A CATEGORY OF ASSOCIATION SCHEMES

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Abstract. We define a category of association schemes and investigate its basic properties. We characterize monomorphisms and epimorphisms in our category. The category is not balanced. The category has kernels, cokernels, and epimorphic images. The category is not an exact category, but we consider exact sequences. Finally, we consider a full subcategory of our category and show that it is equivalent to the category of finite groups.

1. Introduction

Association schemes are one of the central objects in algebraic combinatorics. They are related to many mathematical subjects, for instance, to codes, to combinatorial designs, to distance-regular graphs, finite (permutation) groups, coherent configurations, and to designed experiments in statistics. However, so far, scheme theory has not yet been investigated from an abstract categorical point of view. In this article, we shall define two categories of association schemes. We shall denote these categories by $\mathcal{AS}$ and $\mathcal{AS}_0$, respectively. Similar to the category of sets, our category $\mathcal{AS}$ does not have zero objects. Our category $\mathcal{AS}_0$ of association schemes with base point is modeled on the category of nonempty sets with base point, and this category has zero objects. Some of our results on $\mathcal{AS}_0$ are also valid for $\mathcal{AS}$, but the arguments are more involved. To keep this article as self-contained as possible, we shall occasionally recall definitions from scheme theory which may be found in [4]. Our definition of morphisms in $\mathcal{AS}_0$ is the same as in [4]. In §3, we characterize monomorphisms, epimorphisms, and bimorphisms of $\mathcal{AS}_0$. Among other things we shall see that $\mathcal{AS}_0$ is not balanced in the sense that (in $\mathcal{AS}_0$) bimorphisms are not necessarily isomorphisms. In §4.1, we shall see that $\mathcal{AS}_0$ has kernels and cokernels. We shall prove that subschemes and quotient schemes are kernels and cokernels of morphisms, respectively. In scheme theory, closed subsets and normal closed subsets generalize the group theoretic notions of subgroups and normal subgroups, respectively. It is known that quotient schemes are defined over any closed subset, not only over normal closed subsets. However, in our category $\mathcal{AS}_0$, each closed subset defines a normal subobject. So it is natural that it defines a quotient object. Also, in general, closed subsets may give rise to pairwise non-isomorphic subschemes. But, since the objects in $\mathcal{AS}_0$ have base points, a closed subset determines a unique subscheme. In §4.2, we show that $\mathcal{AS}_0$ has epimorphic images. In §4.3, we consider exact sequences in $\mathcal{AS}_0$, although $\mathcal{AS}_0$ is not an exact category. Finally, in §5, we consider the category of finite groups as a full subcategory of $\mathcal{AS}_0$.

2. Definitions of association schemes and their category

In this section, we will first give the definition of association schemes and state some facts without proofs. The reader is referred to the books of Zieschang [4], [5] and Bannai-Ito [1]. After that, we will define a category of association schemes. For the general theory of categories, we refer to the book of Mitchell [3].

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2.1. **Association schemes.** Let X be a nonempty finite set, and let S be a partition of $X \times X$. The pair (X, S) is called an *association scheme*, or briefly a *scheme*, if the following properties hold [4, Introduction].

(1) $1_X := \{(x, x) \mid x \in X\} \in S$.
(2) For each $s \in S$, $s^* := \{(y, x) \mid (x, y) \in s\} \in S$.
(3) For $s, t, u \in S$, there exists a nonnegative integer a_{stu} such that $\sharp\{z \in X \mid (x, z) \in s,\ (z, y) \in t\} = a_{stu}$ whenever $(x, y) \in u$.

Obviously we have $(s^*)^* = s$.

For the remainder of this subsection, the pair (X, S) will stand for an association scheme. We call an element of X a *point* and an element of S a *relation* of the scheme (X, S). For $x, y \in X$, there exists a uniquely determined $s \in S$ such that $(x, y) \in s$ since S is a partition of $X \times X$. So we can define a surjective map $r : X \times X \to S$ by $(x, y) \in r(x, y)$. When we want to specify S, we use the notation r_S. For $x \in X$ and $s \in S$, we define $xs := \{y \in X \mid (x, y) \in s\}$. Referring to this notation, the equation in the condition (3) is $|xs \cap yt^*| = a_{stu}$. For $x \in X$ and $T \subseteq S$, we also use the notation $xT = \bigcup_{t \in T} xt$. For $x \in X$ and $s \in S$, $a_{ss^*1_X} = |xs \cap x(s^*)^*| = |xs|$. We call this number $a_{ss^*1_X}$ the *valency* of $s \in S$ and write n_s instead of $a_{ss^*1_X}$. It is easy to see that $n_s = n_{s^*}$. The number $|X|$ is called the *order* of (X, S), and $|S| - 1$ is called the *class* of the scheme S.

Example 2.1 (one-point schemes)**.** Let $X = \{x\}$ and $S = \{\{(x, x)\}\}$. Then (X, S) is an association scheme. We call this a *one-point scheme* and denote it by 0_x or 0 (since this is a zero object in our category defined in §2.2).

Example 2.2 (class-one schemes)**.** Let X be a finite set with $|X| > 1$. Put $S = \{\{(x, x) \mid x \in X\}, \{(x, y) \mid x \neq y\}\}$. Then (X, S) is an association scheme. We call this a *class-one scheme*.

