CLIFFORD TYPE THEOREMS FOR ASSOCIATION SCHEMES AND THEIR ALGEBRAIC FUSIONS

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Dedicated to Professor Shigeo Koshitani on his sixtieth birthday

ABSTRACT. We investigate the relationship between complex characters of association schemes and their fusion schemes. We first prove Frobenius reciprocity between the irreducible representations of finite schemes and their fusion schemes, and add a formula on multiplicities. After that, we provide Clifford type theorems for association schemes and their algebraic fusions.

1. INTRODUCTION

Induction and restriction of representations of association schemes have been generalized from finite group theory in [2] and have been investigated repeatedly with respect to their closed subsets. The present paper deals with the question if induction and restriction of representations of association schemes are related similarly if one considers them with respect to fusion schemes instead of closed subsets. We first suggest definitions of induction and restriction of representations of association schemes with respect to their fusion schemes. We then prove Frobenius reciprocity (Theorem 2.4) and give a formula on multiplicities (Theorem 3.3). Finally, we consider Clifford type theorems for algebraic fusions (Theorem 4.3 and Theorem 4.5).

Let (X, S) be an association scheme in the sense of [8] (see also [4]). For $s, t, u \in S$, we denote the intersection number (or the structure constant) by p_{st}^{u}. For $s \in S$, we denote the valency of s by n_s. The adjacency matrix of $s \in S$ will be denoted by σ_s. The complex adjacency algebra $\mathbb{C}S$ is known to be semisimple [8, Theorem 4.1.3]. We denote by $\mathrm{Irr}(S)$ the set of all irreducible complex characters of S.

For a character χ of S, let V_χ be a right $\mathbb{C}S$-module affording χ. Put

$$(\chi, \chi')_S = \dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}S}(V_\chi, V_{\chi'}).$$

Since $\mathbb{C}S$ is semisimple, if $\chi, \chi' \in \mathrm{Irr}(S)$, then $(\chi, \chi')_S = \delta_{\chi\chi'}$.

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Let (X, S) and (X, U) be association schemes. If, for every $s \in S$, there exists $u \in U$ such that $s \subseteq u$, then we say that (X, U) is a *fusion scheme* of (X, S), or (X, S) is a *fission scheme* of (X, U). In this case, the adjacency algebra $\mathbb{C}U$ is naturally a subalgebra of $\mathbb{C}S$.

A map $g : S \to S$ is said to be an *algebraic automorphism* of (X, S) if $p_{s^g t^g}^{u^g} = p_{st}^u$ for all $s, t, u \in S$. We denote the set of all algebraic automorphisms of (X, S) by $\mathrm{AAut}(S)$. Then $\mathrm{AAut}(S)$ becomes a group and we call it the *algebraic automorphism group* of (X, S). Let G be a subgroup of $\mathrm{AAut}(S)$. For $s \in S$, put $s^G = \bigcup_{g \in G} s^g$ and put $S^G = \{s^G \mid s \in S\}$. Then (X, S^G) becomes a fusion scheme of (X, S). The scheme (X, S^G) is called an *algebraic fusion* of (X, S) by the action of G. For algebraic fusions, see [6].

Example 1.1. Let (X, S) be a thin scheme (see [8, Introduction]). We can regard S as a finite group. Then the inner automorphism group $\mathrm{Inn}(S)$ is a subgroup of $\mathrm{AAut}(S)$. The algebraic fusion $(X, S^{\mathrm{Inn}(S)})$ is the group association scheme of S (see [1, Chapter I, Example 2.1 (2)]). The adjacency algebra of $(X, S^{\mathrm{Inn}(S)})$ coincides with the center of the group algebra $\mathbb{C}S$.

2. Frobenius reciprocity

Let A be a ring and B be a subring of A.

For a right A-module V, we can define a right B-module $V\downarrow_B$ by restricting the action of A on V to B. The module $V\downarrow_B$ is called the *restriction* of V to B.

For a right B-module W, we can define a right A-module $W\uparrow^A = W \otimes_B A$. The module $W\uparrow^A$ is called the *induction* of W to A.

