SYMBOLIC EVALUATIONS INSPIRED BY RAMANUJAN'S SERIES FOR $\frac{1}{\pi}$

JOHN MAXWELL CAMPBELL

A DISSERTATION SUBMITTED TO

THE FACULTY OF GRADUATE STUDIES

IN PARTIAL FULFILMENT OF THE REQUIREMENTS

FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

GRADUATE PROGRAM IN MATHEMATICS AND STATISTICS

YORK UNIVERSITY

TORONTO, ONTARIO

April 2022

Abstract

We introduce classes of Ramanujan-like series for $\frac{1}{\pi}$, by devising methods for evaluating harmonic sums involving squared central binomial coefficients, as in the family of Ramanujan-type series indicated below, letting $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ denote the n^{th} harmonic number:

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2 H_n}{16^n(2n-1)} &= \frac{8\ln(2)-4}{\pi}, \\
\sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2 H_n}{16^n(2n-3)} &= \frac{120\ln(2)-68}{27\pi}, \\
\sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2 H_n}{16^n(2n-5)} &= \frac{10680\ln(2)-6508}{3375\pi}, \\
\ldots
\end{aligned}$$

In this direction, our main technique is based on the evaluation of a parameter derivative of a beta-type integral, but we also show how new integration results involving complete elliptic integrals may be used to evaluate Ramanujan-like series for $\frac{1}{\pi}$ containing harmonic numbers.

We present a generalization of the recently discovered harmonic summation formula

$$\sum_{n=1}^{\infty} \binom{2n}{n}^2 \frac{H_n}{32^n} = \frac{\Gamma^2\left(\frac{1}{4}\right)}{4\sqrt{\pi}} \left(1 - \frac{4\ln(2)}{\pi}\right)$$

through creative applications of an integration method that we had previously introduced. We provide explicit closed-form expressions for natural variants of the above series. At the time of our research being conducted, up-to-date versions of Computer Algebra Systems such as Mathematica and Maple could not evaluate our introduced series, such as

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2 H_n}{32^n(n+1)} = 8 - \frac{2\Gamma^2\left(\frac{1}{4}\right)}{\pi^{3/2}} - \frac{4\pi^{3/2} + 16\sqrt{\pi}\ln(2)}{\Gamma^2\left(\frac{1}{4}\right)}.$$

We also introduce a class of harmonic summations for Catalan's constant G and $\frac{1}{\pi}$ such as the series

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2 H_n}{16^n(n+1)^2} = 16 + \frac{32G - 64\ln(2)}{\pi} - 16\ln(2),$$

which we prove through a variation of our previous integration method for constructing $\frac{1}{\pi}$ series.

We also present a new integration method for evaluating infinite series involving alternating harmonic numbers, and we apply a Fourier–Legendre-based technique recently introduced by Campbell et al., to prove new rational double hypergeometric series formulas for expressions involving $\frac{1}{\pi^2}$, especially the constant $\frac{\zeta(3)}{\pi^2}$, which is of number-theoretic interest.

Acknowledgements

I am very grateful to my supervisor, Mike Zabrocki, for so much about the work that I have done under his supervision. It is a tremendous privilege to have worked under the supervision of Dr. Zabrocki as both a Ph.D. candidate and as an undergraduate researcher. The mathematical work of Mike Zabrocki will continue to inspire me, and has had a great and positive impact on my career and on my research interests.

My devotion to honouring my mother and my father, Christine and Terry Campbell, is reflected in how diligently and conscientiously I have worked in terms of my research, my education, and my career. The insights that my parents have provided to me throughout my life will continue to be of great help to me. Through my Christian faith, I seek guidance in my honouring the legacy of my family name, and I have been blessed with the gracious gifts given by the wisdom and the support from my parents and my grandparents, Beryl and Harold Dyer and Shirley and Arthur Campbell.

I am thankful to Nantel Bergeron, Yun Gao, and Alexey Kuznetsov for their responsibilities as members of my supervisory committee. Dr. Bergeron, Dr. Kuznetsov, and Mike have all expressed themselves to me in regard to my mathematical research and my Ph.D. work in ways that are deeply inspiring to me. It should also be acknowledged that Ada Chan had previously been involved in my Ph.D. supervision.

I want to thank many mathematicians with whom, over the years and on many projects, I have collaborated, including: Marco Cantarini, Kwang-Wu Chen, Wenchang Chu, Jacopo D'Aurizio, Paul Levrie, and Jonathan Sondow. It is such a privilege and an honour to have worked with you all.

I am thankful to the external examiner corresponding to my Thesis, Rob Corless, and to the internal-external member for my committee, Tom Kirchner. It was a pleasure to be in contact with you both, and I hope to be in further contact with Dr. Corless in relation to our many similar interests in and pertaining to the field of computer algebra.

I am very grateful to Jeffrey Reid Pettis, one of my best friends. Your successes as and exceptional talents as an educator continue to inspire me, and there is so much about my personal development and social development over the years and decades that is directly due to our friendship. Your unique literary insights have fueled so many of our discussions over the years, and strengthen, on so many levels, the intellectual development of your students. There is so much that I value about your deep good-heartedness and your remarkable senses of compassion and empathy. Veronica Lynne Ramshaw has also been a great friend to me in my Ph.D. years, and this has been a source of such positivity for me.

I am thankful to so many of my colleagues and former classmates at York. Ben Fraser has been such a great friend during our Ph.D. years at York! Thanks to Albi Kazazi, Marco Tosato, and many others for all the good times we have had at York.

Statements of Originality and Authorship Sections 2, 3, 4, and 6 are, respectively, near verbatim copies of the following single-authored published articles that I had authored, with the possibility of minor modifications to some of the wording, layout, etc.: [19], [18], [20], and [17]; all of these single-authored publications had been completed and published while I was accepted/enrolled in my PhD program at York University, and the articles [17], [18], [19], and [20] have not been included as part of any previous theses/dissertations. The above Abstract is based on an amalgamation of the Abstracts for [17], [18], [19], and [20].

Extent of Collaboration/Authorship The main Sections in this Thesis are Sections 2, 3, 4, and 6, which, as stated above, are near verbatim copies of single-authored articles I had written. Section 5 below is a near verbatim copy of introductory/preliminary sections of the article [31] that I had authored with Jacopo D'Aurizio and Jonathan Sondow; the main applications of the techniques of [31] given in the coauthored article [31] are also summarized in Section 5. While Section 5 mainly consists of exposition and preliminary material, it should be noted that Jacopo D'Aurizio is the primary author of the material in the subsections given as Section 5.2 and Section 5.3. A partial list of publications that I have authored/coauthored and that have been completed and accepted while I have been enrolled as a Ph.D. student at York University is given below.

J. M. Campbell, Sums of Fibonacci Numbers Indexed by Integer Parts, to appear in *Fibonacci Quart.*

J. M. Campbell, Special values of Legendre's chi-function and the inverse tangent integral, to appear in *Irish Math. Soc. Bull.*

J. M. Campbell, Products of multiple-index Fibonacci numbers, to appear in *Fibonacci Quart.*

J. M. Campbell, A matrix-based recursion relation for F_{F_n}, to appear in *Fibonacci Quart.*

J. M. Campbell, WZ proofs of identities from Chu and Kılıç, with applications, to appear in *Appl. Math. E-Notes.*

J. M. Campbell and M. Cantarini, A series evaluation technique based on a modified Abel lemma, *Turkish J. Math.* **46** no. 4 (2022), Article 30.

J. M. Campbell, Solution to a problem due to Chu and Kiliç, *Integers* **22** (2022), #A46.

J. M. Campbell and W. Chu, Double series transforms derived from Fourier-Legendre theory, *Commun. Korean Math. Soc.* **37** no. 2 (2022), 551–566.

J. M. Campbell and K.-W. Chen, Explicit identities for infinite families of series involving squared binomial coefficients, *J. Math. Anal. Appl.* **513** no. 2 (2022), 126219.

J. M. Campbell, M. Cantarini, and J. D'Aurizio, Symbolic computations via Fourier–Legendre expansions and fractional operators, *Integral Transforms Spec. Funct.* **33** no. 2 (2022), 157–175.

J. M. Campbell, Some nontrivial two-term dilogarithm identities, *Irish Math. Soc. Bull.* no. 88 (2021), 31–37.

J. M. Campbell, Combinatorial interpretations of primitivity in the algebra of symmetric functions, *Sarajevo J. Math.* **17** no. 2 (2021), 151–165.

J. M. Campbell, A WZ proof for a Ramanujan-like series involving cubed binomial coefficients, *J. Difference Equ. Appl.* **27** no. 10 (2021), 1507–1511.

J. M. Campbell, WZ proofs for lemniscate-like constant evaluations, *Integers* **21** (2021), #A107.

J. M. Campbell and K.-W. Chen, An integration technique for evaluating quadratic harmonic sums, *Aust. J. Math. Anal. Appl.* **18** no. 2 (2021), Art. 15.

J. M. Campbell and W. Chu, Lemniscate-like constants and infinite series, *Math. Slovaca.* **71** no. 4 (2021), 845–848.

W. Chu and J. M. Campbell, Harmonic sums from the Kummer theorem, *J. Math. Anal. Appl.* **501** no. 2 (2021), 125179.

J. M. Campbell, New families of double hypergeometric series for constants involving $1/\pi^2$, *Ann. Polon. Math.* **125** no. 1 (2021), 1–20.

J. M. Campbell, On the visualization of large-order graph distance matrices, *J. Math. Arts* **14** no. 4 (2020), 297–330.

J. M. Campbell, J. D'Aurizio, and J. Sondow, Hypergeometry of the parbelos, *Amer. Math. Monthly* **127** no. 1 (2020), 23–32.

J. M. Campbell, New series involving harmonic numbers and squared central binomial coefficients, *Rocky Mountain J. Math.* **49** no. 8 (2019), 2513–2544.

J. M. Campbell, J. D'Aurizio, and J. Sondow, On the interplay among hypergeometric functions, complete elliptic integrals, and Fourier–Legendre expansions, *J. Math. Anal. Appl.* **479** no. 1 (2019), 90–121.

J. M. Campbell, Series containing squared central binomial coefficients and alternating harmonic numbers, *Mediterr. J. Math.* **16** no. 2 (2019), Article:37.

J. M. Campbell, Ramanujan-like series for $\frac{1}{\pi}$ involving harmonic numbers, *Ramanujan J.* **46** no. 2 (2018), 373–387.

J. M. Campbell, Visualizing large-order groups with computer-generated Cayley tables, *J. Math. Arts* **11** no. 2 (2017), 67–99.

• The above article received the Outstanding Paper Prize in 2016–2017 for The Journal of Mathematics and the Arts.

• A special Announcement on the Outstanding Paper Prize for the above article was printed in The Journal of Mathematics and the Arts, in which it is noted that the final winner out of the Associate Editors' short list was selected by a jury consisting of Ingrid Daubechies, Lynn Gamwell, Douglas Norton, and Laura Taalman. A news item on this prestigious award titled *PhD student in Math receives Outstanding Paper Award from journal* was released via York University's Faculty of Science.

• This article is included in the list of Notable Writings in the book *The Best Writing on Mathematics 2018* from Princeton University Press.

J. M. Campbell and A. Sofo, An integral transform related to series involving alternating harmonic numbers, *Integr. Transf. Spec. F.* **28** no. 7 (2017), 547–559.

J. Campbell, A class of symmetric difference-closed sets related to commuting involutions, *Discrete Math. Theor. Comput. Sci.* **19** no. 1 (2017), dmtcs:3210.

The following research item was also published during my enrolment as a PhD student:

J. M. Campbell, Bipieri tableaux, *Australas. J. Combin.* **66** (2016), 66–103.

Contents

List of Figures

Chapter 1

Introduction

This Thesis is primarily concerned with the study of closed-form expressions from a perspective grounded in special functions theory, and with a particular emphasis being placed on hypergeometric functions and their generalizations. The development of applications of identities involving special functions so as to obtain simplified or explicit or more efficient ways of expressing such functions may be considered as a core part of the study of special functions, and this Thesis is firmly rooted in this area.

Throughout the history of mathematical analysis, a natural problem has arisen continually and ubiquitously: The problem of determining a meaningful evaluation for a given mathematical object or construction that involves a limiting process or operation, as in with definite integrals, infinite series, infinite products, continued fractions, applications of differential operators, etc. While there is some degree of subjectivity as to what should be meant by the expression *closed-form evaluation* [14], this does not take away from how important the aforementioned problem is within mathematical analysis and related areas.

As an example of an especially important problem in the history of mathematics given by finding a closed form for an infinite series, we refer to the *Basel problem*. This refers to the problem of evaluating the infinite series $1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$. Leonhard Euler famously solved the Basel problem in 1734, providing the closed-form evaluation $\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}$. The fact that to this day, in the year 2022, there is no known closed form for the corresponding cubic sum $1 + \frac{1}{2^3} + \frac{1}{3^3} + \cdots$ is representative of the extreme depth about the subject of the evaluation of infinite series. In this Thesis, we mainly focus on introducing techniques for determining closed forms for infinite series and definite integrals, based on our articles in [17, 18, 19, 20].

There are many sources of motivation concerning the study of closed forms for mathematical objects such as infinite series and definite integrals [14]. In this direction, we refer the interested reader to Borwein and Crandall's text *Closed forms: what they are and why we care* [14], and, in this regard, we briefly review some relevant points.

To begin with, given a series or integral that arises in some context or application, symbolically evaluating such an expression in a simplified, meaningful way could give insight into further applications, or connections to other areas in mathematics, and could be of benefit in terms of numerical computation:

> "As mathematical discovery more and more involves extensive computation, the premium on having a closed form increases. The insight provided by discovering a closed form ideally comes at the top of the list, but efficiency of computation will run a good second." [14, p. 55]

Much of the field of computer algebra is devoted to the development of algorithms for computing symbolic forms for an inputted series, integral, etc., with many commercial and industrial applications:

> "It is now the case that much mathematical computation is *hybrid*: mixing numeric and symbolic computation. Indeed, which is which may not be clear to the user if, say, numeric techniques have been used to return a symbolic answer or if a symbolic closed form has been used to make possible a numerical integration. Moving from classical to modern physics, both understanding and effectiveness frequently demand hybrid computation." [14, p. 56]

Another source of motivation behind the study of the application of infinite sum/integral evaluations is in the hope of determining number-theoretic properties about mathematical constants, as in with the irrationality measure or the transcendence of a given constant defined by a series/integral [3, 26]. For example, the problem of determining whether or not Catalan's constant $G = 1 - \frac{1}{3^2} + \frac{1}{5^2} - \cdots$ is irrational is widely considered to be a major unsolved problem in mathematics. The foregoing considerations provide much in the way of inspiration behind the infinite series evaluations that we have introduced and that are given in this Thesis.

As a representative example of a symbolic evaluation for an infinite series given in this Thesis, we highlight the formula

$$\sum_{n=1}^{\infty} \tilde{C}_n^2 H_n = \frac{32G - 64\ln 2}{\pi} - 16\ln 2 + 16 \tag{1.1}$$

that we had proved in [18], letting $\left(\tilde{C}_n : n \in \mathbb{N}\right)$ denote the sequence of normalized Catalan numbers, and writing $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ to denote the n^{th} harmonic number. The summands of the series examined in this Thesis are given by naturally occurring sequences, as in (1.1). At the time of our research being conducted, up-to-date versions of Computer Algebra Systems such as Mathematica and Maple could not evaluate our introduced series/integrals.

This Thesis is mainly devoted to the development of techniques for evaluating series involving harmonic or harmonic-type numbers that may be written as double hypergeometric series and as equivalent integral expressions involving special functions such as the complete elliptic integrals. There is much motivation regarding research based on harmonic sums. This topic dates back to Euler's seminal work on sums of this form. As stated in [73], Euler, after his renowned solution to the Basel problem, introduced closed forms for harmonic sums as in $\sum_{n=1}^{\infty} \frac{H_n^{(m)}}{n^q}$, which are referred to as Euler sums, writing $H_n^{(m)} = 1 + \frac{1}{2^m} + \cdots + \frac{1}{n^m}$ to denote generalized harmonic numbers. The evaluation of Euler-type sums, as in with binomial generalizations and the like of the classical Euler sums, is of importance in special functions theory, number theory, and physics; see [73] and the references therein.

Originally, our methods relied on integration techniques based on variants and generalizations of the special function known as the beta function [18, 19, 20], as we explore in Sections 2–4 below. However, through our joint work with Jacopo D'Aurizio and Jonathan Sondow [30, 31], it was shown how the closed forms put forth in [19, 20] are closely related to Fourier–Legendre (FL) theory, with the FL-based techniques introduced in [31] having been employed in [31] (cf. [33]) in the evaluation of binomial-harmonic series inspired by Ramanujan's famous series for $\frac{1}{\pi}$.

The Legendre polynomials, as introduced by Adrien-Marie Legendre in 1782, form one of the most important families of orthogonal polynomials, and arise very often in the fields of special functions, numerical analysis, approximation theory, number theory, and Fourier analysis, and in many other areas within mathematics, not to mention the prominent role that Legendre polynomials play in physics. If a function f over $[-1, 1]$ may be written as

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x),$$

letting $P_n(x)$ denote the n^{th} Legendre polynomial, then the above expansion is referred to as a *Fourier–Legendre expansion* or a *Fourier–Legendre series*, providing a standard generalization as to what is classically meant by a Fourier series.

There are many basic properties of Legendre polynomials that naturally lend themselves to the study of series involving Pochhammer symbols, binomial coefficients, etc., making note, in particular, of the identity whereby $P_{2n}(0) = \left(-\frac{1}{4}\right)^n \binom{2n}{n}$ for all natural numbers n, together with the generating function identity

$$\frac{1}{\sqrt{1 - 2xz + z^2}} = \sum_{n=0}^{\infty} P_n(x) z^n.$$

Both of these identities were employed by Bauer in 1859 [6] to prove the formula

$$\frac{2}{\pi} = \sum_{n=0}^{\infty} \left(-\frac{1}{64}\right)^n \binom{2n}{n}^3 (4n+1) \tag{1.2}$$

which was famously rediscovered by Ramanujan, as recorded in his second notebook [61].

In this Thesis, we provide techniques for determining evaluations in terms of $\frac{1}{\pi}$ for rational series that involve powers of central binomial coefficients, as in the famous formulas due to Ramanujan that are shown below and in (1.2) [70] (cf. [9, p. 352–364]):

$$\begin{aligned} \frac{4}{\pi} &= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{8n} \binom{2n}{n}^3 (6n+1), \\ \frac{16}{\pi} &= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{12n} \binom{2n}{n}^3 (42n+5). \end{aligned} \tag{1.3}$$

A central object of study in this Thesis is given by new infinite series that involve both squared central binomial coefficients and finite sums as summand factors.

As in [88], we state that much of the interest in the study of series that contain harmonic-type numbers and central binomial coefficients is due to the role that such series play in many different subjects, including the "analysis of algorithms, combinatorics, number theory and elementary particle physics" [88, p. 1]. The number-theoretic significance of terminating/non-terminating sums involving harmonic-type numbers and binomial expressions, as in with the work of Roberto Tauraso et al. in this area [65, 79], motivates the pursuit of the study of such sums in its own right; see also [64].

