DPSU-14-1

Solvable Discrete Quantum Mechanics: q-Orthogonal Polynomials with $|q| = 1$ and Quantum Dilogarithm

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Abstract

Several kinds of q-orthogonal polynomials with $|q| = 1$ are constructed as the main parts of the eigenfunctions of new solvable discrete quantum mechanical systems. Their orthogonality weight functions consist of quantum dilogarithm functions, which are a natural extension of the Euler gamma functions and the q-gamma functions (q-shifted factorials). The dimensions of the orthogonal spaces are finite. These q-orthogonal polynomials are expressed in terms of the Askey-Wilson polynomials and their certain limit forms.

1 Introduction

Several years ago the present authors published "Unified theory of exactly and quasiexactly solvable 'discrete' quantum mechanics. I. Formalism,"[1] (to be referred to as **U**, and its equations are cited as (**U**2.3) etc). It provided a unified framework for the known solvable discrete quantum mechanical systems [2] with pure imaginary shifts (idQM) and with real shifts (rdQM). The Wilson and the Askey-Wilson polynomials [3, 4, 5, 6] are the main parts of the eigenpolynomials of solvable idQM, whereas the (dual) (q-)Hahn and the (q-)Racah polynomials [3, 4, 5, 7] belong to rdQM. The unified theory **U** provides not only a synthesised explanation of the two sufficient conditions of exact solvability, *i.e.* shape invariance [8] and the closure relations [9], but also a systematic method to introduce new solvable and quasi-exactly solvable systems based on sinusoidal coordinates [9].

In this paper we present several new solvable systems in idQM and q-orthogonal polynomials as the main parts of their eigenfunctions. They correspond to the choices of the sinusoidal coordinates $\eta(x) = \cosh x, e^{\pm x}, \sinh x$, whereas that of the Askey-Wilson system is $\cos x$. The new system with $\eta(x) = \cosh x$ is obtained from the Askey-Wilson system by the replacement $x^{\text{AW}} \to -ix$ (3.11), which induces the change of the q-parameter $0 < q^{\text{AW}} < 1 \to q = e^{-i\gamma}$, $0 < \gamma < \pi$, $|q| = 1$. This gives rise to q-orthogonal polynomials with $|q| = 1$ as the main parts of the eigenfunctions. The orthogonality weight functions consist of quantum dilogarithm functions [10]. The weight functions for the orthogonal polynomials of the known solvable idQM are the Euler gamma functions $\Gamma(z)$ for $\eta(x) = x, x^2$ (the continuous Hahn and the Wilson polynomials etc) and the q-gamma functions $\Gamma_q(z) = (1-q)^{1-z}(q;q)_\infty(q^z;q)_\infty^{-1}$ or the q-shifted factorials $(a;q)_\infty$ for $\eta(x) = \cos x, \sin x$ (the Askey-Wilson polynomials etc). The double gamma functions or the quantum dilogarithms are naturally expected for $\eta(x) = \cosh x, e^{\pm x}, \sinh x$, having $|q| = 1$. For example, in the XXZ spin chain with spin one-half which has quantum algebra symmetry, various quantities in the antiferromagnetic regime $\Delta < -1$ $(-1 < q < 0)$ are described by using the q-shifted factorials $(a;q)_\infty$, and those in the gapless regime $|\Delta| \le 1$ $(|q| = 1)$ are described by using the double sine function [11]. The double gamma function and the double sine function are known for more than a century [12, 13, 14] and they are closely related to quantum dilogarithm functions [10, 11].

Another way to understand the appearance of quantum dilogarithm in comparison with the (q-)gamma functions is their *functional equations*. Within the framework of idQM, the orthogonality weight function is obtained as the square of the groundstate wavefunction $\phi_0(x)$, which is the zero mode of the operator $\mathcal{A}$, (2.5) of the factorised Hamiltonian (2.1). As a building block of the groundstate wavefunction, we need a solution $F(x)$ of the following functional equation

$$\frac{F(x-i\frac{\gamma}{2})}{F(x+i\frac{\gamma}{2})} = \frac{v(x+i\frac{\gamma}{2})}{v^*(x-i\frac{\gamma}{2})},$$

where $v(x)$ is a main factor of the potential function $V(x)$. For the eight types of the sinusoidal coordinates introduced in (**U**A.1)–(**U**A.8) and listed in (3.1), the main factors of the potentials are

$$\text{(i)–(ii)}\ \ v(x) = a + ix, \quad \text{(iii)–(iv)}\ \ v(x) = 1 - ae^{ix}, \quad \text{(v)–(viii)}\ \ v(x) = 1 + ae^{x}.$$

If we find an $F_1(x)$ which solves $\frac{F_1(x-i\frac{\gamma}{2})}{F_1(x+i\frac{\gamma}{2})} = v(x+i\frac{\gamma}{2})$, then $F(x) = F_1(x)F_1^*(x)\big(= F^*(x)\big)$ solves the above equation. For the above $v(x)$, we have

$$\text{(i)–(ii)}\ \ F_1(x) = \Gamma(a+ix),\ \gamma = 1, \quad \text{(iii)–(iv)}\ \ F_1(x) = (ae^{ix};e^{\gamma})_\infty^{-1},$$

(v)–(viii) $F_1(x)=\Phi_{\frac{\gamma}{2}}(x+i\frac{\gamma}{2}+\log a)$,

where $\Phi_\gamma(z)$ is the quantum dilogarithm introduced in Appendix B and satisfies the functional equation (B.4).

The present paper is organised as follows. The key components of the foundation paper **U** are recapitulated in section two. Starting from the essence of the discrete quantum mechanics with pure imaginary shifts (idQM) in § 2.1, the construction method of the solvable potentials based on sinusoidal coordinates is summarised in § 2.2. A brief note on the hermiticity of the resulting Hamiltonians is given in § 2.3. Section three is the main part of the paper, providing the details of the four new idQM systems; the factorised potential functions, the eigenpolynomials, the ground state eigenfunctions or the orthogonality weight functions and verifications of the hermiticity. The representative system having the sinusoidal coordinate $\eta(x)=\cosh x$ is discussed in detail in § 3.1. Comments on the 'roots of unity q' cases are given in § 3.1.5. The cases of $\eta(x)=e^{\pm x}$ are discussed briefly in § 3.2. The system with the sinusoidal coordinate $\eta(x)=\sinh x$ is discussed in § 3.3. Comments on the closure relation and shape invariance of the new solvable systems are given in § 3.4 and § 3.5, respectively. The normalisation constants are discussed in § 3.6. By construction the discrete quantum mechanics reduces to the ordinary quantum mechanics in certain limits. In § 3.7 the new solvable idQM systems derived in § 3.1–§ 3.3 are shown to reduce to the known solvable systems with the Morse, hyperbolic Darboux-Pöschl-Teller and hyperbolic symmetric top II potentials [15, 16, 17, 18], at the levels of the Hamiltonian, the ground state wave function and the eigenpolynomials. Other limiting properties are discussed in § 3.8. The final section is for a summary and comments. In Appendix A, various data of the solvable idQM systems corresponding to the eight choices of the sinusoidal coordinates (3.1) are described. In Appendix B, a brief summary of quantum dilogarithm functions is presented.

2 Solvable Discrete Quantum Mechanics

In order to set the stage and to introduce proper notation we recapitulate the essence of the discrete quantum mechanics with pure imaginary shifts (idQM) [2, 6] and the method to construct exactly solvable systems presented in **U**.

2.1 Discrete quantum mechanics with pure imaginary shifts

In idQM [2, 6] the dynamical variables are the real coordinate x ($x_1 < x < x_2$) and the conjugate momentum $p = -i\partial_x$. The Hamiltonian is positive semi-definite due to the factorised form:

$$\mathcal{H} \stackrel{\text{def}}{=} \sqrt{V(x)}\, e^{\gamma p} \sqrt{V^*(x)} + \sqrt{V^*(x)}\, e^{-\gamma p} \sqrt{V(x)} - V(x) - V^*(x) = \mathcal{A}^\dagger \mathcal{A}, \tag{2.1}$$

$$\mathcal{A} \stackrel{\text{def}}{=} i\big(e^{\frac{\gamma}{2}p} \sqrt{V^*(x)} - e^{-\frac{\gamma}{2}p} \sqrt{V(x)}\,\big), \quad \mathcal{A}^\dagger \stackrel{\text{def}}{=} -i\big(\sqrt{V(x)}\, e^{\frac{\gamma}{2}p} - \sqrt{V^*(x)}\, e^{-\frac{\gamma}{2}p}\big). \tag{2.2}$$

Here the potential function $V(x)$ is an analytic function of x and γ is a real constant. The $*$-operation on an analytic function $f(x) = \sum_n a_n x^n$ ($a_n \in \mathbb{C}$) is defined by $f^*(x) \stackrel{\text{def}}{=} f(x^*)^* = \sum_n a_n^* x^n$, in which a_n^* is the complex conjugation of a_n. Obviously $f^{**}(x) = f(x)$ and $f(x)^* = f^*(x^*)$. The inner product of two functions f and g is the integral of the product of two analytic functions $(f, g) \stackrel{\text{def}}{=} \int_{x_1}^{x_2} dx\, f^*(x) g(x)$. The Schrödinger equation

$$\mathcal{H}\phi_n(x) = \mathcal{E}_n \phi_n(x) \quad (n = 0, 1, \ldots), \quad 0 = \mathcal{E}_0 < \mathcal{E}_1 < \cdots, \tag{2.3}$$

is an analytic difference equation with pure imaginary shifts ($e^{\pm\gamma p} f(x) = f(x \mp i\gamma)$). The orthogonality relation reads

$$(\phi_n, \phi_m) = h_n \delta_{nm} \quad (0 < h_n < \infty), \tag{2.4}$$

and the number of the eigenstates ($h_n < \infty$) is finite or infinite. The eigenfunction $\phi_n(x)$ can be chosen 'real', $\phi_n^*(x) = \phi_n(x)$, and we follow this convention. The groundstate wavefunction $\phi_0(x)$ is the zero mode of the operator $\mathcal{A}$, $\mathcal{A}\phi_0(x) = 0$, namely

$$\sqrt{V^*(x - i\tfrac{\gamma}{2})}\, \phi_0(x - i\tfrac{\gamma}{2}) = \sqrt{V(x + i\tfrac{\gamma}{2})}\, \phi_0(x + i\tfrac{\gamma}{2}). \tag{2.5}$$

The higher eigenfunctions have factorised forms,

$$\phi_n(x) = \phi_0(x) P_n\big(\eta(x)\big), \tag{2.6}$$

in which $P_n(\eta)$ is a polynomial of degree n in the sinusoidal coordinate $\eta = \eta(x)$. This form implies that $P_n(\eta)$'s ($n = 0, 1, \ldots$) are orthogonal polynomials (in the ordinary sense) with the weight function $\phi_0(x)^2$. The Schrödinger equation (2.3) becomes

$$\widetilde{\mathcal{H}} P_n\big(\eta(x)\big) = \mathcal{E}_n P_n\big(\eta(x)\big), \tag{2.7}$$

where the similarity transformed Hamiltonian $\widetilde{\mathcal{H}}$ is square root free

$$\widetilde{\mathcal{H}} \stackrel{\text{def}}{=} \phi_0(x)^{-1} \circ \mathcal{H} \circ \phi_0(x) = V(x)(e^{\gamma p} - 1) + V^*(x)(e^{-\gamma p} - 1). \tag{2.8}$$

In this formalism, the orthogonal polynomials $\{P_n(\eta)\}$ and their weight function $\phi_0(x)^2$ are determined by the given form of the potential function $V(x)$. With abuse of language, we call (2.7) a polynomial equation, which is in fact a difference equation for the polynomial $P_n(\eta(x))$.

2.2 Construction of solvable potentials

In **U** a systematic method to obtain exactly and quasi-exactly solvable systems was presented for discrete quantum mechanics. Here we summarise it for idQM, focusing on the exactly solvable case.

As seen above, the Schrödinger equation (2.3) becomes the polynomial equation (2.7) by the similarity transformation with respect to the groundstate wavefunction $\phi_0(x)$. The method presented in **U** is reversing this order:

1) Construct exactly solvable polynomial equation, $\widetilde{\mathcal{H}} P_n\big(\eta(x)\big) = \mathcal{E}_n P_n\big(\eta(x)\big)$, based on the sinusoidal coordinate $\eta(x)$.
2) Find the groundstate wavefunction $\phi_0(x)$, which satisfies (2.5) and ensures the hermiticity of the Hamiltonian.
3) By the similarity transformation $\mathcal{H} = \phi_0(x) \circ \widetilde{\mathcal{H}} \circ \phi_0(x)^{-1}$, exactly solvable quantum system is obtained, $\mathcal{H}\phi_n(x) = \mathcal{E}_n \phi_n(x)$, $\phi_n(x) = \phi_0(x) P_n\big(\eta(x)\big)$.

2.2.1 sinusoidal coordinate

The sinusoidal coordinate $\eta(x)$ for idQM is defined as a 'real' analytic ($\eta^*(x) = \eta(x)$) function of x satisfying the following (**U**2.18), (**U**2.20):

$$\begin{cases} 0) & \eta(x) \text{ and } x \text{ are one-to-one for } x_1 < x < x_2 \\ 1) & \eta(x - i\gamma) + \eta(x + i\gamma) = (2 + r_1^{(1)})\eta(x) + r_{-1}^{(2)} \\ 2) & \big(\eta(x - i\gamma) - \eta(0)\big)\big(\eta(x + i\gamma) - \eta(0)\big) = \big(\eta(x) - \eta(-i\gamma)\big)\big(\eta(x) - \eta(i\gamma)\big) \end{cases} . \tag{2.9}$$

Here $r_1^{(1)}$ and $r_{-1}^{(2)}$ are real parameters and we assume $r_1^{(1)} > -4$ and $\eta(x) \neq \eta(x - i\gamma) \neq \eta(x + i\gamma) \neq \eta(x)$. The two conditions 1) & 2) imply that any symmetric polynomial in $\eta(x - i\gamma)$ and $\eta(x + i\gamma)$ is expressed as a polynomial in $\eta(x)$. In this paper we do not require the condition $\eta(0)=0$, which was imposed in **U** due to a unified presentation for idQM and rdQM. An affine transformed one $\eta^{\text{new}}(x)=a\eta(x)+b$ (a,b: constants, $a\neq 0$) works as a sinusoidal coordinate, and the argument of exact solvability does not change.

2.2.2 potential function

We assume that the potential function $V(x)$ has the following form (**U**2.30)–(**U**2.31):

$$V(x)=\frac{\widetilde{V}(x)}{\bigl(\eta(x-i\gamma)-\eta(x)\bigr)\bigl(\eta(x-i\gamma)-\eta(x+i\gamma)\bigr)}\,, \tag{2.10}$$

$$\widetilde{V}(x)=\sum_{\substack{k,l\geq 0\\ k+l\leq 2}}v_{k,l}\,\eta(x)^k\eta(x-i\gamma)^l, \tag{2.11}$$

where $v_{k,l}$ are real constants ($\sum_{k+l=2}v_{k,l}^2\neq 0$). Due to the properties (2.9), the number of independent real parameters among $\{v_{k,l}\,(k+l\leq 2)\}$ (concerning the functional form of $\widetilde{V}(x)$) is five and one of them corresponds to an overall normalisation. For example, we can take $v_{k,l}$ $(k+l\leq 2,\ l=0,1)$ as independent parameters, since $v_{0,2}$ is redundant as

$$\begin{aligned}\eta(x-i\gamma)^2=(2+r_1^{(1)})\eta(x)\eta(x-i\gamma)-\eta(x)^2+r_{-1}^{(2)}\bigl(\eta(x)+\eta(x-i\gamma)\bigr)\\ -\eta(-i\gamma)\eta(i\gamma)+\eta(0)\bigl(\eta(0)-r_{-1}^{(2)}\bigr).\end{aligned} \tag{2.12}$$

2.2.3 polynomial equation

Our starting point is the similarity transformed Hamiltonian $\widetilde{\mathcal{H}}$:

$$\widetilde{\mathcal{H}}=V(x)(e^{\gamma p}-1)+V^*(x)(e^{-\gamma p}-1).$$

This acts on $\eta(x)^n$ $(n=0,1,\ldots)$ as (**U**2.36)

$$\widetilde{\mathcal{H}}\eta(x)^n=\sum_{m=0}^{n}\eta(x)^m\widetilde{\mathcal{H}}^{\eta}_{m,n}, \tag{2.13}$$

where the explicit forms of the real matrix elements $\widetilde{\mathcal{H}}^{\eta}_{m,n}$ are given in Appendix A.1. The polynomial space $\mathcal{V}_n\stackrel{\text{def}}{=}\text{Span}\bigl[1,\eta(x),\ldots,\eta(x)^n\bigr]$ is invariant under the action of $\widetilde{\mathcal{H}}$:

$$\widetilde{\mathcal{H}}\mathcal{V}_n\subseteq\mathcal{V}_n. \tag{2.14}$$

The matrix $(\widetilde{\mathcal{H}}^{\eta}_{m,n})$ is an upper triangular matrix. The solution of (2.7) is given explicitly in a determinant form (**U**3.2),

$$\mathcal{E}_n=\widetilde{H}^{\eta}_{n,n},\quad P_n\big(\eta(x)\big)\propto\begin{vmatrix}1&\eta(x)&\eta(x)^2&\cdots&\eta(x)^n\\ \mathcal{E}_0-\mathcal{E}_n&\widetilde{H}^{\eta}_{0,1}&\widetilde{H}^{\eta}_{0,2}&\cdots&\widetilde{H}^{\eta}_{0,n}\\ &\mathcal{E}_1-\mathcal{E}_n&\widetilde{H}^{\eta}_{1,2}&\cdots&\widetilde{H}^{\eta}_{1,n}\\ &&\ddots&\ddots&\vdots\\ \text{\huge0}&&&\mathcal{E}_{n-1}-\mathcal{E}_n&\widetilde{H}^{\eta}_{n-1,n}\end{vmatrix}. \tag{2.15}$$

Note that the determinant in (2.15) is 'real' ($f(x)=|\cdots|$, $f^*(x)=f(x)$) since each component of the matrix is 'real'.