For $t, u \in S$, we define the *complex product* tu of t and u by $tu = \{s \in S \mid a_{tus} \neq 0\}$. For $T, U \subseteq S$, we also define the complex product TU by $TU = \bigcup_{t \in T} \bigcup_{u \in U} tu$. The complex multiplication is associative [4, Lemma 1.2.1]. We also use the notation tU for $\{t\}U$, and so on.

Example 2.3 (thin schemes)**.** Let G be a finite group. For $g \in G$, put $[g] = \{(x, y) \in G \times G \mid xg = y\}$ and $[G] = \{[g] \mid g \in G\}$. Then $(G, [G])$ is an association scheme. For every $[g] \in [G]$, we have $n_{[g]} = 1$.

Conversely, assume that $n_s = 1$ for each relation s of an association scheme (X, S). Then S is a finite group with respect to the complex multiplication (in this case, $st = \{u\}$ for some $u \in S$, so we define $st = u$). We call such a scheme a *thin scheme*.

Let T be a nonempty subset of S. We call T a *closed subset* of S if $TT = T$ [4, §1.3]. Note that intersections of closed subsets are closed. For a subset U of S, the intersection of all closed subsets containing U is called the *closed subset generated by* U and written by $\langle U \rangle$. Also $\langle U \rangle = \bigcup_{n=1}^{\infty} U^n$ holds.

Let T be a closed subset of S, and $x \in X$. We put $S_{xT} = \{s \cap (xT \times xT) \mid s \in S,\ s \cap (xT \times xT) \neq \emptyset\}$. Then (xT, S_{xT}) is an association scheme. We call this a *subscheme* of (X, S) with respect to x and T [4, §1.5]. Note that a subscheme depends on the choice of a point $x \in X$.

Again, let T be a closed subset of S. Put $X/T = \{xT \mid x \in X\}$. Then X/T is a partition of X. For $s \in S$, we define a relation s^T on X/T by $s^T = \{(xT, yT) \mid s \in r_S(xT, yT)\}$. Put $S/\!\!/T = \{s^T \mid s \in S\}$. Then $(X/T, S/\!\!/T)$ is an association scheme [4, Theorem 1.5.4]. We call this the *quotient scheme* of (X, S) by T.

Let (X, S) and (X, T) be association schemes. We say that (X, T) is a *fusion scheme* of (X, S) if for every $s \in S$ there exists $t \in T$ such that $s \subseteq t$. In this case, we also say that (X, S) is a *fission scheme* of (X, T).

We will characterize subschemes, quotient schemes, and fusion schemes in our category.

2.2. **Categories.** For a general category $\mathcal{C}$, we write $f \in [M,N]_{\mathcal{C}}$, $f : M \to N$, or $M \xrightarrow{f} N$ to say that f is a morphism in $\mathcal{C}$ from M to N. Let I_M denote the identity morphism of an object M in $\mathcal{C}$.

We define the category $\mathcal{S}_0$ of sets with base points. An object of $\mathcal{S}_0$ is a pair (A, a) of a nonempty set A and an element a of A. A morphism f from (A, a) to (B, b) is a map $f : A \to B$ which satisfies $f(a) = b$. An object (A, a) of $\mathcal{S}_0$ is a zero object if $A = \{a\}$. Note that, in contrast to $\mathcal{S}_0$, the category of all sets has no zero object.

We define the category $\mathcal{AS}$ of association schemes. An object of $\mathcal{AS}$ is an association scheme (X, S). A morphism f from (X, S) to (Y, T) is a map $f : X \cup S \to Y \cup T$ such that $f(X) \subseteq Y$, $f(S) \subseteq T$, and $f(r(x, x')) = r(f(x), f(x'))$ for all $x, x' \in X$. The definition of morphisms is the same as in [4, §1.7].

Similar to the category of sets, the category $\mathcal{AS}$ has no zero object. So we define the category $\mathcal{AS}_0$ of association schemes with base points. An object of $\mathcal{AS}_0$ is (X, S, x), where (X, S) is an association scheme and $x \in X$. A morphism f from (X, S, x) to (Y, T, y) is a morphism in $\mathcal{AS}$ that satisfies $f(x) = y$. We also say that an object (X, S, x) of $\mathcal{AS}_0$ is an association scheme.

Theorem 2.4. *A one-point scheme* 0 *(defined in Example 2.1) is a zero object in* $\mathcal{AS}_0$.

Proof. This is clear by definition. □

We note that the zero morphism 0 from (X, S, x_0) to (Y, T, y_0) is the map defined by $0(x) = y_0$ for every $x \in X$ and $0(s) = 1_Y$ for every $s \in S$.

Now we will give some examples.

Example 2.5. Let (X, S, x) be an association scheme, and T a closed subset of S. Then we can define the subscheme (xT, S_{xT}, x) as in §2.1. In this case, we can define a natural map from $xT \cup S_{xT}$ to $X \cup S$ and this map gives a morphism from (xT, S_{xT}, x) to (X, S, x) in $\mathcal{AS}_0$.

Example 2.6. Let (X, S, x_0) be an association scheme, and T a closed subset of S. Then we can define the quotient scheme $(X/T, S/\!\!/T, x_0T)$ as in §2.1. In this case, we can define a natural map from $X \cup S$ to $X/T \cup S/\!\!/T$ by $x \mapsto xT$ and $s \mapsto s^T$ for $x \in X$ and $s \in S$. Then this map gives a morphism from (X, S, x_0) to $(X/T, S/\!\!/T, x_0T)$ in $\mathcal{AS}_0$.