1.1 Organization of the Thesis

After reviewing background/preliminary material in Sections 1.2–1.4, we proceed to reproduce our published work on *Ramanujan-like series for* $\frac{1}{\pi}$ *involving harmonic numbers* [19] in Section 2, which largely concerns the application of an evaluation method based on integral expressions of the form

$$\int_0^1 \left(\sum_{n=0}^{\infty} (-1)^n \binom{\frac{1}{2}}{n} f(n) \frac{x^{2n} \ln\left(1-x^2\right)}{\sqrt{1-x^2}} \right) dx$$

so as to produce remarkable [87] and Ramanujan-like [19, 87] series for expressions involving $\frac{1}{\pi}$, as in the formula

$$\frac{8\ln(2)-4}{\pi} = \sum_{n=1}^{\infty} \left(\frac{1}{16}\right)^n \binom{2n}{n}^2 \frac{H_n}{2n-1}$$

that we had introduced in [19] (see Remark 2.2.2 below), noting the resemblance to (1.2). Series evaluation methods based on manipulations of integrals involving the complete elliptic integrals are also a subject of focus in Section 2. Afterwards, in the following Section, we reproduce our article [18] on further applications of the techniques from [19] for evaluating series involving products of the form $\binom{2n}{n}^2 H_n$. The concepts from [19] are also generalized in Section 4, in which we reproduce our publication on *Series containing squared central binomial coefficients and alternating harmonic numbers* [20], which introduces a variant of the beta integration-based Lemma from [19] so as to evaluate binomial-harmonic series involving expressions of the form H_{2n} for $n \in \mathbb{N}$, as in the following evaluation:

$$\frac{16G+24-48\ln(2)}{\pi} + 4 - 8\ln(2) = \sum_{n=1}^{\infty} \left(\frac{1}{16}\right)^n \binom{2n}{n}^2 \frac{H_{2n}}{(n+1)^2}.$$

Then, in Section 5, we provide preliminary material on FL theory, reproducing introductory material from our joint works in [30] and [31], and we briefly review the main results from [31]. We then, in Section 6, reproduce our publication [17] on the application of the methods from [31] in the evaluation of double hypergeometric series for $\frac{1}{\pi^2}$.

1.2 Pi and the AGM

This Thesis is focused on the problem of evaluating series involving

$$\left(\frac{1}{4^n}\binom{2n}{n}\right)^2 \Delta(n)$$

in terms of $\frac{1}{\pi}$ or $\frac{1}{\pi^2}$ or special values of the complete elliptic integrals, letting $\Delta(n)$ denote a harmonic-type number or Ramanujan's S-function or related series (see Section 6). Three standard references on Ramanujan's series for the reciprocal of π are given by: Berndt's text [9], the survey paper [5] (cf. [11]), and the classic text *Pi and the AGM* [10]; see [92], as well. In this section, we focus on background material that is covered in *Pi and the AGM* [10] that directly pertains to the main results given in this dissertation.

It is not an exaggeration to state that the history of mathematics is permeated with applications concerning identities and computations involving π. In this regard, naturally occurring phenomena involving limiting operations often can be shown to result in expressions involving π. An especially historically significant instance of this is given by the above referred solution due to Euler of the famous Basel problem. Quite similarly, the rational series for $\frac{1}{\pi}$ introduced by Ramanujan are very significant in the history of research on the closed-form evaluation of infinite series.

The complete elliptic integrals of the first and second kinds are to play an important role throughout much of this Thesis, especially in Sections 2, 5, and 6 below. This leads us to review material from [10, §1] that relates to our uses of the special functions **K** and **E**, as defined below, in the evaluation of infinite series, and that provides a historical context pertaining to our applications of these special functions. We adopt a standard definition of complete elliptic integrals, in accordance with the definitions provided in the Digital Library of Mathematical Functions[1]:

$$\mathbf{K}(k) := \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}, \tag{1.4}$$

$$\mathbf{E}(k) := \int_0^{\pi/2} \sqrt{1-k^2\sin^2\theta}\, d\theta. \tag{1.5}$$

Gauss' *arithmetic-geometric mean* (AGM) iteration is a real-valued function of two positive real variables, defined as follows, and studied by Gauss toward the end of the eighteenth century. Following [10, §1.1], by setting $a_{n+1} := \frac{a_n+b_n}{2}$ and $b_{n+1} := \sqrt{a_n b_n}$, from the famous arithmetic-geometric mean inequality, this gives us that $b_n \geq b_{n+1} \geq a_{n+1} \geq a_n$. We have that

$$a_{n+1} - b_{n+1} = \frac{a_n - 2\sqrt{a_n b_n} + b_n}{2}.$$

This can be shown to give us that $a_{n+1} - b_{n+1} < \frac{1}{2}(a_n - b_n)$ [89]. So, we may set $M(a,b) := \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$, letting $a_0 := a$ and $b_0 := b$ [89]. This gives us a continuous function, since we have that

$$M(a,b) = \frac{(a+b)\pi}{4\mathbf{K}\left(\frac{a-b}{a+b}\right)}; \tag{1.6}$$

see [89].

The AGM iteration is described in [10, p. 1] as 'One of the jewels of classical analysis', and since this iterative procedure is intimately connected with so much about the history of the complete elliptic integral functions and to basic properties about **K** and **E** that are of relevance to our dissertation, it is worthwhile to briefly review the AGM iteration. One of the main reasons as to why this AGM procedure and this

[1] See `https://dlmf.nist.gov/19`.

M-function are of such interest in the field of mathematical analysis comes from the functional equation whereby

$$M(a,b) = M\left(\frac{a+b}{2}, \sqrt{ab}\right). \tag{1.7}$$

Again, following [10, §1.1], we have that

$$\lambda\, M(a,b) = M(\lambda a, \lambda b). \tag{1.8}$$

If we set $a = 1$ in (1.7), this gives us that

$$M(1,b) = M\left(\frac{1+b}{2}, \sqrt{b}\right),$$

which is equivalent to

$$M\,(1,b) = \frac{1+b}{2} M\left(1, \frac{2\sqrt{b}}{1+b}\right).$$

So, again following [10, §1.1], the evaluation of M, in general, reduces to the evaluation of $M(1,b)$, in view of (1.8). From the functional equation whereby

$$f(x) = \frac{1+x}{2} f\left(\frac{2\sqrt{x}}{1+x}\right), \tag{1.9}$$

this can be used to prove the identity involving $\mathbf{K}$ in (1.6); one of the most basic functional relations involving the complete elliptic integral $\mathbf{K}$ is as follows [34, 90]:

$$\mathbf{K}(k) = \frac{1}{1+k}\, \mathbf{K}\left(\frac{2\sqrt{k}}{1+k}\right).$$

Gauss proved the following limiting formula for the AGM:

$$\frac{1}{M(1,x)} = \frac{2}{\pi} \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-(1-x^2)\sin^2\theta}}. \tag{1.10}$$

We remark that the equality

$$\frac{1}{M(1,\sqrt{2})} = \frac{2}{\pi} \int_0^1 \frac{dt}{\sqrt{1-t^4}} \tag{1.11}$$

has motivated much of our recent research [29] based on the results introduced in Section 2. Gauss' proof involves the series expansion

$$\frac{2}{\pi}\int_0^{\pi/2}\frac{d\theta}{\sqrt{1-x^2\sin^2\theta}} = 1+\sum_{i=1}^{\infty}\left(\frac{(2i-1)!}{i!(i-1)!}\right)^2\frac{x^{2i}}{4^{2i-1}}, \tag{1.12}$$

as given by expanding the above integrand as a Maclaurin series and then reversing the order of integration and infinite summation [10, §1.2]. Sections 2–4 and Section 6 below are mainly based on series involving squared, normalized central binomial coefficients as summand factors, and reversing integration and infinite summation also arises very frequently therein.

Elliptic function theory played an important role in nineteenth-century mathematics [10, p. 7] [60]. Some of our main contributions in this Thesis concern developments in the study and application of complete elliptic integrals. The study of elliptic integrals as hypergeometric functions actually forms a prominent area in the study of hypergeometric functions in general, in large part because series expansions as in (1.14) and (1.15) often have applications in physics and engineering, and often arise in many different areas in mathematics.

Many of our results are based in the discipline within special functions theory given by elliptic-type functions as hypergeometric functions. So, as in [10, §1.3], we express the functions in (1.4) and (1.5) with the Maclaurin series for these functions, which form an important subclass of Gaussian hypergeometric series that we use frequently.

We define and denote the Pochhammer symbol so that $(x)_0 = 1$ and $(x)_n = x(x+1)\cdots(x+n-1)$, for $n \in \mathbb{N}$. Generalized hypergeometric series are defined and denoted as below:

$${}_pF_q\left[\begin{matrix} a_1, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_q \end{matrix} \;\middle|\; z\right] = \sum_{n=0}^{\infty}\frac{(a_1)_n(a_2)_n\cdots(a_p)_n}{(b_1)_n(b_2)_n\cdots(b_q)_n}\frac{z^n}{n!}. \tag{1.13}$$

The Maclaurin series expansions for the functions $\mathbf{K}$ and $\mathbf{E}$ defined by the integrals in (1.4) and (1.5) are as below [10, pp. 8–10]:

$$\mathbf{K}(k) = \frac{\pi}{2}\cdot {}_2F_1\left[\begin{matrix} \frac{1}{2}, \frac{1}{2} \\ 1 \end{matrix} \;\middle|\; k^2\right], \tag{1.14}$$

$$\mathbf{E}(k) = \frac{\pi}{2}\cdot {}_2F_1\left[\begin{matrix} \frac{1}{2}, -\frac{1}{2} \\ 1 \end{matrix} \;\middle|\; k^2\right]. \tag{1.15}$$

These power series expansions are very important in this Thesis, for the reasons indicated below.

According to the definition of the Pochhammer symbol, we may rewrite the above Maclaurin series expansions so that

$$\mathbf{K}(k) = \frac{\pi}{2} \sum_{n=0}^{\infty} \left(\frac{1}{16} \right)^n \binom{2n}{n}^2 k^{2n}$$

and

$$\mathbf{E}(k) = -\frac{\pi}{2} \sum_{n=0}^{\infty} \left(\frac{1}{16} \right)^n \binom{2n}{n}^2 \frac{k^{2n}}{2n-1}. \tag{1.16}$$

A main part of Sections 2–4 and Section 6 concerns the development of identities for series involving

$$\left(\frac{1}{16} \right)^n \binom{2n}{n}^2$$

as a summand factor for $n \in \mathbb{N}_0$, writing $\mathbb{N}_0$ in place of $\mathbb{N} \cup \{0\}$. As below, we review the evaluation of $\mathbf{K}\left(\frac{1}{\sqrt{2}}\right)$ and $\mathbf{E}\left(\frac{1}{\sqrt{2}}\right)$.

We are to frequently apply the special functions known as the *gamma function* and the *beta function*, as defined below. This is mainly because: (1) Some of the our main integration methods rely on beta-type integration identities, as in Lemmas 2.2.1 and 4.2.1; (2) We often rely on identities involving harmonic numbers derived from the Γ-function; and (3) Many of the explicit series evaluations given in Section 3 are given in terms of special values of the Γ-function.

We let ψ denote the special function known as the digamma function [69, §9]. This function is defined as follows:

$$\psi(z) = \frac{d}{dz} \ln \Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)}.$$

Let $\gamma = \lim_{n \to \infty} \big(H_n - \ln n \big)$ denote the Euler-Mascheroni constant. The ψ-function admits the following infinite series expansion:

$$\psi(z) = -\gamma + \sum_{n=0}^{\infty} \frac{z-1}{(n+1)(n+z)}.$$

Let $\Re(c)$ denote the real part of a complex number c. The Γ-function is defined by the Euler integral

$$\Gamma(x) := \int_0^{\infty} e^{-t} t^{x-1} \, dt$$

for $\Re(x) > 0$, and generalizes the factorial function, with $\Gamma(n+1) = n!$ for $n \in \mathbb{N}_0$. The beta function is defined so that

$$\beta(x, y) := \int_0^1 t^{x-1} (1-t)^{y-1} \, dt \tag{1.17}$$

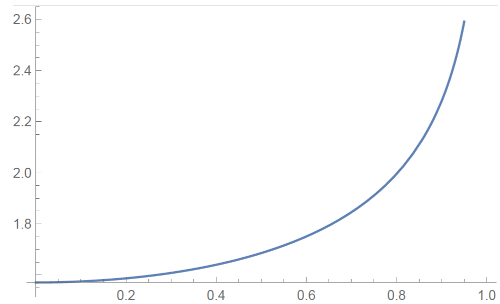

Figure 1.1: A graph of $\mathbf{K}$ on the domain $[0, 1)$ generated by Mathematica.

for $\Re(x), \Re(y) > 0$, and admits the following evaluation:

$$\beta(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)}.$$

As indicated above, some of the main integration identities that we apply are based on generalizations and variants of this special function.

We record the *reflection formula*

$$\Gamma(x)\Gamma(1 - x) = \frac{\pi}{\sin(\pi x)}. \tag{1.18}$$

This also gives us the value $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, which we often make use of, as in Section 2.5, for example. More generally, we frequently make use of half-integer values of the Γ-function, since the Legendre duplication formula [44]

$$\frac{\Gamma(z)\Gamma(z)}{\Gamma(2z)} = 2^{1-2z} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(z)}{\Gamma\left(z + \frac{1}{2}\right)} \tag{1.19}$$

allows us to express $\Gamma\left(z + \frac{1}{2}\right)$ in terms of central binomial coefficients.

We later apply the symbolic evaluations that are highlighted as Theorem 1.9 in *Pi and the AGM* [10]. Apart from the values for $\mathbf{K}\left(\frac{1}{\sqrt{2}}\right)$ and $\mathbf{E}\left(\frac{1}{\sqrt{2}}\right)$ highlighted below, arguments such that $\mathbf{K}$ and $\mathbf{E}$ admit explicit symbolic forms are given by the *elliptic lambda function* [10, §4]. If $\mathbf{K}(\lambda)$ admits an explicit evaluation, then the same must hold for $\mathbf{E}(\lambda)$, and vice-versa; see [10, §5]. Graphs for $\mathbf{K}$ and $\mathbf{E}$ on the domain $[0, 1)$ are given in Figures 1.1 and 1.2.

We briefly review the proof that

$$\mathbf{K}\left(\frac{1}{\sqrt{2}}\right) = \frac{\Gamma^2\left(\frac{1}{4}\right)}{4\sqrt{\pi}} \tag{1.20}$$

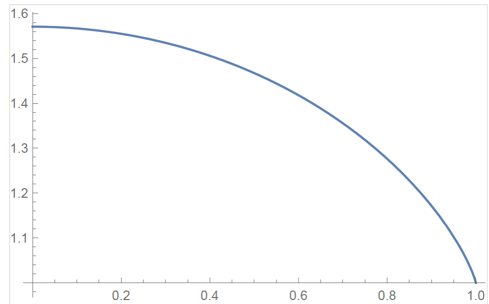

Figure 1.2: A graph of $\mathbf{E}$ on the domain $[0,1]$ generated by Mathematica.

and that

$$\mathbf{E}\left(\frac{1}{\sqrt{2}}\right)=\frac{4\Gamma^2\left(\frac{3}{4}\right)+\Gamma^2\left(\frac{1}{4}\right)}{8\sqrt{\pi}}$$

given as the proof of Theorem 1.7 in [10, p. 25]. By definition of $\mathbf{K}$, we have that:

$$\begin{aligned}\mathbf{K}\left(\frac{1}{\sqrt{2}}\right)&=\int_0^1\frac{dt}{\sqrt{(1-t^2)\left(1-\frac{t^2}{2}\right)}}\\&=\sqrt{2}\int_0^1\frac{dt}{\sqrt{(1-t^2)(2-t^2)}}.\end{aligned}$$

Enforcing the substitution $x^2:=\frac{t^2}{2-t^2}$ so that

$$\mathbf{K}\left(\frac{1}{\sqrt{2}}\right)=\sqrt{2}\int_0^1\frac{dx}{\sqrt{1-x^4}},$$

we again encounter the integral shown in (1.11). Setting $u:=t^4$, we obtain that

$$\mathbf{K}\left(\frac{1}{\sqrt{2}}\right)=\frac{\sqrt{2}}{4}\beta\left(\frac{1}{4},\frac{1}{2}\right).$$

By the reflection formula in (1.18), we have that $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$ and $\Gamma\left(\frac{3}{4}\right)=\frac{\sqrt{2}\pi}{\Gamma\left(\frac{1}{4}\right)}$, giving us the desired result. The proof for $\mathbf{E}\left(\frac{1}{\sqrt{2}}\right)$ is similar, if we again use the substitution $x^2:=\frac{t^2}{2-t^2}$ [10, p. 25].

The series highlighted in (1.2) above due to Ramanujan is proved in [10] as a special case of an identity involving the elliptic alpha function [10, §5]. As we had previously indicated, the formula in (1.2) was proved by Bauer in 1859 using a Fourier–Legendre expansion, and this classical proof is closely related to much of the material in this Thesis on FL theory.

In Section 6, we extend the main techniques in [31] so as to be applicable to double series for constants containing expressions as in $\frac{1}{\pi}$ or $\frac{1}{\pi^2}$. The techniques in Sections 5 and 6 rely heavily on the use of integrals involving $\mathbf{K}$ or $\mathbf{E}$. A number of integrals of this form are involved in *Pi and the AGM* [10], as in the identity [10, p. 188]

$$\int_0^1 \frac{\mathbf{K}(k)}{\sqrt{1-k^2}}\, dx = \mathbf{K}^2\left(\frac{1}{\sqrt{2}}\right).$$

As stated above, the explicit evaluation in (1.20) is of interest in terms of the results introduced in Section 3; see (3.23) and (3.24) below, and the surrounding material.

Let the *generalized complete elliptic integrals of the first kind* be defined so that

$$\mathbf{K}_s(k) = \frac{\pi}{2}\, {}_2F_1\left[\begin{matrix} \frac{1}{2}-s, \frac{1}{2}+s \\ 1 \end{matrix} \middle| \, k^2\right],$$

and similarly for $\mathbf{E}_s(k)$. Applications of generalizations and variants of $\mathbf{K}$ and $\mathbf{E}$ are central to our work. As indicated in [10], the evaluation in (1.20) may also be proved using the identity

$$\frac{2}{\pi}\mathbf{K}_s(h) = {}_2F_1\left[\begin{matrix} \frac{1}{4}-\frac{s}{2}, \frac{1}{4}+\frac{s}{2} \\ 1 \end{matrix} \middle| \, (2hh')^2\right]$$

for $0 \leq h \leq \frac{1}{\sqrt{2}}$, writing $h' = \sqrt{1-h^2}$.

The integer moments of $\mathbf{K}$ and $\mathbf{E}$, as in the integrals

$$\int_0^1 k^n \mathbf{K}(k)\, dk \tag{1.21}$$

and

$$\int_0^1 k^n \mathbf{E}(k)\, dk, \tag{1.22}$$

along with similarly defined integrals, have played an important role in *On the interplay among hypergeometric functions, complete elliptic integrals, and Fourier-Legendre expansions* [31]. Two of the main hypergeometric transforms from [31] were directly based on the Maclaurin and FL expansions of the latter factors in (1.21) and (1.22). Also, the transformation methods in Section 6 are based on generalizations of (1.21) and (1.22).

The integral evaluation

$$\int_0^1 \frac{\mathbf{K}(k)}{1+k}\, dk = \frac{\pi^2}{8}$$

highlighted in [10, §5.6], along with a number of similar integral formulas given in [10, §5.6], are instances of a class of mathematical objects that is central to our work: Informally, we are referring to integrals of the form

$$\int_0^1 \mathbf{K}(x)g(x)\,dx$$

for certain elementary functions g; see Sections 2, 5, and 6 below.