2.2.4 exactly solvable Hamiltonian

The groundstate wavefunction $\phi_0(x)$ is required to obey (2.5) and it should be square integrable. Contrary to the ordinary quantum mechanics in which the groundstate wavefunction satisfies a first order differential equation, the solution of the difference equation (2.5) is not unique. However, the groundstate wavefunction is uniquely determined by the hermiticity requirement of the Hamiltonian up to an overall normalisation, as will be shown shortly in the subsequent subsection § 2.3.

Once the desired groundstate wavefunction $\phi_0(x)$ is obtained, the inverse similarity transformation of $\widetilde{\mathcal{H}}$ with respect to $\phi_0(x)$ gives an exactly solvable idQM,

$$\mathcal{H}=\phi_0(x)\circ\widetilde{\mathcal{H}}\circ\phi_0(x)^{-1},\quad \mathcal{H}\phi_n(x)=\mathcal{E}_n\phi_n(x),\quad \phi_n(x)=\phi_0(x)P_n\big(\eta(x)\big).$$

The existence of the groundstate wavefunction $\phi_0(x)$ and the hermiticity of the Hamiltonian $\mathcal{H}$ restrict the range of parameters $\{v_{k,l}\}$. Depending on the choice of the sinusoidal coordinate $\eta(x)$ (3.1), the number of square-integrable eigenstates $\phi_n(x)$ is finite or infinite.

As explained in Introduction, it is very important to *factorise the potential function* $V(x)$ for solving the zero mode equation (2.5) for $\phi_0(x)$. Then (2.5) can be solved for each factor of $V(x)$ and the multiplication of each factor solution would give $\phi_0(x)$. This type of solution method for $\phi_0(x)$ is characteristic to difference equations and it is very different from the corresponding situation for the differential equations. The factorisation naturally introduces *new parametrisation of the potential function* $V(x)$, which provides inherited parametrisation of the weight function $\phi_0(x)^2$ and the polynomials $\{P_n(\eta)\}$.

The present method also covers the known solvable potentials in idQM; for example, those corresponding to the Wilson and the Askey-Wilson polynomials. In these cases, the natural parameters are not the original $\{v_{k,l}\}$ (2.11) in § 2.2.2 but those obtained by factorisation. As for the new examples of solvable idQM in § 3 the situation is the same. The starting point will be the factorised potential functions (3.4), (3.22), (3.23) and (3.36). The relation among the original parameters $\{v_{k,l}\}$ (2.11) and the factorisation parameters are given in (A.7).

2.3 Hermiticity

The hermiticity of the Hamiltonian ($\mathcal{H} = p^2 + U(z)$) in ordinary quantum mechanics (oQM) is simple but that of idQM is involved [1, 6, 19]. Here we review the hermiticity of the Hamiltonian (2.1) with the eigenfunctions (2.6). It should be stressed that the groundstate wavefunction $\phi_0(x)$ contains square roots but the weight function $\phi_0(x)^2$ is square root free.

Let us consider the functions of the form $f(x) = \phi_0(x)\mathcal{P}\bigl(\eta(x)\bigr)$, where $\mathcal{P}(\eta)$ is a polynomial in η and $\mathcal{P}^*(\eta) = \mathcal{P}(\eta)$. For two functions $f_1(x) = \phi_0(x)\mathcal{P}_1\bigl(\eta(x)\bigr)$ and $f_2(x) = \phi_0(x)\mathcal{P}_2\bigl(\eta(x)\bigr)$ ($\deg \mathcal{P}_1 = n_1$, $\deg \mathcal{P}_2 = n_2$), the hermiticity $(f_1, \mathcal{H}f_2) = (\mathcal{H}f_1, f_2)$ is realised if the following quantity vanishes [19]:

$$
\begin{aligned}
&\int_{-\frac{\gamma}{2}}^{\frac{\gamma}{2}} dy \bigl(G(x_2+iy) - G^*(x_2-iy)\bigr) - \int_{-\frac{\gamma}{2}}^{\frac{\gamma}{2}} dy \bigl(G(x_1+iy) - G^*(x_1-iy)\bigr)\\
&= 2\pi \frac{\gamma}{|\gamma|} \sum_{x_0:\text{pole in } D_\gamma} \operatorname{Res}_{x=x_0}\bigl(G(x) - G^*(x)\bigr),
\end{aligned}
\tag{2.16}
$$

where $G(x)$ and D_γ are

$$
\begin{aligned}
&G(x) = V(x + i\tfrac{\gamma}{2})\phi_0(x + i\tfrac{\gamma}{2})^2\, \mathcal{P}_1\bigl(\eta(x + i\tfrac{\gamma}{2})\bigr)\mathcal{P}_2\bigl(\eta(x - i\tfrac{\gamma}{2})\bigr),\\
&\quad\Bigl(\Rightarrow G^*(x) = V(x + i\tfrac{\gamma}{2})\phi_0(x + i\tfrac{\gamma}{2})^2\, \mathcal{P}_1\bigl(\eta(x - i\tfrac{\gamma}{2})\bigr)\mathcal{P}_2\bigl(\eta(x + i\tfrac{\gamma}{2})\bigr)\Bigr),
\end{aligned}
\tag{2.17}
$$

$$
D_\gamma \overset{\text{def}}{=} \bigl\{x \in \mathbb{C} \bigm| x_1 \le \operatorname{Re} x \le x_2, |\operatorname{Im} x| \le \tfrac{1}{2}|\gamma|\bigr\}. \tag{2.18}
$$

In the next section we will verify the condition (2.16) for concrete examples. The degree of polynomial $\mathcal{P}(\eta)$ may have upper limit due to square integrability.

3 New Examples of Exactly Solvable idQM

The following eight sinusoidal coordinates are presented in Appendix A of **U** (in which affine transformed $\eta(x)$ are used for (iii) and (v)–(vii)) :

$$\begin{aligned}
&\text{(i)}: && \eta(x)=x, && -\infty<x<\infty,\\
&\text{(ii)}: && \eta(x)=x^2, && 0<x<\infty,\\
&\text{(iii)}: && \eta(x)=\cos x, && 0<x<\pi,\\
&\text{(iv)}: && \eta(x)=\sin x, && -\tfrac{\pi}{2}<x<\tfrac{\pi}{2},\\
&\text{(v)}: && \eta(x)=e^{-x}, && -\infty<x<\infty,\\
&\text{(vi)}: && \eta(x)=e^{x}, && -\infty<x<\infty,\\
&\text{(vii)}: && \eta(x)=\cosh x, && 0<x<\infty,\\
&\text{(viii)}: && \eta(x)=\sinh x, && -\infty<x<\infty.
\end{aligned} \tag{3.1}$$

We take γ as

$$\text{(i)–(ii)}:\gamma=1,\quad \text{(iii)–(iv)}:\gamma<0,\quad \text{(v)–(viii)}:0<\gamma<\pi. \tag{3.2}$$

These eight sinusoidal coordinates satisfy (2.9) and (3.53) with

$$\begin{aligned}
&\text{(i)}: r_1^{(1)}=0,\ r_{-1}^{(2)}=0, && \text{(ii)}: r_1^{(1)}=0,\ r_{-1}^{(2)}=-2,\\
&\text{(iii)–(iv)}: r_1^{(1)}=(e^{\frac{\gamma}{2}}-e^{-\frac{\gamma}{2}})^2,\ r_{-1}^{(2)}=0, && \text{(v)–(viii)}: r_1^{(1)}=(e^{i\frac{\gamma}{2}}-e^{-i\frac{\gamma}{2}})^2,\ r_{-1}^{(2)}=0.
\end{aligned} \tag{3.3}$$

The cases (i)–(iii) give the *well-known quantum systems with infinitely many discrete eigenstates*, whose eigenfunctions are described by (i) continuous Hahn, (ii) Wilson and (iii) Askey-Wilson polynomials. In the case (iii) the parameter $q=e^{\gamma}$ of the Askey-Wilson polynomials satisfies $0<q<1$. The case (iv) is nothing but the case (iii) with the shift of the coordinate, $x^{(\text{iv})}=x^{(\text{iii})}-\frac{\pi}{2}$.

The *new examples of solvable idQM* are obtained by the sinusoidal coordinates (v)–(viii). We will explain $\eta(x)=\cosh x$ (vii) case in detail and others briefly. These systems have *finitely many discrete eigenstates*. The explicit forms of the eigenpolynomials (2.15) are listed in (A.8). These eigenpolynomials are given by the Askey-Wilson polynomial with the parameter $q=e^{-i\gamma}$, $|q|=1$ and its limiting forms. Their groundstate wavefunctions are described by the quantum dilogarithm. In other words, the present research presents four kinds of (finitely many) q-orthogonal polynomials with $|q|=1$ and having quantum dilogarithm functions as their weight functions.

For these eight sinusoidal coordinates, the potential function $V(x)$ (2.10)–(2.11) in the original parameters $\{v_{k,l}\}$ are listed in (A.5) and the *factorised forms* are listed in (A.6).

Let us denote the factorisation parameters as $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots)$. We write the parameter dependence explicitly, as $\mathcal{H}(\boldsymbol{\lambda})$, $\mathcal{A}(\boldsymbol{\lambda})$, $\mathcal{E}_n(\boldsymbol{\lambda})$, $\phi_n(x; \boldsymbol{\lambda})$, etc.

3.1 $\eta(x) = \cosh x$

Here we consider the case (vii) $\eta(x) = \cosh x$, $0 < x < \infty$, $0 < \gamma < \pi$, in detail.

3.1.1 factorised potential function

For this choice of the sinusoidal coordinate, the potential function $V(x)$ in the original parameter $\{v_{k,l}\}$ (2.10)–(2.11) is listed in (A.5). The factorised form is shown in (A.6). By choosing the overall scaling factor $A = \frac{\sin\frac{\gamma}{2}\sin\gamma}{|a_1 a_2|}$, we adopt the factorised potential:

$$V(x; \boldsymbol{\lambda}) = e^{i\pi} e^{-i\frac{\gamma}{2}} \frac{a_1^* a_2^*}{|a_1 a_2|} \frac{\prod_{j=1}^{2}(1 + a_j e^x)(1 + a_j^{*\,-1} e^x)}{(e^{2x} - 1)(e^{-i\gamma} e^{2x} - 1)}, \tag{3.4}$$

where $e^{i\pi}$ indicates $\sqrt{e^{i\pi}} = e^{\frac{i}{2}\pi}$ in $\sqrt{V(x; \boldsymbol{\lambda})}$. Here the relation between $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)$ and $a_j \in \mathbb{C}_{\neq 0}$ $(j = 1, 2)$ is

$$a_j \stackrel{\text{def}}{=} e^{-i\gamma\lambda_j}, \quad \lambda_j \stackrel{\text{def}}{=} \alpha_j + i\beta_j, \quad -\pi < \gamma\alpha_j \le \pi, \quad \beta_j \in \mathbb{R}, \quad \alpha \stackrel{\text{def}}{=} \alpha_1 + \alpha_2. \tag{3.5}$$

Note that $\frac{a_1^* a_2^*}{|a_1 a_2|} = e^{i\gamma\alpha}$ and

$$V^*(x; \boldsymbol{\lambda}) = V(-x; \boldsymbol{\lambda}). \tag{3.6}$$

This potential function has four real parameters α_1, α_2, β_1 and β_2 in addition to γ.

3.1.2 eigenpolynomials

The determinant formula for the eigenpolynomial (2.15) gives (A.8) in the new parameters. So we take $P_n(\eta; \boldsymbol{\lambda})$ and $\mathcal{E}_n(\boldsymbol{\lambda})$ as

$$P_n(\eta; \boldsymbol{\lambda}) = e^{-i\frac{\pi}{2}n} e^{i\gamma\frac{3}{4}n(n-1)} e^{i\gamma\alpha n}\, p_n(\eta; -a_1, -a_2, -a_1^{*\,-1}, -a_2^{*\,-1} | e^{-i\gamma}), \tag{3.7}$$

$$\mathcal{E}_n(\boldsymbol{\lambda}) = 4 \sin\tfrac{\gamma}{2}n\, \sin\tfrac{\gamma}{2}(n - 1 + 2\alpha), \tag{3.8}$$

where p_n is the Askey-Wilson polynomial (A.9). By using (A.9) and (A.13)–(A.14), we can show the reality of the polynomial,

$$P_n^*(\eta; \boldsymbol{\lambda}) = P_n(\eta; \boldsymbol{\lambda}). \tag{3.9}$$

Let us prove that $P_n(\eta;\boldsymbol{\lambda})$ (3.7) and $\mathcal{E}_n(\boldsymbol{\lambda})$ (3.8) satisfy (2.7). We know that the Askey-Wilson polynomial satisfies (2.7),

$$
\begin{aligned}
&\frac{\prod_{j=1}^4(1-a_j^{\text{AW}}e^{ix^{\text{AW}}})}{(1-e^{2ix^{\text{AW}}})(1-e^{\gamma^{\text{AW}}}e^{2ix^{\text{AW}}})}\Bigl(p_n\bigl(\cos(x^{\text{AW}}-i\gamma^{\text{AW}});a_1^{\text{AW}},a_2^{\text{AW}},a_3^{\text{AW}},a_4^{\text{AW}}|e^{\gamma^{\text{AW}}}\bigr)\\
&\qquad\qquad\qquad\qquad -p_n(\cos x^{\text{AW}};a_1^{\text{AW}},a_2^{\text{AW}},a_3^{\text{AW}},a_4^{\text{AW}}|e^{\gamma^{\text{AW}}})\Bigr)\\
&+\frac{\prod_{j=1}^4(1-a_j^{\text{AW}}e^{-ix^{\text{AW}}})}{(1-e^{-2ix^{\text{AW}}})(1-e^{\gamma^{\text{AW}}}e^{-2ix^{\text{AW}}})}\Bigl(p_n\bigl(\cos(x^{\text{AW}}+i\gamma^{\text{AW}});a_1^{\text{AW}},a_2^{\text{AW}},a_3^{\text{AW}},a_4^{\text{AW}}|e^{\gamma^{\text{AW}}}\bigr)\\
&\qquad\qquad\qquad\qquad -p_n(\cos x^{\text{AW}};a_1^{\text{AW}},a_2^{\text{AW}},a_3^{\text{AW}},a_4^{\text{AW}}|e^{\gamma^{\text{AW}}})\Bigr)\\
&=(e^{-\gamma^{\text{AW}}n}-1)(1-a_1^{\text{AW}}a_2^{\text{AW}}a_3^{\text{AW}}a_4^{\text{AW}}e^{\gamma^{\text{AW}}(n-1)})p_n(\cos x^{\text{AW}};a_1^{\text{AW}},a_2^{\text{AW}},a_3^{\text{AW}},a_4^{\text{AW}}|e^{\gamma^{\text{AW}}}).
\end{aligned}
\tag{3.10}
$$

We stress that this equation holds for any complex values of the parameters a_j^{AW} and γ^{AW} and the coordinate x^{AW} (except for the zeros of denominators). By substituting

$$
x^{\text{AW}}\to -ix,\quad \gamma^{\text{AW}}\to -i\gamma,\quad a_j^{\text{AW}}\to -a_j\ (j=1,2),\quad a_j^{\text{AW}}\to -a_{j-2}^{*\,-1}\ (j=3,4), \tag{3.11}
$$

into (3.10), we obtain

$$
\begin{aligned}
&\frac{\prod_{j=1}^2(1+a_je^{x})(1+a_j^{*\,-1}e^{x})}{(e^{2x}-1)(e^{-i\gamma}e^{2x}-1)}\Bigl(p_n\bigl(\cosh(x-i\gamma);-a_1,-a_2,-a_1^{*\,-1},-a_2^{*\,-1}|e^{-i\gamma}\bigr)\\
&\qquad\qquad\qquad\qquad -p_n\bigl(\cosh x;-a_1,-a_2,-a_1^{*\,-1},-a_2^{*\,-1}|e^{-i\gamma}\bigr)\Bigr)\\
&+e^{i\gamma}e^{-2i\gamma\alpha}\frac{\prod_{j=1}^2(1+a_j^*e^{x})(1+a_j^{-1}e^{x})}{(e^{2x}-1)(e^{i\gamma}e^{2x}-1)}\Bigl(p_n\bigl(\cosh(x+i\gamma);-a_1,-a_2,-a_1^{*\,-1},-a_2^{*\,-1}|e^{-i\gamma}\bigr)\\
&\qquad\qquad\qquad\qquad -p_n\bigl(\cosh x;-a_1,-a_2,-a_1^{*\,-1},-a_2^{*\,-1}|e^{-i\gamma}\bigr)\Bigr)\\
&=-e^{i\frac{\gamma}{2}}e^{-i\gamma\alpha}\times 4\sin\tfrac{\gamma}{2}n\,\sin\tfrac{\gamma}{2}(n-1+2\alpha)\,p_n\bigl(\cosh x;-a_1,-a_2,-a_1^{*\,-1},-a_2^{*\,-1}|e^{-i\gamma}\bigr).
\end{aligned}
\tag{3.12}
$$

Dividing this by $-e^{i\frac{\gamma}{2}}e^{-i\gamma\alpha}$ and multiplying $e^{-i\frac{\pi}{2}n}e^{i\gamma\frac{3}{4}n(n-1)}e^{i\gamma\alpha n}$ give (2.7).

