet has a definite $\mathcal{P}_1$ as

$$\begin{aligned}\mathcal{P}_1([N,2k]) &= (-1)^k\eta_1^1,\\ \mathcal{P}_1([N,2k+1]) &= (-1)^k\eta_2^1,\\ \mathcal{P}_1(\mathbf{N}) &= +\eta_{\rm v}^1,\\ \mathcal{P}_1(\bar{\mathbf{N}}) &= -\eta_{\rm v}^1.\end{aligned} \tag{44}$$

The same argument holds for U.

Combining the Z_2 projections with P_0 and P_1, $SO(2N)$ is broken down to $SU(m)\times SU(n)\times U(1)^{2-k}$, and $\mathbf{2}_a^{N-1}$ and $\mathbf{2N}$ are decomposed into

$$\begin{aligned}\mathbf{2}_1^{N-1} &= \sum_{k=0}^{[N/2]}\sum_{\ell=0}^{2k}({}_mC_\ell,{}_nC_{2k-\ell}) = \sum_{k=0}^{[N/2]}\sum_{\ell={\rm even}}({}_mC_\ell,{}_nC_{2k-\ell}) + \sum_{k=0}^{[N/2]}\sum_{\ell={\rm odd}}({}_mC_\ell,{}_nC_{2k-\ell}),\\ \mathbf{2}_2^{N-1} &= \sum_{k=0}^{[(N-1)/2]}\sum_{\ell=0}^{2k+1}({}_mC_\ell,{}_nC_{2k-\ell+1}) = \sum_{k=0}^{[(N-1)/2]}\sum_{\ell={\rm even}}({}_mC_\ell,{}_nC_{2k-\ell+1}) + \sum_{k=0}^{[(N-1)/2]}\sum_{\ell={\rm odd}}({}_mC_\ell,{}_nC_{2k-\ell+1}),\\ \mathbf{2N} &= (\mathbf{m},\mathbf{1}) + (\mathbf{1},\mathbf{n}) + (\bar{\mathbf{m}},\mathbf{1}) + (\mathbf{1},\bar{\mathbf{n}}).\end{aligned} \tag{45}$$

The Z_2 parities of each multiplet are listed in Table II. The Z_2 parity assignments for types IIb and IIc are obtained by the exchange of $\mathcal{P}_0$, $\mathcal{P}_1$, and $\mathcal{U}$, i.e., $(\mathcal{P}_0,\mathcal{P}_1,\mathcal{U})^{\rm IIa} = (\mathcal{P}_1,\mathcal{P}_0,\mathcal{U})^{\rm IIb} = (\mathcal{U},\mathcal{P}_0,\mathcal{P}_1)^{\rm IIc}$.

A fermion with spin 1/2 in five dimensions is regarded as a Dirac fermion or a pair of Weyl fermions with opposite chiralities in four dimensions. The representations of each Weyl fermion are decomposed in the same way, but left-handed Weyl fermions and right-handed ones should have opposite Z_2 parities to each other, i.e., $(\mathcal{P}_{0R},\mathcal{P}_{1R};\mathcal{U}_R) = (-\mathcal{P}_{0L},-\mathcal{P}_{1L};-\mathcal{U}_L)$, from the requirement that the kinetic term is invariant under the Z_2 parity transformation. Here, $(\mathcal{P}_{0R},\mathcal{P}_{1R};\mathcal{U}_R)$ and $(\mathcal{P}_{0L},\mathcal{P}_{1L};\mathcal{U}_L)$ are Z_2 parities for right-handed Weyl fermions and left-handed ones, respectively. Zero modes for not only left-handed Weyl fermions but also right-handed ones, having even Z_2 parities, compose chiral fermions in the SM.

TABLE II. Z_2 parities of matter fermions in type IIa.

Representation		$\mathcal{P}_0$	$\mathcal{P}_1$	$\mathcal{U}$
$\mathbf{2}_1^{N-1}$	$({}_mC_\ell,{}_nC_{2k-\ell})_{\ell={\rm even}}$	$+\eta_1^0$	$(-1)^k\eta_1^1$	$(-1)^k\eta_1^0\eta_1^1$
	$({}_mC_\ell,{}_nC_{2k-\ell})_{\ell={\rm odd}}$	$-\eta_1^0$	$(-1)^k\eta_1^1$	$-(-1)^k\eta_1^0\eta_1^1$
$\mathbf{2}_2^{N-1}$	$({}_mC_\ell,{}_nC_{2k-\ell+1})_{\ell={\rm even}}$	$+\eta_2^0$	$(-1)^k\eta_2^1$	$(-1)^k\eta_2^0\eta_2^1$
	$({}_mC_\ell,{}_nC_{2k-\ell+1})_{\ell={\rm odd}}$	$-\eta_2^0$	$(-1)^k\eta_2^1$	$-(-1)^k\eta_2^0\eta_2^1$
$\mathbf{2N}$	$(\mathbf{m},\mathbf{1})$	$+\eta_{\rm v}^0$	$+\eta_{\rm v}^1$	$+\eta_{\rm v}^0\eta_{\rm v}^1$
	$(\mathbf{1},\mathbf{n})$	$-\eta_{\rm v}^0$	$+\eta_{\rm v}^1$	$-\eta_{\rm v}^0\eta_{\rm v}^1$
	$(\bar{\mathbf{m}},\mathbf{1})$	$+\eta_{\rm v}^0$	$-\eta_{\rm v}^1$	$-\eta_{\rm v}^0\eta_{\rm v}^1$
	$(\mathbf{1},\bar{\mathbf{n}})$	$-\eta_{\rm v}^0$	$-\eta_{\rm v}^1$	$+\eta_{\rm v}^0\eta_{\rm v}^1$

In supersymmetry (SUSY) models, the hypermultiplet is the fundamental quantity concerning bulk matter fields in five dimensions. The hypermultiplet is equivalent to a pair of chiral multiplets with opposite gauge quantum numbers such as the representation $\mathbf{R}$ and the conjugate one $\bar{\mathbf{R}}$ in four dimensions. The chiral multiplet with $\bar{\mathbf{R}}$ contains a left-handed Weyl fermion with $\bar{\mathbf{R}}_L$. This Weyl fermion is regarded as a right-handed one with $\mathbf{R}_R$ by the use of the charge conjugation. Hence our analysis works on SUSY models as well as non-SUSY ones.

III. UNIFICATION OF QUARKS AND LEPTONS BASED ON $SO(2N)$

Now let us investigate unification of quarks and leptons in $SO(2N)$ gauge theory on S^1/Z_2. We count the numbers of fermion species coming from a single multiplet $\mathbf{2}_1^{N-1}$ or $\mathbf{2}_2^{N-1}$ based on the survival hypothesis for the following breaking patterns:

$$SO(2N) \to SO(10) \times H_1, \tag{46}$$

$$SO(2N) \to SO(6) \times SO(4) \times H_2 \simeq SU(4)_C \times SU(2)_L \times SU(2)_R \times H_2, \tag{47}$$

$$SO(2N) \to SU(5) \times SU(N-5) \times U(1)^2, \tag{48}$$

where H_1 and H_2 are some product groups such as $SO(2r_1) \times \cdots \times SO(2r_n)$.