1.3 Further preliminaries

The interchange of limiting operations is at the core of so much about the field of mathematical analysis. Since we are to frequently make use of the interchange of the operations of infinite summation and integration, it is appropriate to provide a preliminary framework so as to justify our later applying interchanges of this form. However, in our publications [17, 18, 19, 20, 31], it was not deemed to be necessary to go into detail about such interchanging operations to prove the identities we had introduced. This is because: For a reasonably well-behaved sequence $(a_n : n \in \mathbb{N}_0)$ such that $\sum_{n=0}^{\infty} a_n x^n$ or $\sum_{n=0}^{\infty} a_n P_n(2x-1)$ (see Section 5 below) is expressible in terms of previously known elementary/special functions, verifying the interchangeability of the operators $\int_0^1 \cdot\, dx$ and $\sum_{n=0}^{\infty} \cdot$ is typically relatively straightforward, and the main focus of our work is in special functions theory, as opposed to real analysis.

As described in [55, §1.2], the classical result known as *Tonelli's Theorem* is often useful in the justification of the interchange of the operations of definite integration and infinite summation. As in [55, §1.2], we refer the interested reader to Theorem 6.10 in the classic text [91].

Tonelli's Theorem: Suppose that $f(x,y) \geq 0$ for ordered pairs in $E \times F = \{(x,y) \in \mathbb{R}^{m+n} : x \in E, y \in F\}$. It then follows that the following interchange property holds [55, §1.2] (cf. [91]):

$$\int_E \int_F f(x,y)\,dy\,dx = \int_F \int_E f(x,y)\,dx\,dy.$$

As stated in [55, §1.2],

> "Absolutely convergent series can be considered to be special cases of Lebesgue integrals over the measure space $(0, 1, 2, \ldots)$. Hence, the interchange of summation and integration can be applied to functions which are Lebesgue integrable, including those not necessarily non-negative." [55,

p. 14]

Again, following [55, §1.2], we obtain, as a Corollary, that: For a subset E of a Cartesian power of $\mathbb{R}$, and for functions $f_n \colon E \to [0, \infty)$ for natural numbers n, the following interchange property holds for absolutely convergent integrals and series:

$$\int_E \sum_{n=1}^{\infty} f_n = \sum_{n=1}^{\infty} \int_E f_n.$$

Similarly, again following [55, §1.2], if $\sum_{n=1}^{\infty} A(k,n)$ and $\sum_{k=1}^{\infty} A(k,n)$ are absolutely convergent for natural numbers n and k, then

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} A(k,n) = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} A(k,n)$$

and these series are absolutely convergent.

The famous *Monotone Convergence Theorem* is also useful, for our purposes, in order to justify the exchange of integration and infinite summation.

Monotone Convergence Theorem: Assume that $f_n \colon X \to [0, \infty)$ is a measurable function for each natural number n, writing X to denote a measurable set. Also, suppose that $f_n \to f$ almost everywhere, and that $f_1 \leq f_2 \leq \cdots$. It then follows that

$$\lim_{n \to \infty} \int_X f_n = \int_X f.$$

With regard to the interchange of limiting operations results considered above, we encountered a problem: The situation is somewhat more complicated when dealing with shifted FL expansions, since the sequence of shifted Legendre polynomials is not increasing, and since shifted Legendre polynomials are typically not nonnegative over $[0, 1]$.

Let us illustrate this matter in a concrete way, using one of the main transforms from Section 6, as given in Theorem 6.2.1. In order to prove the motivating example highlighted in Section 6.1.1, we set $f_n = \left(\frac{1}{16}\right)^n \binom{2n}{n}^2$ in Theorem 6.2.1, and we would then have to justify moving the operator $\int_0^1 \cdot \, dx$ inside both of the following infinite sums:

$$\int_0^1 \left(\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{f_n}{2m+1} x^n P_m(2x-1) \right) dx. \tag{1.23}$$

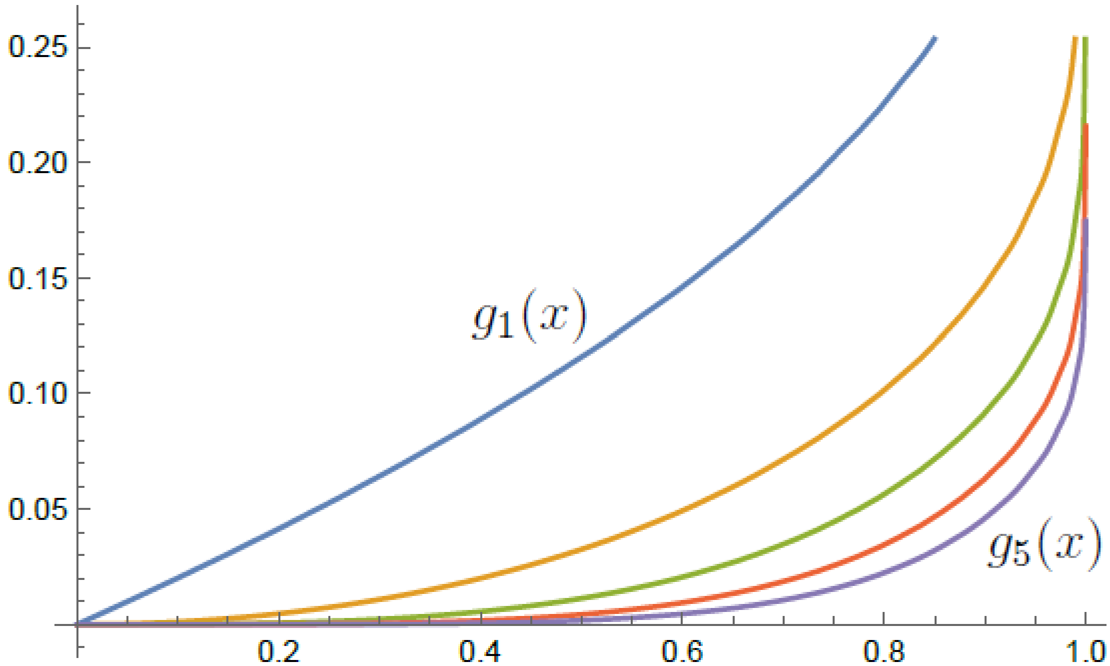

Figure 1.3: Graphs illustrating $g_n(x)$ for $n = 1, \ldots, 5$.

So, we begin by setting

$$g_n(x) := f_n x^n \sum_{m=0}^{\infty} \frac{1}{2m+1} P_m(2x-1)$$

and we want to show that

$$\int_0^1 \left(\sum_{n=0}^{\infty} g_n(x) \right) dx = \sum_{n=0}^{\infty} \left(\int_0^1 g_n(x)\, dx \right).$$

The functions $g_1(x)$, $g_2(x)$, …, $g_5(x)$ are illustrated in Figure 1.3.

We have that $g_n(x) \geq 0$ for $n \in \mathbb{N}$ and for $x \in [0,1]$. So, by Tonelli's Theorem, we obtain that (1.23) may be written as:

$$\sum_{n=0}^{\infty} \left(\int_0^1 \left(\sum_{m=0}^{\infty} \frac{f_n}{2m+1} x^n P_m(2x-1) \right) dx \right).$$

So, we restrict our attention to the outer summand:

$$\int_0^1 \left(\sum_{m=0}^{\infty} \frac{f_n}{2m+1} x^n P_m(2x-1) \right) dx.$$

Write

$$h_{m,n}(x) := \frac{f_n}{2m+1} x^n P_m(2x-1).$$

In this case, we cannot use Tonelli's Theorem to show that

$$\forall n \in \mathbb{N}_0 \ \int_0^1 \left(\sum_{m=0}^{\infty} h_{m,n}(x) \right) dx = \sum_{m=0}^{\infty} \left(\int_0^1 h_{m,n}(x)\, dx \right).$$

Also, the Monotone Convergence Theorem cannot be used in this case: It is not the case that $\sum_{m=0}^{0} h_{m,1}(x) \leq \sum_{m=0}^{1} h_{m,1}(x)$, it is not the case that $\sum_{m=0}^{1} h_{m,1}(x) \leq \sum_{m=0}^{2} h_{m,1}(x)$, etc.

We can show that we may apply the famous *Dominated Convergence Theorem*, as reproduced below.

Dominated Convergence Theorem: Suppose that $f_n \colon \mathbb{R} \to \mathbb{R}$ is a measurable function for each natural number n, and suppose that the sequence of such functions converges pointwise almost everywhere to f. Also, assume that there exists an integrable function g such that $|f_n(x)| \leq g(x)$ for each element $n \in \mathbb{N}$ and for all x. It then follows that f is integrable and that

$$\int_{\mathbb{R}} f = \lim_{n \to \infty} \int_{\mathbb{R}} f_n.$$

See also [75, 85] for some results on the uniform convergence of Fourier–Legendre expansions.

Apart from the concrete examples provided above for (1.23), we may apply the same approach more generally, in this Thesis, to justify the interchange of the operations of integration and infinite summation over expressions involving shifted Legendre polynomials. It was not necessary to justify the interchange of integration and infinite summation in the published research articles [17, 18, 19, 20, 31], so it is reasonable to leave it to the reader to fill in the details as to how we may, as described above, demonstrate how the above results on the interchange of limiting operations may be applied to the results introduced in [17, 18, 19, 20, 31].

1.4 A note on the use of Computer Algebra Systems

In this Thesis, as based on the articles [17], [18], [19], and [20], we may refer to series or integral evaluations being new. In the course of our writing the the articles [17], [18], [19], and [20], at the time, up-to-date versions of Mathematica and Maple, such as Mathematica 11, could not directly evaluate the series/integrals for which our evaluations are described as new, i.e., using the usual commands for inputting series/integrals, and without the use of separate or additional packages; wherever appropriate, we have specified, as below, our use of the 2022 version of Mathematica and Maple 2020. By inputting equivalent formulations of such sums/integrals into Maple or Mathematica, and by using commands such as the Wolfram commands `FunctionExpand` and/or `FullSimplify`, then no evaluation can be produced. For example, in view of Remark 2.2.2 below, let us consider the third series highlighted in the above Abstract, namely:

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2 H_n}{16^n(2n-5)}.$$

Inputting the following into Mathematica 2022, this CAS is not able to provide any evaluation.

```
Sum[Binomial[2 n, n]^2 HarmonicNumber[n]/16^n/(2 n - 5),
{n, 1, Infinity}]
```

By inputting equivalent versions of the input indicated above, Mathematica 2022 is still unable to provide any evaluation. For example, by applying `FunctionExpand` to the above summand, we obtain the equivalent expression suggested by the following Mathematica input, but, again, Mathematica 2022 is not able to provide any evaluation.

```
Sum[(Gamma[1/2 + n]^2*(EulerGamma + PolyGamma[0, 1 + n]))/
((-5 + 2*n)*Pi*Gamma[1 + n]^2), {n, 1, Infinity}]
```

Similarly, by applying an index shift or by altering the lower parameter for the sum under consideration, we still are not able to obtain any evaluation via Mathematica, as in with the following input.

```
Sum[Binomial[2 n, n]^2 HarmonicNumber[n]/16^n/(2 n - 5),
{n, 0, Infinity}]
```

Chapter 2

Ramanujan-like series for $\frac{1}{\pi}$ involving harmonic numbers

2.1 Introduction

The evaluation of summations containing binomial coefficients and harmonic numbers is an interesting topic, and a large amount of mathematical literature is devoted to this area. We consider, in this Section, the general problem of evaluating infinite series involving harmonic numbers and squared central binomial coefficients.

Many harmonic sums involving binomial coefficients may be evaluated by applying differential operators to known hypergeometric identities. In particular, the Ramanujan-like formula

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2 H_n}{16^n (2n-1)^2} = \frac{12 - 16\ln(2)}{\pi}$$

which was noted in 2014 by Junesang Choi [38] and in 2016 by Hongwei Chen [37] may be proven using this method. The author of [37] strongly encourages further exploration of properties of series involving powers of binomial coefficients and harmonic numbers.

Although certain types harmonic summations may be evaluated by applying differential operators to specific hypergeometric identities, it is not, in general, obvious as to how to evaluate a given a harmonic sum involving powers of binomial coefficients. Inspired in part by [37], we introduce, in this Section, a very general method for evaluating series with a summand involving a factor of the form $\binom{2n}{n}^2 H_n$. Using this method, we prove a variety of new Ramanujan-like formulas for $\frac{1}{\pi}$.

The main method introduced in this Section for evaluating harmonic sums containing expressions of the form $\binom{2n}{n}^2$ is based on a parameter derivative of a beta-type integral. This technique is very useful and general, because this method may be used to evaluate series of the form

$$\sum_{n=1}^{\infty} g(n) \binom{2n}{n}^2 H_n$$

for a given function g on $\mathbb{N}$, whereas techniques given by differentiating specific classes of hypergeometric identities will only produce specific results. We also show how a generating function recently considered by Boyadzhiev in [15] may be used to evaluate certain harmonic sums with squared central binomial coefficients, and we also describe a general method for evaluating harmonic sums of the form

$$\sum_{n=1}^{\infty} g(n) \binom{2n}{n}^2 (H_n^2 + H_n^{(2)}).$$

The Ramanujan-type formula

$$\sum_{n=0}^{\infty} \left(-\frac{1}{64}\right)^n \binom{2n}{n}^3 (3(4n+1)H_n - 2) = -\frac{12\ln(2)}{\pi}$$

is proven in [51], and several other Ramanujan-like formulas related to harmonic numbers are proven in [51]. Apart from [37] and [51], it seems that there had not, prior to our work, been very much mathematical literature concerning harmonic sums for $\frac{1}{\pi}$. It is thus natural to use a systematic approach towards the construction of new series of this form.

The formulas for $\frac{1}{\pi}$ introduced in this Section can also be proven using some new formulas for definite integrals involving complete elliptic integrals, as we discuss in Section 2.4. We are interested in the general problem of evaluating integrals involving complete elliptic integrals in part because there are many applications related to elliptic functions and elliptic integrals in various fields connected to physics and engineering. As discussed in [58], many physical problems involve multiple integrals involving elliptic integrals such that once one of the integrations is completed, the resultant integrand contains an elliptic integral.

2.2 Harmonic sums involving squared central binomial coefficients

In this Section, we describe some different techniques for evaluating Ramanujan-like series involving harmonic numbers and expressions of the form $\binom{2n}{n}^2$, and in Section 2.3 we apply our main technique, which is described in Section 2.2.1, by introducing new harmonic series for $\frac{1}{\pi}$.

2.2.1 First strategy

Our first strategy for constructing new Ramanujan-type series for $\frac{1}{\pi}$ is based on the evaluation of integrals of the form $\int_0^1 \frac{x^n \ln(1-x^2)}{\sqrt{1-x^2}} dx$. Integrals of this form may be evaluated in a natural way in terms of harmonic numbers, as indicated in Lemma 2.2.1 below. We remark that a somewhat similar strategy based on the evaluation of integrals involving an expression of the form $\frac{x \ln^{2m-1} x}{1-x}$ was recently used to evaluate certain classes of series involving harmonic numbers in [74].

Lemma 2.2.1. *Let $f\colon \mathbb{N}_0 \to \mathbb{C}$ be such that the series*

$$\sum_{n=0}^{\infty} (-1)^n \binom{\frac{1}{2}}{n} f(n) \frac{x^{2n} \ln\left(1-x^2\right)}{\sqrt{1-x^2}}$$

is integrable on $[0,1]$, and write $g(n) = \frac{f(n)}{16^n (2n-1)}$. Then $\sum_{n=0}^{\infty} g(n) \binom{2n}{n}^2 H_n$ is equal to:

$$\frac{2}{\pi}\left(\int_0^1 \left(\sum_{n=0}^{\infty} (-1)^n \binom{\frac{1}{2}}{n} f(n) \frac{x^{2n} \ln\left(1-x^2\right)}{\sqrt{1-x^2}} \right) dx - \right. \tag{2.1}$$

$$\left. \pi \ln(2) \sum_{n=0}^{\infty} g(n) \binom{2n}{n}^2 \right), \tag{2.2}$$

under the assumption that it is possible to reverse the order of integration and infinite summation in (2.1).

Proof. This follows by applying the identity

$$\int_0^1 \frac{x^{2n} \ln\left(1-x^2\right)}{\sqrt{1-x^2}} dx = -\frac{\pi 2^{-2n} n \Gamma(2n) \left(H_n + 2\ln(2)\right)}{\Gamma^2\left(n+1\right)}, \tag{2.3}$$

by reversing the order of integration and summation in (2.1). □

The above Lemma is especially useful because it is not, in general, otherwise obvious as to how to evaluate a series of the form $\sum_{n=0}^{\infty} g(n) \binom{2n}{n}^2 H_n$.

Since one of the key tools used in this Section is based on the evaluation of integrals of the form $\int_0^1 \frac{x^n \ln(1-x^2)}{\sqrt{1-x^2}} dx$, it is worthwhile to note that this integral is a derivative of the beta function with respect to a parameter. In particular, recalling (1.17), we have that

$$\frac{1}{2} \frac{\partial}{\partial y} \beta\left(\frac{n+1}{2}, y+1\right) = \int_0^1 u^n \left(1-u^2\right)^y \ln\left(1-u^2\right) du,$$

so that

$$\int_0^1 \frac{u^n \ln\left(1-u^2\right)}{\sqrt{1-u^2}}\, du = \frac{\sqrt{\pi}\Gamma\left(\frac{n+1}{2}\right)\left(\psi\left(\frac{1}{2}\right)-\psi\left(\frac{n+2}{2}\right)\right)}{2\Gamma\left(\frac{n+2}{2}\right)}$$

for $n \in \mathbb{N}_0$. So, the Legendre duplication formula, as in (1.19), allows us to express the above evaluation with central binomial coefficients, depending on the parity of n.

2.2.2 Boyadzhiev's generating function

Boyadzhiev recently evaluated the generating function for the sequence

$$\left(\binom{2n}{n} H_n : n \in \mathbb{N}_0\right)$$

and similar sequences in [15], proving that

$$\sum_{n=0}^{\infty} \binom{2n}{n} H_n x^n = \frac{2}{\sqrt{1-4x}} \ln\left(\frac{1+\sqrt{1-4x}}{2\sqrt{1-4x}}\right)$$

for $|x| < \frac{1}{4}$.

Boyadzhiev's generating function may be used to construct certain types of Ramanujan-like series for $\frac{1}{\pi}$ involving squared central binomial coefficients and harmonic numbers. However, in view of our considerations concerning the expression in (2.6), it seems that it is not, in general, possible to use this kind of generating function-based approach, compared to our integration method in Lemma 2.2.1.

To construct a Ramanujan-type series for $\frac{1}{\pi}$ using Boyadzhiev's generating function one may use the following integral identity:

$$\frac{\binom{2n}{n}}{4^n} = \frac{2}{\pi} \int_0^{\infty} \frac{dx}{(x^2+1)^{n+1}}.$$

To illustrate this idea, as well as the technique described in Section 2.2.1, we offer two corresponding proofs of the following Ramanujan-like formula.

Remark 2.2.2. *As of the time the research article [19] was being written, up-to-date Computer Algebra Systems such as Mathematica and Maple could not evaluate the series in Theorem 2.2.3 below. However, the version of Mathematica in 2022 is able to evaluate this series, but not our generalizations of this series such as*

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2 H_n}{16^n (2n-3)}, \tag{2.4}$$

noting that our evaluation for (2.4) *does not follow in a direct way from Theorem 2.2.3 via reindexing* *arguments, as we later explore in Section 3. Maple 2020 is also not able to evaluate* (2.4)*.*

Theorem 2.2.3. $\sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2 H_n}{16^n(2n-1)} = \frac{8\ln(2)-4}{\pi}$.