3.1.3 groundstate

To obtain a quantum system $\mathcal{H}$ from a polynomial system $\widetilde{\mathcal{H}}$, we have to find the groundstate wavefunction which satisfies (2.5) and ensures the hermiticity of the Hamiltonian. First let us consider the zero mode equation of $\mathcal{A}$, (2.5). From the form of $V(x)$ (3.4), its main building block is $1+ae^x$ and we want to have a function $f(x)$ satisfying

$$
\sqrt{1+a^*e^{x-i\frac{\gamma}{2}}}\,f(x-i\tfrac{\gamma}{2})=\sqrt{1+ae^{x+i\frac{\gamma}{2}}}\,f(x+i\tfrac{\gamma}{2})\quad\text{or}\quad \frac{f(x+i\frac{\gamma}{2})^2}{f(x-i\frac{\gamma}{2})^2}=\frac{1+e^{x-i\frac{\gamma}{2}+\log a^*}}{1+e^{x+i\frac{\gamma}{2}+\log a}}.
$$

The quantum dilogarithm function $\Phi_\gamma(z)$ satisfies the functional equation (see Appendix B)

$$\frac{\Phi_\gamma(z+i\gamma)}{\Phi_\gamma(z-i\gamma)}=\frac{1}{1+e^z}. \tag{3.13}$$

From this we have

$$\frac{\Phi_{\frac{\gamma}{2}}(x+i\frac{\gamma}{2}+\log a+i\frac{\gamma}{2})}{\Phi_{\frac{\gamma}{2}}(x+i\frac{\gamma}{2}+\log a-i\frac{\gamma}{2})}=\frac{1}{1+e^{x+i\frac{\gamma}{2}+\log a}},\quad \frac{\Phi_{\frac{\gamma}{2}}(x-i\frac{\gamma}{2}+\log a^*+i\frac{\gamma}{2})}{\Phi_{\frac{\gamma}{2}}(x-i\frac{\gamma}{2}+\log a^*-i\frac{\gamma}{2})}=\frac{1}{1+e^{x-i\frac{\gamma}{2}+\log a^*}}.$$

The following $f(x)$,

$$f(x)=\left(\frac{\Phi_{\frac{\gamma}{2}}(x+i\frac{\gamma}{2}+\log a)}{\Phi_{\frac{\gamma}{2}}(x-i\frac{\gamma}{2}+\log a^*)}\right)^{\frac{1}{2}}=f^*(x),$$

solves the above equation. Using this building block, we adopt the groundstate wavefunction $\phi_0(x;\boldsymbol{\lambda})$ as

$$\begin{aligned}\phi_0(x;\boldsymbol{\lambda})&=e^{(\frac{1}{2}-\alpha-\frac{\pi}{\gamma})x}\sqrt{(e^{2x}-1)(e^{\frac{4\pi}{\gamma}x}-1)}\\&\quad\times\left(\prod_{j=1}^{2}\frac{\Phi_{\frac{\gamma}{2}}\big(x+\gamma\beta_j+i\gamma(\frac{1}{2}-\alpha_j)\big)\Phi_{\frac{\gamma}{2}}\big(x-\gamma\beta_j+i\gamma(\frac{1}{2}-\alpha_j)\big)}{\Phi_{\frac{\gamma}{2}}\big(x+\gamma\beta_j-i\gamma(\frac{1}{2}-\alpha_j)\big)\Phi_{\frac{\gamma}{2}}\big(x-\gamma\beta_j-i\gamma(\frac{1}{2}-\alpha_j)\big)}\right)^{\frac{1}{2}}.\end{aligned} \tag{3.14}$$

By using the properties of $\Phi_\gamma(z)$ presented in Appendix B, we can show that (2.5) and

$$\phi_0^*(x;\boldsymbol{\lambda})=\phi_0(x;\boldsymbol{\lambda}), \tag{3.15}$$

$$\phi_0(-x;\boldsymbol{\lambda})^2=\phi_0(x;\boldsymbol{\lambda})^2. \tag{3.16}$$

Whether this groundstate wavefunction ensures the hermiticity of the Hamiltonian will be clarified in the next § 3.1.4.

As shown in the previous subsection the eigenpolynomials of (vii) and the Askey-Wilson systems (iii) are related to each other under the replacement (3.11). As for the groundstate wavefunctions the correspondence is rather formal. After rewriting Φ_γ in ϕ_0 (3.14) as $\Phi_\gamma(z)=\Phi^{\text{F}}_{\sqrt{\frac{\pi}{\gamma}}}\big(\frac{z}{2\sqrt{\pi\gamma}}\big)^{-1}$ [20] (see (B.1)), we perform the inverse of the replacement (3.11) and use (B.14) to obtain

$$\begin{aligned}&\phi_0(x;\boldsymbol{\lambda})\to\phi_0^{\text{AW}}(x^{\text{AW}};\boldsymbol{\lambda}^{\text{AW}};\gamma^{\text{AW}})\,\mathcal{C}(x^{\text{AW}},\boldsymbol{\lambda}^{\text{AW}},\gamma^{\text{AW}}),\\&\phi_0^{\text{AW}}(x;\boldsymbol{\lambda};\gamma)=\left(\frac{(e^{2ix},e^{-2ix};e^\gamma)_\infty}{\prod_{j=1}^4(a_je^{ix},a_je^{-ix};e^\gamma)_\infty}\right)^{\frac{1}{2}}\quad(a_j=e^{\gamma\lambda_j}),\\&\mathcal{C}(x,\boldsymbol{\lambda},\gamma)=e^{\frac{1}{2}(1-\sum_{j=1}^4\lambda_j)ix+\frac{\pi}{\gamma}x}\sqrt{(e^{2ix}-1)(e^{\frac{4\pi}{\gamma}x}-1)}\\&\qquad\times\left(\frac{\prod_{j=1}^4(e^{ix-\gamma\lambda_j+\gamma},e^{-ix+\gamma\lambda_j};e^\gamma)_\infty}{(e^{2ix},e^{-2ix};e^\gamma)_\infty}\prod_{j=1}^4\frac{(e^{\frac{2\pi}{\gamma}x-2\pi i\lambda_j+\frac{4\pi^2}{\gamma}};e^{\frac{4\pi^2}{\gamma}})_\infty}{(e^{\frac{2\pi}{\gamma}x+2\pi i\lambda_j};e^{\frac{4\pi^2}{\gamma}})_\infty}\right)^{\frac{1}{2}}.\end{aligned} \tag{3.17}$$

Here ϕ_0^{AW} is the groundstate wavefunction of the Askey-Wilson system and $\mathcal{C}$ is a pseudo constant, $\mathcal{C}(x-i\gamma,\boldsymbol{\lambda},\gamma)=\mathcal{C}(x,\boldsymbol{\lambda},\gamma)$. Therefore $\mathcal{C}$ does not contribute to (2.5) and we can discard this.

3.1.4 hermiticity

We restrict the range of the parameters as

$$-\gamma\alpha>\pi+\frac{\gamma}{2},\quad \gamma-\pi<\gamma\alpha_j<0\ \ (j=1,2). \tag{3.18}$$

The eigenpolynomial $P_n(\eta;\boldsymbol{\lambda})$ is expressed in terms of a terminating ${}_4\phi_3$, and the denominator of each term of the expansion (A.10) is $(a_1a_2,a_1a_1^{*\,-1},a_1a_2^{*\,-1};e^{-i\gamma})_k\cdot(e^{-i\gamma};e^{-i\gamma})_k$. We avoid the parameter values of $\{a_j\}$ such that $(a_1a_2,a_1a_1^{*\,-1},a_1a_2^{*\,-1};e^{-i\gamma})_k$ vanishes. Concerning the factor $(e^{-i\gamma};e^{-i\gamma})_k$, see § 3.1.5.

For $x=R+iy$ ($R>0$, $0\leq y\leq\gamma$), the argument z of each $\Phi_{\frac{\gamma}{2}}(z)$ in $\phi_0(x;\boldsymbol{\lambda})$ satisfies $|\text{Im}\,z|<\frac{\gamma}{2}+\pi$, where the integral representation (B.2) and the asymptotic forms (B.12) are valid. From (B.12), the asymptotic behaviour of $\phi_0(R+iy;\boldsymbol{\lambda})$ ($0\leq y\leq\gamma$) at large R is

$$\bigl|\phi_0(R+iy;\boldsymbol{\lambda})\bigr|\simeq\text{const}\times e^{(-\frac12+\alpha+\frac{\pi}{\gamma})R}, \tag{3.19}$$

and those of $V(R+iy;\boldsymbol{\lambda})$ and $P_n\bigl(\eta(R+iy);\boldsymbol{\lambda}\bigr)$ are

$$\bigl|V(R+iy;\boldsymbol{\lambda})\bigr|\simeq\text{const},\quad \bigl|P_n\bigl(\eta(R+iy);\boldsymbol{\lambda}\bigr)\bigr|\simeq\text{const}\times e^{nR}. \tag{3.20}$$

At $x=x_1=0$, $\phi_0(x;\boldsymbol{\lambda})^2$ and $P_n\bigl(\eta(x);\boldsymbol{\lambda}\bigr)$ are regular. Therefore the wavefunction $\phi_n(x;\boldsymbol{\lambda})=\phi_0(x;\boldsymbol{\lambda})P_n\bigl(\eta(x);\boldsymbol{\lambda}\bigr)=\phi_n^*(x;\boldsymbol{\lambda})$ is square integrable $(\phi_n,\phi_n)<\infty$ only for

$$n<\frac12-\alpha-\frac{\pi}{\gamma}. \tag{3.21}$$

The maximal value of n is $n_{\text{max}}(\boldsymbol{\lambda})=[\frac12-\alpha-\frac{\pi}{\gamma}]'$, where $[x]'$ denotes the greatest integer not exceeding and not equal to x.

Let us check the condition (2.16). We can show that

$$\begin{aligned}
&\text{(a)}:\quad \frac{n_1+n_2}{2}<\frac12-\alpha-\frac{\pi}{\gamma}\ \ \Rightarrow\ \int_{-\frac{\gamma}{2}}^{\frac{\gamma}{2}}dy\,G(x_2+iy)=0=\int_{-\frac{\gamma}{2}}^{\frac{\gamma}{2}}dy\,G^*(x_2-iy),\\
&\text{(b)}:\quad \int_{-\frac{\gamma}{2}}^{\frac{\gamma}{2}}dy\,G(x_1+iy)=\int_{-\frac{\gamma}{2}}^{\frac{\gamma}{2}}dy\,G^*(x_1-iy),\\
&\text{(c)}:\quad G(x)\text{ and }G^*(x)\text{ do not have poles in }D_\gamma.
\end{aligned}$$

Here we have used (3.19)–(3.20) for (a), (3.6) and (3.16) for (b) and (B.8)–(B.9) for (c). Thus the condition (2.16) holds for the eigenstates $\phi_n(x;\boldsymbol{\lambda})$ ($n=0,1,\ldots,n_{\text{max}}$), namely the hermiticity of the Hamiltonian is established. It is straightforward to verify that the energy eigenvalues are monotonously increasing: $0=\mathcal{E}_0(\boldsymbol{\lambda})<\mathcal{E}_1(\boldsymbol{\lambda})<\cdots<\mathcal{E}_{n_{\text{max}}}(\boldsymbol{\lambda})$.

3.1.5 comments on $\frac{\gamma}{2\pi}\in\mathbb{Q}$

Let us set $q=e^{-i\gamma}$. If $\frac{\gamma}{2\pi}\in\mathbb{Q}$, there are positive integers m leading to $q^m=1$. The expansion formula of $P_n(\eta;\boldsymbol{\lambda})$ has factors $\frac{1}{(q;q)_k}$ ($1\leq k\leq n$) (see the sentences below (3.18)), namely factors $\frac{1}{1-q^k}$. So, if $q^k=1$ happens for some $1\leq k\leq n$, $P_n(\eta;\boldsymbol{\lambda})$ diverges and it loses its meaning. However, this does not happen for the eigenfunctions $\phi_n(x;\boldsymbol{\lambda})$ ($n=0,1,\ldots,n_{\text{max}}$).

Let us set $\gamma=\frac{M}{N}2\pi$ (N and M are coprime positive integers). The range $0<\gamma<\pi$ means $2M<N$. The smallest positive integer m leading to $q^m=1$ is $m=N$. The ranges of n (3.21) and the parameters (3.5) mean $n<\frac{1}{2}-\alpha-\frac{\pi}{\gamma}<\frac{1}{2}+\frac{\pi}{\gamma}$. These lead to $n<\frac{1}{2}+\frac{\pi}{\gamma}=\frac{1}{2}+\frac{N}{2M}\leq\frac{1}{2}+\frac{N}{2}<N$. Therefore $q^k=1$ ($1\leq k\leq n$) does not happen for $n\leq n_{\text{max}}$.

3.2 $\eta(x)=e^{\pm x}$

Here we consider the cases (v) $\eta(x)=e^{-x}$ and (vi) $\eta(x)=e^{x}$, $-\infty<x<\infty$, $0<\gamma<\pi$. The outline is the same as that in § 3.1 and we will present the results briefly.

The factorised potential functions (A.6) derived from the original parameters form (A.5) read as follows ($A=\frac{4\sin\frac{\gamma}{2}\sin\gamma}{|a_1a_2|}$ in (A.6)):

$$\text{(v)}:\quad V(x;\boldsymbol{\lambda})=e^{i\pi}e^{-i\frac{\gamma}{2}}\frac{a_1^*a_2^*}{|a_1a_2|}\prod_{j=1}^{2}(1+a_j^{*\,-1}e^{x}),\tag{3.22}$$

$$\text{(vi)}:\quad V(x;\boldsymbol{\lambda})=e^{-i\pi}e^{i\frac{\gamma}{2}}\frac{a_1a_2}{|a_1a_2|}\prod_{j=1}^{2}(1+a_j^{-1}e^{-x}),\tag{3.23}$$

where $e^{\pm i\pi}$ indicates $\sqrt{e^{\pm i\pi}}=e^{\pm\frac{i}{2}\pi}$ in $\sqrt{V(x;\boldsymbol{\lambda})}$, and the relation between $\boldsymbol{\lambda}=(\lambda_1,\lambda_2)$ and $a_j\in\mathbb{C}_{\neq0}$ ($j=1,2$) is (3.5). Note that these two systems (v) and (vi) are essentially the same, as they are related by the replacements $x\leftrightarrow-x$ and $V(x)\leftrightarrow V^*(-x)$:

$$V^{(\text{vi})}(x;\boldsymbol{\lambda})=V^{(\text{v})\,*}(-x;\boldsymbol{\lambda}).\tag{3.24}$$

The energy eigenvalue $\mathcal{E}_n(\boldsymbol{\lambda})$ and the corresponding eigenpolynomial $P_n(\eta;\boldsymbol{\lambda})$ are listed in (A.12) and (A.8):

$$\mathcal{E}_n(\boldsymbol{\lambda}) = 4\sin\tfrac{\gamma}{2}n\,\sin\tfrac{\gamma}{2}(n-1+2\alpha), \tag{3.25}$$

$$P_n(\eta;\boldsymbol{\lambda}) = e^{-i\frac{\pi}{2}n}e^{i\gamma\frac{3}{4}n(n-1)}e^{i\gamma\alpha n}\,\tilde{p}_n(\eta;-a_1,-a_2,-a_1^{*\,-1},-a_2^{*\,-1}|e^{-i\gamma}), \tag{3.26}$$

where $\tilde{p}_n$ is defined in (A.11) as obtained from the Askey-Wilson polynomial by a certain limiting procedure. By using (A.11) and (A.16)–(A.17), we can show the 'reality' of P_n (3.9). This eigenpolynomial (3.26) can be obtained from that of the case (vii) (3.7) (with an appropriate overall normalisation) by replacement ($R \in \mathbb{R}$)

$$\text{(v)}: \quad x^{\text{(vii)}} = -x + R, \quad a_j^{\text{(vii)}} = a_j^{*\,-1}e^{R}\ (j=1,2), \tag{3.27}$$

$$\text{(vi)}: \quad x^{\text{(vii)}} = x + R, \qquad a_j^{\text{(vii)}} = a_j e^{-R}\ (j=1,2), \tag{3.28}$$

and the limit $R \to \infty$. The above factorised potential functions (3.22)–(3.23) and the polynomial equation (2.7) can also be obtained from those of the case (vii) (3.4) and (3.12) by the same limit.