A. $SO(2N) \supset SO(10)$

First, we study the symmetry breaking pattern $SO(2N) \to SO(10) \times H_1$. In the case with the breaking pattern $SO(2N) \to SO(10) \times SO(2(N-5))$, Weyl fermions with $\mathbf{2}_{1L}^{N-1}$ and $\mathbf{2}_{1R}^{N-1}$ are decomposed into

$$\begin{aligned} \mathbf{2}_{1L}^{N-1} &= (\mathbf{16}, \mathbf{2}_1^{N-6})_L + (\overline{\mathbf{16}}, \mathbf{2}_2^{N-6})_L, \\ \mathbf{2}_{1R}^{N-1} &= (\mathbf{16}, \mathbf{2}_1^{N-6})_R + (\overline{\mathbf{16}}, \mathbf{2}_2^{N-6})_R, \end{aligned} \tag{49}$$

and the Z_2 parities of each multiplet are given in Table III. If we take $\eta_1^1 = \eta_1^0 = +1$, no mirror particles appear and $(\mathbf{16}, \mathbf{2}_1^{N-6})_L$ survives. Then the number of $\mathbf{16}_L$ is regarded as that of families. Hence we have 2^{N-6} families for the $SO(10)$ multiplets. The same argument holds for the case with $\mathbf{2}_2^{N-1}$.

TABLE III. Z_2 parity assignment for $\mathbf{2}_{1L}^{N-1}$ and $\mathbf{2}_{1R}^{N-1}$ in $SO(10) \times SO(2(N-5))$.

Representation		$\mathcal{P}_0$	$\mathcal{P}_1$	$\mathcal{U}$
$\mathbf{2}_{1L}^{N-1}$	$(\mathbf{16}, \mathbf{2}_1^{N-6})_L$	$+\eta_1^0$	$+\eta_1^1$	$+\eta_1^0\eta_1^1$
	$(\overline{\mathbf{16}}, \mathbf{2}_2^{N-6})_L$	$+\eta_1^0$	$-\eta_1^1$	$-\eta_1^0\eta_1^1$
$\mathbf{2}_{1R}^{N-1}$	$(\mathbf{16}, \mathbf{2}_1^{N-6})_R$	$-\eta_1^0$	$-\eta_1^1$	$+\eta_1^0\eta_1^1$
	$(\overline{\mathbf{16}}, \mathbf{2}_2^{N-6})_R$	$-\eta_1^0$	$+\eta_1^1$	$-\eta_1^0\eta_1^1$

We find that no massless fermions survive in the case in which $H_1 = SO(2r_1) \times SO(2r_2) \times SO(2r_3)$ or $H_1 = SO(2r_1) \times SO(2r_2)$ after the survival hypothesis is imposed.

B. $SO(2N) \supset SU(4)_C \times SU(2)_L \times SU(2)_R$

Next, we study the symmetry breaking pattern $SO(2N) \to G_{\mathrm{PS}} \times H_2$ where G_{PS} is the Pati-Salam gauge group $SU(4)_C \times SU(2)_L \times SU(2)_R$ [29].

In the case with $SO(2N) \to G_{\mathrm{PS}} \times SO(2q) \times SO(2s)$, Weyl fermions with $\mathbf{2}_{1L}^{N-1}$ and $\mathbf{2}_{1R}^{N-1}$ are decomposed into

$$\begin{aligned} \mathbf{2}_{1L}^{N-1} = {} & (\mathbf{4}, \mathbf{2}, \mathbf{1}, \mathbf{2}_1^{q-1}, \mathbf{2}_1^{r-1})_L + (\mathbf{4}, \mathbf{2}, \mathbf{1}, \mathbf{2}_2^{q-1}, \mathbf{2}_2^{r-1})_L + (\bar{\mathbf{4}}, \mathbf{1}, \mathbf{2}, \mathbf{2}_1^{q-1}, \mathbf{2}_1^{r-1})_L + (\bar{\mathbf{4}}, \mathbf{1}, \mathbf{2}, \mathbf{2}_2^{q-1}, \mathbf{2}_2^{r-1})_L, +(\mathbf{4}, \mathbf{1}, \mathbf{2}, \mathbf{2}_1^{q-1}, \mathbf{2}_2^{r-1})_L \\ & + (\mathbf{4}, \mathbf{1}, \mathbf{2}, \mathbf{2}_2^{q-1}, \mathbf{2}_1^{r-1})_L + (\bar{\mathbf{4}}, \mathbf{2}, \mathbf{1}, \mathbf{2}_1^{q-1}, \mathbf{2}_2^{r-1})_L + (\bar{\mathbf{4}}, \mathbf{2}, \mathbf{1}, \mathbf{2}_2^{q-1}, \mathbf{2}_1^{r-1})_L, \\ \mathbf{2}_{1R}^{N-1} = {} & (\mathbf{4}, \mathbf{2}, \mathbf{1}, \mathbf{2}_1^{q-1}, \mathbf{2}_1^{r-1})_R + (\mathbf{4}, \mathbf{2}, \mathbf{1}, \mathbf{2}_2^{q-1}, \mathbf{2}_2^{r-1})_R + (\bar{\mathbf{4}}, \mathbf{1}, \mathbf{2}, \mathbf{2}_1^{q-1}, \mathbf{2}_1^{r-1})_R + (\bar{\mathbf{4}}, \mathbf{1}, \mathbf{2}, \mathbf{2}_2^{q-1}, \mathbf{2}_2^{r-1})_R, +(\mathbf{4}, \mathbf{1}, \mathbf{2}, \mathbf{2}_1^{q-1}, \mathbf{2}_2^{r-1})_R \\ & + (\mathbf{4}, \mathbf{1}, \mathbf{2}, \mathbf{2}_2^{q-1}, \mathbf{2}_1^{r-1})_R + (\bar{\mathbf{4}}, \mathbf{2}, \mathbf{1}, \mathbf{2}_1^{q-1}, \mathbf{2}_2^{r-1})_R + (\bar{\mathbf{4}}, \mathbf{2}, \mathbf{1}, \mathbf{2}_2^{q-1}, \mathbf{2}_1^{r-1})_R, \end{aligned} \tag{50}$$

and the Z_2 parities of each multiplet are given in Table IV. If we take $\eta_1^1 = \eta_1^0 = +1$, $(\mathbf{4}, \mathbf{2}, \mathbf{1}, \mathbf{2}_1^{r-1}, \mathbf{2}_1^{s-1})_L$, $(\bar{\mathbf{4}}, \mathbf{1}, \mathbf{2}, \mathbf{2}_2^{r-1}, \mathbf{2}_2^{s-1})_L$, $(\mathbf{4}, \mathbf{1}, \mathbf{2}, \mathbf{2}_2^{r-1}, \mathbf{2}_1^{s-1})_R$ and $(\bar{\mathbf{4}}, \mathbf{2}, \mathbf{1}, \mathbf{2}_1^{r-1}, \mathbf{2}_2^{s-1})_R$ have zero modes. Hence we have 2^{N-6} families for the G_{PS} multiplets. The same argument holds for the case with $\mathbf{2}_2^{N-1}$.