Proof #1: In Lemma 2.2.1, we set $f(n) = 1$. In this case, the integrand in (2.1) reduces to $\ln\left(1-x^2\right)$. By setting the argument of the Maclaurin series in (1.16) for $\mathbf{E}$ as 1, the elliptic singular value $\mathbf{E}(1) = 1$ allows us to evaluate the series in (2.2), again with $f(n) = 1$. Since $x\ln\left(1-x^2\right) - 2x + 2\tanh^{-1}(x)$ is an antiderivative of the integrand in (2.1) for $f(n) = 1$, we may evaluate the definite integral in (2.1). $\square$

Proof #2: By Boyadzhiev's generating function, we have that

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n} H_n x^{2n-2}}{4^n} = \frac{2\ln\left(\frac{\sqrt{1-x^2}+1}{2\sqrt{1-x^2}}\right)}{x^2\sqrt{1-x^2}}.$$

Integrating both sides of this equality, we have that $\sum_{n=0}^{\infty} \frac{\binom{2n}{n} H_n y^{2n-1}}{4^n(2n-1)}$ is equal to:

$$\frac{2\left(-\sqrt{1-y^2} - \sqrt{1-y^2}\ln\left(\frac{1}{\sqrt{1-y^2}}+1\right) + \sqrt{1-y^2}\ln(2) + 1\right)}{y}.$$

Therefore, $\sum_{n=0}^{\infty} \frac{\binom{2n}{n} H_n 2\left(\frac{1}{x^2+1}\right)^{n+1}}{4^n\pi(2n-1)}$ is equal to

$$\frac{4\left(\sqrt{x^2+1} - \sqrt{x^2}\ln\left(\frac{1}{\sqrt{\frac{x^2}{x^2+1}}}+1\right) + \sqrt{x^2}(\ln(2)-1)\right)}{\pi\left(x^2+1\right)^{3/2}}.$$

Mathematica is able to evaluate the integral

$$\int_0^{\infty} \frac{4\left(\sqrt{x^2+1} - \sqrt{x^2}\ln\left(\frac{1}{\sqrt{\frac{x^2}{x^2+1}}}+1\right) + \sqrt{x^2}(\ln(2)-1)\right)}{\pi\left(x^2+1\right)^{3/2}}\,dx$$

as $\frac{8\ln(2)-4}{\pi}$, and we can prove this by evaluating the corresponding indefinite integral. So, this gives us a proof for the above evaluation of $\sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2 H_n}{16^n(2n-1)}$. $\square$

In Section 2.4, we offer another proof of Theorem 2.2.3, through the use of integrals involving complete elliptic integrals.

Our main strategy introduced in Section 2.2.1 for evaluating Ramanujan-type series for $\frac{1}{\pi}$ involving harmonic numbers is much more general and powerful compared to the alternative technique outlined in

Section 2.2.2. For example, it seems that it would not be feasible to use Boyadzhiev's generating function to prove the following result due to Choi [38] and to Chen [37], although Lemma 2.2.1 may be used to prove the following Theorem.

Theorem 2.2.4. $\sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2 H_n}{16^n (2n-1)^2} = \frac{12-16\ln(2)}{\pi}$ [37, 38].

Proof #1: In Lemma 2.2.1, we set $f(n) = \frac{1}{2n-1}$. This yields the following integrand, again with reference to Lemma 2.2.1:

$$-\frac{\ln\left(1-x^2\right)\left(\sqrt{1-x^2}+x\sin^{-1}(x)\right)}{\sqrt{1-x^2}}. \tag{2.5}$$

According to the Maclaurin series identity

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2 x^n}{16^n (2n-1)^2} = \frac{4\mathbf{E}\left(\sqrt{x}\right)}{\pi} + \frac{2(x-1)\mathbf{K}\left(\sqrt{x}\right)}{\pi}$$

following from (1.16), by taking the indefinite integral of (2.5), we may obtain the desired evaluation via Lemma 2.2.1. □

By Boyadzhiev's generating function, we have that $\sum_{n=0}^{\infty} \frac{\binom{2n}{n} H_n y^{2n-2}}{4^n (2n-1)}$ is equal to

$$\frac{2\left(-\sqrt{1-y^2}-\sqrt{1-y^2}\ln\left(\frac{1}{\sqrt{1-y^2}}+1\right)+\sqrt{1-y^2}\ln(2)+1\right)}{y^2}. \tag{2.6}$$

Integrating the above expression yields an expression involving the polylogarithm function. It does not seem to be possible, e.g., with current CAS software or with known symbolic computation algorithms for indefinite integrals, to evaluate the corresponding integral needed to evaluate the series $\sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2 H_n}{16^n (2n-1)^2}$ using the identity $\frac{\binom{2n}{n}}{4^n} = \frac{2}{\pi}\int_0^{\infty} \frac{dx}{(x^2+1)^{n+1}}$. This illustrates how the strategy introduced in Section 2.2.1 may be used to construct Ramanujan-like series for $\frac{1}{\pi}$ that cannot be evaluated directly following the technique given in Section 2.2.2. We later offer an alternative proof of Theorem 2.2.4 using definite integrals involving complete elliptic integrals. We also offer an alternative proof of the following Theorem using integrals containing complete elliptic integrals.

Theorem 2.2.5. $\sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2 H_n}{16^n (2n-3)} = \frac{120\ln(2)-68}{27\pi}$.

Proof #1: Setting $f(n)$ to be equal to $\frac{2n-1}{2n-3}$ in Lemma 2.2.1, this yields the integrand

$$\frac{1}{3}\left(2x^2+1\right)\ln\left(1-x^2\right). \tag{2.7}$$

The series in (2.2), again with $f(n) = \frac{2n-1}{2n-3}$, is as follows:

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{16^n(2n-3)}.$$

So, the desried evaluation for the binomial-harmonic series under consideration follows by taking an antiderivative of (2.7), and by evaluating the series in (2.2) according to the elliptic integral identity

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2 x^n}{16^n(2n-3)} = -\frac{2(4x+1)\mathbf{E}\left(\sqrt{x}\right)}{9\pi} - \frac{4(1-x)\mathbf{K}\left(\sqrt{x}\right)}{9\pi},$$

by taking $x \to 1$. □

2.2.3 Series involving an expression of the form $H_n^2 + H_n^{(2)}$

In Section 2.5, we show how the evaluation of integrals of the form

$$\int_0^1 \frac{x^n \ln^2\left(1-x^2\right)}{\sqrt{1-x^2}}\, dx$$

may be used to prove new formulas for Ramanujan-like series, such as the formula

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2 \left(H_n^2 + H_n^{(2)}\right)}{16^n(2n-1)} = \frac{4\pi}{3} + \frac{16\left(2\ln(2) - 2\ln^2(2) - 1\right)}{\pi}$$

introduced in [19].

2.3 Ramanujan-type series for $\frac{1}{\pi}$ involving harmonic numbers

We begin by presenting new Ramaujan-like series for $\frac{1}{\pi}$ with a summand of the form $\frac{\binom{2n}{n}^2 H_n}{16^n p_n}$, letting p_n denote a polynomial with integer coefficients. Lemma 2.2.1 may be used to evaluate summations of this form in the case whereby p_n is a polynomial with two different factors, but this case can be reduced to simpler cases by decomposing the appropriate fraction into simpler fractions. So, we omit consideration of summands of the form $\frac{\binom{2n}{n}^2 H_n}{16^n p_n}$ in the case whereby p_n has two different factors.

The Ramanujan-like formulas given in [19], as reproduced in this Section, had apparently not appeared in any mathematical literature prior to [19] concerning summations involving harmonic numbers and binomial coefficients, with the exception of the formula which is given in Theorem 2.2.4. The Ramanujan-like formulas listed below, as in (2.8)–(2.16), may be verified using Lemma 2.2.1. For example, we provide a proof for the equality in (2.8).

We set $f(n) = \frac{2n-1}{n+2}$ in Lemma 2.2.1. In this case, the integrand in (2.1) is as below:

$$\frac{2\left(\sqrt{1-x^2}x^2+2\sqrt{1-x^2}-2\right)\ln\left(1-x^2\right)}{3x^4\sqrt{1-x^2}}.$$

A corresponding antiderivative is as below:

$$\frac{2}{9}\Bigg(-\frac{4\sqrt{1-x^2}}{x}+\frac{\left(\left(4\sqrt{1-x^2}-3\right)x^2+2\left(\sqrt{1-x^2}-1\right)\right)\ln\left(1-x^2\right)}{x^3}+$$
$$\frac{4}{x}+5\ln(1-x)-5\ln(x+1)+8\sin^{-1}(x)\Bigg).$$

As for the required non-harmonic series in (2.2), Ramanujan's series (cf. [1]) of the form

$$S(r) = \sum_{k=0}^{\infty}\left(\frac{1}{16}\right)^k\frac{\binom{2k}{k}^2}{k+r}$$

are considered in Section 6, in which the classical formula for such series for $r \in \mathbb{N}$ is reproduced. The foregoing considerations nicely illustrate how series as in (2.8) are natural but nontrivial extensions of Ramanujan's S-series; this was recently explored, subsequent to the publications included in this Thesis, in our recent coauthored publication [27].

For the sake of brevity, we leave it to the reader to verify the remaining equalities given below, apart from our proof of (2.8):

$$\sum_{n=1}^{\infty}\frac{\binom{2n}{n}^2H_n}{16^n(n+2)} = \frac{16}{9}+\frac{16-80\ln(2)}{9\pi}, \tag{2.8}$$

$$\sum_{n=1}^{\infty}\frac{\binom{2n}{n}^2H_n}{16^n(n+3)} = \frac{256}{225}+\frac{416-1424\ln(2)}{225\pi}, \tag{2.9}$$

$$\sum_{n=1}^{\infty}\frac{\binom{2n}{n}^2H_n}{16^n(n+4)} = \frac{1024}{1225}+\frac{6416-18288\ln(2)}{3675\pi}, \tag{2.10}$$

$$\sum_{n=1}^{\infty}\frac{\binom{2n}{n}^2H_n}{16^n(2n-1)} = \frac{8\ln(2)-4}{\pi}, \tag{2.11}$$

$$\sum_{n=1}^{\infty}\frac{\binom{2n}{n}^2H_n}{16^n(2n-3)} = \frac{120\ln(2)-68}{27\pi}, \tag{2.12}$$

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2 H_n}{16^n(2n-5)} = \frac{-6508+10680\ln(2)}{3375\pi}, \tag{2.13}$$

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2 H_n}{16^n(2n-1)^2} = \frac{12-16\ln(2)}{\pi}, \tag{2.14}$$

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2 H_n}{16^n(2n-3)^2} = \frac{164-176\ln(2)}{27\pi}, \tag{2.15}$$

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2 H_n}{16^n(2n-5)^2} = \frac{70724-67760\ln(2)}{16875\pi}. \tag{2.16}$$

We remark that analogues of Lemma 2.2.1 may be used to evaluate series involving expressions of the form H_{2n}. Explicitly, as an equivalent formulation of the moment formula in (2.3), we find that

$$\int_0^1 \frac{x^{4n}\ln\left(1-x^2\right)}{\sqrt{1-x^2}}\,dx = \pi\left(-2^{-4n-1}\right)\binom{4n}{2n}\left(H_{2n}+2\ln(2)\right),$$

leading us to formulate the following analogue of Lemma 2.2.1: For a sequence $(f_n : n \in \mathbb{N}_0)$ such that it is possible to reverse the order of integration and infinite summation with respect to the expression

$$\int_0^1 \frac{\ln\left(1-x^2\right)\sum_{n=0}^{\infty}\frac{\left(\frac{x^4}{4}\right)^n\binom{2n}{n}f(n)}{2n-1}}{\sqrt{1-x^2}}\,dx,$$

the binomial-harmonic sum

$$\frac{\pi}{2}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{64}\right)^n\binom{2n}{n}\binom{4n}{2n}f(n)H_{2n}}{2n-1}$$

is expressible as

$$-\int_0^1 \frac{\ln\left(1-x^2\right)\sum_{n=0}^{\infty}\frac{\left(\frac{x^4}{4}\right)^n\binom{2n}{n}f(n)}{2n-1}}{\sqrt{1-x^2}}\,dx - \pi\ln(2)\sum_{n=0}^{\infty}\frac{\left(\frac{1}{64}\right)^n\binom{2n}{n}\binom{4n}{2n}f(n)}{2n-1}. \tag{2.17}$$

Our study of summations as in (2.17) formed a basis, subsequent to [19], for our joint article on the *hypergeometry of the parbelos* [30].

Example 2.3.1. *Setting $f(n)=1$ in the above analogue of Lemma 2.2.1, we can show that*

$$\sum_{n=1}^{\infty}\frac{2^{-6n}\binom{2n}{n}\binom{4n}{2n}H_{2n}}{2n-1}$$

is equal to the following expression involving the dilogarithm:

$$-\frac{\pi}{12}+\frac{\mathrm{Li}_2\left(17-12\sqrt{2}\right)}{2\pi}-\frac{\sqrt{2}}{\pi}+\frac{3\ln^2\left(1+\sqrt{2}\right)}{\pi}+\frac{5\sqrt{2}\ln(2)}{\pi}+$$
$$\frac{3\ln\left(\sqrt{2}-1\right)}{\pi}+\frac{\ln\left(\sqrt{2}-1\right)\ln\left(1+\sqrt{2}\right)}{\pi}.$$

We remark that the Ramanujan-like series introduced in this Section may be rewritten as double series using identities such as the following series identity:

$$\sum_{n=1}^{\infty}\frac{1}{n(m+n)}=\frac{\psi^{(0)}(m+1)+\gamma}{m}.$$

For example, the formula

$$\sum_{n=0}^{\infty}\frac{\binom{2n}{n}^2 H_n}{16^n(2n-3)}=\frac{120\ln(2)-68}{27\pi}$$

may be rewritten as the following double hypergeometric series formula:

$$\frac{96\ln(2)-88}{9\pi}=\sum_{m,n\geq 0}\frac{\binom{2m}{m}^2}{16^m(2m-1)(n+1)(m+n+2)}.$$

2.4 Definite integrals involving complete elliptic integrals

We recall the definitions of $\mathbf{K}$ and $\mathbf{E}$ given, respectively, in (1.14) and (1.15). The series for $\frac{1}{\pi}$ given in Section 2.3 are closely related to properties concerning complete elliptic integrals. To illustrate this idea, we begin by showing how the known integral formula

$$\int_0^1 \ln(1-x)\mathbf{K}(\sqrt{x})\,dx = 8\ln(2)-8$$

may be used to construct yet another proof of Theorem 2.2.3.

Proof #3 of Theorem 2.2.3: Using integration by parts with respect to the integral

$$\int_0^1 \ln(1-x)\mathbf{K}(\sqrt{x})\,dx = 8\ln(2)-8,$$

by letting $u = \ln(1-x)$ and $v' = \mathbf{K}(\sqrt{x})$, with

$$v(x) = 2x\mathbf{K}(\sqrt{x}) - 2\mathbf{K}(\sqrt{x}) + 2\mathbf{E}(\sqrt{x}) - 2,$$

we have that $8\ln(2) - 8$ is equal to

$$\int_0^1 \frac{-2 + 2\mathbf{E}(\sqrt{x}) - 2\mathbf{K}(\sqrt{x}) + 2x\mathbf{K}(\sqrt{x})}{1-x}\, dx.$$

Therefore,

$$8\ln(2) - 8 = -2\int_0^1 \frac{1 - \mathbf{E}\left(\sqrt{x}\right)}{1-x}\, dx - 2\int_0^1 \mathbf{K}\left(\sqrt{x}\right)\, dx.$$

So, we have that

$$2 - 4\ln(2) = \int_0^1 \frac{1 - \mathbf{E}(\sqrt{x})}{1-x}\, dx. \tag{2.18}$$

So, since

$$1 - \mathbf{E}(\sqrt{x}) = \frac{1}{2}\pi \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(-\frac{1}{2}\right)_n (1 - x^n)}{(1)_n n!},$$

we have that

$$\frac{1 - \mathbf{E}(\sqrt{x})}{1-x} = \frac{1}{2}\pi \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(-\frac{1}{2}\right)_n \left(\frac{1-x^n}{1-x}\right)}{(1)_n n!}.$$

By integrating both sides of this equality, we find that

$$2 - 4\ln(2) = \frac{1}{2}\pi \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(-\frac{1}{2}\right)_n H_n}{(1)_n n!},$$

as desired. □

The equality given in (2.18) turns out to be unexpectedly useful, since (2.18) may be used to construct an alternative proof of Theorem 2.2.4, as shown below. The formula in (2.18) can also be used to construct another proof of Theorem 2.2.5.

Proof #2 of Theorem 2.2.4: Rewriting the equality $\int_0^1 \mathbf{K}(\sqrt{x})\, dx = 2$ as

$$\int_0^1 \left(-\frac{\mathbf{K}(\sqrt{x})}{2(-1+x)} + \frac{x\mathbf{K}(\sqrt{x})}{2(-1+x)} \right) dx = 1,$$

from (2.18), we have that

$$\int_0^1 \left(\frac{-1+\mathbf{E}(\sqrt{x})}{-1+x} - \frac{\mathbf{K}(\sqrt{x})}{2(-1+x)} + \frac{x\mathbf{K}(\sqrt{x})}{2(-1+x)} \right) dx = 3 - 4\ln(2).$$

Rewrite this equality as follows:

$$\int_0^1 \frac{\pi \left(\frac{2(x-1)\mathbf{K}(\sqrt{x})}{\pi} + \frac{4\mathbf{E}(\sqrt{x})}{\pi} \right) - 4}{\pi(x-1)} dx = \frac{12 - 16\ln(2)}{\pi}.$$

This equality may be rewritten as follows:

$$\int_0^1 \frac{-4 + \pi\, {}_2F_1\left[\begin{matrix} -\frac{1}{2}, -\frac{1}{2} \\ 1 \end{matrix} \middle| \, x \right]}{\pi(-1+x)} dx = \frac{12 - 16\ln(2)}{\pi}.$$

But since

$$\sum_{n=0}^{\infty} \frac{16^{-n}\left(1-x^n\right)\binom{2n}{n}^2}{(-1+2n)^2(1-x)} = \frac{\pi\, {}_2F_1\left[\begin{matrix} -\frac{1}{2}, -\frac{1}{2} \\ 1 \end{matrix} \middle| \, x \right] - 4}{\pi(x-1)},$$

by integrating both sides of the above equality appropriately we thus have that

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2 H_n}{16^n(2n-1)^2} = \frac{12 - 16\ln(2)}{\pi},$$

as desired. □

Proof #2 of Theorem 2.2.5: Begin by evaluating $\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2\left(\frac{1-x^n}{1-x}\right)}{16^n(2n-3)}$ as

$$\frac{10 - 3\pi\, {}_3F_2\left[\begin{matrix} -\frac{3}{2}, \frac{1}{2}, \frac{1}{2} \\ -\frac{1}{2}, 1 \end{matrix} \middle| \, x \right]}{9\pi(-1+x)},$$

which is equal to

$$\frac{10 - 3\pi\left(\frac{4(1-x)\mathbf{K}(\sqrt{x})}{3\pi} + \frac{2(4x+1)\mathbf{E}(\sqrt{x})}{3\pi}\right)}{9\pi(x-1)}.$$

So, the infinite series given in Theorem 2.2.5 is equal to the following integral:

$$\int_0^1 \frac{10 - 3\pi\left(\frac{2(1+4x)\mathbf{E}(\sqrt{x})}{3\pi} + \frac{4(1-x)\mathbf{K}(\sqrt{x})}{3\pi}\right)}{9\pi(-1+x)}\, dx.$$

Therefore, the series given in Theorem 2.2.5 is equal to:

$$\frac{8}{9\pi} + \int_0^1 \left(\frac{10}{9\pi(-1+x)} - \frac{2\mathbf{E}(\sqrt{x})}{9\pi(-1+x)} - \frac{8x\mathbf{E}(\sqrt{x})}{9\pi(-1+x)}\right) dx,$$

which is equal to:

$$\frac{8}{9\pi} - \frac{2}{9\pi}\int_0^1 \frac{1-\mathbf{E}(\sqrt{x})}{1-x}\, dx - \frac{8}{9\pi}\int_0^1 \frac{1-x\mathbf{E}(\sqrt{x})}{1-x}\, dx.$$

We can find a closed-form evaluation of the above expression using (2.18), since $\frac{1-x\mathbf{E}(\sqrt{x})}{1-x} = \mathbf{E}(\sqrt{x}) + \frac{1-\mathbf{E}(\sqrt{x})}{1-x}$. □

Similar approaches may be used to construct alternative proofs for the other formulas for $\frac{1}{\pi}$ given in Section 2.3. We remark that new formulas for integrals containing complete elliptic integrals, such as the formula

$$\int_0^1 \ln^2(1-x) d\mathbf{E}(\sqrt{x}) = 8 - \frac{2\pi^2}{3} + 16(\ln(2)-1)\ln(2),$$

may be proven by rewriting the harmonic numbers in the series given in Section 2.3 and Section 2.5 using integrals such as $\int_0^1 x^m \ln(1-x)\, dx$. Many integrals of this form can also be proven using integration by parts and by applying known results concerning complete elliptic integrals, so we omit a full investigation of these integrals.