We adopt the following groundstate wavefunctions $\phi_0(x;\boldsymbol{\lambda})$:

$$\text{(v)}: \quad \phi_0(x;\boldsymbol{\lambda}) = e^{(\frac{1}{2}-\alpha-\frac{\pi}{\gamma})x}\biggl(\prod_{j=1}^{2}\frac{\Phi_{\frac{\gamma}{2}}\bigl(x-\gamma\beta_j+i\gamma(\frac{1}{2}-\alpha_j)\bigr)}{\Phi_{\frac{\gamma}{2}}\bigl(x-\gamma\beta_j-i\gamma(\frac{1}{2}-\alpha_j)\bigr)}\biggr)^{\frac{1}{2}}, \tag{3.29}$$

$$\text{(vi)}: \quad \phi_0(x;\boldsymbol{\lambda}) = e^{-(\frac{1}{2}-\alpha-\frac{\pi}{\gamma})x}\biggl(\prod_{j=1}^{2}\frac{\Phi_{\frac{\gamma}{2}}\bigl(-x-\gamma\beta_j+i\gamma(\frac{1}{2}-\alpha_j)\bigr)}{\Phi_{\frac{\gamma}{2}}\bigl(-x-\gamma\beta_j-i\gamma(\frac{1}{2}-\alpha_j)\bigr)}\biggr)^{\frac{1}{2}}. \tag{3.30}$$

Reflecting the relationship of the potentials (3.24) these two groundstate wavefunctions are also related:

$$\phi_0^{\text{(vi)}}(x;\boldsymbol{\lambda}) = \phi_0^{\text{(v)}}(-x;\boldsymbol{\lambda}). \tag{3.31}$$

The zero mode equation (2.5) and the 'reality' of ϕ_0 (3.15) can be derived from the properties of $\Phi_\gamma(z)$ presented in Appendix B. These groundstate wavefunctions (3.29)–(3.30) can be obtained from that of the case (vii) (3.14) (with an appropriate overall normalisation) by the replacement (3.27)–(3.28) and $R \to \infty$ limit.

We restrict the range of the parameters as

$$-\gamma\alpha > \pi + \frac{\gamma}{2}, \quad \gamma - \pi < \gamma\alpha_j < 0 \ \ (j=1,2). \tag{3.32}$$

We avoid the parameter values of $\{a_j\}$ such that the denominators of P_n vanish. Since (v) and (vi) are essentially the same, we will argue the case (v) below. For $x = \pm R + iy$ ($R > 0$,

$0 \le y \le \gamma$), the asymptotic behaviour of $\phi_0(\pm R+iy;\boldsymbol{\lambda})$ at large R is

$$\bigl|\phi_0(R+iy;\boldsymbol{\lambda})\bigr| \simeq \text{const}\times e^{(-\frac12-\frac{\pi}{\gamma})R},\quad \bigl|\phi_0(-R+iy;\boldsymbol{\lambda})\bigr| \simeq \text{const}\times e^{-(\frac12-\alpha-\frac{\pi}{\gamma})R}, \tag{3.33}$$

and those of $V(\pm R+iy;\boldsymbol{\lambda})$ and $P_n\bigl(\eta(\pm R+iy);\boldsymbol{\lambda}\bigr)$ are

$$\begin{aligned} \bigl|V(R+iy;\boldsymbol{\lambda})\bigr| &\simeq \text{const}\times e^{2R}, & \bigl|V(-R+iy;\boldsymbol{\lambda})\bigr| &\simeq \text{const},\\ \bigl|P_n\bigl(\eta(R+iy);\boldsymbol{\lambda}\bigr)\bigr| &\simeq \text{const}, & \bigl|P_n\bigl(\eta(-R+iy);\boldsymbol{\lambda}\bigr)\bigr| &\simeq \text{const}\times e^{nR}. \end{aligned} \tag{3.34}$$

The wavefunction $\phi_n(x;\boldsymbol{\lambda})=\phi_0(x;\boldsymbol{\lambda})P_n\bigl(\eta(x);\boldsymbol{\lambda}\bigr)=\phi_n^*(x;\boldsymbol{\lambda})$ is square integrable $(\phi_n,\phi_n)<\infty$ only for

$$n<\frac12-\alpha-\frac{\pi}{\gamma}. \tag{3.35}$$

The maximal value of n is $n_{\max}(\boldsymbol{\lambda})=[\frac12-\alpha-\frac{\pi}{\gamma}]'$. We can show that

$$\begin{aligned} &(\text{a}):\quad \int_{-\frac{\gamma}{2}}^{\frac{\gamma}{2}} dy\, G(x_2+iy)=0=\int_{-\frac{\gamma}{2}}^{\frac{\gamma}{2}} dy\, G^*(x_2-iy),\\ &(\text{b}):\quad \frac{n_1+n_2}{2}<\frac12-\alpha-\frac{\pi}{\gamma}\ \Rightarrow\ \int_{-\frac{\gamma}{2}}^{\frac{\gamma}{2}} dy\, G(x_1+iy)=0=\int_{-\frac{\gamma}{2}}^{\frac{\gamma}{2}} dy\, G^*(x_1-iy),\\ &(\text{c}):\quad G(x)\ \text{and}\ G^*(x)\ \text{do not have poles in}\ D_\gamma. \end{aligned}$$

Here (3.33)–(3.34) are used for (a) and (b), and (B.8)–(B.9) for (c). Thus the condition (2.16) holds for the eigenstates $\phi_n(x;\boldsymbol{\lambda})$ $(n=0,1,\ldots,n_{\max})$, namely the hermiticity of the Hamiltonian is established. We can verify that $0=\mathcal{E}_0(\boldsymbol{\lambda})<\mathcal{E}_1(\boldsymbol{\lambda})<\cdots<\mathcal{E}_{n_{\max}}(\boldsymbol{\lambda})$.

As in the $\cosh x$ example § 3.1.5, the 'rational' $\frac{\gamma}{2\pi}\in\mathbb{Q}$ does not cause any trouble for the eigenfunctions $\phi_n(x;\boldsymbol{\lambda})$ $(n=0,1,\ldots,n_{\max})$.

3.3 $\eta(x)=\sinh x$

Here we consider the case (viii) $\eta(x)=\sinh x$, $-\infty<x<\infty$, $0<\gamma<\pi$. The outline is the same as that in § 3.1 and we will present the results briefly.

The factorised potential functions (A.6) derived from the original parameters form (A.5) read ($A=(-1)^{K+1}\frac{\sin\frac{\gamma}{2}\sin\gamma}{|a_1a_2|}$ in (A.6)):

$$V(x;\boldsymbol{\lambda})=e^{i\pi K}e^{-i\frac{\gamma}{2}}\frac{a_1^*a_2^*}{|a_1a_2|}\frac{\prod_{j=1}^2(1+a_je^x)(1-a_j^{*\,-1}e^x)}{(1+e^{2x})(1+e^{-i\gamma}e^{2x})}\quad (K=\pm1,0), \tag{3.36}$$

where $e^{i\pi K}$ indicates $\sqrt{e^{i\pi K}}=e^{\frac{i}{2}\pi K}$ in $\sqrt{V(x;\boldsymbol{\lambda})}$, and the relation between $\boldsymbol{\lambda}=(\lambda_1,\lambda_2)$ and $a_j\in\mathbb{C}_{\neq0}$ $(j=1,2)$ is (3.5). The three choices of K correspond to three different parameter ranges given below, see (3.42).

The energy eigenvalue $\mathcal{E}_n(\boldsymbol{\lambda})$ and the corresponding eigenpolynomial $P_n(\eta;\boldsymbol{\lambda})$ are listed in (A.12) and (A.8):

$$\mathcal{E}_n(\boldsymbol{\lambda}) = (-1)^{K+1}4\sin\tfrac{\gamma}{2}n\,\sin\tfrac{\gamma}{2}(n-1+2\alpha), \tag{3.37}$$

$$P_n(\eta;\boldsymbol{\lambda}) = e^{-i\frac{\pi}{2}n}e^{i\gamma\frac{3}{4}n(n-1)}e^{i\gamma\alpha n}\,(-i)^n p_n(i\eta; ia_1, ia_2, -ia_1^{*\,-1}, -ia_2^{*\,-1}|e^{-i\gamma}), \tag{3.38}$$

where p_n is the Askey-Wilson polynomial (A.9). The 'reality' of P_n (3.9) can be shown based on the properties (A.13)–(A.15) and the expansion formula of the Askey-Wilson polynomial (A.9). The polynomial equation (2.7) can be obtained from that of the Askey-Wilson case (3.10) by substitution

$$x^{\text{AW}} \to -ix+\frac{\pi}{2},\quad \gamma^{\text{AW}} \to -i\gamma,\quad a_j^{\text{AW}} \to ia_j\ (j=1,2),\quad a_j^{\text{AW}} \to -ia_{j-2}^{*\,-1}\ (j=3,4). \tag{3.39}$$

We adopt the following groundstate wavefunction $\phi_0(x;\boldsymbol{\lambda})$:

$$\phi_0(x;\boldsymbol{\lambda}) = e^{(\frac{1}{2}-\alpha-K\frac{\pi}{\gamma})x}\sqrt{1+e^{2x}} \times\biggl(\prod_{j=1}^{2}\frac{\Phi_{\frac{\gamma}{2}}\bigl(x+\gamma\beta_j+i\gamma(\frac{1}{2}-\alpha_j)\bigr)\Phi_{\frac{\gamma}{2}}\bigl(x-\gamma\beta_j+i\gamma(\frac{1}{2}-\alpha_j^-)\bigr)}{\Phi_{\frac{\gamma}{2}}\bigl(x+\gamma\beta_j-i\gamma(\frac{1}{2}-\alpha_j)\bigr)\Phi_{\frac{\gamma}{2}}\bigl(x-\gamma\beta_j-i\gamma(\frac{1}{2}-\alpha_j^-)\bigr)}\biggr)^{\frac{1}{2}}, \tag{3.40}$$

where α_j^- $(j=1,2)$ is defined by

$$\gamma\alpha_j^- \stackrel{\text{def}}{=} \begin{cases} \gamma\alpha_j+\pi & \text{for } -\pi<\gamma\alpha_j\le 0 \\ \gamma\alpha_j-\pi & \text{for } \quad 0<\gamma\alpha_j\le\pi \end{cases}, \quad \alpha^- \stackrel{\text{def}}{=} \alpha_1^-+\alpha_2^-, \tag{3.41}$$

and it satisfies $-\pi<\gamma\alpha_j^-\le\pi$. We can verify the zero mode equation (2.5) and the 'reality' of ϕ_0 (3.15) based on the properties of the quantum dilogarithm function $\Phi_\gamma(z)$ presented in Appendix B.

We restrict the range of the parameters as

$$-\gamma\alpha > K\pi+\frac{\gamma}{2},\quad \begin{cases} K=1: & \gamma-\pi<\gamma\alpha_j<0\ \ (j=1,2) \\ K=-1: & \gamma<\gamma\alpha_j<\pi\ \ (j=1,2) \\ K=0: & \gamma-\pi<\gamma\alpha_1<0,\ \gamma<\gamma\alpha_2<\pi \\ & \text{or } \gamma-\pi<\gamma\alpha_2<0,\ \gamma<\gamma\alpha_1<\pi \end{cases}. \tag{3.42}$$

We avoid the parameter values of $\{a_j\}$ such that the denominators of P_n vanish. For $x=\pm R+iy$ ($R>0$, $0\le y\le\gamma$), the asymptotic behaviour of $\phi_0(\pm R+iy;\boldsymbol{\lambda})$ at large R is

$$\bigl|\phi_0(R+iy;\boldsymbol{\lambda})\bigr| \simeq \text{const}\times e^{(-\frac{1}{2}+\alpha^--\frac{K\pi}{\gamma})R},\quad \bigl|\phi_0(-R+iy;\boldsymbol{\lambda})\bigr| \simeq \text{const}\times e^{-(\frac{1}{2}-\alpha-\frac{K\pi}{\gamma})R}, \tag{3.43}$$

and those of $V(\pm R+iy;\boldsymbol{\lambda})$ and $P_n\bigl(\eta(\pm R+iy);\boldsymbol{\lambda}\bigr)$ are

$$\bigl|V(\pm R+iy;\boldsymbol{\lambda})\bigr|\simeq\text{const},\quad \bigl|P_n\bigl(\eta(\pm R+iy);\boldsymbol{\lambda}\bigr)\bigr|\simeq\text{const}\times e^{nR}. \tag{3.44}$$

The wavefunction $\phi_n(x;\boldsymbol{\lambda})=\phi_0(x;\boldsymbol{\lambda})P_n\bigl(\eta(x);\boldsymbol{\lambda}\bigr)=\phi_n^*(x;\boldsymbol{\lambda})$ is square integrable $(\phi_n,\phi_n)<\infty$ only for

$$n<\frac{1}{2}-\alpha-\frac{K\pi}{\gamma}. \tag{3.45}$$

The maximal value of n is $n_{\max}(\boldsymbol{\lambda})=[\frac{1}{2}-\alpha-\frac{K\pi}{\gamma}]'$. We can show that

$$\begin{aligned}
&(\text{a}):\quad \frac{n_1+n_2}{2}<\frac{1}{2}-\alpha^-+\frac{K\pi}{\gamma}\quad\Rightarrow\quad\int_{-\frac{\gamma}{2}}^{\frac{\gamma}{2}}dy\,G(x_2+iy)=0=\int_{-\frac{\gamma}{2}}^{\frac{\gamma}{2}}dy\,G^*(x_2-iy),\\
&(\text{b}):\quad \frac{n_1+n_2}{2}<\frac{1}{2}-\alpha-\frac{K\pi}{\gamma}\quad\Rightarrow\quad\int_{-\frac{\gamma}{2}}^{\frac{\gamma}{2}}dy\,G(x_1+iy)=0=\int_{-\frac{\gamma}{2}}^{\frac{\gamma}{2}}dy\,G^*(x_1-iy),\\
&(\text{c}):\quad G(x)\text{ and }G^*(x)\text{ do not have poles in }D_\gamma.
\end{aligned}$$

Here various asymptotic behaviours (3.43)–(3.44) are used for (a) and (b), and the positions of the poles and zeros of the quantum dilogarithm (B.8)–(B.9) for (c). Thus the hermiticity condition (2.16) holds for the eigenstates $\phi_n(x;\boldsymbol{\lambda})$ $(n=0,1,\ldots,n_{\max})$, namely the Hamiltonian is hermitian. We can verify that $0=\mathcal{E}_0(\boldsymbol{\lambda})<\mathcal{E}_1(\boldsymbol{\lambda})<\cdots<\mathcal{E}_{n_{\max}}(\boldsymbol{\lambda})$.

As in § 3.1.5, the case $\frac{\gamma}{2\pi}\in\mathbb{Q}$ does not cause any trouble for the eigenstates $\phi_n(x;\boldsymbol{\lambda})$ $(n=0,1,\ldots,n_{\max})$, because we have $n<\frac{1}{2}-\alpha-\frac{K\pi}{\gamma}<\frac{1}{2}+\frac{\pi}{\gamma}$ due to (3.45).

3.4 Closure relation

The closure relation is a sufficient condition for exact solvability of quantum systems whose eigenfunctions have the factorised form with the sinusoidal coordinate (2.6). It is a commutator relation between the Hamiltonian $\mathcal{H}$ (or $\widetilde{\mathcal{H}}$) and the sinusoidal coordinate $\eta(x)$ [9]:

$$[\mathcal{H},[\mathcal{H},\eta]]=\eta R_0(\mathcal{H})+[\mathcal{H},\eta]R_1(\mathcal{H})+R_{-1}(\mathcal{H}), \tag{3.46}$$

$$\text{or}\quad [\widetilde{\mathcal{H}},[\widetilde{\mathcal{H}},\eta]]=\eta R_0(\widetilde{\mathcal{H}})+[\widetilde{\mathcal{H}},\eta]R_1(\widetilde{\mathcal{H}})+R_{-1}(\widetilde{\mathcal{H}}), \tag{3.47}$$

where $R_i(z)$ are polynomials in z with real coefficients $r_i^{(j)}$,

$$R_1(z)=r_1^{(1)}z+r_1^{(0)},\quad R_0(z)=r_0^{(2)}z^2+r_0^{(1)}z+r_0^{(0)},\quad R_{-1}(z)=r_{-1}^{(2)}z^2+r_{-1}^{(1)}z+r_{-1}^{(0)}. \tag{3.48}$$

The constants $r_1^{(1)}$ and $r_{-1}^{(2)}$ have appeared in (2.9). The closure relation (3.46) allows us to obtain the exact Heisenberg operator solution for $\eta(x)$, and the annihilation and creation operators $a^{(\pm)}$ are extracted from this exact Heisenberg operator solution [9]. Roughly speaking, the three terms in r.h.s of (3.46) correspond to the three term recurrence relations of the corresponding orthogonal polynomials.