Proposition 2.1 ([7, Chapter 1, Theorem 11.3 (i)])**.** *Let A be a ring and B be a subring of A. For a right A-module V and a right B-module W, we have*

$$\mathrm{Hom}_B(W, V\downarrow_B) \cong \mathrm{Hom}_A(W\uparrow^A, V).$$

Remark 2.2. In the definition of the induction $W\uparrow^A = W \otimes_B A$, A is considered as a (B, A)-bimodule. The restriction $V\downarrow_B$ is $V \otimes_A A$, where A is considered as an (A, B)-bimodule. So they are similar operations.

Let (X, S) be an association scheme, and let T be a closed subset. For $x \in X$, we can define a subscheme (xT, T_{xT}) (see [8, Section 1.5]). The adjacency algebra $\mathbb{C}(T_{xT})$ is isomorphic to the subalgebra $\mathbb{C}T = \bigoplus_{t \in T} \mathbb{C}\sigma_t$ of $\mathbb{C}S$. So we often identify them. We will write $\mathrm{Irr}(T)$ for $\mathrm{Irr}(T_{xT})$. We will use notations $\varphi\uparrow^S$ and $\chi\downarrow_T$ for the characters of the induction and the restriction of modules. The next theorem is well-known and an immediate consequence from Proposition 2.1.

Theorem 2.3 (Frobenius reciprocity [2, Theorem 5.2])**.** *Let (X, S) be an association scheme, and let T be a closed subset of S. Let $\varphi \in \mathrm{Irr}(T)$ and let $\chi \in \mathrm{Irr}(S)$. Then $(\varphi, \chi{\downarrow_T})_T = (\varphi{\uparrow^S}, \chi)_S$.*

Now, let (X, S) be an association scheme, and let (X, U) be a fusion scheme of (X, S). Then the adjacency algebra $\mathbb{C}U$ is a subalgebra of $\mathbb{C}S$. So we can define inductions and restrictions for this case. Again, by Proposition 2.1, we have Frobenius reciprocity also for fusion schemes.

Theorem 2.4 (Frobenius reciprocity for fusion schemes)**.** *Let (X, S) be an association scheme, and let (X, U) be a fusion scheme of (X, S). Let $\varphi \in \mathrm{Irr}(U)$ and let $\chi \in \mathrm{Irr}(S)$. Then $(\varphi, \chi{\downarrow_U})_U = (\varphi{\uparrow^S}, \chi)_S$.*

For $\chi \in \mathrm{Irr}(S)$, we denote by e_χ the central primitive idempotent of $\mathbb{C}S$ corresponding to χ. The next lemma will be used in Section 4.

Lemma 2.5. *Let (X, S) be an association scheme, and let (X, U) be a fusion scheme of (X, S). Let $\chi \in \mathrm{Irr}(S)$, and let $\varphi \in \mathrm{Irr}(U)$. Then $(\varphi{\uparrow^S}, \chi)_S \neq 0$ if and only if $e_\varphi e_\chi \neq 0$.*

Proof. Let V be a simple right $\mathbb{C}S$ module affording χ, and let W be a simple right $\mathbb{C}U$ module affording φ. First we note that $e_\varphi \mathbb{C}U \cong mW$ and $e_\chi \mathbb{C}S \cong nV$ where $m = \dim_{\mathbb{C}} W$ and $n = \dim_{\mathbb{C}} V$. By [7, Chapter 1, Theorem 4.3 (i)], we have

$$\begin{aligned}
(\varphi{\uparrow^S},\ \chi)_S &= \dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}S}(W{\uparrow^S},\ V) \\
&= \frac{1}{mn} \dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}S}(e_\varphi \mathbb{C}U \otimes_{\mathbb{C}U} \mathbb{C}S,\ e_\chi \mathbb{C}S) \\
&= \frac{1}{mn} \dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}S}(e_\varphi \mathbb{C}S,\ e_\chi \mathbb{C}S) \\
&= \frac{1}{mn} \dim_{\mathbb{C}} e_\chi \mathbb{C}S e_\varphi = \frac{1}{mn} \dim_{\mathbb{C}} \mathbb{C}S e_\chi e_\varphi.
\end{aligned}$$

So the statement holds. □

3. Standard modules and multiplicities

In [3, Theorem 5.1], a formula on inductions and multiplicities was given. In this section, we will give an alternative proof of [3, Theorem 5.1] and give a similar formula on inductions from fusion schemes.

Usually, multiplicities of characters are defined only for irreducible characters (see [8, Section 4.1]). In [3], the definition was extended to arbitrary characters.