2.5 Summations with squared harmonic numbers

Given the variety of integration results and Ramanujan-type series related to Lemma 2.2.1, it is natural to consider generalizations of the strategy introduced in Section 2.2.1. We begin by considering the integral

$$\int_0^1 \frac{x^n \ln^2\left(1-x^2\right)}{\sqrt{1-x^2}}\, dx,$$

which is equal to the following expression for $n > -5$.

$$\frac{1}{4n\Gamma\left(\frac{n}{2}+1\right)\Gamma\left(\frac{n}{2}\right)}\left(\sqrt{\pi}\Gamma\left(\frac{n+1}{2}\right)\left(4\Gamma\left(\frac{n}{2}+1\right)\left(\psi^{(0)}\left(\frac{n}{2}+1\right)^2 - \psi^{(1)}\left(\frac{n}{2}+1\right)\right) + n\Gamma\left(\frac{n}{2}\right)\left(4(\gamma + 2\ln(2))\psi^{(0)}\left(\frac{n}{2}+1\right) + \pi^2 + 2\gamma^2 + 8\ln^2(2) + 8\gamma\ln(2)\right)\right)\right).$$

Theorem 2.5.1. $\sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2 \left(H_n^2 + H_n^{(2)}\right)}{16^n (2n-1)} = \frac{4\pi}{3} + \frac{16\left(2\ln(2) - 2\ln^2(2) - 1\right)}{\pi}$.

Proof. Since

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n} \binom{\frac{1}{2}}{n} \ln^2\left(1-x^2\right)}{\sqrt{1-x^2}} = \ln^2\left(1-x^2\right),$$

we have that

$$\sum_{n=0}^{\infty} (-1)^n \binom{\frac{1}{2}}{n} \left(\int_0^1 \frac{x^{2n} \ln^2\left(1-x^2\right)}{\sqrt{1-x^2}} \, dx \right) = 8 - \frac{\pi^2}{3} + 4\ln^2(2) - 8\ln(2).$$

Expanding the above summand using the above evaluation of the expression $\int_0^1 \frac{x^n \ln^2\left(1-x^2\right)}{\sqrt{1-x^2}} \, dx$ together with Theorem 2.2.3 may be used to evaluate $\sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2 \left(H_n^2 + H_n^{(2)}\right)}{16^n (2n-1)}$. □

The following Propositions may be proven by analogy with our proof of Theorem 2.5.1. Choi [38] had proved an equivalent form of Proposition 2.5.2 in 2014 by differentiating Gauss' ${}_2F_1(1)$-identity.

Proposition 2.5.2. $\sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2 \left(H_n^2 + H_n^{(2)}\right)}{16^n (2n-1)^2} = \frac{64 + 64\ln^2(2) - 96\ln(2)}{\pi} - \frac{8\pi}{3}$ [38].

Proposition 2.5.3. $\sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2 \left(H_n^2 + H_n^{(2)}\right)}{16^n (2n-3)} = \frac{20\pi}{27} + \frac{16\left(-53 - 90\ln^2(2) + 102\ln(2)\right)}{81\pi}$.

Proposition 2.5.4. $\sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2 \left(H_n^2 + H_n^{(2)}\right)}{16^n (2n-3)^2} = \frac{32(268 + 9\ln(2)(22\ln(2) - 41))}{243\pi} - \frac{88\pi}{81}$.

It is often useful to translate summations such as the series given in the above Propositions using integral identities such as $\int_0^1 n x^{n-1} \ln^2(1-x) \, dx = (H_n)^2 + H_n^{(2)}$. For example, we have that

$$\int_0^1 \mathbf{K}(\sqrt{x}) \ln^2(1-x) \, dx = 48 - \frac{4\pi^2}{3} + 32(\ln(2) - 2)\ln(2) \tag{2.19}$$

from Proposition 2.5.2, together with the aforementioned integral identity. We remark that such integration results often may be used to prove formulas for double series involving harmonic numbers. For example, we can show that

$$\sum_{m,n \geq 0} \frac{\binom{2m}{m}^2 H_n}{16^m (n+1)(m+n+2)} = \frac{48 + 32(\ln(2) - 2)\ln(2)}{\pi} - \frac{4\pi}{3}$$

using (2.19) together with the identity $\sum_{n=0}^{\infty} \frac{x^{n+1} H_n}{n+1} = \frac{1}{2}\ln^2(1-x)$.

2.6 Variations

Recall that the main techniques introduced in this Section for constructing series for $\frac{1}{\pi}$ involving harmonic numbers are based upon the evaluation of the following integrals:

$$\int_0^1 \frac{x^n \ln(1-x^2)}{\sqrt{1-x^2}}\, dx,$$
$$\int_0^1 \frac{x^n \ln^2\left(1-x^2\right)}{\sqrt{1-x^2}}\, dx.$$

Intuitively, our approach towards constructing new series for $\frac{1}{\pi}$ involved the integration of "variations" of the binomial expansion of $\sqrt{1-x^2}$ using the above integrals (see Lemma 2.2.1 and Section 2.5), so that the expression $\frac{1}{\sqrt{1-x^2}}$ would in some sense "cancel" with an expression similar to $\sqrt{1-x^2}$, thus yielding a relatively simple logarithmic integral being equal to π times a Ramanujan-like series involving harmonic numbers. It is natural to consider simple variations of this strategy, based on the evaluation of definite integrals of the form $\int_0^1 \frac{x^n e(x)}{\sqrt{1-x^2}}\, dx$ and similar integrals, letting $e(x)$ denote an elementary function.

Example 2.6.1. *The interesting formula*

$$\frac{\pi - \tanh(\pi)}{\sqrt{\pi}} = \sum_{n=0}^{\infty} \frac{\binom{2n}{n}\left((-i-n)_{-\frac{1}{2}} + (i-n)_{-\frac{1}{2}}\right)}{4^n}$$

can be shown to hold by evaluating

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^n \binom{-\frac{1}{2}}{n} \sin(\ln(x))}{\sqrt{1-x}} = -\frac{\sin(\ln(x))}{x-1},$$

and integrating both sides of this equation by evaluating $\int_0^1 \frac{x^n \sin(\ln(x))}{\sqrt{1-x}}\, dx$.

Example 2.6.2. *We can show that*

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}\left(H_{-1-i-n} - H_{-1+i-n}\right)}{4^n(2n-1)}$$

is equal to

$$\frac{4\cosh(\pi)\left(\Gamma(1+i)\Gamma\left(\frac{3}{2}-i\right) - \Gamma(1-i)\Gamma\left(\frac{3}{2}+i\right)\right)}{5\sqrt{\pi}}$$

by evaluating

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^n \binom{\frac{1}{2}}{n} \sin(\ln(x))}{-1+x} = -\frac{\sin(\ln(x))}{\sqrt{1-x}},$$

and integrating both sides of this equality, by evaluating $\int_0^1 \frac{x^n \sin(\ln(x))}{-1+x}\,dx$.

Example 2.6.3. *The equality*

$$-\frac{4i\coth(\pi)}{5\sqrt{\pi}} = \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{4^n(2n-1)} \left(\frac{1}{(-i-n)_{\frac{1}{2}}} - \frac{1}{(i-n)_{\frac{1}{2}}} \right)$$

can be shown to hold by evaluating

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n} \binom{\frac{1}{2}}{n} \sin^2(\ln(x))}{\sqrt{1-x^2}} = \sin^2(\ln(x))$$

and integrating both sides of this equation.

We currently leave it as an open problem to further investigate variations of the techniques given in Section 2.2.

2.7 Conclusion

The evaluation of integrals involving complete elliptic integrals based on the evaluation of harmonic sums given by analogues of Lemma 2.2.1 may be an interesting area to explore. Also, we currently leave the natural problem of evaluating Ramanujan-like series involving higher powers of central binomial coefficients and harmonic numbers as an open problem.

Recall that one of the main tools used in this Section is based on the following parameter derivative for a beta-type integral:

$$\frac{1}{2}\frac{\partial}{\partial y}\beta\left(\frac{n+1}{2}, y+1\right) = \int_0^1 u^n \left(1-u^2\right)^y \ln\left(1-u^2\right) du.$$

It would be interesting to further explore the use of other kinds of parameter derivatives of beta-type integrals in the construction of Ramanujan-like series involving $\frac{1}{\pi}$ and definite integrals involving complete elliptic integrals.

While there are many new formulas which can be obtained by applying parameter derivatives to hypergeometric identities, the main technique indicated in Section 2.2.1 can be applied in a very general way to compute series involving products of the form $\binom{2n}{n}^2 H_n$, as discussed in Section 2.1. However, we encourage the exploration of new formulas for $\frac{1}{\pi}$ which can be obtained from parameter derivatives of hypergeometric identities.

Chapter 3

Further series involving harmonic numbers and squared central binomial coefficients

3.1 Introduction

In [37, 38], it was noted that the Ramanujan-like formula

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2 H_n}{16^n (2n-1)^2} = \frac{12 - 16\ln(2)}{\pi} \tag{3.1}$$

may be proved through an application of a differential operator to Gauss' ${}_2F_1(1)$-identity. A similar strategy was also applied to prove an equivalent formulation of the equation

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2 H_n}{(n+1)16^n} = 4 - \frac{16\ln(2)}{\pi} \tag{3.2}$$

in 2017 in [62]. Both (3.1) and (3.2) are special cases of our very useful result highlighted as Lemma 2.2.1, as introduced in [19] and reproduced in Section 2. While applying parameter derivatives to classical hypergeometric identities only produces specific results on harmonic summations containing squared central binomial coefficients as a factor in the summand, Lemma 2.2.1 may be applied much more generally. The power of Lemma 2.2.1 motivates the exploration of further applications of techniques from [19], as well as investigations on applications of analogues and variants of Lemma 2.2.1.

3.1.1 Background

The problem of determining explicit symbolic evaluations for summations containing entries of harmonic-type sequences and central binomial coefficients is a deep and interesting subject that has been explored through the use of many different kinds of classical analysis-based techniques. In [46], a variety of infinite summations involving generalized harmonic numbers and central binomial coefficients are evaluated through the use of beta-like integrals. In [39], a more abstract way of "depicting" harmonic-like numbers is used, writing

$$H_n = \sigma_1\left(1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}\right)$$

and

$$\frac{H_n^2 - H_n^{(2)}}{2} = \sigma_2\left(1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}\right),$$

letting

$$\sigma_m\left(x_1, x_2, \ldots, x_n\right) = \sum_{1 \leq k_1 < k_2 < \cdots < k_m \leq n} x_{k_1} x_{k_2} \cdots x_{k_m}$$

denote the elementary symmetric function of order m. The authors in [39] mainly explore the symbolic evaluation of infinite series involving central binomial coefficients as well as expressions such as $\sigma_m\left(1, \frac{1}{3^2}, \ldots, \frac{1}{(2n-1)^2}\right)$ and $\sigma_m\left(1, \frac{1}{2^2}, \ldots, \frac{1}{(n-1)^2}\right)$. The results put forth in [39] are nicely representative of how mathematical problems concerning the symbolic computation of series involving harmonic-like numbers can be closely connected with seemingly unrelated subjects in the theory of symmetric functions and in number theory.

The series expansions for powers of the inverse sine mapping proved in [13] involve central binomial coefficients and "nested" harmonic-type multisums, including

$$\sum_{n_1=1}^{k-1} \frac{1}{(2n_1)^2} \sum_{n_2=1}^{n_1-1} \frac{1}{(2n_2)^2} \cdots \sum_{n_N=1}^{n_{N-1}-1} \frac{1}{(2n_N)^2}.$$

Motivated in part by the main results from [13], and in particular the classical infinite series identity

$$\frac{2}{3}\left(\arcsin\left(\frac{z}{2}\right)\right)^4 = \sum_{n=1}^{\infty} \frac{H_{n-1}^{(2)}}{n^2 \binom{2n}{n}} z^{2n}$$

known to Ramanujan [8], the authors in [65] determine congruences for

$$p\sum_{n=1}^{p-1}\frac{H_{n-1}(2)}{n^d\binom{2n}{n}}t^n \pmod p$$

for prime numbers p and for special values of d, generalizing congruence results given by Zhi-Wei Sun in [77], in which the formula

$$\frac{\pi^3}{48}=\sum_{n=1}^{\infty}\frac{2^nH_{n-1}^{(2)}}{n\binom{2n}{n}}$$

is also introduced. This discussion further illustrates how researching new kinds of subjects concerning summations containing harmonic-like numbers together with central binomial coefficients can lead to unexpected results in both applied analysis and number theory, and surprising connections between these disciplines.

In [51], new hypergeometric identities related to Ramanujan-like series for $\frac{1}{\pi}$ are proved using WZ-pairs, and an equivalent formulation of the Ramanujan-type [79] formula

$$\sum_{n=1}^{\infty}\left(-\frac{1}{64}\right)^n\binom{2n}{n}^3(4n+1)H_n=\frac{32\Gamma^2\left(\frac{1}{8}\right)}{3\Gamma^2\left(-\frac{1}{8}\right)\Gamma^2\left(\frac{1}{4}\right)}-\frac{4\ln(2)}{\pi}$$

is employed in the derivation of one of the main identities given in [51]. The related formula

$$\sum_{n=0}^{\infty}\left(-\frac{1}{1024}\right)^n\binom{2n}{n}^5(2-5(4n+1)H_n)=\frac{1024(15\ln(2)-2\pi)}{3\Gamma^4\left(-\frac{1}{4}\right)}$$

was recently proved in [79] through the use of a parameter derivative applied to a classical ${}_6F_5(-1)$ series identity. These results, along with Sun's work on binomial-harmonic sums as in [76, 78], strongly inspire us to explore new techniques for computing series with summands with a factor of the form H_n and fixed powers of $\binom{2n}{n}$ in the numerator.

The formula

$$\sum_{n=1}^{\infty}\binom{2n}{n}^3\frac{H'_{2n}}{2^{8n}}=\frac{\Gamma^6\left(\frac{1}{3}\right)(8\sqrt{3}\ln 2-3\pi)}{24\cdot 2^{2/3}\pi^4} \tag{3.3}$$

is proved in [82] through the use of special values of the multi-dimensional integral

$$W_n(s):=\int_{[0,1]^n}\left|\sum_{k=1}^{n}e^{2\pi x_k i}\right|^s d\mathbf{x}, \tag{3.4}$$

which, as discussed in [82], is used in the analysis of uniform planar random walks in the case whereby every step is a unit step, with the direction being random. In particular, the definite integral in (3.4) is equal to the s^{th} moment of the distance in a given random walk, measured from the origin of the plane after a total of $n \in \mathbb{N}$ steps are taken. The delightful binomial-harmonic series given in (3.3) is proved through an identity for $W_3(s)$, which shows how series with binomial powers and entries in harmonic-like sequences can have direct applications in the theory of random walks, further motivating the exploration of new applications of the main techniques introduced in [19].

The application of parameter derivatives to hypergeometric identities to prove new results on binomial series containing harmonic numbers was recently discussed in [66]. In [66], formulas for evaluating harmonic summations of the following forms are proved using this method:

$$\sum_{n=1}^{\infty} \frac{(2a)_n (1-2a)_n}{(n!)^2} \frac{H_n}{2^n},$$
$$\sum_{n=1}^{\infty} \frac{(2a)_n (2b)_n}{n! \left(a+b+\frac{1}{2}\right)_n} \frac{H_n}{n+1}.$$

Letting the parameter a given in the former series be equal to $\frac{1}{4}$, we obtain the series

$$\sum_{n=1}^{\infty} \binom{2n}{n}^2 \frac{H_n}{32^n}, \tag{3.5}$$

which is evaluated in terms of the gamma function in [66]. Since Lemma 2.2.1 is very useful for evaluating many different kinds of summations involving factors of the form $\binom{2n}{n}^2 H_n$, whereas the summation techniques considered in [66] are given by differentiating specific hypergeometric identities, it is natural to consider the problem of evaluating generalizations of the series in (3.5), through the use of the integration method given in [19], which is described in Section 3.1.2. In particular, for a rational function $r(n)$, it is not obvious as to how to compute generalizations of (3.5) of the form

$$\sum_{n=1}^{\infty} \binom{2n}{n}^2 \frac{H_n \cdot r(n)}{32^n} \tag{3.6}$$

following the strategies outlined in [66], or using the generating functions for sequences involving products of harmonic numbers and central binomial coefficients given in [15].

The infinite series in (3.5) also recently appeared in [79]. In [79], it is noted that by Bailey's theorem, we have that

$${}_2F_1\left[\begin{matrix} a, 1-a \\ c \end{matrix} \,\middle|\, \frac{1}{2}\right] = \frac{\Gamma\left(\frac{c}{2}\right)\Gamma\left(\frac{c+1}{2}\right)}{\Gamma\left(\frac{a+c}{2}\right)\Gamma\left(\frac{1-a+c}{2}\right)}, \tag{3.7}$$

and that (3.5) may be computed in terms of the gamma function by applying the operator $\frac{\partial}{\partial c}$ to both sides of (3.7) in the case whereby $a = \frac{1}{2}$. A formula for a p-adic analogue of (3.5) is also proved in [79], and many supercongruences for finite sums with central binomial coefficients and harmonic-type numbers are established. The lovely formula

$$\sum_{n=1}^{\infty} \binom{2n}{n}^3 \frac{H_n}{64^n} = \frac{512\pi(\pi - 3\ln(2))}{3\Gamma^4\left(-\frac{1}{4}\right)}$$

is also proved in [79] following the "usual" method of applying a parameter derivative to both sides of a known hypergeometric identity; in this case, the classical result known as Dixon's theorem is used. Instead of applying partial derivative operators to classical hypergeometric series identities, we make use, in this Section, of something of an inverse approach by showing how Lemma 2.2.1 may be used in a very general way to evaluate series involving harmonic numbers and squared central binomial coefficients.