In the case with $SO(2N) \to G_{\mathrm{PS}} \times SO(2(N-5))$, Weyl fermions with $\mathbf{2}_{1L}^{N-1}$ and $\mathbf{2}_{1R}^{N-1}$ are decomposed into

$$\begin{aligned} \mathbf{2}_{1L}^{N-1} &= (\mathbf{4}, \mathbf{2}, \mathbf{1}, \mathbf{2}_1^{N-6})_L + (\bar{\mathbf{4}}, \mathbf{1}, \mathbf{2}, \mathbf{2}_1^{N-6})_L \\ &\quad + (\mathbf{4}, \mathbf{1}, \mathbf{2}, \mathbf{2}_2^{N-6})_L + (\bar{\mathbf{4}}, \mathbf{2}, \mathbf{1}, \mathbf{2}_2^{N-6})_L, \\ \mathbf{2}_{1R}^{N-1} &= (\mathbf{4}, \mathbf{2}, \mathbf{1}, \mathbf{2}_1^{N-6})_R + (\bar{\mathbf{4}}, \mathbf{1}, \mathbf{2}, \mathbf{2}_1^{N-6})_R \\ &\quad + (\mathbf{4}, \mathbf{1}, \mathbf{2}, \mathbf{2}_2^{N-6})_R + (\bar{\mathbf{4}}, \mathbf{2}, \mathbf{1}, \mathbf{2}_2^{N-6})_R, \end{aligned} \tag{51}$$

and the Z_2 parities of each multiplet are given in Table V. If we take $\eta_1^1 = \eta_1^0 = +1$, $(\mathbf{4}, \mathbf{2}, \mathbf{1}, \mathbf{2}_1^{N-6})_L$ and $(\mathbf{4}, \mathbf{1}, \mathbf{2}, \mathbf{2}_2^{N-6})_R$ have zero modes. Then the number of such pairs is regarded as that of families. Hence we have 2^{N-6} families for the G_{PS} multiplets. The same argument holds for the case with $\mathbf{2}_2^{N-1}$.

TABLE IV. Z_2 parity assignment for $\mathbf{2}_{1L}^{N-1}$ and $\mathbf{2}_{1R}^{N-1}$ in $G_{\mathrm{PS}} \times SO(2q) \times SO(2s)$.

Representation		$\mathcal{P}_0$	$\mathcal{P}_1$	$\mathcal{U}$
$\mathbf{2}_{1L}^{N-1}$	$(\mathbf{4}, \mathbf{2}, \mathbf{1}, \mathbf{2}_1^{r-1}, \mathbf{2}_1^{s-1})_L$	$+\eta_1^0$	$+\eta_1^1$	$+\eta_1^0\eta_1^1$
	$(\mathbf{4}, \mathbf{2}, \mathbf{1}, \mathbf{2}_2^{r-1}, \mathbf{2}_2^{s-1})_L$	$+\eta_1^0$	$-\eta_1^1$	$-\eta_1^0\eta_1^1$
	$(\bar{\mathbf{4}}, \mathbf{1}, \mathbf{2}, \mathbf{2}_1^{r-1}, \mathbf{2}_1^{s-1})_L$	$+\eta_1^0$	$-\eta_1^1$	$-\eta_1^0\eta_1^1$
	$(\bar{\mathbf{4}}, \mathbf{1}, \mathbf{2}, \mathbf{2}_2^{r-1}, \mathbf{2}_2^{s-1})_L$	$+\eta_1^0$	$+\eta_1^1$	$+\eta_1^0\eta_1^1$
	$(\mathbf{4}, \mathbf{1}, \mathbf{2}, \mathbf{2}_1^{r-1}, \mathbf{2}_2^{s-1})_L$	$-\eta_1^0$	$+\eta_1^1$	$-\eta_1^0\eta_1^1$
	$(\mathbf{4}, \mathbf{1}, \mathbf{2}, \mathbf{2}_2^{r-1}, \mathbf{2}_1^{s-1})_L$	$-\eta_1^0$	$-\eta_1^1$	$+\eta_1^0\eta_1^1$
	$(\bar{\mathbf{4}}, \mathbf{2}, \mathbf{1}, \mathbf{2}_1^{r-1}, \mathbf{2}_2^{s-1})_L$	$-\eta_1^0$	$-\eta_1^1$	$+\eta_1^0\eta_1^1$
	$(\bar{\mathbf{4}}, \mathbf{2}, \mathbf{1}, \mathbf{2}_2^{r-1}, \mathbf{2}_1^{s-1})_L$	$-\eta_1^0$	$+\eta_1^1$	$-\eta_1^0\eta_1^1$
$\mathbf{2}_{1R}^{N-1}$	$(\mathbf{4}, \mathbf{2}, \mathbf{1}, \mathbf{2}_1^{r-1}, \mathbf{2}_1^{s-1})_R$	$-\eta_1^0$	$-\eta_1^1$	$+\eta_1^0\eta_1^1$
	$(\mathbf{4}, \mathbf{2}, \mathbf{1}, \mathbf{2}_2^{r-1}, \mathbf{2}_2^{s-1})_R$	$-\eta_1^0$	$+\eta_1^1$	$-\eta_1^0\eta_1^1$
	$(\bar{\mathbf{4}}, \mathbf{1}, \mathbf{2}, \mathbf{2}_1^{r-1}, \mathbf{2}_1^{s-1})_R$	$-\eta_1^0$	$+\eta_1^1$	$-\eta_1^0\eta_1^1$
	$(\bar{\mathbf{4}}, \mathbf{1}, \mathbf{2}, \mathbf{2}_2^{r-1}, \mathbf{2}_2^{s-1})_R$	$-\eta_1^0$	$-\eta_1^1$	$+\eta_1^0\eta_1^1$
	$(\mathbf{4}, \mathbf{1}, \mathbf{2}, \mathbf{2}_1^{r-1}, \mathbf{2}_2^{s-1})_R$	$+\eta_1^0$	$-\eta_1^1$	$-\eta_1^0\eta_1^1$
	$(\mathbf{4}, \mathbf{1}, \mathbf{2}, \mathbf{2}_2^{r-1}, \mathbf{2}_1^{s-1})_R$	$+\eta_1^0$	$+\eta_1^1$	$+\eta_1^0\eta_1^1$
	$(\bar{\mathbf{4}}, \mathbf{2}, \mathbf{1}, \mathbf{2}_1^{r-1}, \mathbf{2}_2^{s-1})_R$	$+\eta_1^0$	$+\eta_1^1$	$+\eta_1^0\eta_1^1$
	$(\bar{\mathbf{4}}, \mathbf{2}, \mathbf{1}, \mathbf{2}_2^{r-1}, \mathbf{2}_1^{s-1})_R$	$+\eta_1^0$	$-\eta_1^1$	$-\eta_1^0\eta_1^1$

TABLE V. Z_2 parity assignment for $\mathbf{2}_{1L}^{N-1}$ and $\mathbf{2}_{1R}^{N-1}$ in $G_{\mathrm{PS}} \times SO(2(N-5))$.