Example 2.7. Let (X, T, x_0) be a fusion scheme of an association scheme (X, S, x_0). We define a map f from $X \cup S$ to $X \cup T$ as follows. For $x \in X$, define $f(x) = x$. For $s \in S$, there exists a unique $t \in T$ such that $s \subseteq t$. So define $f(s) = t$. Then f is a morphism from (X, S, x_0) to (X, T, x_0) in $\mathcal{AS}_0$.

We define two elementary functors. The covariant functor $P : \mathcal{AS}_0 \to \mathcal{S}_0$ is defined by $P(X, S, x) = (X, x)$ and, for a morphism $f : (X, S, x) \to (Y, T, y)$ in $\mathcal{AS}_0$, $P(f) = f|_X$, the restriction of f to X. The covariant functor $R : \mathcal{AS}_0 \to \mathcal{S}_0$ is defined by $R(X, S, x) = (S, 1_X)$ and, for a morphism $f : (X, S, x) \to (Y, T, y)$ in $\mathcal{AS}_0$, $R(f) = f|_S$, the restriction of f to S. Now we note that $f(1_X) = f(r(x, x)) = r(f(x), f(x)) = 1_Y$. So $R(f)$ is a morphism in $\mathcal{S}_0$.

Lemma 2.8. *The functor* $P : \mathcal{AS}_0 \to \mathcal{S}_0$ *is faithful.*

Proof. Suppose that $f, g \in [(X, S, x), (Y, T, y)]_{\mathcal{AS}_0}$ and $P(f) = P(g)$. For $s \in S$, there exist $x, x' \in X$ such that $r(x, x') = s$. Then $f(s) = f(r(x, x')) = r(f(x), f(x')) = r(g(x), g(x')) = g(r(x, x')) = g(s)$. So $f = g$ holds. □

To simplify our description, we will abuse the following notations in this article. For an object $M = (X, S, x)$ in $\mathcal{AS}_0$, $P(M)$ and $R(M)$ mean X and S, respectively, though they are precisely (X, x) and $(S, 1_X)$.

3. Morphisms

In this section, we will consider basic properties of morphisms in $\mathcal{AS}_0$. The following result is the main theorem in this section. We use the functors P and R defined in §2.2.

Theorem 3.1. *For a morphism f in $\mathcal{AS}_0$, the following statements hold.*

(1) *f is an epimorphism if and only if so is $P(f)$.*
(2) *f is a monomorphism if and only if so is $P(f)$.*
(3) *If f is an epimorphism, then so is $R(f)$.*
(4) *If $R(f)$ is a monomorphism, then so is f.*

The converses of (3) and (4) do not hold. We will give such examples in Example 3.5. Note that epimorphisms in $\mathcal{S}_0$ are surjections and monomorphisms in $\mathcal{S}_0$ are injections as in the category of sets.

Proof of Theorem 3.1 (1). If $P(f)$ is an epimorphism, then f is an epimorphism by Lemma 2.8.

Let $f : (X, S, x_0) \to (Y, T, y_0)$ be a morphism in $\mathcal{AS}_0$ and suppose that $P(f)$ is not an epimorphism. Then there exists $y_1 \in Y - f(X)$. Put $Z = Y \cup \{z\}$ for some $z \notin Y$ and consider the class-one scheme $(Z, \{1_Z, Z \times Z - 1_Z\}, y_0)$. Define $g_1 : (Y, T, y_0) \to (Z, \{1_Z, Z \times Z - 1_Z\}, y_0)$ by $g_1(y) = y$ for every $y \in Y$, $g_1(1_Y) = 1_Z$, and $g_1(t) = Z \times Z - 1_Z$ for every $t \in T - \{1_Y\}$. Then g_1 is a morphism in $\mathcal{AS}_0$. Define $g_2 : (Y, T, y_0) \to (Z, \{1_Z, Z \times Z - 1\}, y_0)$ by $g_2(y) = y$ for every $y \in Y - \{y_1\}$, $g_2(y_1) = z$, $g_1(1_Y) = 1_Z$, and $g_1(t) = Z \times Z - 1_Z$ for every $t \in T - \{1_Y\}$. Then g_2 is a morphism in $\mathcal{AS}_0$. Now $g_1 \neq g_2$ and $g_1 f = g_2 f$. So f is not an epimorphism. □

To prove Theorem 3.1 (2), we will show the following two lemmas.

Lemma 3.2. *Let $f : (X, S, x_0) \to (Y, T, y_0)$ be a morphism in $\mathcal{AS}_0$. Then $|f^{-1}(y)| = |f^{-1}(y_0)|$ holds for any $y \in f(X)$. Especially, we have $|X| = \sharp\{x \in X \mid f(x) = y_0\}|f(X)|$.*

Proof. Suppose that $y \in f(X)$ and choose $x \in f^{-1}(y)$. For $x' \in X$, $f(x) = f(x')$ if and only if $1_Y = r(f(x), f(x')) = f(r(x, x'))$. Since $\sharp\{x' \in X \mid r(x, x') = s\} = n_s$ for $s \in S$, we have $|f^{-1}(y)| = \sharp\{x' \in X \mid f(x') = f(x)\} = \sum_{s \in f^{-1}(1_Y)} n_s$. This number is independent of the choice of $y \in f(X)$. So the lemma holds. □