Exactly solvable Hamiltonians presented in § 2.2 satisfy the closure relation (3.46). The explicit expressions of $r_i^{(j)}$ in terms of $v_{k,l}$ are given in (**U**3.16)–(**U**3.18). For the new examples (v)–(viii), the coefficients $r_i^{(j)}$ in (3.48) in terms of the factorisation parameters are

$$
\begin{aligned}
&r_{-1}^{(2)}=0,\quad r_0^{(2)}=r_1^{(1)},\quad r_0^{(1)}=2r_1^{(0)},\quad r_1^{(1)}=(e^{i\frac{\gamma}{2}}-e^{-i\frac{\gamma}{2}})^2,\\
&r_1^{(0)}=-(e^{i\frac{\gamma}{2}}-e^{-i\frac{\gamma}{2}})^2(e^{-i\frac{\gamma}{2}}e^{i\gamma\alpha}+e^{i\frac{\gamma}{2}}e^{-i\gamma\alpha})\times C,\\
&r_0^{(0)}=(e^{i\frac{\gamma}{2}}-e^{-i\frac{\gamma}{2}})^2(e^{i\gamma\alpha}-e^{-i\gamma\alpha})(e^{-i\gamma}e^{i\gamma\alpha}-e^{i\gamma}e^{-i\gamma\alpha}),\\
&r_{-1}^{(1)}=-(e^{i\frac{\gamma}{2}}-e^{-i\frac{\gamma}{2}})^2\big((e^{-i\frac{\gamma}{2}}e^{i\gamma\alpha_1}+e^{i\frac{\gamma}{2}}e^{-i\gamma\alpha_1})B_2+(e^{-i\frac{\gamma}{2}}e^{i\gamma\alpha_2}+e^{i\frac{\gamma}{2}}e^{-i\gamma\alpha_2})B_1\big)\times C,\\
&r_{-1}^{(0)}=(e^{i\frac{\gamma}{2}}-e^{-i\frac{\gamma}{2}})^2(e^{-i\gamma}e^{i\gamma\alpha}-e^{i\gamma}e^{-i\gamma\alpha})\big((e^{i\gamma\alpha_1}-e^{-i\gamma\alpha_1})B_2+(e^{i\gamma\alpha_2}-e^{-i\gamma\alpha_2})B_1\big),
\end{aligned}
\tag{3.49}
$$

where B_j and C are given by

$$
B_j=\begin{cases} e^{-\gamma\beta_j} & :\text{(v), (vi)}\\ \frac{1}{2}(e^{-\gamma\beta_j}+e^{\gamma\beta_j}) & :\text{(vii)}\\ \frac{1}{2}(e^{-\gamma\beta_j}-e^{\gamma\beta_j}) & :\text{(viii)}\end{cases},\qquad C=\begin{cases} 1 & :\text{(v), (vi), (vii)}\\ (-1)^{K+1} & :\text{(viii)}\end{cases}. \tag{3.50}
$$

3.5 Shape invariance

The shape invariance [8] is also a sufficient condition for exact solvability of quantum systems, but its eigenfunctions are not restricted to the form (2.6). The shape invariance condition is [6]

$$
\mathcal{A}(\boldsymbol{\lambda})\mathcal{A}(\boldsymbol{\lambda})^\dagger=\kappa\mathcal{A}(\boldsymbol{\lambda}')^\dagger\mathcal{A}(\boldsymbol{\lambda}')+\mathcal{E}_1(\boldsymbol{\lambda}), \tag{3.51}
$$

$$
\text{or}\quad \mathcal{A}(\boldsymbol{\lambda})\mathcal{A}(\boldsymbol{\lambda})^\dagger=\kappa\mathcal{A}(\boldsymbol{\lambda}+\boldsymbol{\delta})^\dagger\mathcal{A}(\boldsymbol{\lambda}+\boldsymbol{\delta})+\mathcal{E}_1(\boldsymbol{\lambda}), \tag{3.52}
$$

where κ is a real positive parameter and $\boldsymbol{\lambda}'$ is uniquely determined by $\boldsymbol{\lambda}$. In concrete examples, with properly chosen parametrisation $\boldsymbol{\lambda}$, $\boldsymbol{\lambda}'$ has a simple additive form $\boldsymbol{\lambda}'=\boldsymbol{\lambda}+\boldsymbol{\delta}$. As shown in (**U**3.46)–(**U**3.47), the energy spectrum and the excited state wavefunctions are determined by the data of the groundstate wavefunction $\phi_0(x;\boldsymbol{\lambda})$ and the energy of the first excited state $\mathcal{E}_1(\boldsymbol{\lambda})$.

The exactly solvable Hamiltonians presented in § 2.2 satisfy the shape invariance condition (3.51), since each of the sinusoidal coordinate $\eta(x)$ (3.1) satisfies on top of (2.9) one more condition

$$\eta(x)-\eta(0)=[\![\tfrac12]\!]\bigl(\eta(x-i\tfrac{\gamma}{2})+\eta(x+i\tfrac{\gamma}{2})-\eta(-i\tfrac{\gamma}{2})-\eta(i\tfrac{\gamma}{2})\bigr), \tag{3.53}$$

where $[\![\frac12]\!]$ is defined in (A.3). The explicit formulas of $\boldsymbol{\lambda}'$ are given in (**U**3.51)–(**U**3.55).

For the new examples (v)–(viii), the data of shape invariance is

$$\boldsymbol{\lambda}=(\lambda_1,\lambda_2)=(\alpha_1+i\beta_1,\alpha_2+i\beta_2),\quad \boldsymbol{\delta}=(\tfrac12,\tfrac12),\quad \kappa=1. \tag{3.54}$$

Note that the shift $\boldsymbol{\lambda}\to\boldsymbol{\lambda}+\boldsymbol{\delta}$ corresponds to $a_j\to a_jq^{\frac12}$ ($q=e^{-i\gamma}$), which has the same form as the Askey-Wilson case. Let us introduce the auxiliary function $\varphi(x)$,

$$\varphi(x)\stackrel{\text{def}}{=}\begin{cases} e^{-x} & :\text{(v)}\\ e^{x} & :\text{(vi)}\\ 2\sinh x & :\text{(vii)}\\ 2\cosh x & :\text{(viii)}\end{cases}. \tag{3.55}$$

Then $V(x;\boldsymbol{\lambda})$ and $\phi_0(x;\boldsymbol{\lambda})$ satisfy

$$V(x;\boldsymbol{\lambda}+\boldsymbol{\delta})=\kappa^{-1}\frac{\varphi(x-i\gamma)}{\varphi(x)}V(x-i\tfrac{\gamma}{2};\boldsymbol{\lambda}), \tag{3.56}$$

$$\phi_0(x;\boldsymbol{\lambda}+\boldsymbol{\delta})=\varphi(x)\sqrt{V(x+i\tfrac{\gamma}{2};\boldsymbol{\lambda})}\,\phi_0(x+i\tfrac{\gamma}{2};\boldsymbol{\lambda}). \tag{3.57}$$

The forward and backward shift operators are defined by

$$\mathcal{F}(\boldsymbol{\lambda})\stackrel{\text{def}}{=}\phi_0(x;\boldsymbol{\lambda}+\boldsymbol{\delta})^{-1}\circ\mathcal{A}(\boldsymbol{\lambda})\circ\phi_0(x;\boldsymbol{\lambda})=i\varphi(x)^{-1}\bigl(e^{\frac{\gamma}{2}p}-e^{-\frac{\gamma}{2}p}\bigr), \tag{3.58}$$

$$\mathcal{B}(\boldsymbol{\lambda})\stackrel{\text{def}}{=}\phi_0(x;\boldsymbol{\lambda})^{-1}\circ\mathcal{A}(\boldsymbol{\lambda})^{\dagger}\circ\phi_0(x;\boldsymbol{\lambda}+\boldsymbol{\delta})=-i\bigl(V(x;\boldsymbol{\lambda})e^{\frac{\gamma}{2}p}-V^*(x;\boldsymbol{\lambda})e^{-\frac{\gamma}{2}p}\bigr)\varphi(x), \tag{3.59}$$

$$\widetilde{\mathcal{H}}(\boldsymbol{\lambda})=\mathcal{B}(\boldsymbol{\lambda})\mathcal{F}(\boldsymbol{\lambda}), \tag{3.60}$$

which are square root free. The actions of the operators $\mathcal{A}(\boldsymbol{\lambda})$ and $\mathcal{A}(\boldsymbol{\lambda})^{\dagger}$ on the eigenfunctions are

$$\mathcal{A}(\boldsymbol{\lambda})\phi_n(x;\boldsymbol{\lambda})=f_n(\boldsymbol{\lambda})\phi_{n-1}\bigl(x;\boldsymbol{\lambda}+\boldsymbol{\delta}\bigr), \tag{3.61}$$

$$\mathcal{A}(\boldsymbol{\lambda})^{\dagger}\phi_{n-1}\bigl(x;\boldsymbol{\lambda}+\boldsymbol{\delta}\bigr)=b_{n-1}(\boldsymbol{\lambda})\phi_n(x;\boldsymbol{\lambda}), \tag{3.62}$$

and those of $\mathcal{F}$ and $\mathcal{B}$ on the polynomials are

$$\mathcal{F}(\boldsymbol{\lambda})P_n\bigl(\eta(x);\boldsymbol{\lambda}\bigr)=f_n(\boldsymbol{\lambda})P_{n-1}\bigl(\eta(x);\boldsymbol{\lambda}+\boldsymbol{\delta}\bigr), \tag{3.63}$$

$$\mathcal{B}(\boldsymbol{\lambda})P_{n-1}\bigl(\eta(x);\boldsymbol{\lambda}+\boldsymbol{\delta}\bigr)=b_{n-1}(\boldsymbol{\lambda})P_n\bigl(\eta(x);\boldsymbol{\lambda}\bigr), \tag{3.64}$$

where the real constants $f_n(\boldsymbol{\lambda})$ and $b_{n-1}(\boldsymbol{\lambda})$ are the factors of the energy eigenvalue $\mathcal{E}_n(\boldsymbol{\lambda}) = f_n(\boldsymbol{\lambda})b_{n-1}(\boldsymbol{\lambda})$ and we set $P_{-1}(\eta;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} 0$, $b_{-1}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} 0$. For (v)–(viii), the coefficients $f_n(\boldsymbol{\lambda})$ and $b_{n-1}(\boldsymbol{\lambda})$ are

$$f_n(\boldsymbol{\lambda}) = \mathcal{E}_n(\boldsymbol{\lambda}) \times \begin{cases} -1 & : \text{(v)} \\ 1 & : \text{(vi)}, \text{(vii)} \\ (-1)^{K+1} & : \text{(viii)} \end{cases}, \quad b_{n-1}(\boldsymbol{\lambda}) = \begin{cases} -1 & : \text{(v)} \\ 1 & : \text{(vi)}, \text{(vii)} \\ (-1)^{K+1} & : \text{(viii)} \end{cases}. \tag{3.65}$$

We remark that the relations (3.63)–(3.64) for (vii) and (viii) are obtained from those for the Askey-Wilson case by the replacements (3.11) and (3.39), and the cases (v) and (vi) are obtained from (vii) by the $R \to \infty$ limit with (3.27) and (3.28).

3.6 Normalisation constants

The shape invariance gives a recurrence relation of the normalisation constants $h_n(\boldsymbol{\lambda})$ in (2.4). Since the relations (3.61)–(3.62) imply

$$\begin{aligned} b_{n-1}(\boldsymbol{\lambda})h_n(\boldsymbol{\lambda}) &= \big(b_{n-1}(\boldsymbol{\lambda})\phi_n(x;\boldsymbol{\lambda}), \phi_n(x;\boldsymbol{\lambda})\big) = \big(\mathcal{A}(\boldsymbol{\lambda})^\dagger\phi_{n-1}(x;\boldsymbol{\lambda}+\boldsymbol{\delta}), \phi_n(x;\boldsymbol{\lambda})\big) \\ &= \big(\phi_{n-1}(x;\boldsymbol{\lambda}+\boldsymbol{\delta}), \mathcal{A}(\boldsymbol{\lambda})\phi_n(x;\boldsymbol{\lambda})\big) = \big(\phi_{n-1}(x;\boldsymbol{\lambda}+\boldsymbol{\delta}), f_n(\boldsymbol{\lambda})\phi_{n-1}(x;\boldsymbol{\lambda}+\boldsymbol{\delta})\big) \\ &= f_n(\boldsymbol{\lambda})h_{n-1}(\boldsymbol{\lambda}+\boldsymbol{\delta}), \end{aligned}$$

we obtain

$$h_n(\boldsymbol{\lambda}) = \frac{f_n(\boldsymbol{\lambda})}{b_{n-1}(\boldsymbol{\lambda})} h_{n-1}(\boldsymbol{\lambda}+\boldsymbol{\delta}) \quad (1 \le n \le n_{\max}), \tag{3.66}$$

namely,

$$h_n(\boldsymbol{\lambda}) = \prod_{k=1}^{n} \frac{f_k\big(\boldsymbol{\lambda}+(n-k)\boldsymbol{\delta}\big)}{b_{k-1}\big(\boldsymbol{\lambda}+(n-k)\boldsymbol{\delta}\big)} \cdot h_0(\boldsymbol{\lambda}+n\boldsymbol{\delta}) \quad (0 \le n \le n_{\max}). \tag{3.67}$$

The three term recurrence relations for the orthogonal polynomials $P_n(\eta;\boldsymbol{\lambda})$ ($\deg P_n(\eta) = n$, $n \in \mathbb{Z}_{\ge 0}$) read:

$$\eta P_n(\eta;\boldsymbol{\lambda}) = A_n(\boldsymbol{\lambda})P_{n+1}(\eta;\boldsymbol{\lambda}) + B_n(\boldsymbol{\lambda})P_n(\eta;\boldsymbol{\lambda}) + C_n(\boldsymbol{\lambda})P_{n-1}(\eta;\boldsymbol{\lambda}). \tag{3.68}$$

Let us set $P_n(\eta;\boldsymbol{\lambda}) = c_n(\boldsymbol{\lambda})P_n^{\text{monic}}(\eta;\boldsymbol{\lambda}) = c_n(\boldsymbol{\lambda})\eta^n +$ (lower order terms), which gives the relation $A_n(\boldsymbol{\lambda}) = \frac{c_n(\boldsymbol{\lambda})}{c_{n+1}(\boldsymbol{\lambda})}$ because $\eta P_n^{\text{monic}}(\eta;\boldsymbol{\lambda}) = P_{n+1}^{\text{monic}}(\eta;\boldsymbol{\lambda}) + \cdots$. The explicit forms of $A_n(\boldsymbol{\lambda})$, $B_n(\boldsymbol{\lambda})$, $C_n(\boldsymbol{\lambda})$ and $c_n(\boldsymbol{\lambda})$ for $P_n(\eta;\boldsymbol{\lambda})$ can be read from (A.18)–(A.21) and their definitions (3.7), (3.26) and (3.38). The three term recurrence relations also give a recurrence relation of the normalisation constants $h_n(\boldsymbol{\lambda})$. Since (3.68) and (2.6) imply (we suppress $\boldsymbol{\lambda}$)

$$(\phi_n, \eta\phi_{n-1}) = (\phi_n, A_{n-1}\phi_n + B_{n-1}\phi_{n-1} + C_{n-1}\phi_{n-2}) = A_{n-1}h_n$$

$$= (\eta\phi_n, \phi_{n-1}) = (A_n\phi_{n+1} + B_n\phi_n + C_n\phi_{n-1}, \phi_{n-1}) = C_n h_{n-1},$$

we obtain

$$\frac{h_n(\boldsymbol{\lambda})}{h_{n-1}(\boldsymbol{\lambda})} = \frac{c_n(\boldsymbol{\lambda})}{c_{n-1}(\boldsymbol{\lambda})} C_n(\boldsymbol{\lambda}) \quad (1 \le n \le n_{\max}), \tag{3.69}$$

namely,

$$h_n(\boldsymbol{\lambda}) = \frac{c_n(\boldsymbol{\lambda})}{c_0(\boldsymbol{\lambda})} \prod_{k=1}^{n} C_k(\boldsymbol{\lambda}) \cdot h_0(\boldsymbol{\lambda}) \quad (0 \le n \le n_{\max}). \tag{3.70}$$

Eqs. (3.67) and (3.70) imply that $h_0(\boldsymbol{\lambda})$ should satisfy

$$\frac{h_0(\boldsymbol{\lambda} + n\boldsymbol{\delta})}{h_0(\boldsymbol{\lambda})} = \frac{c_n(\boldsymbol{\lambda})}{c_0(\boldsymbol{\lambda})} \prod_{k=1}^{n} \frac{f_k\big(\boldsymbol{\lambda} + (n-k)\boldsymbol{\delta}\big)}{b_{k-1}\big(\boldsymbol{\lambda} + (n-k)\boldsymbol{\delta}\big)} C_k(\boldsymbol{\lambda}) \quad (0 \le n \le n_{\max}). \tag{3.71}$$