Representation		$\mathcal{P}_0$	$\mathcal{P}_1$	$\mathcal{U}$
$\mathbf{2}_{1L}^{N-1}$	$(\mathbf{4}, \mathbf{2}, \mathbf{1}, \mathbf{2}_1^{(N-6)})_L$	$+\eta_1^0$	$+\eta_1^1$	$+\eta_1^0\eta_1^1$
	$(\bar{\mathbf{4}}, \mathbf{1}, \mathbf{2}, \mathbf{2}_1^{(N-6)})_L$	$+\eta_1^0$	$-\eta_1^1$	$-\eta_1^0\eta_1^1$
	$(\mathbf{4}, \mathbf{1}, \mathbf{2}, \mathbf{2}_2^{(N-6)})_L$	$-\eta_1^0$	$-\eta_1^1$	$+\eta_1^0\eta_1^1$
	$(\bar{\mathbf{4}}, \mathbf{2}, \mathbf{1}, \mathbf{2}_2^{(N-6)})_L$	$-\eta_1^0$	$+\eta_1^1$	$-\eta_1^0\eta_1^1$
$\mathbf{2}_{1R}^{N-1}$	$(\mathbf{4}, \mathbf{2}, \mathbf{1}, \mathbf{2}_1^{(N-6)})_R$	$-\eta_1^0$	$-\eta_1^1$	$+\eta_1^0\eta_1^1$
	$(\bar{\mathbf{4}}, \mathbf{1}, \mathbf{2}, \mathbf{2}_1^{(N-6)})_R$	$-\eta_1^0$	$+\eta_1^1$	$-\eta_1^0\eta_1^1$
	$(\mathbf{4}, \mathbf{1}, \mathbf{2}, \mathbf{2}_2^{(N-6)})_R$	$+\eta_1^0$	$+\eta_1^1$	$+\eta_1^0\eta_1^1$
	$(\bar{\mathbf{4}}, \mathbf{2}, \mathbf{1}, \mathbf{2}_2^{(N-6)})_R$	$+\eta_1^0$	$-\eta_1^1$	$-\eta_1^0\eta_1^1$

C. $SO(2N) \supset SU(5)$

Finally, we study the symmetry breaking pattern $SO(2N) \to SU(5) \times SU(N-5) \times U(1)^2$. In this case, Weyl fermions with $\mathbf{2}_{1L}^{N-1}$ and $\mathbf{2}_{1R}^{N-1}$ are decomposed into

$$\begin{aligned} \mathbf{2}_{1L}^{N-1} &= \sum_{k=0}^{[N/2]} \sum_{\ell=\text{even}} ({}_5C_\ell, {}_{N-5}C_{2k-\ell})_L + \sum_{k=0}^{[N/2]} \sum_{\ell=\text{odd}} ({}_5C_\ell, {}_{N-5}C_{2k-\ell})_L, \\ \mathbf{2}_{1R}^{N-1} &= \sum_{k=0}^{[N/2]} \sum_{\ell=\text{even}} ({}_5C_\ell, {}_{N-5}C_{2k-\ell})_R + \sum_{k=0}^{[N/2]} \sum_{\ell=\text{odd}} ({}_5C_\ell, {}_{N-5}C_{2k-\ell})_R, \end{aligned} \tag{52}$$

and the Z_2 parities of each multiplet are given in Table VI.

Using the equivalence of $(\mathbf{5}_R)^c$ and $(\overline{\mathbf{10}}_R)^c$ with $\bar{\mathbf{5}}_L$ and $\mathbf{10}_L$, respectively, the numbers of species $\mathbf{1}$, $\mathbf{10}_L$, and $\bar{\mathbf{5}}_L$ are given by

$$n_{\mathbf{1}} = \sum_{k=\text{even}} {}_{N-5}C_{2k} + \sum_{k=\text{odd}} {}_{N-5}C_{2k-5}, \tag{53}$$

$$n_{\mathbf{10}_L} = \sum_{k=\text{even}} {}_{N-5}C_{2k-2} + \sum_{k=\text{odd}} {}_{N-5}C_{2k-3}, \tag{54}$$

$$n_{\bar{\mathbf{5}}_L} = \sum_{k=\text{even}} {}_{N-5}C_{2k-4} + \sum_{k=\text{odd}} {}_{N-5}C_{2k-1}, \tag{55}$$

in the case with $\eta_1^0 = \eta_1^1 = +1$ and

$$n_{\mathbf{1}} = \sum_{k=\text{odd}} {}_{N-5}C_{2k} + \sum_{k=\text{even}} {}_{N-5}C_{2k-5}, \tag{56}$$

$$n_{\mathbf{10}_L} = \sum_{k=\text{odd}} {}_{N-5}C_{2k-2} + \sum_{k=\text{even}} {}_{N-5}C_{2k-3}, \tag{57}$$

$$n_{\bar{\mathbf{5}}_L} = \sum_{k=\text{odd}} {}_{N-5}C_{2k-4} + \sum_{k=\text{even}} {}_{N-5}C_{2k-1}, \tag{58}$$

in the case with $\eta_1^0 = -\eta_1^1 = +1$. Here $n_{\mathbf{1}}$ is the total number of $SU(5)$ singlets $\mathbf{1}$. They are regarded as the so-called right-handed neutrinos which can obtain heavy Majorana masses among themselves as well as the Dirac masses with left-handed neutrinos. Some of them can be involved in a seesaw mechanism [30].

TABLE VI. Z_2 parity assignment for $\mathbf{2}_{1L}^{N-1}$ and $\mathbf{2}_{1R}^{N-1}$ in $SU(5) \times SU(N-5) \times U(1)^2$.

Representation		$\mathcal{P}_0$	$\mathcal{P}_1$	$\mathcal{U}$
$\mathbf{1}_L$	$({}_5C_0, {}_{N-5}C_{2k})_L$	$+\eta_1^0$	$(-1)^k\eta_1^1$	$+(-1)^k\eta_1^0\eta_1^1$
$\mathbf{1}_R$	$({}_5C_0, {}_{N-5}C_{2k})_R$	$-\eta_1^0$	$(-1)^k\eta_1^1$	$-(-1)^k\eta_1^0\eta_1^1$
$\mathbf{5}_L$	$({}_5C_1, {}_{N-5}C_{2k-1})_L$	$-\eta_1^0$	$(-1)^k\eta_1^1$	$-(-1)^k\eta_1^0\eta_1^1$
$\mathbf{5}_R$	$({}_5C_1, {}_{N-5}C_{2k-1})_R$	$+\eta_1^0$	$(-1)^k\eta_1^1$	$+(-1)^k\eta_1^0\eta_1^1$
$\mathbf{10}_L$	$({}_5C_2, {}_{N-5}C_{2k-2})_L$	$+\eta_1^0$	$(-1)^k\eta_1^1$	$+(-1)^k\eta_1^0\eta_1^1$
$\mathbf{10}_R$	$({}_5C_2, {}_{N-5}C_{2k-2})_R$	$-\eta_1^0$	$(-1)^k\eta_1^1$	$-(-1)^k\eta_1^0\eta_1^1$
$\overline{\mathbf{10}}_L$	$({}_5C_3, {}_{N-5}C_{2k-3})_L$	$-\eta_1^0$	$(-1)^k\eta_1^1$	$-(-1)^k\eta_1^0\eta_1^1$
$\overline{\mathbf{10}}_R$	$({}_5C_3, {}_{N-5}C_{2k-3})_R$	$+\eta_1^0$	$(-1)^k\eta_1^1$	$+(-1)^k\eta_1^0\eta_1^1$
$\bar{\mathbf{5}}_L$	$({}_5C_4, {}_{N-5}C_{2k-4})_L$	$+\eta_1^0$	$(-1)^k\eta_1^1$	$+(-1)^k\eta_1^0\eta_1^1$
$\bar{\mathbf{5}}_R$	$({}_5C_4, {}_{N-5}C_{2k-4})_R$	$-\eta_1^0$	$(-1)^k\eta_1^1$	$-(-1)^k\eta_1^0\eta_1^1$
$\bar{\mathbf{1}}_L$	$({}_5C_5, {}_{N-5}C_{2k-5})_L$	$-\eta_1^0$	$(-1)^k\eta_1^1$	$-(-1)^k\eta_1^0\eta_1^1$
$\bar{\mathbf{1}}_R$	$({}_5C_5, {}_{N-5}C_{2k-5})_R$	$+\eta_1^0$	$(-1)^k\eta_1^1$	$+(-1)^k\eta_1^0\eta_1^1$