In this Section, we offer generalizations of the formula

$$\sum_{n=1}^{\infty} \binom{2n}{n}^2 \frac{H_n}{32^n} = \frac{8\sqrt{\pi}(\pi - 4\ln(2))}{\Gamma^2\left(-\frac{1}{4}\right)} \tag{3.8}$$

that had been noted by Tauraso in 2018 in [79] and by Nicholson in 2018 in [66] by showing how creative applications of the integration strategy we had previously introduced in [19] can be used to evaluate series of the following forms for $z \in \mathbb{Z}_{>0}$:

$$\sum_{n\in\mathbb{N}} \frac{\binom{2n}{n}^2 H_n}{32^n(n+z)}, \quad \sum_{n\in\mathbb{N}} \frac{\binom{2n}{n}^2 H_n}{32^n(2n-2z+1)}, \quad \sum_{n\in\mathbb{N}} \frac{\binom{2n}{n}^2 H_n}{32^n(2n-2z+1)^2}.$$

In Section 3.1.2, we briefly review some preliminary results. In Section 3.2, we offer a new proof of the evaluation for (3.5) using the integral transform from [19], to illustrate the idea of applying this integration technique with respect to summations of the form given in (3.6), and we prove the following new formulas:

$$\sum_{n=1}^{\infty} \left(\frac{1}{32}\right)^n \frac{\binom{2n}{n}^2 H_n}{n+1} = 8 - \frac{2\Gamma^2\left(\frac{1}{4}\right)}{\pi^{3/2}} - \frac{4\pi^{3/2} + 16\sqrt{\pi}\ln(2)}{\Gamma^2\left(\frac{1}{4}\right)},$$

$$\sum_{n=1}^{\infty} \left(\frac{1}{32}\right)^n \frac{\binom{2n}{n}^2 H_n}{2n-1} =$$
$$\frac{(4\ln(2) - \pi)\Gamma^2\left(\frac{1}{4}\right)}{8\pi^{3/2}} + \frac{\sqrt{\pi}(\pi + 4\ln(2) - 4)}{\Gamma^2\left(\frac{1}{4}\right)},$$

$$\sum_{n=1}^{\infty}\left(\frac{1}{32}\right)^{n}\frac{\binom{2n}{n}^{2}H_n}{(2n-1)^2}=$$
$$\frac{\Gamma^2\left(\frac{1}{4}\right)(\pi-4\ln(2))}{8\pi^{3/2}}-\frac{2\sqrt{\pi}(\pi+4\ln(2)-6)}{\Gamma^2\left(\frac{1}{4}\right)}.$$

3.1.2 Preliminaries

The equality whereby

$$2\sum_{n=1}^{\infty}\frac{(2a)_n(2b)_n}{n!\left(a+b+\frac{1}{2}\right)_n}\frac{H_n}{2^n}=\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{n!\left(a+b+\frac{1}{2}\right)_n}H_n,$$

is proved in [66], for elements a and b in the set of complex numbers such that $a+b-\frac{1}{2}\notin\mathbb{Z}_{<0}$, and this identity is used to prove that the series $\sum_{n=1}^{\infty}\frac{(2a)_n(1-2a)_n}{(n!)^2}\frac{H_n}{2^n}$ is equal to

$$\frac{\sqrt{\pi}}{2\Gamma(1-a)\Gamma\left(a+\frac{1}{2}\right)}\left(\psi\left(1-a\right)+\psi\left(a+\frac{1}{2}\right)-\psi\left(1\right)-\psi\left(\frac{1}{2}\right)\right), \tag{3.9}$$

letting $\psi(x)=\frac{\Gamma'(x)}{\Gamma(x)}$ denote the digamma function, thus leading to the evaluation of (3.5) noted in Theorem 3.2.1 below. We present a generalization of this Theorem, using the main method applied in [19].

As noted above, the formula in (3.8), which has motivated much of the work put forth in the present Section, also appears in [79] and is proved through a straightforward application of Bailey's theorem. We offer a new proof of (3.8) that is significantly different compared to the proofs of this result from both [79] and [66]. Variants of our proof of (3.8) may be used to greatly generalize the formula in (3.8) through the use of Lemma 2.2.1.

Lemma 2.2.1 often allows us to express an infinite series of the form

$$\sum_{n=1}^{\infty}g(n)\binom{2n}{n}^{2}H_n \tag{3.10}$$

in a convenient way in terms of a relatively "manageable" definite integral over an elementary function. As discussed above, this is very useful because it is not obvious in general how to symbolically compute series of the form noted in (3.10) by applying parameter derivatives to known hypergeometric identities.

We remark that throughout the course of our present work, expressions such as "closed form" are meant to include evaluations involving the gamma function. We also recall the series expansions for **K** and **E** shown in (1.14) and (1.15), respectively.

3.2 Motivating examples

As we previously noted, the following Theorem follows immediately from the formula for (3.9) given in [66]. We offer a new proof of this result, to illustrate how the main technique in [19] can be used to evaluate series as in (3.6).

Theorem 3.2.1. $\sum_{n=1}^{\infty} \binom{2n}{n}^2 \frac{H_n}{32^n} = \frac{\Gamma^2\left(\frac{1}{4}\right)}{4\sqrt{\pi}} \left(1 - \frac{4\ln(2)}{\pi}\right)$ [66, 79].

Proof. Letting $f(n)$ be equal to $2^{-n}(2n-1)$ in Lemma 2.2.1, we find that the series given in the above Theorem is equal to:

$$-\frac{2\sqrt{2}}{\pi} \int_0^1 \frac{\ln\left(1-x^2\right)}{\sqrt{1-x^2}\sqrt{2-x^2}}\,dx - \frac{\Gamma^2\left(\frac{1}{4}\right)\ln(2)}{\pi^{3/2}}.$$

Using the substitution $u = 1 - x^2$ in the above integrand, we find that

$$\sum_{n=1}^{\infty} \binom{2n}{n}^2 \frac{H_n}{32^n} = -\frac{\sqrt{2}}{\pi} \int_0^1 \frac{\ln(u)}{\sqrt{1-u^2}\sqrt{u}}\,du - \frac{\Gamma^2\left(\frac{1}{4}\right)\ln(2)}{\pi^{3/2}}.$$

The substitution of the Maclaurin series for $\frac{1}{\sqrt{1-u^2}}$ in the above integrand may be used to symbolically compute the above integral. In particular, since

$$\sum_{n=0}^{\infty} (-1)^n \binom{-\frac{1}{2}}{n} u^{2n-\frac{1}{2}} \ln(u) = \frac{\ln(u)}{\sqrt{u}\sqrt{1-u^2}},$$

we have that

$$\int_0^1 \frac{\ln(u)}{\sqrt{u}\sqrt{1-u^2}}\,du = -4\sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n \frac{\binom{2n}{n}}{(4n+1)^2}.$$

We can show that the finite sum

$$\sum_{n=0}^{m} \frac{\left(\frac{1}{4}\right)^n \binom{2n}{n}}{(4n+1)^2}$$

is equal to the following expansion for $m \in \mathbb{N}$, through a simple inductive argument involving a telescoping series.

$$\frac{4^{-m-1}}{(4m+5)^2\Gamma\left(-\frac{1}{4}\right)} \cdot \Bigg(-4^m \pi^{3/2}(4m+5)^2\Gamma\left(\frac{1}{4}\right) -$$
$$\Gamma\left(-\frac{1}{4}\right)\binom{2m+2}{m+1} {}_4F_3\left[\begin{matrix} 1, m+\frac{5}{4}, m+\frac{5}{4}, m+\frac{3}{2} \\ m+2, m+\frac{9}{4}, m+\frac{9}{4} \end{matrix} \,\middle|\, 1\right]\Bigg)$$

Rewrite the above expression as indicated below:

$$-\frac{1}{4}\cdot\frac{\pi^{3/2}\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(-\frac{1}{4}\right)}-\frac{\binom{2m+2}{m+1}}{4^{m+1}(4m+5)^2}\cdot {}_4F_3\left[\begin{matrix}1, m+\frac{5}{4}, m+\frac{5}{4}, m+\frac{3}{2}\\ m+2, m+\frac{9}{4}, m+\frac{9}{4}\end{matrix}\;\middle|\; 1\right].$$

It can be shown that the latter term in the above expression approaches 0 as m approaches infinity, which gives us the closed-form evaluation

$$-\frac{\pi^{3/2}\Gamma\left(\frac{1}{4}\right)}{4\Gamma\left(-\frac{1}{4}\right)}=\sum_{n=0}^{\infty}\frac{\left(\frac{1}{4}\right)^n\binom{2n}{n}}{(4n+1)^2}, \tag{3.11}$$

yielding the desired result. □

To show how Lemma 2.2.1 may be applied in a nontrivial way to evaluate classes of variants of the summation given in Theorem 3.2.1, we consider the problem of evaluating the following natural analogue of the harmonic summation given in the above Theorem:

$$\sum_{n=1}^{\infty}\frac{\binom{2n}{n}^2 H_n}{32^n(n+1)}. \tag{3.12}$$

It is not obvious as to how to evaluate this series following the techniques given in [66]. Computer Algebra Systems such as Mathematica 2022 and Maple 2020 are not able to provide any kind of closed-form evaluation for the series in (3.12), and it is not obvious as to how to apply known integral formulas for harmonic numbers to evaluate this sum. For example, through an application of the formula

$$\int_0^1\frac{1-x^n}{1-x}\,dx=H_n,$$

we see that the summation in (3.12) may be expressed as

$$\int_0^1\frac{\frac{8\mathbf{E}\left(\sqrt{\frac{x}{2}}\right)+4(x-2)\mathbf{K}\left(\sqrt{\frac{x}{2}}\right)}{\pi x}+\sqrt{\frac{2}{\pi}}\frac{\Gamma\left(-\frac{1}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}}{x-1}\,dx,$$

but it is not at all obvious how to compute the difficult integral given above. Similarly, through the use of the integral identity whereby

$$H_n=-n\cdot\int_{x\in[0,1]}\ln(1-x)\cdot x^{n-1}\,dx$$

for $n\in\mathbb{N}_0$, we find that the problem of symbolically computing (3.12) is equivalent to the difficult problem of evaluating the following integrals:

$$-\frac{1}{16}\int_0^1 {}_2F_1\left[\begin{matrix}\frac{3}{2},\frac{3}{2} \\ 3\end{matrix}\,\middle|\, \frac{x}{2}\right]\ln(1-x)\,dx =$$

$$\frac{4}{\pi}\int_0^1\left(\frac{\mathbf{E}\left(\sqrt{\frac{x}{2}}\right)}{x^2}+\frac{\left(\frac{x}{2}-2\right)\mathbf{K}\left(\sqrt{\frac{x}{2}}\right)}{x^2}\right)\ln(1-x)\,dx.$$

In a similar fashion, it appears that it would be infeasible to make use of known integral formulas for central binomial coefficients or Catalan-type numbers to determine a closed-form expression for the infinite series in (3.12). To illustrate this assertion, if we factor out the expression $C_n = \frac{1}{n+1}\binom{2n}{n}$ in the summand in this series and replace this factor with a standard integral expression for the Catalan number C_n of order n, we see that the infinite sum in (3.12) may be expressed as

$$\frac{2\sqrt{2}}{\pi}\int_0^4\sqrt{\frac{4-x}{x(8-x)}}\ln\left(\frac{1}{2}+\sqrt{\frac{2}{8-x}}\right)\,dx,$$

which cannot be evaluated by CAS software such as the 2022 version of Mathematica and Maple 2020. Similarly, if we substitute a Wallis-type integral into the summand in (3.12), this would yield a very recalcitrant integral such as that given below:

$$\frac{1}{\pi}\int_0^{2\pi}\left(4\sec^2(t)\left(\sqrt{1-\frac{\cos^2(t)}{2}}\ln\left(2\sqrt{1-\frac{\cos^2(t)}{2}}\right)-\right.\right.$$
$$\left.\left.\left(\sqrt{1-\frac{\cos^2(t)}{2}}+1\right)\ln\left(\sqrt{1-\frac{\cos^2(t)}{2}}+1\right)+\ln(2)\right)\right)\,dt.$$

The above discussion shows that it is not feasible to use standard or conventional integration methods to evaluate

$$\sum_{n=1}^{\infty}\frac{\binom{2n}{n}^2 H_n}{32^n(n+1)},$$

which shows, in part, why determining the closed-form evaluation given below is challenging. Our proof of the following Theorem nicely illustrates how Lemma 2.2.1 may be applied in a creative way to produce a simple closed-form evaluation that does not follow directly from this integration Lemma.

Theorem 3.2.2. *The series*

$$\sum_{n=1}^{\infty}\left(\frac{1}{32}\right)^n\frac{\binom{2n}{n}^2 H_n}{n+1}$$

is equal to

$$8-\frac{2\Gamma^2\left(\frac{1}{4}\right)}{\pi^{3/2}}-\frac{4\pi^{3/2}+16\sqrt{\pi}\ln(2)}{\Gamma^2\left(\frac{1}{4}\right)}.$$

Proof. Letting $f(n)=\left(\frac{1}{2}\right)^n\frac{2n-1}{n+1}$, by Lemma 2.2.1, we have that the series $\sum_{n=0}^{\infty}\left(\frac{1}{32}\right)^n\frac{\binom{2n}{n}^2 H_n}{n+1}$ is equal to the following expression:

$$\frac{4}{\pi}\int_0^1\frac{\left(\sqrt{4-2x^2}-2\right)\ln\left(1-x^2\right)}{x^2\sqrt{1-x^2}}\,dx-\frac{16\sqrt{\pi}\ln(2)}{\Gamma^2\left(\frac{1}{4}\right)}.$$

So, the problem of computing the series given in Theorem 3.2.2 is equivalent to the problem of evaluating the following integrals:

$$\int_0^1\frac{\left(\sqrt{4-2x^2}-2\right)\ln\left(1-x^2\right)}{x^2\sqrt{1-x^2}}\,dx=\int_0^1\frac{\left(\sqrt{2}\sqrt{1+u}-2\right)\ln(u)}{2(1-u)^{3/2}\sqrt{u}}\,du.$$

Rewrite the expression

$$\int_0^1\frac{\left(-2+\sqrt{2}\sqrt{1+u}\right)\ln(u)}{(1-u)^{3/2}\sqrt{u}}\,du$$

as

$$4\pi+\sqrt{2}\int_0^1\frac{\sqrt{1+u}\ln(u)}{(1-u)^{3/2}\sqrt{u}}\,du. \tag{3.13}$$

Our strategy for computing (3.13) in closed form is to find a formula for the Maclaurin series coefficients for the expression $\frac{\sqrt{1+u}}{(1-u)^{3/2}}$ in the integrand in (3.13), then multiply each term in the corresponding series expansion by $\frac{\ln(u)}{\sqrt{u}}$, and then integrate term-by-term. This may appear to be a very roundabout way of determining the symbolic value of the definite integral in (3.13), but it is not at all clear what kinds of integration methods could be successfully applied to find (3.13) in closed form. From the equality

$$\frac{\sqrt{1+u}}{(1-u)^{3/2}}=\sum_{n=0}^{\infty}\frac{u^{2n}(1+2u+4n(1+u))\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}\Gamma(n+1)}, \tag{3.14}$$

we have that

$$\frac{\sqrt{1+u}}{(1-u)^{3/2}}=\sum_{n=0}^{\infty}\frac{(4n+1)u^{2n}\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}\Gamma(n+1)}+\sum_{n=0}^{\infty}\frac{4u^{2n+1}\Gamma\left(n+\frac{3}{2}\right)}{\sqrt{\pi}\Gamma(n+1)}.$$

The evaluation in (3.14) as well as the equivalent evaluation in the next displayed equation both follow from the generating function for central binomial coefficients. We claim that the definite integral in (3.13) is such that the following equalities hold, but proving this is nontrivial.

$$\int_0^1 \frac{\sqrt{1+u}\ln(u)}{(1-u)^{3/2}\sqrt{u}}\,du = -4\sum_{n=0}^{\infty}\frac{(4n+1)\Gamma\left(n+\frac{1}{2}\right)}{(4n+1)^2\sqrt{\pi}\Gamma(n+1)} -$$

$$16\sum_{n=0}^{\infty}\frac{\Gamma\left(n+\frac{3}{2}\right)}{(4n+3)^2\sqrt{\pi}\Gamma(n+1)}$$

$$= -\frac{1}{16}\sqrt{\frac{\pi}{2}}\Gamma^2\left(-\frac{1}{4}\right) - \frac{\Gamma^2\left(\frac{1}{4}\right)}{\sqrt{2\pi}}.$$

We may rewrite the first series as

$$-4\sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^n \frac{\binom{2n}{n}}{4n+1}. \tag{3.15}$$

Using the generating function for the sequence

$$\left(\binom{2n}{n} : n \in \mathbb{N}_0\right), \tag{3.16}$$

we have that

$$-\frac{4}{\sqrt{1-x^4}} = -4\sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^n x^{4n}\binom{2n}{n}.$$

We have that $\int -\frac{4}{\sqrt{1-x^4}}\,dx$ may be evaluated as $-4F\left(\sin^{-1}(x)\middle| -1\right)$, letting F denote the elliptic integral of the first kind, using the Mathematica notation for this function. Taking limits as $x \to 0$ and $x \to 1$, we obtain the expression $-4\mathbf{K}(\sqrt{-1})$. From the formula

$$L = 2\int_0^1 \frac{dx}{\sqrt{1-x^4}} = 2\mathbf{K}\left(\sqrt{-1}\right)$$

for the lemniscate constant L [80], which is half of the arc length

$$s = \frac{1}{\sqrt{2\pi}}\Gamma^2\left(\frac{1}{4}\right)$$

of a lemniscate with parameter $a = 1$ [80], we obtain the desired evaluation for the sum in (3.15). Now, rewrite the series

$$-16\sum_{n=0}^{\infty}\frac{\Gamma\left(n+\frac{3}{2}\right)}{(4n+3)^2\sqrt{\pi}\Gamma(n+1)}$$

as below:

$$4\sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^n\left(\frac{1}{(4n+3)^2} - \frac{1}{4n+3}\right)\binom{2n}{n}.$$

From the generating function for (3.16), we see that the equivalence

$$4\sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^n\frac{\binom{2n}{n}}{4n+3}=\int_0^1\frac{4x^2}{\sqrt{1-x^4}}\,dx$$

holds. The value of the integral

$$\int_0^1\frac{x^2}{\sqrt{1-x^4}}\,dx$$

is also a lemniscate constant [80], and the following evaluation is known to hold [80]:

$$\int_0^1\frac{x^2}{\sqrt{1-x^4}}\,dx=\frac{\sqrt{2}\pi^{3/2}}{\Gamma^2\left(\frac{1}{4}\right)}.$$

The desired evaluation for the remaining series

$$4\sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^n\frac{1}{(4n+3)^2}\binom{2n}{n}$$

may be proved in essentially the same way compared with the method that was used to obtain the evaluation presented in (3.11): We may inductively prove that the partial sum

$$\sum_{n=0}^{m}\frac{\left(\frac{1}{4}\right)^n\binom{2n}{n}}{(4n+3)^2}$$

may be evaluated as

$$\frac{1}{4^{m+1}(4m+7)^2\Gamma\left(\frac{1}{4}\right)}\Bigg(4^{m-1}(\pi-4)\sqrt{\pi}(4m+7)^2\Gamma\left(-\frac{1}{4}\right)-$$
$$\Gamma\left(\frac{1}{4}\right)\binom{2m+2}{m+1}{}_4F_3\left[\left.\begin{matrix}1,m+\frac{3}{2},m+\frac{7}{4},m+\frac{7}{4}\\ m+2,m+\frac{11}{4},m+\frac{11}{4}\end{matrix}\,\right|\,1\right]\Bigg)$$

and we may evaluate the limit of the above expression as $m\to\infty$. We thus obtain the desired result. □

As discussed in the Introduction, Lemma 2.2.1 is much more versatile compared to the use of specific hypergeometric identities to determine results on series involving $\binom{2n}{n}^2H_n$. To further illustrate this idea, we offer a complete proof of the closed-form evaluation for the infinite series

$$\sum_{n\in\mathbb{N}}\frac{\binom{2n}{n}^2H_n}{32^n(2n-1)}$$

given below in Theorem 3.2.5. We begin with the integration results given in the following two Lemmas.