Let (X, S) be an association scheme. Let η be a character of S afforded by a right $\mathbb{C}S$-module V. Put

$$m_\eta = \dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}S}(V, \mathbb{C}X),$$

where $\mathbb{C}X$ is the standard $\mathbb{C}S$-module (see [8, Section 4.1]). We call m_η the *multiplicity* of η. If η is irreducible, then m_η is just the multiplicity defined in [8, Section 4.1]. Also, if $\eta = \sum_{\chi\in\mathrm{Irr}(S)} a_\chi \chi$ is an irreducible decomposition of η, then $m_\eta = \sum_{\chi\in\mathrm{Irr}(S)} a_\chi m_\chi$. So the definition is equivalent to that in [3].

Now, we consider the restriction of the standard module to a subscheme or a fusion scheme. Let (X, S) be an association scheme and let T be a closed subset of S. Put

$$X = x_1T \cup \cdots \cup x_rT$$

a coset decomposition (see [8, Section 1.3]), where $r = n_S/n_T$. Then

$$\mathbb{C}X\downarrow_T = \mathbb{C}(x_1T) \oplus \cdots \oplus \mathbb{C}(x_rT).$$

Now $\mathbb{C}(x_iT)$ is the standard $\mathbb{C}(T_{x_iT})$-module by $\mathbb{C}T \cong \mathbb{C}(T_{x_iT})$. The multiplicities of irreducible characters are determined only by structure constants. So we have $\mathbb{C}(x_iT) \cong \mathbb{C}(x_1T)$ for every i. Now we have

$$\mathbb{C}X\downarrow_T \cong \frac{n_S}{n_T}\,\mathbb{C}(x_1T)$$

as a right $\mathbb{C}T$-module.

Theorem 3.1 ([3, Theorem 5.1])**.** *Let (X, S) be an association scheme and let T be a closed subset of S. Let φ be a character of T. Then*

$$m_{\varphi\uparrow^S} = \frac{n_S}{n_T} m_\varphi.$$

Proof. Let W be a right $\mathbb{C}T$-module affording φ. Choose $x \in X$ arbitrarily. Then

$$\begin{aligned} m_{\varphi\uparrow^S} &= \dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}S}(W\uparrow^S,\ \mathbb{C}X) = \dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}T}(W,\ \mathbb{C}X\downarrow_T) \\ &= \frac{n_S}{n_T} \dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}T}(W,\ \mathbb{C}(xT)) = \frac{n_S}{n_T} m_\varphi. \end{aligned}$$

Now the theorem is proved. □

Remark 3.2. Let K be an algebraically closed field of positive characteristic. In this case, $K(x_iT)$ is not necessarily isomorphic to $K(x_1T)$ as a right KT-module.

We will give an example. Let us consider two non-isomorphic schemes (X_1, T_1) and (X_2, T_2) such that they are algebraically isomorphic but their modular standard modules are non-isomorphic. For example, such examples were considered in [5]. Define a scheme similar to the wreath product of them by the scheme of order 2. Namely, put T_1 and T_2 in diagonal parts and fill the remaining parts by a single relation. Then this scheme satisfies the above condition, because standard modules are non-isomorphic.

Let (X,S) be an association scheme and let (X,U) be a fusion scheme of (X,S). Then clearly $\mathbb{C}X\downarrow_U$ is the standard $\mathbb{C}U$-module. So we have the next theorem.

Theorem 3.3. *Let (X,S) be an association scheme and let (X,U) be a fusion scheme of (X,U). Let φ be a character of U. Then*

$$m_{\varphi\uparrow^S} = m_\varphi.$$

Proof. The proof of Theorem 3.3 is similar to the proof of Theorem 3.1. □

Corollary 3.4. *Let (X,S) be an association scheme and let (X,U) be a fusion scheme of (X,U). Suppose that $(\chi,\varphi\uparrow^S)_S \neq 0$ for $\chi \in \mathrm{Irr}(S)$ and $\varphi \in \mathrm{Irr}(U)$. Then $m_\varphi \geq m_\chi$.*

Proof. By Theorem 3.3 and that χ appears in $\varphi\uparrow^S$, we have $m_\varphi = m_{\varphi\uparrow^S} \geq m_\chi$. □

4. Algebraic fusions

Let (X,S) be an association scheme, and let G be a subgroup of the algebraic automorphism group $\mathrm{AAut}(S)$. Then G also acts on the adjacency algebra $\mathbb{C}S$ as algebra automorphisms by $(\sum_{s\in S} a_s\sigma_s)^g = \sum_{s\in S} a_s\sigma_{s^g}$ for $g \in G$ and $\sum_{s\in S} a_s\sigma_s \in \mathbb{C}S$. By definition, we have the following lemma.