In the same way, Weyl fermions with $\mathbf{2}_{2L}^{N-1}$ and $\mathbf{2}_{2R}^{N-1}$ are decomposed into

$$\begin{aligned} \mathbf{2}_{2L}^{N-1} &= \sum_{k=0}^{[(N-1)/2]} \sum_{\ell=\text{even}} ({}_5C_\ell, {}_{N-5}C_{2k-\ell+1})_L + \sum_{k=0}^{[(N-1)/2]} \sum_{\ell=\text{odd}} ({}_5C_\ell, {}_{N-5}C_{2k-\ell+1})_L, \\ \mathbf{2}_{2R}^{N-1} &= \sum_{k=0}^{[(N-1)/2]} \sum_{\ell=\text{even}} ({}_5C_\ell, {}_{N-5}C_{2k-\ell+1})_R + \sum_{k=0}^{[(N-1)/2]} \sum_{\ell=\text{odd}} ({}_5C_{\ell+1}, {}_{N-5}C_{2k-\ell})_R, \end{aligned} \tag{59}$$

and the Z_2 parities of each multiplet are given in Table VII. The numbers of species $\mathbf{1}$, $\mathbf{10}_L$, and $\bar{\mathbf{5}}_L$ are given by

$$n_{\mathbf{1}} = \sum_{k=\text{even}} {}_{N-5}C_{2k+1} + \sum_{k=\text{odd}} {}_{N-5}C_{2k-4}, \tag{60}$$

$$n_{\mathbf{10}_L} = \sum_{k=\text{even}} {}_{N-5}C_{2k-1} + \sum_{k=\text{odd}} {}_{N-5}C_{2k-2}, \tag{61}$$

$$n_{\bar{\mathbf{5}}_L} = \sum_{k=\text{even}} {}_{N-5}C_{2k-3} + \sum_{k=\text{odd}} {}_{N-5}C_{2k}, \tag{62}$$

in the case with $\eta_2^0 = \eta_2^1 = +1$ and

$$n_{\mathbf{1}} = \sum_{k=\text{odd}} {}_{N-5}C_{2k+1} + \sum_{k=\text{even}} {}_{N-5}C_{2k-4}, \tag{63}$$

$$n_{\mathbf{10}_L} = \sum_{k=\text{odd}} {}_{N-5}C_{2k-1} + \sum_{k=\text{even}} {}_{N-5}C_{2k-2}, \tag{64}$$

TABLE VII. Z_2 parity assignment for $\mathbf{2}_{2L}^{N-1}$ and $\mathbf{2}_{2R}^{N-1}$ in $SU(5)\times SU(N-5)\times U(1)^2$.

Representation		$\mathcal{P}_0$	$\mathcal{P}_1$	$\mathcal{U}$
$\mathbf{1}_L$	$({}_5C_0, {}_{N-5}C_{2k+1})_L$	$+\eta_2^0$	$(-1)^k\eta_2^1$	$+(-1)^k\eta_2^0\eta_2^1$
$\mathbf{1}_R$	$({}_5C_0, {}_{N-5}C_{2k+1})_R$	$-\eta_2^0$	$(-1)^k\eta_2^1$	$-(-1)^k\eta_2^0\eta_2^1$
$\mathbf{5}_L$	$({}_5C_1, {}_{N-5}C_{2k})_L$	$-\eta_2^0$	$(-1)^k\eta_2^1$	$-(-1)^k\eta_2^0\eta_2^1$
$\mathbf{5}_R$	$({}_5C_1, {}_{N-5}C_{2k})_R$	$+\eta_2^0$	$(-1)^k\eta_2^1$	$+(-1)^k\eta_2^0\eta_2^1$
$\mathbf{10}_L$	$({}_5C_2, {}_{N-5}C_{2k-1})_L$	$+\eta_2^0$	$(-1)^k\eta_2^1$	$+(-1)^k\eta_2^0\eta_2^1$
$\mathbf{10}_R$	$({}_5C_2, {}_{N-5}C_{2k-1})_R$	$-\eta_2^0$	$(-1)^k\eta_2^1$	$-(-1)^k\eta_2^0\eta_2^1$
$\overline{\mathbf{10}}_L$	$({}_5C_3, {}_{N-5}C_{2k-2})_L$	$-\eta_2^0$	$(-1)^k\eta_2^1$	$-(-1)^k\eta_2^0\eta_2^1$
$\overline{\mathbf{10}}_R$	$({}_5C_3, {}_{N-5}C_{2k-2})_R$	$+\eta_2^0$	$(-1)^k\eta_2^1$	$+(-1)^k\eta_2^0\eta_2^1$
$\bar{\mathbf{5}}_L$	$({}_5C_4, {}_{N-5}C_{2k-3})_L$	$+\eta_2^0$	$(-1)^k\eta_2^1$	$+(-1)^k\eta_2^0\eta_2^1$
$\bar{\mathbf{5}}_R$	$({}_5C_4, {}_{N-5}C_{2k-3})_R$	$-\eta_2^0$	$(-1)^k\eta_2^1$	$-(-1)^k\eta_2^0\eta_2^1$
$\bar{\mathbf{1}}_L$	$({}_5C_5, {}_{N-5}C_{2k-4})_L$	$-\eta_2^0$	$(-1)^k\eta_2^1$	$-(-1)^k\eta_2^0\eta_2^1$
$\bar{\mathbf{1}}_R$	$({}_5C_5, {}_{N-5}C_{2k-4})_R$	$+\eta_2^0$	$(-1)^k\eta_2^1$	$+(-1)^k\eta_2^0\eta_2^1$

TABLE VIII. The numbers of $\mathbf{1}$, $\mathbf{10}_L$, and $\bar{\mathbf{5}}_L$ for $SO(14)$.

	$\eta_1^0=\eta_1^1=+1$	$\eta_1^0=-\eta_1^1=+1$	$\eta_2^0=\eta_2^1=+1$	$\eta_2^0=-\eta_2^1=+1$
$n_{\mathbf{1}}$	3	1	3	1
$n_{\mathbf{10}_L}$	1	3	1	3
$n_{\bar{\mathbf{5}}_L}$	3	1	3	1

TABLE IX. The numbers of $\mathbf{1}$, $\mathbf{10}_L$, and $\bar{\mathbf{5}}_L$ for $SO(16)$.

	$\eta_1^0=\eta_1^1=+1$	$\eta_1^0=-\eta_1^1=+1$	$\eta_2^0=\eta_2^1=+1$	$\eta_2^0=-\eta_2^1=+1$
$n_{\mathbf{1}}$	4	4	6	2
$n_{\mathbf{10}_L}$	4	4	2	6
$n_{\bar{\mathbf{5}}_L}$	4	4	6	2

TABLE X. The numbers of $\mathbf{1}$, $\mathbf{10}_L$, and $\bar{\mathbf{5}}_L$ for $SO(18)$.