The explicit forms of $h_n(\boldsymbol{\lambda})$ are conjectured as

$$\begin{aligned}
\text{(v)}, \text{(vi)}: \quad h_n(\boldsymbol{\lambda}) &= 2\pi \prod_{k=0}^{n-1} 4 \sin\tfrac{\gamma}{2}(k+1) \sin\gamma(n+\alpha-1-\tfrac{k}{2}) \\
&\quad\times \frac{\Phi_{\frac{\gamma}{2}}\big(i(\pi - \frac{\gamma}{2})\big)}{\Phi_{\frac{\gamma}{2}}\big(-i(3\pi + \gamma(2n+2\alpha-\frac{1}{2}))\big)} \cdot \prod_{j=1}^{2} \Phi_{\frac{\gamma}{2}}\big(-i(\pi + \gamma(n+2\alpha_j-\tfrac{1}{2}))\big) \\
&\quad\times \prod_{\epsilon=\pm1} \Phi_{\frac{\gamma}{2}}\big(\epsilon\gamma(\beta_1-\beta_2) - i(\pi + \gamma(n+\alpha-\tfrac{1}{2}))\big) \qquad (3.72) \\
&\quad\times e^{i\frac{\gamma}{2}\left((\alpha_1-\alpha_2)^2 - (\beta_1-\beta_2)^2 - (n+\alpha)(\frac{2\pi}{\gamma}+1) - \frac{8\pi^2}{3\gamma^2} - \frac{\pi}{\gamma} + \frac{1}{3}\right)} \times e^{-(\beta_1+\beta_2)(\pi+\gamma(n+\alpha-\frac{1}{2}))},
\end{aligned}$$

$$\begin{aligned}
\text{(vii)}: \quad h_n(\boldsymbol{\lambda}) &= 2\pi \prod_{k=0}^{n-1} 4 \sin\tfrac{\gamma}{2}(k+1) \sin\gamma(n+\alpha-1-\tfrac{k}{2}) \\
&\quad\times \frac{\Phi_{\frac{\gamma}{2}}\big(i(\pi - \frac{\gamma}{2})\big)}{\Phi_{\frac{\gamma}{2}}\big(-i(3\pi + \gamma(2n+2\alpha-\frac{1}{2}))\big)} \cdot \prod_{j=1}^{2} \Phi_{\frac{\gamma}{2}}\big(-i(\pi + \gamma(n+2\alpha_j-\tfrac{1}{2}))\big) \\
&\quad\times \prod_{\epsilon_1,\epsilon_2=\pm1} \Phi_{\frac{\gamma}{2}}\big(\gamma(\epsilon_1\beta_1+\epsilon_2\beta_2) - i(\pi + \gamma(n+\alpha-\tfrac{1}{2}))\big) \qquad (3.73) \\
&\quad\times e^{i\frac{\gamma}{2}\left((n+\alpha-1)^2 + (\alpha_1-\alpha_2)^2 - 2(\beta_1^2+\beta_2^2) - 2(\frac{1}{2}+\frac{\pi}{\gamma})^2\right)},
\end{aligned}$$

$$\begin{aligned}
\text{(viii)}: \quad h_n(\boldsymbol{\lambda}) &= 2\pi \prod_{k=0}^{n-1} (-1)^{K+1} 4 \sin\tfrac{\gamma}{2}(k+1) \sin\gamma(n+\alpha-1-\tfrac{k}{2}) \\
&\quad\times \frac{\Phi_{\frac{\gamma}{2}}\big(i(\pi - \frac{\gamma}{2})\big)}{\Phi_{\frac{\gamma}{2}}\big(-i((1+2K)\pi + \gamma(2n+2\alpha-\frac{1}{2}))\big)} \\
&\quad\times \prod_{j=1}^{2} \Phi_{\frac{\gamma}{2}}\big(-i(-\tfrac{\alpha_j}{|\alpha_j|}\pi + \gamma(n+2\alpha_j-\tfrac{1}{2}))\big) \qquad (3.74) \\
&\quad\times \prod_{\epsilon=\pm1} \Phi_{\frac{\gamma}{2}}\big(\epsilon\gamma(\beta_1-\beta_2) - i((1+K-K^2)\pi + \gamma(n+\alpha-\tfrac{1}{2}))\big)
\end{aligned}$$

$$\times \prod_{\epsilon=\pm 1} \Phi_{\frac{\gamma}{2}}\bigl(\epsilon\gamma(\beta_1+\beta_2) - i(K(K+1)\pi + \gamma(n+\alpha-\tfrac{1}{2}))\bigr)$$
$$\times\, e^{i\frac{\gamma}{2}\left((n+\alpha)^2+2(n+\alpha)(\frac{K\pi}{\gamma}-1)+(\alpha_1-\alpha_2)(\alpha_1-\alpha_2-\frac{\pi}{\gamma}(\frac{\alpha_1}{|\alpha_1|}-\frac{\alpha_2}{|\alpha_2|}))-2(\beta_1^2+\beta_2^2)+\frac{1}{2}-\frac{\pi}{\gamma}(1+2K)+\frac{\pi^2}{\gamma^2}\right)}$$
$$\times\, e^{\pi(\frac{\alpha_1}{|\alpha_1|}\beta_1+\frac{\alpha_2}{|\alpha_2|}\beta_2)},$$

which are supported by numerical calculation. We can check that they satisfy the properties (3.66)–(3.67) and (3.69)–(3.71).

3.7 Limit to oQM

In an appropriate $\gamma \to 0$ limit, idQM reduces to oQM [21]. Let us take the parameters as

$$\begin{aligned}
\text{(v)},\text{(vi)}:&\ \ \alpha_1 = -\tfrac{\pi}{2\gamma} - h_1,\ \ \alpha_2 = -\tfrac{\pi}{2\gamma} - h + h_1 + \tfrac{1}{2},\ \ e^{\gamma\beta_j} = (\gamma\beta'_j)^{-1},\\
\text{(vii)}:&\ \ \alpha_1 = -\tfrac{\pi}{\gamma} + \tfrac{1}{2}(g+\tfrac{1}{2}),\ \ \alpha_2 = \tfrac{1}{2}(-h+\tfrac{1}{2}),\\
\text{(viii)}:&\ \ \alpha_1 = -\tfrac{\pi}{2\gamma} - h_1,\ \ \alpha_2 = -\tfrac{\pi}{2\gamma} - h + h_1 + \tfrac{1}{2},\ \ K=1.
\end{aligned} \tag{3.75}$$

Here we assume that g, h, h_1 and β'_j in (v)–(vi) and β_j in (vii)–(viii) are independent of γ. Note that h_1 is a redundant parameter and $\beta'_j > 0$. By taking $\gamma \to 0$ limit of $\mathcal{A}(\boldsymbol{\lambda})$, we obtain

$$\begin{aligned}
\text{(v)}:&\ \ \lim_{\gamma\to 0}\tfrac{1}{\gamma^2}\mathcal{H}(\boldsymbol{\lambda}) = \mathcal{H}^{\text{M}}, &&\beta'_1+\beta'_2=\mu,\\
\text{(vi)}:&\ \ \lim_{\gamma\to 0}\tfrac{1}{\gamma^2}\mathcal{H}(\boldsymbol{\lambda})\big|_{x\to -x} = \mathcal{H}^{\text{M}}, &&\beta'_1+\beta'_2=\mu,\\
\text{(vii)}:&\ \ \lim_{\gamma\to 0}\tfrac{4}{\gamma^2}\mathcal{H}(\boldsymbol{\lambda})\big|_{x\to 2x} = \mathcal{H}^{\text{hDPT}}, &&\\
\text{(viii)}:&\ \ \lim_{\gamma\to 0}\tfrac{1}{\gamma^2}\mathcal{H}(\boldsymbol{\lambda}) = \mathcal{H}^{\text{hst}}, &&\beta_1+\beta_2=\mu.
\end{aligned} \tag{3.76}$$

Here $\mathcal{H}^{\text{M}}$, $\mathcal{H}^{\text{hDPT}}$ and $\mathcal{H}^{\text{hst}}$ are the Hamiltonians for Morse, hyperbolic Darboux-Pöschl-Teller and hyperbolic symmetric top II (see for example [17, 18]),

$$\begin{aligned}
\mathcal{H}^{\text{M}} &= p^2 + \mu^2 e^{2x} - \mu(2h+1)e^x + h^2, && -\infty < x < \infty,\\
\mathcal{H}^{\text{hDPT}} &= p^2 + \frac{g(g-1)}{\sinh^2 x} - \frac{h(h+1)}{\cosh^2 x} + (h-g)^2, && 0 < x < \infty,\\
\mathcal{H}^{\text{hst}} &= p^2 + \frac{-h(h+1)+\mu^2+\mu(2h+1)\sinh x}{\cosh^2 x} + h^2, && -\infty < x < \infty,
\end{aligned} \tag{3.77}$$

and $\mathcal{H}^{\text{hst}}$ with $\mu = 0$ is the Hamiltonian for the soliton potential. It is easy to show that the eigenvalues $\mathcal{E}_n(\boldsymbol{\lambda})$ have the corresponding limits, too. Considering the above limiting forms of the Hamiltonians (3.76), we know that the limit of the eigenfunctions $\phi_n(x;\boldsymbol{\lambda})$ should be

$\lim_{\gamma\to 0}\gamma^{(\text{some power})}\times\phi_n(x;\boldsymbol{\lambda})\propto\phi_n^{\text{oQM}}(x)$. In fact this can be verified by direct calculation with the use of the formulas in Appendix B and A.5,

$$\begin{aligned}
\text{(v)}:\ \ &\lim_{\gamma\to 0}\phi_0(x;\boldsymbol{\lambda})=\phi_0^{\text{M}}(x),\quad \phi_0^{\text{M}}(x)=e^{hx-\mu e^x},\\
&\lim_{\gamma\to 0}\gamma^{-n}P_n\bigl(\eta(x);\boldsymbol{\lambda}\bigr)=(2\mu)^n n!\,P_n^{\text{M}}(e^{-x}),\ \ P_n^{\text{M}}(\eta)=(2\mu\eta^{-1})^{-n}L_n^{(2h-2n)}(2\mu\eta^{-1}),\\
\text{(vii)}:\ \ &\lim_{\gamma\to 0}\phi_0(x;\boldsymbol{\lambda})\bigr|_{x\to 2x}=2^{g-h}\phi_0^{\text{hDPT}}(x),\\
&\quad\phi_0^{\text{hDPT}}(x)=\bigl(\tfrac{\cosh 2x-1}{2}\bigr)^{\frac{g}{2}}\bigl(\tfrac{\cosh 2x+1}{2}\bigr)^{-\frac{h}{2}}=\bigl(\sinh x\bigr)^g\bigl(\cosh x\bigr)^{-h},\\
&\lim_{\gamma\to 0}\gamma^{-n}P_n\bigl(\eta(x);\boldsymbol{\lambda}\bigr)\bigr|_{x\to 2x}=(-1)^n2^{2n}n!P_n^{\text{hDPT}}(\cosh 2x),\\
&\quad P_n^{\text{hDPT}}(\eta)=P_n^{(g-\frac12,-h-\frac12)}(\eta),\\
\text{(viii)}:\ \ &\lim_{\gamma\to 0}\phi_0(x;\boldsymbol{\lambda})=2^{-h}e^{-\frac12\pi\mu}\phi_0^{\text{hst}}(x),\quad \phi_0^{\text{hst}}(x)=\bigl(\cosh x\bigr)^{-h}e^{-\mu\tan^{-1}\sinh x},\\
&\lim_{\gamma\to 0}\gamma^{-n}P_n\bigl(\eta(x);\boldsymbol{\lambda}\bigr)=2^{2n}n!\,P_n^{\text{hst}}(\sinh x),\ \ P_n^{\text{hst}}(\eta)=i^{-n}P_n^{(-h-\frac12-i\mu,-h-\frac12+i\mu)}(i\eta).
\end{aligned}\tag{3.78}$$

Here $L_n^{(\alpha)}(\eta)$ and $P_n^{(\alpha,\beta)}(\eta)$ are the Laguerre and the Jacobi polynomials in η of degree n, respectively. The limit of $n_{\max}(\boldsymbol{\lambda})$ also gives the corresponding one. The case (vi) can be obtained from (v) by the replacement $x\to -x$.

3.8 Other limits

It is well known that the idQM system of the Wilson polynomial can be obtained from that of the Askey-Wilson polynomial by the replacements $x^{\text{AW}}=\frac{\pi}{L}x^{\text{W}}$ and $\gamma^{\text{AW}}=-\frac{\pi}{L}$ and taking $L\to\infty$ ($q=e^{\gamma^{\text{AW}}}\to 1$) limit [6]. Let us consider similar $q\to 1$ limits of (vii) and (vi) cases.

Let us set

$$x=\frac{x'}{R},\quad \gamma=\frac{1}{R},\quad \lambda_j=-\pi R+\lambda'_j\ \ (j=1,2),\tag{3.79}$$

where $R>0$ and we assume that λ'_j is R-independent. In the $R\to\infty$ ($q=e^{-i\gamma}\to 1$) limit we have

$$\begin{aligned}
\text{(vii)}:\quad &\lim_{R\to\infty}R^2\,V(x;\boldsymbol{\lambda})=\frac{\prod_{j=1}^2(\lambda'_j+ix')(\lambda'^*_j+ix')}{2ix'(2ix'+1)}=V^{\text{W}}(x';\boldsymbol{\lambda}^{\text{W}}),\\
\text{(vi)}:\quad &\lim_{R\to\infty}R^2\,V(x;\boldsymbol{\lambda})=\prod_{j=1}^2(\lambda'_j+ix')=V^{\text{cH}}(x';\boldsymbol{\lambda}^{\text{cH}}).
\end{aligned}\tag{3.80}$$

Here V^{W} and V^{cH} are the potential functions of the Wilson and the continuous Hahn systems, respectively and $\boldsymbol{\lambda}^{\text{W}}=(\lambda'_1,\lambda'_2,\lambda'^*_1,\lambda'^*_2)$ and $\boldsymbol{\lambda}^{\text{cH}}=(\lambda'_1,\lambda'_2)$ [6]. Thus the Wilson and the continuous Hahn systems are obtained from (vii) and (vi), respectively. We can verify the limits of the corresponding eigenpolynomials

$$\begin{aligned}
&\text{(vii)}: \quad \lim_{R\to\infty} R^{3n} P_n\big(\eta(x);\boldsymbol{\lambda}\big) = (-1)^n W_n(x'^2; \lambda'_1, \lambda'_2, \lambda_1'^*, \lambda_2'^*),\\
&\text{(vi)}: \quad \lim_{R\to\infty} R^{2n} P_n\big(\eta(x);\boldsymbol{\lambda}\big) = n!\, p_n(x'; \lambda'_1, \lambda'_2, \lambda_1'^*, \lambda_2'^*), \qquad (3.81)\\
&\text{(vii)}\,\text{(vi)}: \quad \lim_{R\to\infty} R^2 \mathcal{E}_n(\boldsymbol{\lambda}) = n(n+\lambda'_1+\lambda'_2+\lambda_1'^*+\lambda_2'^*-1), \quad \lim_{R\to\infty} n_{\max}(\boldsymbol{\lambda}) = \infty,
\end{aligned}$$

where $W_n(\eta; a_1, a_2, a_3, a_4)$ and $p_n(\eta; a_1, a_2, a_3, a_4)$ are the Wilson and the continuous Hahn polynomials, respectively [5].

The q-deformation of the continuous Hahn polynomial is known as the continuous q-Hahn polynomial $p_n(\eta; a_1, a_2, a_3, a_4; q)$ $(-1 < \eta < 1,\ 0 < q < 1)$ which is the Askey-Wilson polynomial with different parameters [5]. Our $\tilde{p}_n(\eta; a_1, a_2, a_3, a_4|q)$ $(0 < \eta < \infty,\ |q| = 1)$ (A.11) gives another q-deformation of the continuous Hahn polynomial.

4 Summary and Comments

Several kinds of q-orthogonal polynomials with $|q| = 1$ have been constructed. Their weight functions consist of products of quantum dilogarithm functions, which are natural generalisation of the Euler gamma functions and q-gamma functions. The total number of mutually orthogonal polynomials is finite.

In other words, several new examples of exactly solvable systems in discrete quantum mechanics with pure imaginary shifts have been derived based on the sinusoidal coordinates $\eta(x) = \cosh x, e^{\pm x}, \sinh x$. The method called "unified theory of exactly and quasi-exactly solvable discrete quantum mechanics" was developed by the present authors several years ago [1]. These new systems have finitely many discrete eigenstates. Their eigenpolynomials are given by the Askey-Wilson polynomial and its certain limiting forms with the parameter $q = e^{-i\gamma}$, $|q| = 1$. The groundstate wavefunctions are described by the quantum dilogarithm. In appropriate limits they reduce to known solvable systems in the ordinary quantum mechanics: Morse, hyperbolic Darboux-Pöschl-Teller and hyperbolic symmetric top II potentials, which also have finitely many discrete eigenstates.

Several comments are in order. The listed parameter ranges of these new systems, (3.18), (3.32) and (3.42), are rather conservative. These could be extended with scrutiny. We have considered the factorisations (A.6), but $\widetilde{V}(x)$ (A.5) may have different factorised forms for less generic parameters. For example, the following factorisation is allowed for (vii):

$$\widetilde{V}(x) = e^{-2x} A \prod_{j=1}^{4} (e^x + e^{i\theta_j}) \quad (A > 0,\ \theta_j \in \mathbb{R}). \tag{4.1}$$

The four parameters (a_1, a_2, a_3, a_4) of the Askey-Wilson polynomial $p_n(\cos x; a_1, a_2, a_3, a_4|q)$ with $0 < q < 1$ and $|a_j| < 1$ are either (a) $a_1, a_2 \in \mathbb{C}$, $a_3 = a_1^*$, $a_4 = a_2^*$ or (b) $a_1, a_2, a_3, a_4 \in \mathbb{R}$ [5]. Our factorisation (A.6) corresponds to (a) and (4.1) corresponds to (b). It is an interesting problem to clarify whether the well-defined quantum systems are obtained for such less generic cases as (4.1). By construction, the q-value of the present theory is limited as $q = e^{-i\gamma}$, $0 < \gamma < \pi$. Recently there are interesting developments related to q-orthogonal polynomials with $q = -1$ [22]. So far, we have not been able to extend our theory to $q = -1$ or $\gamma = \pi$, for which the integral representation of the quantum dilogarithm $\Phi_\gamma(z)$ (B.2) has double poles. The fact that the new solvable systems have only finitely many discrete eigenstates mean that scattering problems can be formulated. It is an interesting challenge to calculate the reflection and transmission amplitudes of these systems. The Hamiltonians of the known exactly solvable idQM systems, *e.g.* the Wilson (ii) and the Askey-Wilson (iii) have discrete symmetries, which are essential for multi-indexed deformations [19, 23, 24] for these systems through Darboux transformations [25, 26]. It is interesting to find discrete symmetries and (psudo)virtual state wavefunctions for (v)-(viii).

After completing this work, we were informed by R. Askey of the sieved orthogonal polynomials [27, 28]. They could be considered as q-orthogonal polynomials with q being a root of unity, as they are obtained from q-orthogonal polynomials, *e.g.* q-ultraspherical polynomials [3, 4, 5, 29], by certain limiting procedures.