Lemma 3.2.3. *The integral*

$$\int \frac{\sqrt{1+u}\ln(u)}{\sqrt{1-u}\sqrt{u}}\,du \tag{3.17}$$

may be evaluated as below:

$$\sum_{n=0}^{\infty} \frac{2u^{2n+\frac{1}{2}}\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}\Gamma(n+1)}\left(\frac{\ln(u)}{4n+1}+\frac{u\ln(u)}{4n+3}-\frac{2}{(4n+1)^2}-\frac{2u}{(4n+3)^2}\right).$$

Proof. Differentiating the above series with respect to u, we obtain the infinite series

$$\sum_{n=0}^{\infty} \frac{u^{2n-\frac{1}{2}}(u+1)\Gamma\left(n+\frac{1}{2}\right)\ln(u)}{\sqrt{\pi}\Gamma(n+1)},$$

which is equivalent to

$$\frac{(u+1)\ln(u)}{\sqrt{u}}\cdot\sum_{n=0}^{\infty}\left(\frac{u}{2}\right)^{2n}\binom{2n}{n}.$$

Using the generating function for the integer sequence $\left(\binom{2n}{n} : n \in \mathbb{N}_0\right)$, we obtain the desired result. $\square$

Lemma 3.2.4. *The evaluation*

$$\int_0^1 \frac{\sqrt{1+u}\ln(u)}{\sqrt{1-u}\sqrt{u}}\,du = \frac{(\pi-4)\Gamma^2\left(-\frac{1}{4}\right)}{16\sqrt{2\pi}}+\frac{\pi^{3/2}\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(-\frac{1}{4}\right)}$$

holds.

Proof. Consider the summand given by the series expansion in Lemma 3.2.3. The limit of this summand as u approaches 0 vanishes, and the limit as u approchaes 1 is equal to

$$-\frac{8\left(16n^2+16n+5\right)\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}(4n+1)^2(4n+3)^2\Gamma(n+1)}.$$

This shows us that the integral

$$\int_0^1 \frac{\sqrt{1+u}\ln(u)}{\sqrt{1-u}\sqrt{u}}\,du$$

may be written as

$$-8\cdot\sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^n\frac{\left(16n^2+16n+5\right)\binom{2n}{n}}{(4n+1)^2(4n+3)^2},$$

which, in turn, must be equal to

$$-4\sum_{n=0}^{\infty}\frac{\left(\frac{1}{4}\right)^n\binom{2n}{n}}{(4n+1)^2}-4\sum_{n=0}^{\infty}\frac{\left(\frac{1}{4}\right)^n\binom{2n}{n}}{(4n+3)^2}. \tag{3.18}$$

We had previously evaluated the first series in (3.11). A method for evaluating the latter series is presented in the above proof for Theorem 3.2.2. □

Theorem 3.2.5. *The following equality holds:*

$$\sum_{n=1}^{\infty}\frac{\binom{2n}{n}^2 H_n}{32^n(2n-1)}=\frac{\Gamma^2\left(\frac{1}{4}\right)(4\ln(2)-\pi)}{8\pi^{3/2}}+\frac{\sqrt{\pi}(\pi+4\ln(2)-4)}{\Gamma^2\left(\frac{1}{4}\right)}.$$

Proof. We observe that

$$\sum_{n=0}^{\infty}\left(\frac{1}{32}\right)^n\frac{\binom{2n}{n}^2 H_n}{2n-1}$$

is equal to

$$\frac{2}{\pi}\int_0^1\frac{\sqrt{2-x^2}\ln\left(1-x^2\right)}{\sqrt{2}\sqrt{1-x^2}}\,dx+\frac{4\sqrt{\pi}\ln(2)}{\Gamma^2\left(\frac{1}{4}\right)}+\frac{\Gamma^2\left(\frac{1}{4}\right)\ln(2)}{2\pi^{3/2}} \tag{3.19}$$

from Lemma 2.2.1, and we thus observe that the series given in Theorem 3.2.5 is also equal to

$$\frac{1}{\sqrt{2}\pi}\int_0^1\frac{\sqrt{1+u}\ln(u)}{\sqrt{1-u}\sqrt{u}}\,du+\frac{4\sqrt{\pi}\ln(2)}{\Gamma^2\left(\frac{1}{4}\right)}+\frac{\Gamma^2\left(\frac{1}{4}\right)\ln(2)}{2\pi^{3/2}}.$$

From Lemma 3.2.4, we obtain the desired result. □

Theorem 3.2.6. *The series*

$$\sum_{n=1}^{\infty}\left(\frac{1}{32}\right)^n\frac{\binom{2n}{n}^2 H_n}{(2n-1)^2}$$

is equal to

$$\frac{\Gamma^2\left(\frac{1}{4}\right)(\pi-4\ln(2))}{8\pi^{3/2}}-\frac{2\sqrt{\pi}(\pi+4\ln(2)-6)}{\Gamma^2\left(\frac{1}{4}\right)}.$$

Proof. A direct application of Lemma 2.2.1 shows that the series given in the above Theorem is equal to the following expression:

$$-\frac{\sqrt{2}}{\pi}\int_0^1\frac{\left(\sqrt{2-x^2}+x\sin^{-1}\left(\frac{x}{\sqrt{2}}\right)\right)\ln\left(1-x^2\right)}{\sqrt{1-x^2}}\,dx+$$
$$\sqrt{\frac{2}{\pi}}\ln(2)\left(\frac{\Gamma\left(-\frac{1}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}+\frac{2\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(-\frac{1}{4}\right)}\right).$$

Expanding the above integrand, we again encounter the integral

$$\int_0^1 \frac{\sqrt{2-x^2}\ln\left(1-x^2\right)}{\sqrt{1-x^2}}\,dx$$

which we had previously seen in (3.19). So, we find that the infinite sum given in Theorem 3.2.6 is also equal to the following:

$$-\frac{\sqrt{2}}{\pi}\int_0^1 \frac{x\sin^{-1}\left(\frac{x}{\sqrt{2}}\right)\ln\left(1-x^2\right)}{\sqrt{1-x^2}}\,dx+ \\ \frac{\Gamma^2\left(\frac{1}{4}\right)(\pi-4\ln(2))}{8\pi^{3/2}}-\frac{\sqrt{\pi}(-4+\pi+8\ln(2))}{\Gamma^2\left(\frac{1}{4}\right)}.$$

By rewriting the above integral as

$$\int_0^1 \frac{\sin^{-1}\left(\frac{\sqrt{1-u}}{\sqrt{2}}\right)\ln(u)}{2\sqrt{u}}\,du \tag{3.20}$$

and then substituting the expression $\frac{\sqrt{1-u}}{\sqrt{2}}$ into the Maclaurin series for the inverse sine, we see that the definite integral given in (3.20) is also equal to

$$-\frac{\pi}{4\sqrt{2}}\sum_{n=0}^{\infty}\left(\frac{1}{32}\right)^n\frac{\binom{2n}{n}^2 H_{n+1}}{n+1}+\frac{\sqrt{\pi}\Gamma\left(-\frac{1}{4}\right)\ln(4)}{4\Gamma\left(\frac{1}{4}\right)}$$

and we may thus apply Theorem 3.2.2 to yield the desired result. □

3.3 Generalizations and variations

We begin by noting that one of the key ingredients in our proof of Theorem 3.2.2 was based on the use of a Maclaurin-type series for the expression

$$\frac{\sqrt{1+u}}{(1-u)^{3/2}} \tag{3.21}$$

for $u \in [0,1)$, and that the determination of a suitable power series expansion for this expression was nontrivial in that it is not obvious as to how to find explicit formulas for the Taylor series coefficients for (3.21) without already knowing the formula we had introduced in (3.14). For our strategy in computing the Maclaurin series coefficients for (3.21), we had planned to make use of known results on the series expansion for

$$\sqrt{\frac{1+u}{1-u}}$$

and then "interpret" the left-hand factor in the left-hand side of

$$\frac{1}{1-u}\sqrt{\frac{1+u}{1-u}} = \frac{\sqrt{1+u}}{(1-u)^{3/2}}$$

as a partial sum operator. The preceding discussion nicely illustrates how successful applications of Lemma 2.2.1 often require creative manipulations of generating functions.

It seems that applying the proof technique for Theorem 3.2.2 to try to determine the value of

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2 H_n}{32^n(n+2)} \tag{3.22}$$

may be cumbersome or infeasible. However, by means of a re-indexing argument, we see that there is a connection between (3.22) and Theorem 3.2.2, Theorem 3.2.5, and Theorem 3.2.6.

Theorem 3.3.1. *The series*

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2 H_n}{32^n(n+2)}$$

is equal to

$$\frac{64}{9} - \frac{\Gamma^2\left(\frac{1}{4}\right)(\pi - 4\ln(2) + 18)}{9\pi^{3/2}} + \frac{\sqrt{\frac{2}{\pi}}\Gamma\left(-\frac{1}{4}\right)(9\pi + 36\ln(2) - 16)}{18\Gamma\left(\frac{1}{4}\right)}.$$

Proof. Apply the re-indexing technique indicated below:

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2 H_n}{32^n(n+2)} \\
&= \sum_{n=2}^{\infty} \left(\frac{1}{32}\right)^{n-1} \frac{\binom{2n-2}{n-1}^2 H_{n-1}}{n+1} \\
&= \sum_{n=2}^{\infty} \left(\frac{1}{32}\right)^{n-1} \frac{\binom{2n-2}{n-1}^2 \left(H_n - \frac{1}{n}\right)}{n+1} \\
&= \sum_{n=2}^{\infty} \left(\frac{32^{1-n}\binom{2n-2}{n-1}^2 H_n}{n+1} - \frac{32^{1-n}\binom{2n-2}{n-1}^2}{n(n+1)}\right) \\
&= \frac{1}{2} + \frac{8\sqrt{\frac{2}{\pi}}\Gamma\left(\frac{1}{4}\right)}{9\Gamma\left(-\frac{1}{4}\right)} + 32\sum_{n=2}^{\infty} \frac{32^{-n}\binom{2n-2}{n-1}^2 H_n}{n+1} \\
&= \frac{1}{2} + \frac{8\sqrt{\frac{2}{\pi}}\Gamma\left(\frac{1}{4}\right)}{9\Gamma\left(-\frac{1}{4}\right)} + 8\sum_{n=2}^{\infty} \frac{2^{-5n} n^2 \binom{2n}{n}^2 H_n}{(n+1)(2n-1)^2} \\
&= \frac{1}{2} + \frac{8\sqrt{\frac{2}{\pi}}\Gamma\left(\frac{1}{4}\right)}{9\Gamma\left(-\frac{1}{4}\right)} +
\end{aligned}$$

$$8\sum_{n=2}^{\infty}2^{-5n}\binom{2n}{n}^2 H_n\left(\frac{1}{9(n+1)}+\frac{1}{6(2n-1)^2}+\frac{5}{18(2n-1)}\right).$$

We now have that the desired result follows immediately from Theorem 3.2.2, Theorem 3.2.5, and Theorem 3.2.6. □

To compute a given series of the form

$$\sum_{n=1}^{\infty}\frac{\binom{2n}{n}^2 H_n}{32^n(n+z)}$$

in closed form for $z \in \mathbb{Z}_{>0}$, we make use of the inductive approach described below. We begin by re-writing the above summation as suggested below:

$$\begin{aligned}
&\sum_{n=1}^{\infty}\frac{\binom{2n}{n}^2 H_n}{32^n(n+z)}\\
&=\sum_{n=1}^{\infty}\frac{\binom{2n}{n}^2 H_n}{32^n((n+1)+z-1)}\\
&=\sum_{n=2}^{\infty}\frac{\binom{2n-2}{n-1}^2 H_{n-1}}{32^{n-1}(n+z-1)}\\
&=8\sum_{n=2}^{\infty}\frac{2^{-5n}n^2\binom{2n}{n}^2 H_{n-1}}{(2n-1)^2(n+z-1)}\\
&=8\sum_{n=2}^{\infty}\frac{2^{-5n}n^2\binom{2n}{n}^2\left(H_n-\frac{1}{n}\right)}{(2n-1)^2(n+z-1)}\\
&=8\sum_{n=2}^{\infty}\frac{2^{-5n}n^2\binom{2n}{n}^2 H_n}{(2n-1)^2(n+z-1)}-8\sum_{n=2}^{\infty}\frac{2^{-5n}n\binom{2n}{n}^2}{(2n-1)^2(n+z-1)}.
\end{aligned}$$

Hypergeometric series of the form

$$\sum_{n}\left(\frac{1}{32}\right)^n\frac{n\binom{2n}{n}^2}{(2n-1)^2(n+m)}$$

always have closed-form evaluations for $m \in \mathbb{N}$, as may be verified by writing

$$\sum_{n=0}^{\infty}\frac{n\binom{2n}{n}^2 x^{n+m-1}}{32^n(2n-1)^2}=\frac{1}{8}x^m{}_2F_1\left[\begin{matrix}\frac{1}{2},\frac{1}{2}\\ 2\end{matrix}\,\middle|\,\frac{x}{2}\right] \tag{3.23}$$

and evaluating the above expressions as

$$\frac{1}{8}x^m\left(\frac{4\left(1-\frac{2}{x}\right)\mathbf{K}\left(\sqrt{\frac{x}{2}}\right)}{\pi}+\frac{8\mathbf{E}\left(\sqrt{\frac{x}{2}}\right)}{\pi x}\right) \tag{3.24}$$

and then using known results on moments of complete elliptic integrals. Now, we observe that we may expand the factor

$$\frac{n^2}{(2n-1)^2(n+z-1)}$$

from the summand in the series

$$\sum_n \frac{n^2}{(2n-1)^2(n+z-1)} \left(\frac{1}{32}\right)^n \binom{2n}{n}^2 H_n$$

as follows:

$$\frac{n^2}{(2n-1)^2(n+z-1)} =$$
$$\left(\frac{z-1}{2z-1}\right)^2 \cdot \frac{1}{n+z-1}+$$
$$\frac{4z-3}{2(2z-1)^2} \cdot \frac{1}{2n-1}+$$
$$\frac{1}{2(2z-1)} \cdot \frac{1}{(2n-1)^2}.$$

So, in our attempts to compute

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2 H_n}{32^n(n+z)}$$

in closed form, we see that this problem amounts to the symbolic computation of the following series:

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2 H_n}{32^n(n+(z-1))}, \quad \sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2 H_n}{32^n(2n-1)}, \quad \sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2 H_n}{32^n(2n-1)^2}.$$

So, we see that Theorem 3.2.2 may be regarded as the "base case" for our inductive technique, with Theorem 3.2.5 and Theorem 3.2.6 providing the required evaluations for the latter two sums given above, thus highlighting the utility of the Theorems given in Section 3.2. Through an application of the technique described above, we obtain the following new results:

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2 H_n}{32^n(n+3)} =$$
$$\frac{2048}{225} - \frac{4\sqrt{\pi}(297\pi + 1188\ln(2) - 800)}{225\Gamma^2\left(\frac{1}{4}\right)} -$$
$$\frac{2\Gamma^2\left(\frac{1}{4}\right)(313 + 25\pi - 100\ln(2))}{225\pi^{3/2}},$$

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2 H_n}{32^n (n+4)} =$$
$$\frac{16384}{1225} + \frac{\Gamma^2\left(\frac{1}{4}\right)(-15974 - 1425\pi + 5700\ln(2))}{3675\pi^{3/2}} - \frac{4\sqrt{\pi}(-7984 + 2401\pi + 9604\ln(2))}{1225\Gamma^2\left(\frac{1}{4}\right)},$$

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2 H_n}{32^n (n+5)} =$$
$$\frac{2097152}{99225} + \frac{2\Gamma^2\left(\frac{1}{4}\right)(-1071611 - 98550\pi + 394200\ln(2))}{297675\pi^{3/2}} - \frac{4\sqrt{\pi}(-391872 + 102851\pi + 411404\ln(2))}{33075\Gamma^2\left(\frac{1}{4}\right)}.$$

We encounter computational obstacles in attempting to apply the same kind of inductive procedure with respect to summations of the form

$$\sum_{n\in\mathbb{N}} \frac{\binom{2n}{n}^2 H_n}{32^n (2n-2z+1)}$$

for $z \in \mathbb{Z}_{>0}$. This illustrates how the problem of computing generalizations of (3.5) can be very complicated and often requires a degree of ingenuity in the application of Lemma 2.2.1. The problem of evaluating

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2 H_n}{32^n (2n-3)} \tag{3.25}$$

is especially interesting, since there is an unexpected corollary of the symbolic evaluation for the infinite summation given above that we discuss in Section 3.5.

Suppose that we were to attempt to evaluate the series in (3.25) using Theorem 3.2.5, following the inductive strategy that had been employed in our generalization of the proof for Theorem 3.3.1. So, we must re-index (3.25) as demonstrated below:

$$\sum_{n=1}^{\infty} \frac{32^{-n}\binom{2n}{n}^2 H_n}{2n-3}$$
$$\sum_{n=0}^{\infty} \frac{32^{-n-1}\binom{2n+2}{n+1}^2 H_{n+1}}{2n-1}$$
$$\frac{1}{8}\sum_{n=0}^{\infty} \frac{2^{-5n}(2n+1)^2\binom{2n}{n}^2\left(H_n + \frac{1}{n+1}\right)}{(n+1)^2(2n-1)}$$

$$\frac{1}{8}\sum_{n=0}^{\infty}\frac{2^{-5n}(2n+1)^2\binom{2n}{n}^2 H_n}{(n+1)^2(2n-1)}+\frac{1}{8}\sum_{n=0}^{\infty}\frac{2^{-5n}(2n+1)^2\binom{2n}{n}^2}{(n+1)^3(2n-1)}.$$

However, it is not clear as to how to evaluate the sum

$$\sum_{n=0}^{\infty}\left(\frac{1}{32}\right)^n\frac{\binom{2n}{n}^2(2n+1)^2}{(n+1)^3(2n-1)}$$

in closed form. Moreover, if we apply partial fraction decomposition with respect to the factor

$$\frac{(2n+1)^2}{(n+1)^2(2n-1)}$$

in the "re-indexed" harmonic summation given above, we obtain the expression

$$\frac{16}{9(2n-1)}+\frac{10}{9(n+1)}-\frac{1}{3(n+1)^2},$$

but it is not at all clear as to how the series

$$\sum_{n\in\mathbb{N}}\frac{\binom{2n}{n}^2 H_n}{32^n(n+1)^2}$$

could be evaluated, even through an application of Lemma 2.2.1, since this would require the symbolic computation of a difficult integral such as

$$\int_0^1\frac{\ln(u)\ln\left(1+\frac{\sqrt{1+u}}{\sqrt{2}}\right)}{(1-u)^{3/2}\sqrt{u}}\,du$$

and it is not clear as to how to apply the Maclaurin series substitution strategy employed to evaluate (3.13).