	$\eta_1^0=\eta_1^1=+1$	$\eta_1^0=-\eta_1^1=+1$	$\eta_2^0=\eta_2^1=+1$	$\eta_2^0=-\eta_2^1=+1$
$n_{\mathbf{1}}$	6	10	10	6
$n_{\mathbf{10}_L}$	10	6	6	10
$n_{\bar{\mathbf{5}}_L}$	6	10	10	6

$$n_{\bar{\mathbf{5}}_L} = \sum_{k=\mathrm{odd}} {}_{N-5}C_{2k-3} + \sum_{k=\mathrm{even}} {}_{N-5}C_{2k}, \tag{65}$$

in the case with $\eta_2^0=-\eta_2^1=+1$.

As examples, the numbers of species $\mathbf{1}$, $\mathbf{10}_L$, and $\bar{\mathbf{5}}_L$ for $SO(14)$, $SO(16)$, and $SO(18)$ are listed in Tables VIII, IX, and X.

IV. CONCLUSIONS AND DISCUSSIONS

We have studied the possibility of family unification on the basis of $SO(2N)$ gauge theory on the five-dimensional space-time, $M^4\times S^1/Z_2$. We have found that several $SO(10)$, $SU(4)\times SU(2)_L\times SU(2)_R$, or $SU(5)$ multiplets come from a single bulk multiplet of $SO(2N)$ after the orbifold breaking and obtained the numbers of species. As a result, we have found that there is no single bulk spinor multiplet of $SO(2N)$ on $M^4\times S^1/Z_2$, which leads to three families of $SO(10)$, $SU(4)\times SU(2)_L\times SU(2)_R$, or $SU(5)$ multiplets only as zero modes using the orbifold breaking with the Z_2 projection such as $P=\sigma_0\otimes I_{m,n}$ and/or $\sigma_2\otimes I_{m,n}$. Other multiplets including brane fields are necessary to compose complete three families of quarks and leptons which we observe. Specifically, four families of $SO(10)$ or $SU(4)\times SU(2)_L\times SU(2)_R$ multiplets are obtained as zero modes from a single spinor bulk fermion of $SO(16)$ through particular breaking patterns with suitable Z_2 parity assignments. In this case, a mirror brane family would be necessary to couple to members of the fourth family and to give them a heavy mass of order $O(100)$ GeV or bigger than that because they have not been discovered. The magnitude of masses for members of the fourth family is not predicted in our framework.

Most of our results are unique to models on $M^4 \times S^1/Z_2$, but some of them can be generalized to models on other types of space-time. For example, we have the no-go theorem such that there is no single bulk spinor multiplet of $SO(2N)$ on a higher-dimensional space-time, which leads to three families of $SO(10)$ or $SU(4) \times SU(2)_L \times SU(2)_R$ multiplets only as zero modes through the orbifold breaking mechanism using the projection operators such as $\sigma_0 \otimes I_{m,n}$ and/or $\sigma_2 \otimes I_{m,n}$. This comes from the group theoretical feature that even numbers of **16** of $SO(10)$ or (**4**, **2**, **1**) and (**4**, **1**, **2**) of $SU(4) \times SU(2)_L \times SU(2)_R$ always appear after the breakdown of $SO(2N)$ with the above projection operators. Our results including the above no-go theorem can be a starting point for the construction of a more realistic model.

There are several open questions, which are left for future work.

The unwanted matter degrees of freedom can be successfully made massive thanks to the orbifolding. However, some extra gauge fields remain massless even after the symmetry breaking due to the Hosotani mechanism. In most cases, this kind of non-Abelian gauge subgroup plays the role of family symmetry. These massless degrees of freedom must be made massive by further breaking of the family symmetry. Here, we point out that brane fields can be key to the solutions. Most models have chiral anomalies at the four-dimensional boundaries and we have a choice to introduce appropriate brane fields to cancel these anomalies. Further, scalar components of some brane superfields can play a role of Higgs fields for the breakdown of extra gauge symmetries including non-Abelian gauge symmetries. As a result, extra massless fields including the family gauge bosons can be massive.

In general, there appear D-term contributions to scalar masses in supersymmetric models after the breakdown of such extra gauge symmetries and the D-term contributions lift the mass degeneracy. [31–33]. The mass degeneracy for each squark and slepton species in the first two families is favorable for suppressing flavor-changing neutral current (FCNC) processes. The dangerous FCNC processes can be avoided if the sfermion masses in the first two families are rather large or the fermion and its superpartner mass matrices are aligned. The requirement of degenerate masses would yield a constraint on the D-term condensations and/or SUSY breaking mechanism unless other mechanisms work. If we consider the Scherk-Schwarz mechanism [34] for $N = 1$ SUSY breaking, the D-term condensations can vanish for the gauge symmetries broken at the orbifold breaking scale, because of a universal structure of the soft SUSY breaking parameters. The D-term contributions have been studied in the framework of $SU(N)$ orbifold GUTs [35].

Fermion mass hierarchy and generation mixings can also occur through the Froggatt-Nielsen mechanism [36] on the breakdown of extra gauge symmetries and the suppression of brane-localized Yukawa coupling constants among brane weak Higgs doublets and bulk matters with the volume suppression factor [37].

The orbifold GUT is more naturally realized in warped space; see e.g. [38] for a review. The Hosotani mechanism has been studied in warped space [39] and it has been applied on the gauge-Higgs unification [40]. It would be interesting to look for the orbifold family unification based on warped space and/or other types of orbifolds.[3]

It has been shown that $SO(1, D-1)$ space-time symmetry can lead to family structure [43]. Hence it would be interesting to study theories on a higher-dimensional space-time with a view to the synergy of $SO(2N)$ gauge symmetry and $SO(1, D-1)$ space-time symmetry.

Furthermore, it would be interesting to study cosmological implications of the class of models presented in this paper, see e.g. [44] and references therein for useful articles toward this direction.

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APPENDIX A: GAUGE INVARIANCE AND EQUIVALENCE RELATIONS

We discuss the gauge invariance on S^1/Z_2. Given the BCs (P_0, P_1, U), there still remains residual gauge invariance. Under the gauge transformation with $\Omega(x, y)$, A_M transforms as

$$A_M \to A'_M = \Omega A_M \Omega^{-1} - \frac{i}{g}\Omega \partial_M \Omega^{-1}, \qquad (A1)$$

where g is a gauge coupling and A'_M satisfies, instead of (25),

$$\begin{aligned} s_0\colon A'_\mu(x, -y) &= P'_0 A'_\mu(x, y) P'^{-1}_0, \\ A'_y(x, -y) &= -P'_0 A'_y(x, y) P'^{-1}_0, \\ s_1\colon A'_\mu(x, 2\pi R - y) &= P'_1 A'_\mu(x, y) P'^{-1}_1, \\ A'_y(x, 2\pi R - y) &= -P'_1 A'_y(x, y) P'^{-1}_1, \\ t\colon A'_M(x, y + 2\pi R) &= U' A'_M(x, y) U'^{-1}. \end{aligned} \qquad (A2)$$

The P'_0, P'_1, and U' are given by

[3] Equivalence classes of BCs in $SU(N)$ gauge theory have been studied based on six-dimensional space-time including T^2/Z_2 in Ref. [41] and other two-dimensional orbifolds in Ref. [42].

$$\begin{aligned} P_0' &= \Omega(x,-y)P_0\Omega^{-1}(x,y),\\ P_1' &= \Omega(x,2\pi R-y)P_1\Omega^{-1}(x,y),\\ U' &= \Omega(x,y+2\pi R)U\Omega^{-1}(x,y), \end{aligned} \tag{A3}$$

where we assume that P_0', P_1', and U' are independent of y.