Acknowledgements

S. O. thanks M. Jimbo for useful discussion of quantum dilogarithm. We thank R. Askey for enlightening discussion on sieved orthogonal polynomials. S. O. and R. S. are supported in part by Grant-in-Aid for Scientific Research from the Ministry of Education, Culture, Sports, Science and Technology (MEXT), No.25400395 and No.22540186, respectively.

A Various Data

A.1 Matrix elements $\widetilde{\mathcal{H}}^{\eta}_{m,n}$

The sinusoidal coordinate $\eta(x)$ satisfying (2.9) has the property (**U**2.21),

$$\frac{\eta(x-i\gamma)^{n+1}-\eta(x+i\gamma)^{n+1}}{\eta(x-i\gamma)-\eta(x+i\gamma)}=\sum_{k=0}^{n}g_n^{(k)}\eta(x)^{n-k}\quad(n\in\mathbb{Z}_{\geq 0}),\tag{A.1}$$

where $g_n^{(k)}$ are real constants and we set $g_n^{(k)}=0$ unless $0\leq k\leq n$. The matrix elements $\widetilde{\mathcal{H}}^{\eta}_{m,n}$ (2.13) are given by (**U**2.35)–(**U**2.36):

$$\widetilde{\mathcal{H}}^{\eta}_{m,n}=\sum_{j=\max(n-2-m,0)}^{n-m}e_{n-m,j,n},\quad e_{m,j,n}\stackrel{\text{def}}{=}\sum_{l=0}^{2-m+j}v_{2-m+j-l,l}\sum_{r=0}^{n-1}g_{n+l-r-2}^{(j)}.\tag{A.2}$$

A.2 Explicit forms of $g_n^{(k)}$

For the eight sinusoidal coordinates (3.1) with (3.2) and (3.3), the explicit forms of $[\![n]\!]$ (which was denoted as $[n]$ in (**U**2.25)) and $g_n^{(k)}$ in (A.1) are the following:

<u>$[\![n]\!]$</u> :

$$[\![n]\!]=\begin{cases}n & \text{for (i)–(ii)}\\ \dfrac{e^{-\gamma n}-e^{\gamma n}}{e^{-\gamma}-e^{\gamma}} & \text{for (iii)–(iv)}\\ \dfrac{e^{i\gamma n}-e^{-i\gamma n}}{e^{i\gamma}-e^{-i\gamma}} & \text{for (v)–(viii)}\end{cases}.\tag{A.3}$$

<u>$g_n^{(k)}$</u> :

$$\begin{aligned}
\text{(i)}:\quad & g_n^{(k)}=\theta(k:\text{even})\,(-1)^{\frac{k}{2}}\binom{n+1}{k+1},\\
\text{(ii)}:\quad & g_n^{(k)}=\frac{(-1)^k}{2}\binom{2n+2}{2k+1},\\
\text{(v)},\text{(vi)}:\quad & g_n^{(k)}=[\![n+1]\!]\,\delta_{k\,0},\\
\text{(iii)},\text{(iv)},\text{(vii)}:\quad & g_n^{(k)}=\theta(k:\text{even})\,\frac{(n+1)!}{2^k}\sum_{r=0}^{\frac{k}{2}}\binom{n-k+r}{r}\frac{(-1)^r\,[\![n-k+1+2r]\!]}{(\frac{k}{2}-r)!\,(n-\frac{k}{2}+1+r)!},\\
\text{(viii)}:\quad & g_n^{(k)}=(-1)^{\frac{k}{2}}\times\big(\text{RHS of the above equation}\big).
\end{aligned}\tag{A.4}$$

Here $\theta(P)$ is a step function for a proposition P ; $\theta(P)=1$ for P : true, $\theta(P)=0$ for P : false.

A.3 Explicit forms of $V(x)$

For the eight sinusoidal coordinates (3.1) with (3.2), the potential function $V(x)$ (2.10)–(2.11) becomes

$$
\begin{aligned}
&\text{(i)}: V(x)=-\tfrac{1}{2}\widetilde{V}(x),\ \widetilde{V}(x)=\sum_{k=0}^{2}\tilde{v}_kx^k,\ \tilde{v}_2\in\mathbb{R},\ \tilde{v}_k\in\mathbb{C}\ (k\neq 2),\\
&\text{(ii)}: V(x)=\frac{\frac{1}{2}\widetilde{V}(x)}{2ix(2ix+1)},\ \widetilde{V}(x)=\sum_{k=0}^{4}\tilde{v}_kx^k,\ i^k\tilde{v}_k\in\mathbb{R},\\
&\text{(iii)}: V(x)=\frac{e^{\frac{\gamma}{2}}}{\sinh\frac{\gamma}{2}\sinh\gamma}\frac{e^{2ix}\,\widetilde{V}(x)}{(1-e^{2ix})(1-e^{\gamma}e^{2ix})},\ \widetilde{V}(x)=\sum_{k=-2}^{2}\tilde{v}_ke^{ikx},\ \tilde{v}_k\in\mathbb{R},\\
&\text{(iv)}: V(x)=\frac{-e^{\frac{\gamma}{2}}}{\sinh\frac{\gamma}{2}\sinh\gamma}\frac{e^{2ix}\,\widetilde{V}(x)}{(1+e^{2ix})(1+e^{\gamma}e^{2ix})},\ \widetilde{V}(x)=\sum_{k=-2}^{2}\tilde{v}_ke^{ikx},\ i^k\tilde{v}_k\in\mathbb{R},\\
&\text{(v)}: V(x)=\frac{-e^{-i\frac{\gamma}{2}}}{4\sin\frac{\gamma}{2}\sin\gamma}\,e^{2x}\,\widetilde{V}(x),\ \widetilde{V}(x)=\sum_{k=0}^{2}\tilde{v}_ke^{-kx},\ \tilde{v}_0\in\mathbb{R},\ \tilde{v}_k\in\mathbb{C}\ (k\neq 0),\qquad\text{(A.5)}\\
&\text{(vi)}: V(x)=\frac{-e^{i\frac{\gamma}{2}}}{4\sin\frac{\gamma}{2}\sin\gamma}\,e^{-2x}\,\widetilde{V}(x),\ \widetilde{V}(x)=\sum_{k=0}^{2}\tilde{v}_k^*e^{kx},\ \tilde{v}_0\in\mathbb{R},\ \tilde{v}_k\in\mathbb{C}\ (k\neq 0),\\
&\text{(vii)}: V(x)=\frac{-e^{-i\frac{\gamma}{2}}}{\sin\frac{\gamma}{2}\sin\gamma}\,\frac{e^{2x}\,\widetilde{V}(x)}{(1-e^{2x})(1-e^{-i\gamma}e^{2x})},\ \widetilde{V}(x)=\sum_{k=-2}^{2}\tilde{v}_ke^{kx},\ \tilde{v}_{-k}=\tilde{v}_k^*\in\mathbb{C},\\
&\text{(viii)}: V(x)=\frac{-e^{-i\frac{\gamma}{2}}}{\sin\frac{\gamma}{2}\sin\gamma}\,\frac{e^{2x}\,\widetilde{V}(x)}{(1+e^{2x})(1+e^{-i\gamma}e^{2x})},\ \widetilde{V}(x)=\sum_{k=-2}^{2}\tilde{v}_ke^{kx},\ \tilde{v}_{-k}=(-1)^k\tilde{v}_k^*\in\mathbb{C}.
\end{aligned}
$$

The transformation from $\{v_{k,l}\ (k+l\leq 2)\}$ to $\{\mathrm{Re}\,\tilde{v}_k,\mathrm{Im}\,\tilde{v}_k\}$ is a real linear map of rank 5. The condition $\sum_{k+l=2}v_{k,l}^2\neq 0$ becomes: (i): $(\tilde{v}_2,\mathrm{Im}\,\tilde{v}_1)\neq(0,0)$, (ii): $(\tilde{v}_4,\tilde{v}_3)\neq(0,0)$, (iii)–(iv): $(\tilde{v}_2,\tilde{v}_{-2})\neq(0,0)$, (v)–(viii): $\tilde{v}_2\neq 0$.

A.4 Factorised forms of $\widetilde{V}(x)$

For the generic values of the original parameters $\{v_{k,l}\}$, the above $\widetilde{V}(x)$ have the following factorised forms:

$$
\begin{aligned}
&\text{(i)}: \widetilde{V}(x)=(-i)^2A\prod_{j=1}^{2}(a_j+ix),\qquad \text{(ii)}:\ \widetilde{V}(x)=A\prod_{j=1}^{4}(a_j+ix),\\
&\text{(iii)}: \widetilde{V}(x)=e^{-i2x}A\prod_{j=1}^{4}(1-a_je^{ix}),\quad \text{(iv)}:\ \widetilde{V}(x)=(-i)^2e^{-i2x}A\prod_{j=1}^{4}(1-ia_je^{ix}),
\end{aligned}
$$

$$(\mathrm{v}):\widetilde{V}(x)=A\prod_{j=1}^{2}(1+a_j^*e^{-x}),\qquad (\mathrm{vi}):\ \widetilde{V}(x)=A\prod_{j=1}^{2}(1+a_je^{x}),\tag{A.6}$$
$$(\mathrm{vii}):\widetilde{V}(x)=A\prod_{j=1}^{2}(1+a_je^{x})(1+a_j^*e^{-x}),\quad (\mathrm{viii}):\ \widetilde{V}(x)=A\prod_{j=1}^{2}(1+a_je^{x})(1-a_j^*e^{-x}),$$

where A is a real overall scaling parameter and $a_j\in\mathbb{C}$ are the new parameters. For (ii)–(iv), the four parameters a_j obey $\{a_1^*,\ldots,a_4^*\}=\{a_1,\ldots,a_4\}$ (as a set). For less generic cases, $\widetilde{V}(x)$ may have different factorised forms.

We present the relations among the original real parameters $v_{k,l}$ in $V(x)$ (2.10)–(2.11) and the new complex parameters a_j after factorisation in (A.6). Since $v_{0,2}$ is redundant (see (2.12)), we set $v_{0,2}=0$. Then $v_{j,k}$ are expressed in terms of a_j and the overall scaling parameter A:

$$\begin{aligned}
(\mathrm{i}):\ &v_{0,0}=-A\,\mathrm{Re}(a_1a_2),\ v_{0,1}=A\,\mathrm{Im}(a_1a_2),\ v_{1,0}=A\,\mathrm{Im}(a_1+a_2-a_1a_2),\\
&v_{1,1}=A\,\mathrm{Re}(a_1+a_2),\ v_{2,0}=A\bigl(1-\mathrm{Re}(a_1+a_2)\bigr),\\
(\mathrm{ii}):\ &v_{0,0}=-\tfrac{1}{2}A(b_3-2b_4),\ v_{0,1}=-\tfrac{1}{2}Ab_3,\ v_{1,0}=\tfrac{1}{2}A(b_1-2b_2+b_3),\ v_{1,1}=\tfrac{1}{2}Ab_1,\\
&v_{2,0}=\tfrac{1}{2}A(2-b_1),\\
(\mathrm{iii}):\ &v_{0,0}=-A(1-b_2+b_4),\ v_{0,1}=\frac{-2A(b_1-b_3)}{e^{-\gamma}-e^{\gamma}},\ v_{1,0}=\frac{2A(e^{\gamma}b_1-e^{-\gamma}b_3)}{e^{-\gamma}-e^{\gamma}},\\
&v_{1,1}=\frac{4A(1-b_4)}{e^{-\gamma}-e^{\gamma}},\ v_{2,0}=\frac{-4A(e^{\gamma}-e^{-\gamma}b_4)}{e^{-\gamma}-e^{\gamma}},\\
(\mathrm{iv}):\ &v_{j,k}=(-1)^{j+k}v_{j,k}^{(\mathrm{iii})},\\
(\mathrm{v}),(\mathrm{vi}):\ &v_{0,0}=A,\ v_{0,1}=\frac{-A(a_1+a_2-a_1^*-a_2^*)}{e^{i\gamma}-e^{-i\gamma}},\ v_{1,0}=\frac{A\bigl(e^{i\gamma}(a_1+a_2)-e^{-i\gamma}(a_1^*+a_2^*)\bigr)}{e^{i\gamma}-e^{-i\gamma}},\\
&v_{1,1}=\frac{-A(a_1a_2-a_1^*a_2^*)}{e^{i\gamma}-e^{-i\gamma}},\ v_{2,0}=\frac{A(e^{i\gamma}a_1a_2-e^{-i\gamma}a_1^*a_2^*)}{e^{i\gamma}-e^{-i\gamma}},\\
(\mathrm{vii}):\ &v_{0,0}=A\bigl((1-a_1a_2)(1-a_1^*a_2^*)+(a_1+a_2)(a_1^*+a_2^*)\bigr),\\
&v_{0,1}=\frac{-2A\bigl((a_1-a_1^*)(1+a_2a_2^*)+(a_2-a_2^*)(1+a_1a_1^*)\bigr)}{e^{i\gamma}-e^{-i\gamma}},\\
&v_{1,0}=\frac{2A\bigl(e^{i\gamma}(a_1(1+a_2a_2^*)+a_2(1+a_1a_1^*))-e^{-i\gamma}(a_1^*(1+a_2a_2^*)+a_2^*(1+a_1a_1^*))\bigr)}{e^{i\gamma}-e^{-i\gamma}},\\
&v_{1,1}=\frac{-4A(a_1a_2-a_1^*a_2^*)}{e^{i\gamma}-e^{-i\gamma}},\ v_{2,0}=\frac{4A(e^{i\gamma}a_1a_2-e^{-i\gamma}a_1^*a_2^*)}{e^{i\gamma}-e^{-i\gamma}},\\
(\mathrm{viii}):\ &v_{0,0}=A\bigl((1+a_1a_2)(1+a_1^*a_2^*)-(a_1+a_2)(a_1^*+a_2^*)\bigr),\\
&v_{0,1}=\frac{-2A\bigl((a_1-a_1^*)(1-a_2a_2^*)+(a_2-a_2^*)(1-a_1a_1^*)\bigr)}{e^{i\gamma}-e^{-i\gamma}},
\end{aligned}\tag{A.7}$$

$$v_{1,0}=\frac{2A\big(e^{i\gamma}(a_1(1-a_2a_2^*)+a_2(1-a_1a_1^*))-e^{-i\gamma}(a_1^*(1-a_2a_2^*)+a_2^*(1-a_1a_1^*))\big)}{e^{i\gamma}-e^{-i\gamma}},$$
$$v_{1,1}=\frac{-4A(a_1a_2-a_1^*a_2^*)}{e^{i\gamma}-e^{-i\gamma}},\ \ v_{2,0}=\frac{4A(e^{i\gamma}a_1a_2-e^{-i\gamma}a_1^*a_2^*)}{e^{i\gamma}-e^{-i\gamma}},$$

where b_j's in (ii)–(iv) are

$$b_1\stackrel{\text{def}}{=}\sum_{j=1}^4 a_j,\ \ b_2\stackrel{\text{def}}{=}\sum_{1\le j<k\le 4}a_ja_k,\ \ b_3\stackrel{\text{def}}{=}\sum_{1\le j<k<l\le 4}a_ja_ka_l,\ \ b_4\stackrel{\text{def}}{=}a_1a_2a_3a_4.$$

A.5 Explicit forms of determinant (2.15) (eigenpolynomial)

For the eight sinusoidal coordinates (3.1) with (3.2), the determinant expression of the eigenpolynomials (2.15) can be evaluated explicitly in terms of the factorisation parameters $\{a_j\}$:

$$\begin{aligned}
&\Big(\frac{A}{2}\Big)^n(n!)^2\times p_n\big(\eta(x)\,;a_1,a_2,a_1^*,a_2^*\big)\quad :\text{(i)},\\
&\Big(\frac{-A}{2}\Big)^n n!\times W_n\big(\eta(x)\,;a_1,a_2,a_3,a_4\big)\quad :\text{(ii)},\\
&\Big(\frac{A}{\sinh\frac{\gamma}{2}\sinh\gamma}\Big)^n\prod_{k=1}^n\sinh\tfrac{-k\gamma}{2}\cdot e^{-\frac{3}{4}\gamma n(n-1)}\times\left\{\begin{array}{ll}p_n\big(\eta(x)\,;a_1,a_2,a_3,a_4|e^{\gamma}\big)&:\text{(iii)}\\(-1)^np_n\big(-\eta(x)\,;a_1,a_2,a_3,a_4|e^{\gamma}\big)&:\text{(iv)}\end{array}\right.,\\
&\Big(\frac{A|a_1a_2|}{\sin\frac{\gamma}{2}\sin\gamma}\Big)^n\prod_{k=1}^n\sin\tfrac{k\gamma}{2}\times e^{-i\frac{\pi}{2}n}e^{i\frac{3}{4}\gamma n(n-1)}\Big(\frac{a_1^*a_2^*}{|a_1a_2|}\Big)^n\\
&\qquad\times\left\{\begin{array}{ll}\tilde{p}_n\big(\eta(x)\,;-a_1,-a_2,-a_1^{*\,-1},-a_2^{*\,-1}|e^{-i\gamma}\big)\times 2^{-n}&:\text{(v), (vi)}\\p_n\big(\eta(x)\,;-a_1,-a_2,-a_1^{*\,-1},-a_2^{*\,-1}|e^{-i\gamma}\big)&:\text{(vii)}\\(-i)^np_n\big(i\eta(x)\,;ia_1,ia_2,-ia_1^{*\,-1},-ia_2^{*\,-1}|e^{-i\gamma}\big)&:\text{(viii)}\end{array}\right..
\end{aligned}\tag{A.8}$$