Lemma 3.3. *Let $f : (X, S, x_0) \to (Y, T, y_0)$ be a morphism in $\mathcal{AS}_0$. Then $f^{-1}(1_Y)$ is a closed subset of S.*

Proof. Suppose that $s_1, s_2 \in f^{-1}(1_Y)$ and $u \in s_1 s_2$. There exist $x_1, x_2, x_3 \in X$ such that $r(x_1, x_2) = s_1$, $r(x_2, x_3) = s_2$, and $r(x_1, x_3) = u$. Then $1_Y = f(s_1) = f(r(x_1, x_2)) = r(f(x_1), f(x_2))$. This means that $f(x_1) = f(x_2)$. Similarly $f(x_2) = f(x_3)$. So we have $f(x_1) = f(x_3)$. Now $f(u) = f(r(x_1, x_3)) = r(f(x_1), f(x_3)) = 1_Y$ and $u \in f^{-1}(1_Y)$. This means that $f^{-1}(1_Y)$ is a closed subset of S. □

Lemma 3.3 is a special case of [4, Lemma 1.7.2 (ii)].

Proof of Theorem 3.1 (2). If $P(f)$ is a monomorphism, then f is a monomorphism by Lemma 2.8.

Let $f : (X, S, x_0) \to (Y, T, y_0)$ be a morphism in $\mathcal{AS}_0$ and suppose that $P(f)$ is not a monomorphism. Then $|f^{-1}(y)| > 1$ for some $y \in Y$. By Lemma 3.2, $|f^{-1}(y_0)| > 1$. Choose $x_0 \neq x \in f^{-1}(y_0)$. Then $r(x_0, x) \neq 1_X$ and $f(r(x_0, x)) = r(f(x_0), f(x)) = 1_Y$. So $f^{-1}(1_Y) \neq \{1_X\}$. By Lemma 3.3, $f^{-1}(1_Y)$ is a closed subset of S.

Put $U = f^{-1}(1_Y)$ and consider the subscheme $(x_0 U, S_{x_0 U}, x_0)$. Let g be a morphism from $(x_0 U, S_{x_0 U}, x_0)$ to (X, S, x_0) defined in Example 2.5. Then $fg = 0 = f0$ and $g \neq 0$ by $U \neq \{1_X\}$. This means that f is not a monomorphism. □

Theorem 3.1 (2) and the next lemma show Theorem 3.1 (4).

Lemma 3.4. *Let $f : (X, S, x_0) \to (Y, T, y_0)$ be a morphism in $\mathcal{AS}_0$. Suppose $f^{-1}(1_Y) = \{1_X\}$. Then $P(f)$ is a monomorphism. Especially, if $R(f)$ is a monomorphism, then so is $P(f)$.*

Proof. Suppose that $f(x) = f(x')$ for $x, x' \in X$. Then $f(r(x, x')) = r(f(x), f(x')) = 1_Y$. By the assumption, we have $r(x, x') = 1_X$ and $x = x'$. This means that $P(f)$ is a monomorphism. □

Now we will show the last part of the theorem.

Proof of Theorem 3.1 (3). Let $f : (X, S, x_0) \to (Y, T, y_0)$ be an epimorphism in $\mathcal{AS}_0$. Let $t \in T$. There exist $y, y' \in Y$ such that $t = r(y, y')$. By Theorem 3.1 (1), $P(f)$ is an epimorphism. So there exist $x, x' \in X$ such that $f(x) = y$ and $f(x') = y'$. Then $f(r(x, x')) = r(f(x), f(x')) = r(y, y') = t$. This shows that $R(f)$ is an epimorphism. □

Example 3.5. (1) Let X and Y be finite sets with $1 < |X| < |Y|$, $x_0 \in X$, and $y_0 \in Y$. Any injection $f : X \to Y$ that satisfies $f(x_0) = y_0$ induces a morphism $\widetilde{f}$ from the class-one scheme $(X, \{1_X, X \times X - 1_X\}, x_0)$ to the class-one scheme $(Y, \{1_Y, Y \times Y - 1_Y\}, y_0)$ in $\mathcal{AS}_0$. Then $R(\widetilde{f})$ is an epimorphism and $P(\widetilde{f})$ is not.

(2) Let (X, T, x_0) be a fusion scheme of (X, S, x_0), and let $f : (X, S, x_0) \to (X, T, x_0)$ be the morphism defined in Example 2.7. Suppose $|S| > |T|$. Then $P(f)$ is a monomorphism and $R(f)$ is not.

Now we will give a characterization of a bimorphism, that is an epimorphism and a monomorphism, in $\mathcal{AS}_0$.

Theorem 3.6. *Let $f : (X, S, x_0) \to (Y, T, y_0)$ be a morphism in $\mathcal{AS}_0$. Then the following statements are equivalent.*

(1) *f is a bimorphism.*
(2) *$P(f)$ is a bijection.*
(3) *$P(f)$ is a bijection and, for every $s \in S$, there exists $t \in T$ such that $(f(x), f(x')) \in t$ whenever $(x, x') \in s$.*

Proof. By Theorem 3.1, (1) and (2) are equivalent and obviously (3) implies (2). By the definition of a morphism, it is also clear that (2) implies (3). □

The above theorem says that a bimorphism in $\mathcal{AS}_0$ is essentially the morphism defined in Example 2.7. So we call a bimorphism in $\mathcal{AS}_0$ a *fusion scheme*.

Obviously a morphism f is an isomorphism if and only if both $P(f)$ and $R(f)$ are bijections.