Theories with different BCs should be equivalent with regard to physical content if they are connected by gauge transformations. The key observation is that the physics should not depend on the gauge chosen. The equivalence is guaranteed in the Hosotani mechanism [24] and the two sets of BCs are equivalent:

$$(P_0,P_1,U)\sim(P_0',P_1',U'). \tag{A4}$$

The equivalence relation (A4) defines equivalence classes of the BCs.

The physical symmetry is understood from the analysis including the Wilson line phases as follows. The Wilson line phases are phases of WU given by

$$WU = \mathbf{P}\exp\left(ig\int_C A_y(x,y)dy\right)\cdot U, \tag{A5}$$

where $\mathbf{P}$ is path ordering along a noncontractible loop on S^1. The eigenvalues of WU are gauge invariant and become physical degrees of freedom. The dynamical phases are given by $\theta^b = 2\pi R g A_y^b$ related to the generators T^b which anticommute with (P_0,P_1), i.e., $\{T^b,P_0\}=\{T^b,P_1\}=0$. They correspond to the parts of A_y with even Z_2 parities. The physical vacuum is given by the configuration of θ^b which minimizes the effective potential. Suppose that the effective potential is minimized at $\langle A_y\rangle$ such that $W\equiv\exp(ig2\pi R\langle A_y\rangle)\neq I$ with (P_0,P_1,U). Perform the gauge transformation with $\Omega=\exp[ig(y+\alpha)\langle A_y\rangle]$, which brings $\langle A_y\rangle$ to $\langle A_y'\rangle=0$. Then the BCs change to

$$\begin{aligned}(P_0^{\mathrm{sym}}, &P_1^{\mathrm{sym}}, U^{\mathrm{sym}})\\ &\equiv (P_0',P_1',U')\\ &= (e^{2ig\alpha\langle A_y\rangle}P_0, e^{2ig(\alpha+\pi R)\langle A_y\rangle}P_1, e^{ig2\pi R\langle A_y\rangle}U(=WU)).\end{aligned} \tag{A6}$$

Since $\langle A_y'\rangle$ vanishes in the new gauge, the unbroken symmetry is spanned by the generators T^a which commute with $(P_0^{\mathrm{sym}},P_1^{\mathrm{sym}})$, i.e., $[T^a,P_0^{\mathrm{sym}}]=[T^a,P_1^{\mathrm{sym}}]=0$.

Let us derive the equivalence relations among BCs based on types I and II. We consider $SO(4)$ gauge theory. For the gauge transformation with $\Omega(y)$ given by

$$\Omega(y)=\exp[i(a\sigma_1\otimes\tau_2+b\sigma_3\otimes\tau_2)y/2\pi R], \tag{A7}$$

we find the equivalence relations:

$$\begin{aligned}
\text{type I: } &(\sigma_0\otimes\tau_3,\sigma_0\otimes\tau_3)\sim(\sigma_0\otimes\tau_3,\exp[i(a\sigma_1\otimes\tau_2+b\sigma_3\otimes\tau_2)]\sigma_0\otimes\tau_3),\\
\text{type IIa: } &(\sigma_0\otimes\tau_3,\sigma_2\otimes\tau_0)\sim(\sigma_0\otimes\tau_3,\exp[i(a\sigma_1\otimes\tau_2+b\sigma_3\otimes\tau_2)]\sigma_2\otimes\tau_0),\\
\text{type IIb: } &(\pm\sigma_2\otimes\tau_0,\sigma_0\otimes\tau_3)\sim(\pm\sigma_2\otimes\tau_0,\exp[i(a\sigma_1\otimes\tau_2+b\sigma_3\otimes\tau_2)]\sigma_0\otimes\tau_3),\\
\text{type IIc: } &(\pm\sigma_2\otimes\tau_0,\sigma_2\otimes\tau_0)\sim(\pm\sigma_2\otimes\tau_0,\exp[i(a\sigma_1\otimes\tau_2+b\sigma_3\otimes\tau_2)]\sigma_2\otimes\tau_0),
\end{aligned} \tag{A8}$$

where $a,b\in\mathbb{R}$ and τ_i $(i=1,2,3)$ are also Pauli matrices. When $\sqrt{a^2+b^2}=\pi \bmod 2\pi$, the equivalence relations become as

$$\begin{aligned}
\text{type I: } &(\sigma_0\otimes\tau_3,\sigma_0\otimes\tau_3)\sim(\sigma_0\otimes\tau_3,-\sigma_0\otimes\tau_3),\\
\text{type IIa: } &(\sigma_0\otimes\tau_3,\sigma_2\otimes\tau_0)\sim(\sigma_0\otimes\tau_3,-\sigma_2\otimes\tau_0),\\
\text{type IIb: } &(\pm\sigma_2\otimes\tau_0,\sigma_0\otimes\tau_3)\sim(\pm\sigma_2\otimes\tau_0,-\sigma_0\otimes\tau_3),\\
\text{type IIc: } &(\pm\sigma_2\otimes\tau_0,\sigma_2\otimes\tau_0)\sim(\pm\sigma_2\otimes\tau_0,-\sigma_2\otimes\tau_0).
\end{aligned} \tag{A9}$$

Because $(\pm\sigma_2\otimes\tau_0,-\sigma_0\otimes\tau_3)$ equals $(\pm\sigma_2\otimes\tau_0,\sigma_0\otimes\tau_3)$, we obtain no relation concerning Eq. (A7) for type IIb. Using (A9), the following relations in $SO(2N)$ gauge theory are derived:

$$\begin{aligned}
[p,q;r,s]^{\mathrm{I}} &\sim [p-1,q+1;r+1,s-1]^{\mathrm{I}} \quad \text{for } p,s\geq 1,\\
&\sim [p+1,q-1;r-1,s+1]^{\mathrm{I}} \quad \text{for } q,r\geq 1,
\end{aligned} \tag{A10}$$

$$\begin{aligned}
[p,q;r,s]^{\mathrm{IIa}} &\sim [p-1,q+1;r-1,s+1]^{\mathrm{IIa}} \quad \text{for } p,s\geq 1,\\
&\sim [p+1,q-1;r+1,s-1]^{\mathrm{IIa}} \quad \text{for } q,r\geq 1,
\end{aligned} \tag{A11}$$

$$\begin{aligned}
[p,q;r,s]^{\mathrm{IIc}} &\sim [p-2,q+2;r,s]^{\mathrm{IIc}} \quad \text{for } p\geq 2,\\
&\sim [p+2,q-2;r,s]^{\mathrm{IIc}} \quad \text{for } q\geq 2,\\
&\sim [p,q;r-2,s+2]^{\mathrm{IIc}} \quad \text{for } r\geq 2,\\
&\sim [p,q;r+2,s-2]^{\mathrm{IIc}} \quad \text{for } s\geq 2.
\end{aligned} \tag{A12}$$