Here $p_n(\eta;a_1,a_2,a_3,a_4)$ in (i) and $W_n(\eta;a_1,a_2,a_3,a_4)$ in (ii) are the continuous Hahn and the Wilson polynomials respectively, and $p_n(\eta;a_1,a_2,a_3,a_4|q)$ in (iii)–(iv) and (vii)–(viii) is the Askey-Wilson polynomial [5],

$$\begin{aligned}
&p_n(\cos x;a_1,a_2,a_3,a_4|q)\\
&\stackrel{\text{def}}{=}a_1^{-n}(a_1a_2,a_1a_3,a_1a_4;q)_n\ {}_4\phi_3\Big(\begin{matrix}q^{-n},\ a_1a_2a_3a_4q^{n-1},\ a_1e^{ix},\ a_1e^{-ix}\\ a_1a_2,\ a_1a_3,\ a_1a_4\end{matrix}\Big|\,q\,;q\Big),
\end{aligned}\tag{A.9}$$

where the basic hypergeometric series ${}_r\phi_s$ is

$$\begin{aligned}
&{}_r\phi_s\Big(\begin{matrix}a_1,\ \cdots,a_r\\ b_1,\ \cdots,b_s\end{matrix}\Big|\,q\,;z\Big)\stackrel{\text{def}}{=}\sum_{n=0}^{\infty}\frac{(a_1,\ \cdots,a_r\,;q)_n}{(b_1,\ \cdots,b_s\,;q)_n}(-1)^{(1+s-r)n}q^{\frac{1}{2}(1+s-r)n(n-1)}\frac{z^n}{(q\,;q)_n},\\
&(a_1,\ \cdots,a_r\,;q)_n\stackrel{\text{def}}{=}\prod_{j=1}^r(a_j\,;q)_n,\quad (a\,;q)_n\stackrel{\text{def}}{=}\prod_{k=1}^n(1-aq^{k-1}).
\end{aligned}\tag{A.10}$$

The polynomial $\tilde{p}_n(\eta;a_1,a_2,a_3,a_4|q)$ in (v)–(vi) is defined as a limit of the Askey-Wilson polynomial:

$$\begin{aligned}\tilde{p}_n(\eta;a_1,a_2,a_3,a_4|q) &\stackrel{\text{def}}{=} \lim_{t\to 0} t^n p_n\bigl(\tfrac{\eta}{2t};ta_1,ta_2,\tfrac{a_3}{t},\tfrac{a_4}{t}|q\bigr)\\ &= a_1^{-n}(a_1a_3,a_1a_4;q)_n\ {}_3\phi_2\Bigl(\genfrac{}{}{0pt}{}{q^{-n},\ a_1a_2a_3a_4q^{n-1},\ a_1\eta}{a_1a_3,\ a_1a_4}\Bigm| q\,;q\Bigr),\end{aligned} \tag{A.11}$$

which can be regarded as a q-deformation of the continuous Hahn polynomial $p_n(\eta;a_1,a_2,a_3,a_4)$ [5],

$$p_n(\eta;a_1,a_2,a_3,a_4) = i^n\frac{(a_1+a_3,a_1+a_4)_n}{n!}\ {}_3F_2\Bigl(\genfrac{}{}{0pt}{}{-n,\ n+a_1+a_2+a_3+a_4-1,\ a_1+i\eta}{a_1+a_3,\ a_1+a_4}\Bigm| 1\Bigr).$$

The corresponding energy eigenvalues $\mathcal{E}_n$ are

$$\begin{aligned}
\text{(i)}:\quad &\mathcal{E}_n = \tfrac{1}{2}A\times n(n+a_1+a_2+a_1^*+a_2^*-1),\\
\text{(ii)}:\quad &\mathcal{E}_n = \tfrac{1}{2}A\times n(n+a_1+a_2+a_3+a_4-1),\\
\text{(iii)},\text{(iv)}:\quad &\mathcal{E}_n = \frac{Ae^{\frac{1}{2}\gamma}}{\sinh\frac{\gamma}{2}\sinh\gamma}\times(e^{-\gamma n}-1)(1-a_1a_2a_3a_4e^{\gamma(n-1)}),\\
\text{(v)},\text{(vi)}:\quad &\mathcal{E}_n = \frac{A|a_1a_2|}{4\sin\frac{\gamma}{2}\sin\gamma}\times 4\sin\tfrac{\gamma}{2}n\cdot\frac{1}{2i}\Bigl(e^{i\frac{\gamma}{2}(n-1)}\frac{|a_1a_2|}{a_1a_2}-e^{-i\frac{\gamma}{2}(n-1)}\frac{|a_1a_2|}{a_1^*a_2^*}\Bigr),\\
\text{(vii)},\text{(viii)}:\quad &\mathcal{E}_n = \frac{A|a_1a_2|}{\sin\frac{\gamma}{2}\sin\gamma}\times 4\sin\tfrac{\gamma}{2}n\cdot\frac{1}{2i}\Bigl(e^{i\frac{\gamma}{2}(n-1)}\frac{|a_1a_2|}{a_1a_2}-e^{-i\frac{\gamma}{2}(n-1)}\frac{|a_1a_2|}{a_1^*a_2^*}\Bigr).
\end{aligned} \tag{A.12}$$

We present some properties of the Askey-Wilson polynomial p_n [5] and $\tilde{p}_n$:

$$p_n(\eta;a_1,a_2,a_3,a_4|q):\ \text{symmetric in } (a_1,a_2,a_3,a_4), \tag{A.13}$$

$$p_n(\eta;a_1,a_2,a_3,a_4|q^{-1}) = (-1)^n(a_1a_2a_3a_4)^nq^{-\frac{3}{2}n(n-1)}p_n(\eta;a_1^{-1},a_2^{-1},a_3^{-1},a_4^{-1}|q), \tag{A.14}$$

$$p_n(-\eta;a_1,a_2,a_3,a_4|q) = (-1)^np_n(\eta;-a_1,-a_2,-a_3,-a_4|q), \tag{A.15}$$

$$\tilde{p}_n(\eta;a_1,a_2,a_3,a_4|q):\ \text{symmetric under } a_1\leftrightarrow a_2 \text{ or } a_3\leftrightarrow a_4, \tag{A.16}$$

$$\tilde{p}_n(\eta;a_1,a_2,a_3,a_4|q^{-1}) = (-1)^n(a_1a_2a_3a_4)^nq^{-\frac{3}{2}n(n-1)}\tilde{p}_n(\eta;a_3^{-1},a_4^{-1},a_1^{-1},a_2^{-1}|q), \tag{A.17}$$

$$p_n(\eta;a_1,a_2,a_3,a_4|q) = c_n\eta^n + (\text{lower order terms}),\quad c_n = 2^n(a_1a_2a_3a_4q^{n-1};q)_n, \tag{A.18}$$

$$\tilde{p}_n(\eta;a_1,a_2,a_3,a_4|q) = \tilde{c}_n\eta^n + (\text{lower order terms}),\quad \tilde{c}_n = (a_1a_2a_3a_4q^{n-1};q)_n. \tag{A.19}$$

The three term recurrence relations of p_n and $\tilde{p}_n$ are

$$\eta p_n(\eta) = A_np_{n+1}(\eta) + B_np_n(\eta) + C_np_{n-1}(\eta),\quad p_n(\eta) = p_n(\eta;a_1,a_2,a_3,a_4|q),$$

$$A_n=\frac{1-b_4q^{n-1}}{2(1-b_4q^{2n-1})(1-b_4q^{2n})},\quad b_4=a_1a_2a_3a_4, \tag{A.20}$$

$$B_n=\frac{a_1+a_1^{-1}}{2}-\frac{(1-b_4q^{n-1})\prod_{j=2}^4(1-a_1a_jq^n)}{2a_1(1-b_4q^{2n-1})(1-b_4q^{2n})}-\frac{a_1(1-q^n)\prod_{2\le j<k\le 4}(1-a_ja_kq^{n-1})}{2(1-b_4q^{2n-2})(1-b_4q^{2n-1})},$$

$$C_n=\frac{(1-q^n)\prod_{1\le j<k\le 4}(1-a_ja_kq^{n-1})}{2(1-b_4q^{2n-2})(1-b_4q^{2n-1})},$$

$$\eta\tilde{p}_n(\eta)=\tilde{A}_n\tilde{p}_{n+1}(\eta)+\tilde{B}_n\tilde{p}_n(\eta)+\tilde{C}_n\tilde{p}_{n-1}(\eta),\quad \tilde{p}_n(\eta)=\tilde{p}_n(\eta;a_1,a_2,a_3,a_4|q),$$

$$\tilde{A}_n=\frac{1-b_4q^{n-1}}{(1-b_4q^{2n-1})(1-b_4q^{2n})},\quad b_4=a_1a_2a_3a_4, \tag{A.21}$$

$$\tilde{B}_n=a_1^{-1}-\frac{(1-b_4q^{n-1})\prod_{j=3}^4(1-a_1a_jq^n)}{a_1(1-b_4q^{2n-1})(1-b_4q^{2n})}+\frac{a_1a_3a_4q^{n-1}(1-q^n)\prod_{j=3}^4(1-a_2a_jq^{n-1})}{(1-b_4q^{2n-2})(1-b_4q^{2n-1})},$$

$$\tilde{C}_n=\frac{-a_3a_4q^{n-1}(1-q^n)\prod_{j=1}^2\prod_{k=3}^4(1-a_ja_kq^{n-1})}{(1-b_4q^{2n-2})(1-b_4q^{2n-1})}.$$

B Quantum Dilogarithm Function

For readers' convenience, we present some properties of the quantum dilogarithm function (see [10, 11, 20] and references therein), which are needed for calculations in the text. There are several definitions for 'quantum dilogarithm function' in the literature. For example, the Faddeev's quantum dilogarithm given in [20] is defined by ($b\in\mathbb{C}$, $\mathrm{Re}\,b\neq 0$),

$$\Phi_b^{\mathrm{F}}(z)=\exp\Bigl(-\int_{\mathbb{R}+i0}\frac{e^{-2izt}}{4\sinh bt\,\sinh b^{-1}t}\frac{dt}{t}\Bigr)\quad \bigl(|\mathrm{Im}\,z|<\bigl|\tfrac{1}{2}\mathrm{Re}\,(b+b^{-1})\bigr|\bigr). \tag{B.1}$$

By analytic continuation, it is defined on the entire z-plane. Our quantum dilogarithm $\Phi_\gamma(z)$ given below is $\Phi_\gamma(z)=\Phi^{\mathrm{F}}_{\sqrt{\frac{\pi}{\gamma}}}\bigl(\frac{z}{2\sqrt{\pi\gamma}}\bigr)^{-1}$. We assume $\gamma>0$ in this Appendix.

Our definition is the following. The quantum dilogarithm function $\Phi_\gamma(z)$ is a meromorphic function, which is defined for $|\mathrm{Im}\,z|<\gamma+\pi$ by the integral representation (B.2) and analytically continued to the whole complex plane by the functional equation (B.4).

definition:

$$\Phi_\gamma(z)=\exp\Bigl(\int_{\mathbb{R}+i0}\frac{e^{-izt}}{4\sinh\gamma t\,\sinh\pi t}\frac{dt}{t}\Bigr)\qquad (|\mathrm{Im}\,z|<\gamma+\pi). \tag{B.2}$$

By taking the integration contour as $(-\infty,-\rho)+C_\rho+(\rho,\infty)$, where C_ρ is a semicircle with radius ρ ($0<\rho<\min(\frac{\pi}{\gamma},1)$) in the upper half plane, the integral can be evaluated as

$$\int_{\mathbb{R}+i0}\frac{e^{-izt}}{4\sinh\gamma t\,\sinh\pi t}\frac{dt}{t}=\int_\rho^\infty\frac{\sin zt}{2i\sinh\gamma t\,\sinh\pi t}\frac{dt}{t}+\int_0^\pi\frac{e^{-iz\rho e^{i\theta}}}{4i\sinh\gamma\rho e^{i\theta}\,\sinh\pi\rho e^{i\theta}}d\theta, \tag{B.3}$$

which is independent of the choice of ρ.

functional relations:

$$\frac{\Phi_\gamma(z+i\gamma)}{\Phi_\gamma(z-i\gamma)} = \frac{1}{1+e^z}, \tag{B.4}$$

$$\frac{\Phi_\gamma(z+i\pi)}{\Phi_\gamma(z-i\pi)} = \frac{1}{1+e^{\frac{\pi}{\gamma}z}}, \tag{B.5}$$

$$\Phi_\gamma^*(z) = \frac{1}{\Phi_\gamma(z)}, \tag{B.6}$$

$$\Phi_\gamma(z)\Phi_\gamma(-z) = \exp\Big(\frac{i}{4\gamma}\big(z^2 + \frac{\gamma^2+\pi^2}{3}\big)\Big). \tag{B.7}$$

poles and zeros:

$$\text{poles of } \Phi_\gamma(z) : z = i\big((2n_1-1)\gamma + (2n_2-1)\pi\big) \quad (n_1, n_2 \in \mathbb{Z}_{\geq 1}), \tag{B.8}$$

$$\text{zeros of } \Phi_\gamma(z) : z = -i\big((2n_1-1)\gamma + (2n_2-1)\pi\big) \quad (n_1, n_2 \in \mathbb{Z}_{\geq 1}). \tag{B.9}$$

For $\frac{\gamma}{\pi} \notin \mathbb{Q}$, these poles (B.8) and zeros (B.9) are simple.

series expansions: ($|\mathrm{Im}\, z| < \gamma + \pi$)

$$\Phi_\gamma(z) = \begin{cases} \exp\Big(\frac{i}{4\gamma}\Big(z^2 + \frac{\gamma^2+\pi^2}{3}\Big) + i\sum_{n=1}^{\infty}\frac{(-1)^n}{2n}\Big(\frac{e^{-\frac{\pi z}{\gamma}n}}{\sin\frac{\pi^2 n}{\gamma}} + \frac{e^{-zn}}{\sin\gamma n}\Big)\Big) & (\mathrm{Re}\, z > 0) \\ \exp\Big(-i\sum_{n=1}^{\infty}\frac{(-1)^n}{2n}\Big(\frac{e^{\frac{\pi z}{\gamma}n}}{\sin\frac{\pi^2 n}{\gamma}} + \frac{e^{zn}}{\sin\gamma n}\Big)\Big) & (\mathrm{Re}\, z < 0) \\ \lim_{\varepsilon\to+0}\Phi_\gamma(\pm\varepsilon + z) & (\mathrm{Re}\, z = 0) \end{cases}$$

$$\text{for } \frac{\gamma}{\pi} \notin \mathbb{Q}, \tag{B.10}$$

$$\Phi_\gamma(z) = \begin{cases} \exp\Big(\frac{i}{4\gamma}\Big(z^2 + \frac{\gamma^2+\pi^2}{3}\Big) + i\sum_{\substack{n=1 \\ n\not\equiv 0\ (\mathrm{mod}\ M)}}^{\infty}\frac{(-1)^n}{2n}\frac{e^{-\frac{\pi z}{\gamma}n}}{\sin\frac{\pi^2 n}{\gamma}} + i\sum_{\substack{n=1 \\ n\not\equiv 0\ (\mathrm{mod}\ N)}}^{\infty}\frac{(-1)^n}{2n}\frac{e^{-zn}}{\sin\gamma n}\Big) & (\mathrm{Re}\, z > 0) \\ \exp\Big(-i\sum_{\substack{n=1 \\ n\not\equiv 0\ (\mathrm{mod}\ M)}}^{\infty}\frac{(-1)^n}{2n}\frac{e^{\frac{\pi z}{\gamma}n}}{\sin\frac{\pi^2 n}{\gamma}} - i\sum_{\substack{n=1 \\ n\not\equiv 0\ (\mathrm{mod}\ N)}}^{\infty}\frac{(-1)^n}{2n}\frac{e^{zn}}{\sin\gamma n}\Big) & (\mathrm{Re}\, z < 0) \\ \lim_{\varepsilon\to+0}\Phi_\gamma(\pm\varepsilon + z) & (\mathrm{Re}\, z = 0) \end{cases}$$

$$\text{for } \gamma = \frac{M}{N}\pi \text{ (N and M are positive integers and coprime).} \tag{B.11}$$

asymptotic forms: ($|\mathrm{Im}\, z| < \gamma + \pi$)

$$\Phi_\gamma(z) \simeq \begin{cases} \exp\Big(\frac{i}{4\gamma}\big(z^2 + \frac{\gamma^2+\pi^2}{3}\big)\Big) & (\mathrm{Re}\, z \to \infty) \\ 1 & (\mathrm{Re}\, z \to -\infty) \end{cases}. \tag{B.12}$$

$\gamma \to 0$ limit:

$$\Phi_\gamma(z) = \exp\Bigl(\frac{1}{2i\gamma}\mathrm{Li}_2(-e^z) + \mathrm{O}(\gamma)\Bigr), \tag{B.13}$$

where the dilogarithm function $\mathrm{Li}_2(z)$ is defined by $\mathrm{Li}_2(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^2}$ ($|z| < 1$) and analytic continuation.

For $\Phi_b^{\mathrm{F}}(z)$ in (B.1), we have
infinite product form of $\Phi_b^{\mathrm{F}}(z)$: ($\mathrm{Im}\, b^2 > 0$)

$$\Phi_b^{\mathrm{F}}(z) = \frac{(-e^{2\pi b^{-1}z - \pi i b^{-2}}; e^{-2\pi i b^{-2}})_\infty}{(-e^{2\pi bz + \pi i b^2}; e^{2\pi i b^2})_\infty}. \tag{B.14}$$

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