Lemma 4.1. *The adjacency algebra $\mathbb{C}S^G$ of the algebraic fusion scheme (X,S^G) coincides with the algebra of fixed points $(\mathbb{C}S)^G = \{\alpha \in \mathbb{C}S \mid \alpha^g = \alpha\}$.*

Let $\chi \in \mathrm{Irr}(S)$ and let $g \in G$. Define χ^g by

$$\chi^g(\sigma_s) = \chi((\sigma_s)^{g^{-1}}) = \chi(\sigma_{s^{g^{-1}}}).$$

Since g acts on $\mathbb{C}S$ as an algebra automorphism, χ^g is also an irreducible character of S. So G also acts on $\mathrm{Irr}(S)$.

Lemma 4.2. *Let $\chi \in \mathrm{Irr}(S)$ and let $g \in G$. Then the following statements hold.*

(1) $\chi^g(1) = \chi(1)$.
(2) $m_{\chi^g} = m_\chi$.
(3) $(e_\chi)^g = e_{\chi^g}$.

Proof. (1) is by definition, (2) is by [8, Theorem 4.1.5 (ii)], and (3) is by [8, Lemma 4.1.4 (ii)]. □

The next theorem is one of the main results in this article.

Theorem 4.3. *Let G be a subgroup of the algebraic automorphism group* $\mathrm{AAut}(S)$ *of an association scheme* (X,S)*. Let* $\varphi \in \mathrm{Irr}(S^G)$ *and let* $\chi \in \mathrm{Irr}(S)$*. Put* $T = \{g \in G \mid \chi^g = \chi\}$*, the stabilizer of* χ *in* G*. Suppose that* $(\varphi{\uparrow}^S,\ \chi)_S \neq 0$*. Then*

$$\varphi{\uparrow}^S = (\varphi{\uparrow}^S,\ \chi)_S \sum_{g \in T\backslash G} \chi^g.$$

Proof. By definition, we have $\varphi{\uparrow}^S = \sum_{\xi \in \mathrm{Irr}(S)} (\varphi{\uparrow}^S,\ \xi)_S \xi$. It is enough to show that $(\varphi{\uparrow}^S,\ \chi)_S = (\varphi{\uparrow}^S,\ \chi^g)_S$ for every $g \in G$ and $(\varphi{\uparrow}^S,\ \xi)_S = 0$ for $\xi \notin \{\chi^g \mid g \in G\}$.

Since φ is irreducible, by Theorem 2.4 and [8, Theorem 4.1.5 (ii)], we have

$$(\varphi{\uparrow}^S,\ \chi)_S = (\varphi,\ \chi{\downarrow}_{S^G})_{S^G} = \frac{m_\varphi}{n_S \varphi(1)} \sum_{u \in S^G} \chi(\sigma_u)\varphi(\sigma_{u^*}).$$

For $u \in S^G$, $\sigma_u \in \mathbb{C}S^G = (\mathbb{C}S)^G$ by Lemma 4.1. So we have $\chi^g(\sigma_u) = \chi(\sigma_u)$ for every $g \in G$. Now we can say that $(\varphi{\uparrow}^S,\ \chi)_S = (\varphi{\uparrow}^S,\ \chi^g)_S$ for every $g \in G$.

Put $e = \sum_{g \in T\backslash G} e_{\chi^g}$. Then e is a central idempotent of $\mathbb{C}S$ and in $(\mathbb{C}S)^G = \mathbb{C}S^G$. So e is a central idempotent of $\mathbb{C}S^G$. By Lemma 2.5, $ee_\varphi \neq 0$. Since a central idempotent of $\mathbb{C}S^G$ can be uniquely decomposed into a sum of central primitive idempotents of $\mathbb{C}S^G$ (see [7, Chapter 1, Theorem 4.6]), $ee_\varphi = e_\varphi$. Suppose that $\xi \in \mathrm{Irr}(S)$ and $\xi \notin \{\chi^g \mid g \in G\}$. Then $ee_\xi = 0$. So $e_\xi e_\varphi = e_\xi e e_\varphi = 0$. This means that $(\varphi{\uparrow}^S,\ \xi)_S = 0$. The proof is completed. □