For another gauge transformation with $\Omega(y)$ given by

$$\Omega(y)=\exp[i(a\sigma_2\otimes\tau_1+b\sigma_0\otimes\tau_2)y/2\pi R], \tag{A13}$$

we find the equivalence relations:

$$\begin{aligned}\text{type I: } &(\sigma_0\otimes\tau_3,\sigma_0\otimes\tau_3)\sim(\sigma_0\otimes\tau_3,\exp[i(a\sigma_2\otimes\tau_1 + b\sigma_0\otimes\tau_2)]\sigma_0\otimes\tau_3),\\ \text{type IIa: } &(\sigma_0\otimes\tau_3,\sigma_2\otimes\tau_3)\sim(\sigma_0\otimes\tau_3,\exp[i(a\sigma_2\otimes\tau_1 + b\sigma_0\otimes\tau_2)]\sigma_2\otimes\tau_3),\\ \text{type IIb: } &(\sigma_2\otimes\tau_3,\sigma_0\otimes\tau_3)\sim(\sigma_2\otimes\tau_3,\exp[i(a\sigma_2\otimes\tau_1 + b\sigma_0\otimes\tau_2)]\sigma_0\otimes\tau_3),\\ \text{type IIc: } &(\sigma_2\otimes\tau_3,\sigma_2\otimes\tau_3)\sim(\sigma_2\otimes\tau_3,\exp[i(a\sigma_2\otimes\tau_1 + b\sigma_0\otimes\tau_2)]\sigma_2\otimes\tau_3).\end{aligned}\tag{A14}$$

When $\sqrt{a^2+b^2}=\pi \bmod 2\pi$, the equivalence relations become as

$$\begin{aligned}\text{type I: } &(\sigma_0\otimes\tau_3,\sigma_0\otimes\tau_3)\sim(\sigma_0\otimes\tau_3,-\sigma_0\otimes\tau_3),\\ \text{type IIa: } &(\sigma_0\otimes\tau_3,\sigma_2\otimes\tau_3)\sim(\sigma_0\otimes\tau_3,-\sigma_2\otimes\tau_3),\\ \text{type IIb: } &(\sigma_2\otimes\tau_3,\sigma_0\otimes\tau_3)\sim(\sigma_2\otimes\tau_3,-\sigma_0\otimes\tau_3),\\ \text{type IIc: } &(\sigma_2\otimes\tau_3,\sigma_2\otimes\tau_3)\sim(\sigma_2\otimes\tau_3,-\sigma_2\otimes\tau_3).\end{aligned}\tag{A15}$$

Using (A5), the following relations in $SO(2N)$ gauge theory are derived:

$$\begin{aligned}[p,q;r,s]^{\mathrm{I}}&\sim[p-1,q+1;r+1,s-1]^{\mathrm{I}}\quad \text{for } p,s\geq 1,\\ &\sim[p+1,q-1;r-1,s+1]^{\mathrm{I}}\quad \text{for } q,r\geq 1,\end{aligned}\tag{A16}$$

$$\begin{aligned}[p,q;r,s]^{\mathrm{IIa}}&\sim[p-1,q+1;r+1,s-1]^{\mathrm{IIa}}\quad \text{for } p,s\geq 1,\\ &\sim[p+1,q-1;r-1,s+1]^{\mathrm{IIa}}\quad \text{for } q,r\geq 1,\end{aligned}\tag{A17}$$

$$\begin{aligned}[p,q;r,s]^{\mathrm{IIb}}&\sim[p-1,q+1;r+1,s-1]^{\mathrm{IIb}}\quad \text{for } p,s\geq 1,\\ &\sim[p+1,q-1;r-1,s+1]^{\mathrm{IIb}}\quad \text{for } q,r\geq 1,\end{aligned}\tag{A18}$$

$$\begin{aligned}[p,q;r,s]^{\mathrm{IIc}}&\sim[p-1,q+1;r+1,s-1]^{\mathrm{IIc}}\quad \text{for } p,s\geq 1,\\ &\sim[p+1,q-1;r-1,s+1]^{\mathrm{IIc}}\quad \text{for } q,r\geq 1.\end{aligned}\tag{A19}$$

APPENDIX B: S^1/Z_2 ORBIFOLD BREAKING OF $SO(2N+1)$

We study the orbifold symmetry breaking in $SO(2N+1)$. Because $SO(2N+1)\supset SO(2N)$, the generators of $SO(2N+1)$ are written as

$$\begin{pmatrix}\text{“}so(2N)\text{”} & (*)\\ (*)^t & 0\end{pmatrix},\tag{B1}$$

where $so(2N)$ represents generators of $SO(2N)$ and $(*)$ are $2N\times 1$ matrix.

As an example, let us take the following representation matrices:

$$P_0=\begin{pmatrix}\sigma_0\otimes I_N & 0\\ 0 & -1\end{pmatrix},\qquad P_1=\begin{pmatrix}\sigma_0\otimes I_{m,n} & 0\\ 0 & \eta\end{pmatrix},\tag{B2}$$

where $\eta=\pm 1$. Then we obtain the breaking pattern:

$$SO(2N+1)\to SO(2m)\times SO(2n),\tag{B3}$$

and the Z_2 parities for gauge bosons A_μ^α are assigned as

$$\begin{aligned}\mathbf{N(2N+1)}&=(\mathbf{m(2m-1)},\mathbf{1})^{++;+}+(\mathbf{1},\mathbf{n(2n-1)})^{++;+}\\ &\quad+(\mathbf{2m},\mathbf{2n})^{+-;-}+(\mathbf{2m},\mathbf{1})^{-\mp;\pm}\\ &\quad+(\mathbf{1},\mathbf{2n})^{-\pm;\mp}.\end{aligned}\tag{B4}$$

There is one spinor representation $\mathbf{2}^N$ in $SO(2N+1)$, which is decomposed into

$$\begin{aligned}\mathbf{2}_L^N&=(\mathbf{2}_1^{m-1},\mathbf{2}_1^{n-1})_L^{++;+}+(\mathbf{2}_2^{m-1},\mathbf{2}_2^{n-1})_L^{+-;-}\\ &\quad+(\mathbf{2}_1^{m-1},\mathbf{2}_2^{n-1})_L^{-\pm;\mp}+(\mathbf{2}_2^{m-1},\mathbf{2}_1^{n-1})_L^{-\mp;\pm},\end{aligned}\tag{B5}$$

$$\begin{aligned}\mathbf{2}_R^N&=(\mathbf{2}_1^{m-1},\mathbf{2}_1^{n-1})_R^{--;-}+(\mathbf{2}_2^{m-1},\mathbf{2}_2^{n-1})_R^{-+;+}\\ &\quad+(\mathbf{2}_1^{m-1},\mathbf{2}_2^{n-1})_R^{+\mp;\pm}+(\mathbf{2}_2^{m-1},\mathbf{2}_1^{n-1})_R^{+\pm;\mp},\end{aligned}\tag{B6}$$

where we take an appropriate intrinsic Z_2 parity assignment. Using the above assignment, we find 2^{N-5} families with $\eta=+1$ and no family with $\eta=-1$ for $SO(10)$ multiplets $\mathbf{16}_L$ after the breaking $SO(2N+1)\to SO(10)\times SO(2(N-5))$.

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