4. Kernels, cokernels, images, and exact sequences

4.1. **Kernels and cokernels.** We call a morphism defined in Example 2.5 a *subscheme*. Of course, a monomorphism equivalent to such a morphism is also said to be a subscheme. We call a morphism defined in Example 2.6 a *quotient scheme*. An epimorphism equivalent to such a morphism is also said to be a quotient scheme.

We will prove the following theorem.

Theorem 4.1. (1) *The category $\mathcal{AS}_0$ has kernels.*
(2) *The category $\mathcal{AS}_0$ has cokernels.*
(3) *A morphism f in $\mathcal{AS}_0$ is a normal subobject (a kernel of some morphism) if and only if f is a subscheme.*
(4) *A morphism f in $\mathcal{AS}_0$ is a conormal quotient object (a cokernel of some morphism) if and only if f is a quotient scheme.*

We begin with some lemmas.

Lemma 4.2. *Let $f : (X, S, x_0) \to (Y, T, y_0)$ be a morphism in $\mathcal{AS}_0$. Then $x_0 f^{-1}(1_Y) = \{x \in X \mid f(x) = y_0\}$.*

Proof. This is clear by the proof of Lemma 3.3. □

Lemma 4.3. *Let $f : (X, S, x_0) \to (Y, T, y_0)$ be a morphism in $\mathcal{AS}_0$. Put $U = f^{-1}(1_Y)$. Then the subscheme $\iota : (x_0 U, S_{x_0 U}, x_0) \to (X, S, x_0)$ is a kernel of f.*

Proof. By Lemma 4.2, $f\iota = 0$ is clear.

Suppose that a morphism $g : (Z, V, z_0) \to (X, S, x_0)$ in $\mathcal{AS}_0$ satisfies $fg = 0$. For $z \in Z$, $fg(z) = y_0$. So $g(z) \in x_0U$. Thus we can define a map $h' : Z \to x_0U$ by $h'(z) = g(z)$. Now we define a map $h'' : V \to S_{x_0U}$ by $h''(r_V(z_1, z_2)) = r_{S_{x_0U}}(h'(z_1), h'(z_2))$. It is necessary to check that h'' is well-defined. Assume $r_V(z_1, z_2) = r_V(z_3, z_4)$. Then $r_S(h'(z_1), h'(z_2)) = r_S(g(z_1), g(z_2)) = g(r_V(z_1, z_2)) = g(r_V(z_3, z_4)) = r_S(g(z_3), g(z_4)) = r_S(h'(z_3), h'(z_4))$. Since $R(\iota)$ is an injection, we have $r_{S_{x_0U}}(h'(z_1), h'(z_2)) = r_{S_{x_0U}}(h'(z_3), h'(z_4))$. This means that h'' is well-defined. Now the pair of maps (h, h') defines a morphism $h : (Z, V, z_0) \to (X_0U, S_{x_0U}, x_0)$. By the construction of h, $\iota h = g$ is clear. The uniqueness of h is also clear since ι is a monomorphism. Now ι is a kernel of f. □

This lemma shows Theorem 4.1 (1) and a half of (3), a kernel of a morphism in $\mathcal{AS}_0$ is a subscheme. To prove Theorem 4.1 (2), we will show two lemmas. Lemma 4.4 follows from [4, Lemma 1.7.1].

Lemma 4.4. *Let $f : (X, S, x_0) \to (Y, T, y_0)$ be a morphism in $\mathcal{AS}_0$, and let $s_1, s_2 \cdots, s_\ell \in S$. Then $f(s_1 s_2 \cdots s_\ell) \subseteq f(s_1)f(s_2)\cdots f(s_\ell)$, where the products are complex products.*

Proof. Suppose $u \in s_1 s_2 \cdots s_\ell$. There exists $x = x_1, x_2, \cdots x_\ell, x_{\ell+1} = x'$ such that $r(x, x') = u$ and $r(x_i, x_{i+1}) = s_i$ for $i = 1, 2, \cdots, \ell$. So $r(f(x_i), f(x_{i+1})) = f(r(x_i, x_{i+1})) = f(s_i)$ for $i = 1, 2, \cdots, \ell$. Now $f(u) = f(r(x, x')) = r(f(x), f(x')) \in f(s_1)f(s_2)\cdots f(s_\ell)$ and the lemma holds. □

Lemma 4.5. *Let $f : (X, S, x_0) \to (Y, T, y_0)$ be a morphism in $\mathcal{AS}_0$. Put $U = \langle f(S) \rangle$. Then the quotient scheme $\pi : (Y, T, y_0) \to (Y/U, T/\!\!/U, y_0U)$ is a cokernel of f.*

Proof. It is clear that $\pi f = 0$. Suppose that $gf = 0$ for a morphism $g : (Y, T, y_0) \to (Z, V, z_0)$ in $\mathcal{AS}_0$.

We define a map $h' : Y/U \to Z$ by $h'(yU) = g(y)$. We will check that h' is well-defined. Assume $yU = y'U$. Then there exists $u \in U$ such that $r(y, y') = u$. Since U is generated by $f(S)$, there exist $u_1, u_2, \cdots, u_\ell \in f(S)$ such that $u \in u_1 u_2 \cdots u_\ell$. There exist $y = y_1, y_2, \cdots, y_\ell, y_{\ell+1} = y'$ such that $r(y_i, y_{i+1}) = u_i$ for $i = 1, 2, \cdots, \ell$. By $gf = 0$, $g(u_i) \in g(f(S)) = \{1_Z\}$. So $r(g(y_i), g(y_{i+1})) = g(r(y_i, y_{i+1})) = g(u_i) = 1_Z$. This means that $g(y_i) = g(y_{i+1})$. So $g(y) = g(y')$ holds and h' is well-defined.