Corollary 4.4. *Under the same assumption in Theorem 4.3, we have*

$$m_\varphi = |G : T|(\varphi{\uparrow}^S,\ \chi)_S\ m_\chi.$$

Proof. This is clear by Theorem 4.3, Lemma 4.2 (2), and Theorem 3.3. □

The equation in Theorem 4.3 seems to be a dual of a well-known formula in Clifford theory for group characters [7, Chapter 3, Theorem 3.8 (i)]. In Clifford theory, the restriction of an irreducible character to a normal subgroup is a sum of conjugate characters with the same multiplicities. The next theorem is also a dual of some parts of [7, Chapter 3, Theorem 3.8].

Theorem 4.5. *Let G be a subgroup of the algebraic automorphism group* $\mathrm{AAut}(S)$ *of an association scheme* (X,S)*. Fix* $\chi \in \mathrm{Irr}(S)$ *and* *put* $T = \{g \in G \mid \chi^g = \chi\}$*, the stabilizer of* χ *in* G*. Put*

$$\begin{aligned}\mathcal{A} &= \{\eta \in \mathrm{Irr}(S^T) \mid (\chi{\downarrow_{S^T}},\ \eta)_{S^T} \neq 0\},\\ \mathcal{B} &= \{\varphi \in \mathrm{Irr}(S^G) \mid (\chi{\downarrow_{S^G}},\ \varphi)_{S^G} \neq 0\}.\end{aligned}$$

Then the following statements hold.

(1) *We can define a bijection* $\kappa : \mathcal{A} \to \mathcal{B}$ *by* $\kappa(\eta) = \eta{\downarrow_{S^G}}$ *for* $\eta \in \mathcal{A}$*. Especially,* $|\mathcal{A}| = |\mathcal{B}|$*.*

(2) *For* $\eta \in \mathcal{A}$*, we have* $(\chi{\downarrow_{S^T}}, \eta)_{S^T} = (\chi{\downarrow_{S^G}}, \kappa(\eta))_{S^G}$*.*

Proof. Put $t = |G : T|$.

Let $\eta \in \mathcal{A}$. Fix an irreducible constituent φ of $\eta{\downarrow_{S^G}}$. Then $\varphi \in \mathcal{B}$. By Theorem 4.3,

$$\varphi{\uparrow^S} = a \sum_{g \in T\backslash G} \chi^g, \tag{4.1}$$

where $a = (\chi{\downarrow_{S^G}}, \varphi)_{S^G}$ and

$$\eta{\uparrow^S} = b\chi \tag{4.2}$$

where $b = (\chi{\downarrow_{S^T}}, \eta)_{S^T}$. Since η is an irreducible constituent of $\varphi{\uparrow^{S^T}}$, we have

$$b = (\eta{\uparrow^S}, \chi)_S \leq ((\varphi{\uparrow^{S^T}}){\uparrow^S}, \chi)_S = (\varphi{\uparrow^S}, \chi)_S = a. \tag{4.3}$$

By Lemma 2.5, $e_\eta e_\varphi \neq 0$. So there exists a primitive idempotent f_η of $\mathbb{C}S^T$ such that $f_\eta e_\eta = f_\eta$ and $f_\eta e_\varphi \neq 0$. Then, for any $g \in G$, $(f_\eta)^g e_\varphi = (f_\eta e_\varphi)^g \neq 0$. Note that χ is the only irreducible character of S which appears in $\eta{\uparrow^S}$. So we have $e_\eta e_\chi = e_\eta$. Since $(f_\eta)^g \in (e_\chi \mathbb{C}S)^g = e_{\chi^g}\mathbb{C}S$, $(f_\eta)^g (f_\eta)^h = 0$ if $Tg \neq Th$. Similarly $(e_\eta)^g (e_\eta)^h = 0$ if $Tg \neq Th$.