We define a map $h'' : T/\!\!/U \to V$ by $h''(t^U) = g(t)$. We will check that h'' is well-defined. Suppose $t^U = (t')^U$. Then there exist $u, u' \in U$ such that $t' \in utu'$. By Lemma 4.4, $g(t') \subseteq g(utu') \subseteq g(u)g(t)g(u')$ holds. Similar to the above argument, $g(u) = g(u') = 1_Z$ holds. So $g(t) = g(t')$ holds and h'' is well-defined.

Using h' and h'', we define $h : Y/U \cup T/\!\!/U \to Z \cup V$. Then $h''(r(yU, y'U)) = g(r(y, y')) = r(g(y), g(y')) = r(h'(yU), h'(y'U))$ and this means that h is a morphism in $\mathcal{AS}_0$.

The uniqueness of such h is clear since π is an epimorphism. Now π is a cokernel of f. □

This lemma shows Theorem 4.1 (2) and a half of (4), a cokernel of a morphism in $\mathcal{AS}_0$ is a quotient scheme.

The next lemma is clear by the above arguments and it shows the rests of Theorem 4.1.

Lemma 4.6. *The following statements hold.*

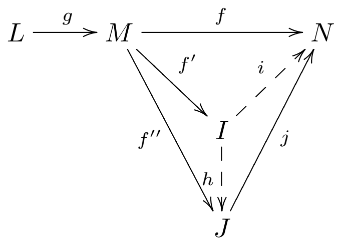
(1) *For a zero morphism $0 : (X, S, x_0) \to (Y, T, y_0)$, a kernel of 0 is $I_{(X,S,x_0)}$ and a cokernel of 0 is $I_{(Y,T,y_0)}$.*
(2) *For a morphism f, a kernel of f is a zero morphism if and only if f is a monomorphism.*
(3) *If ι is a subscheme and g is a cokernel of ι, then ι is a kernel of g.*
(4) *If π is a quotient scheme and g is a kernel of π, then π is a cokernel of g.*

Remark. The category $\mathcal{AS}_0$ is neither a normal category nor a conormal category. Actually, any non-isomorphic fusion scheme is neither a normal subobject nor a conormal quotient object.

4.2. **Images.** In this subsection, we will prove the following theorem.

Theorem 4.7. *The category $\mathcal{AS}_0$ has epimorphic images.*

Proof. Let $f : M \to N$ be a morphism in $\mathcal{AS}_0$. Let $g : L \to M$ be a kernel of f, and let $f' : M \to I$ be a cokernel of g. Since $fg = 0$, there exists a unique morphism $i : I \to N$ such that $f = if'$. We will show that i is an image of f. Then the statement holds since f' is an epimorphism.



Suppose that a monomorphism $j : J \to N$ and a morphism $f'' : M \to J$ in $\mathcal{AS}_0$ satisfy $jf'' = f$. Since $0 = fg = jf''g$ and j is a monomorphism, $f''g = 0$ holds. There exists a unique morphism $h : I \to J$ such that $hf' = f''$ since f' is a cokernel of g. Since $if' = f = jf'' = jhf'$ and f' is an epimorphism, $i = jh$ holds. The uniqueness of such h follows from that j is a monomorphism.

To show that i is a monomorphism, we will prove that $P(i)$ is a monomorphism. Assume that $i(x) = i(x')$ for $x, x' \in P(I)$. Since f' is an epimorphism, there exist $y, y' \in P(M)$ such that $f'(y) = x$ and $f'(y') = x'$. Then $1 = r(i(x), i(x')) = i(r(x, x')) = if'(r(y, y')) = f(r(y, y'))$. So $r(y, y') \in g(L)$ and $1 = f'(r(y, y')) = r(f'(y), f'(y')) = r(x, x')$. This means that $x = x'$ and $P(i)$ is a monomorphism. So i is a monomorphism and it is an image of f. □

We consider images of monomorphisms and epimorphisms.

Proposition 4.8. *If f is a monomorphism, then f is an image of f. If f is an epimorphism, then an image of f is a fusion scheme.*

Proof. If $f : M \to N$ is a monomorphism, then a kernel of f is a zero morphism $0 : 0 \to M$. So I_M is a cokernel of the kernel of f and f is an image of f.

If f is an epimorphism, then the image of f is a monomorphism and an epimorphism. So it is a fusion scheme. □

This proposition says that every epimorphism is a composition of a quotient scheme and a fusion scheme.

We shall remark that an image is obtained by taking a cokernel of a kernel. So this is closely related to the "homomorphism theorem" [4, Theorem 1.7.5].

4.3. **Exact sequences.** We consider a sequence of morphisms

$$\cdots \to M_{i-1} \xrightarrow{f_{i-1}} M_i \xrightarrow{f_i} M_{i+1} \to \cdots$$

in $\mathcal{AS}_0$. We say that the sequence is *exact at M_i* if $\mathrm{Ker}(f_i) = \mathrm{Im}(f_{i-1})$ holds. We note that a kernel or an image of a morphism is not uniquely determined. They are unique up to equivalence of monomorphisms. We use the notation $\mathrm{Ker}(f_i) = \mathrm{Im}(f_{i-1})$ for the meaning that "an image of f_{i-1} is a kernel of f_i", and we use similar notations for the cokernel $\mathrm{Coker}(f)$, and so on. The sequence is said to be an *exact sequence* if it is exact at every object in the sequence. If the sequence is bounded above or below, then we do not consider the exactness at the end of the sequence.

Theorem 4.9. *In $\mathcal{AS}_0$, the following statements hold.*

(1) $0 \to M \xrightarrow{f} N$ *is an exact sequence if and only if f is a monomorphism.*
(2) $M \xrightarrow{f} N \to 0$ *is an exact sequence if and only if f is a quotient scheme.*

(3) $0 \to M \xrightarrow{f} N \to 0$ *is an exact sequence if and only if* f *is an isomorphism.*

(4) $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ *is an exact sequence if and only if* $f = \mathrm{Ker}(g)$ *and* $g = \mathrm{Coker}(f)$.

Proof. (1) By Lemma 4.6 (1), $\mathrm{Im}(0 \to M) = (0 \to M)$. So the sequence is exact if and only if $\mathrm{Ker}(f) = (0 \to M)$. This condition is equivalent to that f is a monomorphism by Lemma 4.6 (2).

(2) By Lemma 4.6 (1), $\mathrm{Ker}(N \to 0) = I_N$. So the sequence is exact if and only if $\mathrm{Im}(f) = I_N$. This condition is equivalent to $f = \mathrm{Coker}(\mathrm{Ker}(f))$. This means that f is a quotient scheme.

(3) By (1) and (2), f is a monomorphism and a quotient scheme if the sequence is exact. Then f is an isomorphism. The converse is clear.

(4) Suppose that $f = \mathrm{Ker}(g)$ and $g = \mathrm{Coker}(f)$. Then $\mathrm{Ker}(g) = f = \mathrm{Im}(f)$ by Lemma 4.6. This shows that the sequence is exact.

Suppose that the sequence is exact. Then f is a monomorphism by (1) and g is a quotient scheme by (2). By Lemma 4.6, $f = \mathrm{Im}(f) = \mathrm{Ker}(g)$. In this case, $g = \mathrm{Coker}(f)$ since g is a quotient scheme. □

An exact sequence in $\mathcal{AS}_0$ of the form

$$0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$$

is called a *short exact sequence* and has already been considered in [2].

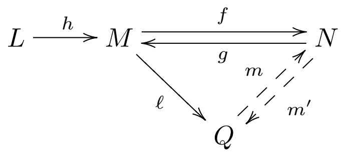
A morphism $f : M \to N$ is said to be a *retraction* if there exists $g : N \to M$ such that $fg = I_N$. We show that a retraction in $\mathcal{AS}_0$ is a quotient scheme and so it defines a short exact sequence.

Proposition 4.10. *Let* $f : M \to N$ *be a retraction in* $\mathcal{AS}_0$. *Then* f *is a quotient scheme and*

$$0 \to L \xrightarrow{\mathrm{Ker}(f)} M \xrightarrow{f} N \to 0$$

is an exact sequence.

Proof. Let $g : N \to M$ be a morphism with the property $fg = I_N$. Let $\mathrm{Ker}(f) = h : L \to N$, and let $\mathrm{Coker}(h) = \ell : M \to Q$. Then there exists a unique morphism $m : Q \to N$ such that $f = m\ell$. Put $m' = \ell g$.



Then $mm' = m\ell g = fg = I_N$. So $mI_Q = m = I_N m = mm'm$ holds. Since m is an image of f, m is a monomorphism. Thus $I_Q = m'm$ holds. This means that m is an isomorphism and f is equivalent to a quotient scheme ℓ as an epimorphism. By Theorem 4.9 (4), we have an exact sequence. □

A morphism $f : M \to N$ is said to be a *coretraction* if there exists $g : N \to M$ such that $gf = I_M$. The dual statement of Proposition 4.10 does not hold. In $\mathcal{AS}_0$, a coretraction is not necessarily a subscheme.

Example 4.11. Put $X = \{x_0, x_1\}$, $Y = \{y_0, y_1, y_2, y_3\}$, $t = \{(y_0, y_1), (y_1, y_0), (y_2, y_3), (y_3, y_2)\} \subseteq Y \times Y$, and $t' = Y \times Y - 1_Y - t$. Define association schemes $(X, \{1_X, X \times X - 1_X\}, x_0)$ and $(Y, \{1_Y, t, t'\}, y_0)$. The maps $f : X \to Y$ and $g : Y \to X$ defined by $f(x_0) = y_0$, $f(x_1) = y_2$, $g(y_0) = x_0$, $g(y_1) = x_0$, $g(y_2) = x_1$, and $g(y_3) = x_1$ define morphisms $\widetilde{f}$ and $\widetilde{g}$ in $\mathcal{AS}_0$. Then $\widetilde{g}\widetilde{f} = I_{(X,\{1_X, X\times X - 1_X\}, x_0)}$ holds but $\widetilde{f}$ is not a subscheme.

5. The category of finite groups

In this section, we will consider the full subcategory $\mathcal{AS}_0^{\mathrm{thin}}$ of $\mathcal{AS}_0$ whose objects are thin schemes. We will show that it is equivalent to the category of finite groups.