We claim that $\eta{\downarrow_{S^G}} = \varphi$. Since f_η is a primitive idempotent in $\mathbb{C}S^T$, $f_\eta \mathbb{C}S^T$ is a simple $\mathbb{C}S^T$-module affording η. So $f_\eta \mathbb{C}S^T$ contains a $\mathbb{C}S^G$-submodule V which affords φ. Fix a non-zero element $v \in V$. Since V is simple, $V = v\mathbb{C}S^G$. To prove $\eta{\downarrow_{S^G}} = \varphi$, it is enough to show that $f_\eta \mathbb{C}S^T = v\mathbb{C}S^G$. It is clear that $f_\eta \mathbb{C}S^T \supseteq v\mathbb{C}S^G$. Since $f_\eta \mathbb{C}S^T$ is a simple $\mathbb{C}S^T$-module, we have $f_\eta \mathbb{C}S^T = v\mathbb{C}S^T$. So we will show $v\mathbb{C}S^G \supseteq v\mathbb{C}S^T$. Since $v \in f_\eta \mathbb{C}S^T$, we have $v = f_\eta v = f_\eta e_\eta v = f_\eta v e_\eta = v e_\eta$. Also $v(e_\eta)^g = v e_\eta (e_\eta)^g = 0$ if $g \in G - T$. Suppose $x \in \mathbb{C}S^T$. By the above arguments, we have

$$vx = ve_\eta x = \sum_{g \in T\backslash G} v(e_\eta)^g x^g = v \sum_{g \in T\backslash G} (e_\eta x)^g \in v\mathbb{C}S^G.$$

Now we have shown that $v\mathbb{C}S^G \supseteq v\mathbb{C}S^T$ and $\eta{\downarrow_{S^G}} = \varphi$. So the map $\kappa : \mathcal{A} \to \mathcal{B}$ is defined.

Since $(f_\eta)^g(f_\eta)^h = 0$ if $Tg \neq Th$, $\sum_{g\in T\backslash G}(f_\eta)^g$ is an idempotent and contained in $\mathbb{C}S^G$. Now, since $(\sum_{g\in T\backslash G}(f_\eta)^g)e_\varphi \neq 0$, a primitive idempotent decomposition of $\sum_{g\in T\backslash G}(f_\eta)^g$ (see [7, Chapter 1, Section 4]) in $\mathbb{C}S^G$ contains a primitive idempotent f_φ such that $f_\varphi e_\varphi \neq 0$. We can see that

$$\mathbb{C}X(\sum_{g\in T\backslash G}(f_\eta)^g) = \bigoplus_{g\in T\backslash G}\mathbb{C}X(f_\eta)^g \supseteq \mathbb{C}Xf_\varphi. \tag{4.4}$$

Note that

$$\dim_{\mathbb{C}}\mathbb{C}X(f_\eta)^g = \dim_{\mathbb{C}}\mathrm{Hom}_{\mathbb{C}S^{g^{-1}Tg}}((f_\eta)^g\mathbb{C}S^{g^{-1}Tg}, \mathbb{C}X) = m_{\eta^g},$$

where $\eta^g \in \mathrm{Irr}(S^{g^{-1}Tg})$. Similar to the proof of Lemma 4.2, we can show that $m_{\eta^g} = m_\eta$. Comparing the dimensions of both sides of (4.4), we have

$$tm_\eta \geq m_\varphi. \tag{4.5}$$

Combining (4.1), (4.2), (4.3), (4.5), and by Theorem 3.3, we have

$$atm_\chi = m_{\varphi\uparrow^S} = m_\varphi \leq tm_\eta = btm_\chi \leq atm_\chi. \tag{4.6}$$

We can conclude that $a = b$, namely $(\chi\downarrow_{S^T}, \eta)_{S^T} = (\chi\downarrow_{S^G}, \varphi)_{S^G}$. The statement (2) holds.

If $\eta' \in \mathcal{A}$ is an irreducible constituent of $\varphi\uparrow^{S^T}$ and $\eta \neq \eta'$, then

$$\begin{aligned} a &= (\chi, \varphi\uparrow^S)_S = (\chi, (\varphi\uparrow^{S^T})\uparrow^S)_S \\ &\geq (\chi, \eta\uparrow^S)_S + (\chi, \eta'\uparrow^S)_S > a \end{aligned}$$

and this is a contradiction. Now κ is injective.

Let $\varphi \in \mathcal{B}$. Since $(\chi\downarrow_{S^G}, \varphi)_{S^G} \neq 0$, there is $\eta \in \mathcal{A}$ such that $(\eta\downarrow_{S^G}, \varphi)_{S^G} \neq 0$. This means κ is surjective. Now the statement (1) holds. □

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