Recall the definition of thin schemes in Example 2.3. Let G be a finite group. For $g \in G$, put $[g] = \{(x,y) \in G \times G \mid xg = y\}$ and $[G] = \{[g] \mid g \in G\}$. Then $(G, [G])$ is a thin scheme.

Proposition 5.1. *Let G be a finite group. In $\mathcal{AS}_0$, $(G, [G], \alpha)$ and $(G, [G], \beta)$ are isomorphic for any $\alpha, \beta \in G$.*

Proof. Define $f : G \to G$ by $f(x) = \beta\alpha^{-1}x$ and $f : [G] \to [G]$ by the identity. Then $f : G \cup [G] \to G \cup [G]$ is an isomorphism from $(G, [G], \alpha)$ to $(G, [G], \beta)$ in $\mathcal{AS}_0$. □

Proposition 5.2. *The category $\mathcal{AS}_0^{\mathrm{thin}}$ is equivalent to the category of finite groups.*

Proof. It is enough to show that a morphism in $\mathcal{AS}_0^{\mathrm{thin}}$ is just a group homomorphism.

Let $f : G \to H$ be a group homomorphism. By Proposition 5.1, we may choose objects $(G, [G], 1_G)$ and $(H, [H], 1_H)$ in $\mathcal{AS}_0^{\mathrm{thin}}$. We define $f' : G \cup [G] \to H \cup [H]$ by $f'(g) = f(g)$ and $f'([g]) = [f(g)]$ for $g \in G$. Then $f'(r(x,y)) = f'([x^{-1}y]) = [f(x^{-1}y)]$ and $r(f'(x), f'(y)) = r(f(x), f(y)) = [f(x)^{-1}f(y)] = [f(x^{-1}y)]$. So f' is a morphism in $\mathcal{AS}_0^{\mathrm{thin}}$.

Let $f : (G, [G], \alpha) \to (H, [H], \beta)$ be a morphism in $\mathcal{AS}_0^{\mathrm{thin}}$. For $g, g' \in G$, $r(g^{-1}, 1_G) = [g]$ and $r(1_G, g') = [g']$. So $[g][g'] = r(g^{-1}, g') = [gg']$. Thus f defines a group homomorphism from G to H. □

We will define some known notions by universal properties in our categories. The definitions are slightly different from the original ones, but they are essentially the same.

Example 5.3 (thin radicals)**.** A morphism $f : M \to N$ in $\mathcal{AS}_0$ is called the *thin radical* of N if the following properties hold :

(1) M is thin,
(2) f is a subscheme, and
(3) if M' is thin and $f' : M' \to N$ is a subscheme, then there exists a unique morphism $g : M' \to M$ such that $fg = f'$.

In [4, §2.3], the thin radical is defined as a closed subset. But in our definition, it is a subscheme. In general, a subscheme determines a closed subset. So in this sense, our definition is equivalent to that in [4, §2.3].

Example 5.4 (thin residues)**.** A morphism $f : M \to N$ in $\mathcal{AS}_0$ is called the *thin residue* of M if the following properties hold :

(1) N is thin,
(2) f is a quotient scheme, and
(3) if N' is thin and $f' : M \to N'$ is a quotient scheme, then there exists a unique morphism $g : N \to N'$ such that $gf = f'$.

In [4, §2.3], the thin residue is also defined as a closed subset. But in our definition, it is a quotient scheme. In general, a closed subset determines a quotient scheme. In this sense, our definition is equivalent to that in [4, §2.3].

Example 5.5 (schurian schemes)**.** An association scheme (X, S, x_0) is said to be *schurian* if it is a quotient scheme of a thin scheme. In this case, we also say that (X, S) is schurian.

Schurian schemes are studied in [5, Chap. 6], for example.

The next theorem is a well known fact but we will give a categorical proof to it.

Theorem 5.6. *Let (X, S) be a schurian scheme. Then, for any $x, x' \in X$, (X, S, x) is isomorphic to (X, S, x') in $\mathcal{AS}_0$.*

Proof. There exists a short exact sequence

$$0 \longrightarrow (Y,T,y_0) \xrightarrow{h} (G,[G],1_G) \xrightarrow{f} (X,S,x_0) \longrightarrow 0$$

for some thin scheme $(G,[G],1_G)$ and some $x_0 \in X$. It is enough to show that, for any $x \in G$, (X,S,x_0) is isomorphic to (X,S,x). Since $P(f)$ is surjective, there exists $a \in G$ such that $f(a) = x$. By Proposition 5.1, there exists an isomorphism $g : (G,[G],1_G) \to (G,[G],a)$ such that $R(g) : [G] \to [G]$ is the identity map. Define $f' : (G,[G],a) \to (X,S,x)$ by $f' = f$ as a map from $G \cup [G]$ to $G \cup [G]$. Then f' is a morphism in $\mathcal{AS}_0$.

$$\begin{CD} (Y,T,y_0) @>h>> (G,[G],1_G) @>f>> (X,S,x_0) \\ @. @VgVV @VVkV \\ @. (G,[G],a) @>>f'> (X,S,x) \end{CD}$$

Since $f = \mathrm{Coker}(h)$ and $f'gh = 0$, there exists $k : (X,S,x_0) \to (X,S,x)$ such that $kf = f'g$. Since $f'g$ is an epimorphism, so is k. Now $P(k)$ is a surjection from a finite set X to itself. Thus $P(k)$ is a bijection. The morphism k is an isomorphism. □

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