To evaluate series of the form

$$\sum_{n\in\mathbb{N}}\frac{\binom{2n}{n}^2 H_n}{32^n(2n-2z+1)}$$

for $z\in\mathbb{Z}_{>0}$, we may apply the following procedure inspired by our proofs for Theorem 3.2.2 and Theorem 3.2.5.

1. Let $f(n)=\frac{2n-1}{2^n(2n-2z+1)}$, apply Lemma 2.2.1, and evaluate the corresponding hypergeometric series

$$\sum_{n\in\mathbb{N}}\frac{\binom{2n}{n}^2}{32^n(2n-2z+1)}$$

in closed form;

2. Through the application of Lemma 2.2.1 noted above, we obtain an integrand of the form

$$\sqrt{\frac{2-x^2}{1-x^2}} \ln\left(1-x^2\right) p(x)$$

for a polynomial $p(x)$ with algebraic coefficients. Apply the substitution $u = 1 - x^2$;

3. We thus obtain an integrand of the form

$$\sqrt{\frac{1+u}{1-u}} \cdot \frac{\ln(u)}{\sqrt{u}} \cdot q(u)$$

for a polynomial $q(u)$ with algebraic coefficients. Replace the expression $\sqrt{\frac{1+u}{1-u}}$ with its Maclaurin series

$$1 + \sum_{n=1}^{\infty} 2^{1-n} u^n \binom{n-1}{\left\lfloor \frac{n-1}{2} \right\rfloor}$$

in the above integrand and integrate term-by-term.

Using the above procedure, we obtain the following results, thus illustrating the versatility of Lemma 2.2.1:

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2 H_n}{32^n(2n-3)} =$$
$$\frac{\sqrt{\pi}(\pi + 4\ln(2) - 4)}{3\Gamma^2\left(\frac{1}{4}\right)} - \frac{\Gamma^2\left(\frac{1}{4}\right)(15\pi - 60\ln(2) + 8)}{216\pi^{3/2}},$$

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2 H_n}{32^n(2n-5)} =$$
$$\frac{11\pi^{3/2}}{75\Gamma^2\left(\frac{1}{4}\right)} - \frac{\Gamma^2\left(\frac{1}{4}\right)(32 + 51\pi - 204\ln(2))}{1080\pi^{3/2}} + \frac{4\sqrt{\pi}(55\ln(2) - 51)}{375\Gamma^2\left(\frac{1}{4}\right)},$$

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2 H_n}{32^n(2n-7)}$$
$$\frac{\sqrt{\pi}(65\pi + 260\ln(2) - 212)}{875\Gamma^2\left(\frac{1}{4}\right)} + \frac{\Gamma^2\left(\frac{1}{4}\right)(5796\ln(2) - 856 - 1449\pi)}{41160\pi^{3/2}}.$$

Using a similar algorithm, we may evaluate series of the form

$$\sum_{n\in\mathbb{N}} \frac{\binom{2n}{n}^2 H_n}{32^n(2n-2z+1)^2}$$

in closed form for $z \in \mathbb{Z}_{>0}$, and we leave it as an exercise to formalize this idea. For example, using an analogue of the procedure described above, we obtain the following result:

$$\sum_{n=0}^{\infty} \left(\frac{1}{32}\right)^n \frac{\binom{2n}{n}^2 H_n}{(2n-3)^2} = \frac{4\sqrt{\pi}(7-\pi-4\ln(2))}{9\Gamma^2\left(\frac{1}{4}\right)} + \frac{\Gamma^2\left(\frac{1}{4}\right)(8+7\pi-28\ln(2))}{216\pi^{3/2}}.$$

3.4 Ramanujan-type formulas

Inspired by the method we had applied to generalize Theorem 3.3.1, we consider the use of similar strategies to determine explicit evaluations for new $\frac{1}{\pi}$ series with summands given by the product of

$$\frac{\binom{2n}{n}^2 H_n}{16^n}$$

and a rational function $r(n)$. These were the main kinds of mathematical objects under investigation in Section 2, as opposed to series involving summand factors of the form

$$\frac{\binom{2n}{n}^2 H_n}{32^n},$$

which served as something of a basis for Section 3.2 and Section 3.3. To illustrate this idea, we begin by considering the problem of finding a symbolic evaluation for the simple and natural-looking series

$$\sum_{n=1}^{\infty} \frac{C_n^2 H_n}{16^n},$$

letting C_n denote the n^{th} Catalan number, as above. Summations of the form

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2 H_n}{16^n(n+z)^2} \tag{3.26}$$

for $z \in \mathbb{Z}_{>0}$ had not been discussed in Section 2; series of the form in (3.26) cannot be evaluated through a direct or straightforward application of Lemma 2.2.1. In contrast to this integration Lemma, we make use of a recursive approach to prove the following result, which serves as the base case for an inductive generalization.

Theorem 3.4.1. *The following equation holds:*

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2 H_n}{16^n (n+1)^2} = 16 + \frac{32G - 64\ln(2)}{\pi} - 16\ln(2).$$

Proof. We begin by making use of the following result that had been introduced in [19]:

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2 H_n}{16^n (2n-3)} = \frac{-68 + 120\ln(2)}{27\pi}.$$

Recall that the above result follows from Lemma 2.2.1 by letting $f(n) = \frac{2n-1}{2n-3}$. Now, apply the following re-indexing argument:

$$\begin{aligned}
&\frac{-68+120\ln(2)}{27\pi} = \\
&\sum_{n=1}^{\infty} \frac{16^{-n}\binom{2n}{n}^2 H_n}{2n-3} = \\
&\sum_{n=0}^{\infty} \frac{16^{-(n+1)}\binom{2n+2}{n+1}^2 H_{n+1}}{2n-1} = \\
&\frac{1}{4}\sum_{n=0}^{\infty} \frac{4^{-2n}(2n+1)^2\binom{2n}{n}^2 H_{n+1}}{(n+1)^2(2n-1)} = \\
&\frac{1}{4}\sum_{n=0}^{\infty} 4^{-2n}\binom{2n}{n}^2 H_n\left(-\frac{1}{3(n+1)^2} + \frac{10}{9(n+1)} + \frac{16}{9(2n-1)}\right) + \\
&\frac{1}{4}\sum_{n=0}^{\infty} 4^{-2n}\binom{2n}{n}^2\Bigg(-\frac{1}{3(n+1)^3} + \frac{10}{9(n+1)^2} - \\
&\frac{16}{27(n+1)} + \frac{32}{27(2n-1)}\Bigg) = \\
&-\frac{1}{12}\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2 H_n}{16^n(n+1)^2} - \frac{1}{12}\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{16^n(n+1)^3} - \frac{24\ln(2)-40}{27\pi}.
\end{aligned}$$

We remark that for the final equality given above, we implicitly made use of Lemma 2.2.1 in the case whereby $f(n) = 1$. So, we have shown that the problem of symbolically computing the infinite series given in the above Theorem is equivalent to the remarkably simpler problem of computing the hypergeometric series given below:

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{16^n (n+1)^3}. \tag{3.27}$$

Making use of the Catalan number integral formula whereby

$$\frac{\binom{2n}{n}}{n+1} = \frac{1}{2\pi} \int_0^4 x^n \sqrt{\frac{4-x}{x}}\, dx,$$

we find that the series in (3.27) is also equal to

$$\frac{1}{2\pi} \int_0^4 -8\sqrt{4-x} \left(\frac{1}{x}\right)^{3/2} \Bigg(-2 + \sqrt{4-x} +$$
$$2\ln(2) - 2\ln\left(1 + \sqrt{1 - \frac{x}{4}}\right)\Bigg)\, dx.$$

We note that the underlying indefinite integral

$$\int -\frac{4\sqrt{4-x}\left(\frac{1}{x}\right)^{3/2}\left(-2 + \sqrt{4-x} + 2\ln(2) - 2\ln\left(1+\sqrt{1-\frac{x}{4}}\right)\right)}{\pi}\, dx$$

may be evaluated as below; this may be verified by differentiating the following expression:

$$\frac{8}{\pi}\sqrt{\frac{1}{x}}\Bigg(-4i\sqrt{x}\operatorname{Li}_2\left(e^{-2i\sin^{-1}\left(\frac{1}{2}\sqrt{\sqrt{4-x}+2}\right)}\right) + x - 4\sqrt{4-x} -$$
$$4i\sqrt{x}\ln\left(\frac{1}{2}\left(\sqrt{2-\sqrt{4-x}} + i\sqrt{\sqrt{4-x}+2}\right)\right) \cdot \ln\left(\sqrt{4-x}+2\right) -$$
$$2\sqrt{4-x}\ln\left(\sqrt{4-x}+2\right) + \sqrt{4-x}\ln(16) -$$
$$4i\sqrt{x}\sin^{-1}\left(\frac{1}{2}\sqrt{\sqrt{4-x}+2}\right)^2 + 2\sqrt{x}\sin^{-1}\left(\frac{\sqrt{4-x}}{2}\right) -$$
$$4\sqrt{x}\sin^{-1}\left(\frac{1}{2}\sqrt{\sqrt{4-x}+2}\right) \cdot$$
$$\left(-1 + 2\ln\left(1 - e^{-2i\sin^{-1}\left(\frac{1}{2}\sqrt{\sqrt{4-x}+2}\right)}\right)\right) -$$
$$\sqrt{x}\ln(16)\sin^{-1}\left(\frac{\sqrt{4-x}}{2}\right) + 8\Bigg).$$

Taking the limit as $x \to 0$ and $x \to 4$ shows that the above definite integral must be equal to as $-\frac{32G}{\pi} - 16 + \frac{48}{\pi} + 16\ln(2)$, thus completing our proof. □

By analogy with our generalization of Theorem 3.3.1, we make use of the following procedure to compute infinite series of the form

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2 H_n}{16^n (n+z)^2}$$

for $z \in \mathbb{Z}_{>0}$. Begin by rewriting (3.26) as follows:

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2 H_n}{16^n (n+z)^2} = \\
&\sum_{n=2}^{\infty} \frac{\binom{2n-2}{n-1}^2 H_{n-1}}{16^{n-1} (n+z-1)^2} = \\
&4\sum_{n=2}^{\infty} \left(\frac{1}{16}\right)^n \frac{n^2 \binom{2n}{n}^2 H_n}{(2n-1)^2 (n+z-1)^2} - \\
&4\sum_{n=2}^{\infty} \left(\frac{1}{16}\right)^n \frac{n \binom{2n}{n}^2}{(2n-1)^2 (n+z-1)^2}.
\end{aligned}$$

For each $z \in \mathbb{Z}_{>0}$, we may evaluate

$$\sum_{n} \left(\frac{1}{16}\right)^n \frac{n \binom{2n}{n}^2}{(2n-1)^2 (n+z-1)^2},$$

using a recursive approach (see [1] and Section 6 below for closely related material on Ramanujan's S-function). Through the use of partial fraction decomposition, expand the quotient

$$\frac{n^2}{(2n-1)^2 (n+z-1)^2}$$

as suggested below:

$$\begin{aligned}
&-\frac{2z-2}{(2z-1)^3} \cdot \frac{1}{n+z-1} \\
&+\frac{1}{(2z-1)^2} \cdot \frac{1}{(2n-1)^2} \\
&+\frac{(z-1)^2}{(2z-1)^2} \cdot \frac{1}{(n+z-1)^2} \\
&+\frac{4z-4}{(2z-1)^3} \cdot \frac{1}{2n-1}.
\end{aligned}$$

Starting with Theorem 3.4.1 as the base case, we may thus evaluate (3.26) recursively since series of the form

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2 H_n}{16^n (n+m)}$$

for $m \in \mathbb{N}$ may be evaluated directly through Lemma 2.2.1. Following this algorithm, we obtain the new results indicated below:

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2 H_n}{16^n (n+2)^2} = \frac{112}{27} - \frac{64\ln(2)}{9} + \frac{16(13+24G-44\ln(2))}{27\pi},$$

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2 H_n}{16^n (n+3)^2} = \frac{6272}{3375} - \frac{1024\ln(2)}{225} + \frac{23632+30720G-54208\ln(2)}{3375\pi},$$

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2 H_n}{16^n (n+4)^2} = \frac{129536}{128625} - \frac{4096\ln(2)}{1225} + \frac{783408+860160G-1484864\ln(2)}{128625\pi}.$$

3.5 Conclusion

Our explorations on the symbolic computation of series with summands containing

$$\left(\frac{1}{32}\right)^n \binom{2n}{n}^2 H_n$$

as a factor have led us to a surprising discovery concerning a new series involving alternating harmonic numbers, as elaborated below.

By Lemma 2.2.1, by letting $f(n) = \left(\frac{1}{2}\right)^n \frac{2n-1}{2n-3}$, we find that the infinite series

$$\sum_{n=0}^{\infty} \left(\frac{1}{32}\right)^n \frac{\binom{2n}{n}^2 H_n}{2n-3} \tag{3.28}$$

is equal to the following:

$$\frac{2}{\pi}\int_0^1 \frac{\left(\sqrt{2-x^2}+x^2\sqrt{2-x^2}\right)\ln\left(1-x^2\right)}{3\sqrt{2}\sqrt{1-x^2}}\,dx - \frac{\Gamma\left(-\frac{1}{4}\right)\ln(2)}{3\sqrt{2\pi}\Gamma\left(\frac{1}{4}\right)} - \frac{10\sqrt{\frac{2}{\pi}}\Gamma\left(\frac{1}{4}\right)\ln(2)}{9\Gamma\left(-\frac{1}{4}\right)}.$$

Interestingly, if the Mathematica command `Simplify` is applied to this expression, Mathematica produces the following output, letting regularized hypergeometric functions be denoted with ${}_p\tilde{F}_q$:

$$\frac{1}{18\sqrt{2}\pi}\Bigg(-\frac{40\sqrt{\pi}\Gamma\left(\frac{1}{4}\right)\ln(2)}{\Gamma\left(-\frac{1}{4}\right)}-\frac{6\sqrt{\pi}\Gamma\left(-\frac{1}{4}\right)\ln(2)}{\Gamma\left(\frac{1}{4}\right)}+$$

$$3\Bigg(4\gamma\mathbf{E}(\sqrt{-1})+\pi\Bigg(-\sqrt{\pi}\Bigg(2\frac{\partial}{\partial x}{}_3\tilde{F}_2\left[\begin{matrix}-\frac{1}{2},\frac{1}{2},\frac{1}{2}\\ 1,x\end{matrix}\,\middle|\,-1\right]\Bigg|_{x=\frac{1}{2}}+$$

$$\frac{\partial}{\partial x}{}_3\tilde{F}_2\left[\begin{matrix}-\frac{1}{2},\frac{1}{2},\frac{1}{2}\\ 2,x\end{matrix}\,\middle|\,-1\right]\Bigg|_{x=\frac{1}{2}}+$$

$$2\frac{\partial}{\partial x}{}_2F_1\left[\begin{matrix}-\frac{1}{2},\frac{1}{2}\\ x\end{matrix}\,\middle|\,-1\right]\Bigg|_{x=1}+$$

$$\frac{\partial}{\partial x}{}_2\tilde{F}_1\left[\begin{matrix}-\frac{1}{2},\frac{1}{2}\\ x\end{matrix}\,\middle|\,-1\right]\Bigg|_{x=2}\Bigg)\Bigg)\Bigg)\Bigg).$$

From the above output, together with the symbolic calculation for (3.28) that we had provided in Section 3.3, we have that the series

$$\sum_{n=1}^{\infty}\left(-\frac{1}{16}\right)^n\binom{2n}{n}^2\frac{(2n+3)H'_{2n}}{(2n-1)(n+1)} \tag{3.29}$$

is conjecturally equal to

$$\frac{5\Gamma^2\left(\frac{1}{4}\right)(\pi-4\ln(2))}{12\sqrt{2}\pi^{3/2}}+\frac{\Gamma\left(-\frac{1}{4}\right)(3\pi+12\ln(2)-20)}{12\sqrt{\pi}\Gamma\left(\frac{1}{4}\right)}-\frac{2}{3}.$$

This is very interesting because it is unexpected that Lemma 2.2.1 can also be applied to determine new series involving alternating harmonic numbers, and this suggests that there may be a deep connection between Lemma 2.2.1 and the main integration technique from [32]. How can we obtain new classes of series containing harmonic-like numbers of the form H'_{2n} as in (3.29) by showing how the definite integral in Lemma 2.2.1 can be expressed in terms of parameter derivatives of hypergeometric expressions of the form ${}_pF_q(-1)$, as above?

Chapter 4

Series containing squared central binomial coefficients and alternating harmonic numbers

4.1 Introduction

As we had discussed above, there appears to be an interesting connection between series involving

$$\left(\frac{1}{32}\right)^n \binom{2n}{n}^2 H_n$$

and series containing factors of the form

$$\left(-\frac{1}{16}\right)^n \binom{2n}{n}^2 H'_{2n}. \tag{4.1}$$

We had considered, as above, that Lemma 2.2.1 cannot be applied directly to evaluate the series

$$\sum_{n=1}^{\infty} \left(\frac{1}{32}\right)^n \frac{\binom{2n}{n}^2 H_n}{2n-3}, \tag{4.2}$$

in the sense that it is unclear how the required integral expression given by this method may be evaluated. The above experimentally discovered formula for (3.29) motivates the study of summations containing (4.1) as a factor. We introduce an interesting integration technique to evaluate series of this form, offering something of a partial solution to an open problem given in [18].

As we had previously considered, it is surprising that Lemma 2.2.1 can be used to evaluate (3.29), since this Lemma is specifically "designed" for series containing H_n for $n \in \mathbb{N}$, as opposed to series involving even-indexed alternating harmonic numbers. The evaluation of generalizations and variations of (3.29) is left as an open problem in [18], and serves as a basis for the research in this Section, inspiring us to construct an integration technique that allows us to evaluate series of this form, by analogy with Lemma 2.2.1.

4.2 Main results

The main integration technique introduced in this Section is given in Lemma 4.2.1 below. This Lemma provides us with a remarkably simple way of evaluating series of the form noted in (4.3), which are otherwise often very difficult to symbolically compute. The Lemma given below may be regarded as a direct analogue of Lemma 2.2.1.

Lemma 4.2.1. *For a sequence $(f_n)_{n\geq 0}$ such that the series*

$$\sum_{n=0}^{\infty} \left(\frac{1}{16}\right)^n H'_{2n} \binom{2n}{n}^2 \frac{f_n}{n+1} \tag{4.3}$$

converges, the above summation is equal to

$$\frac{4}{\pi} \int_0^1 \sum_{n=0}^{\infty} (-1)^n x^{2n} \sqrt{1-x^2} \binom{-\frac{1}{2}}{n} f_n \ln(x)\, dx \tag{4.4}$$

$$+ \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{1}{16}\right)^n \frac{\binom{2n}{n}^2 (2\ln(2)(n+1)+1)}{(n+1)^2} f_n, \tag{4.5}$$

under the assumption that the sequence f is such that it is possible to reverse the order of integration and summation in (4.4)

Proof. We apply the identity

$$\int_0^1 x^{2n} \sqrt{1-x^2} \ln(x)\, dx = \frac{\sqrt{\pi}\left(H_{n-\frac{1}{2}} - H_{n+1}\right)\Gamma\left(n+\frac{1}{2}\right)}{8\Gamma(n+2)},$$

by reversing the order of integration and summation in (4.4) and simplifying, and we obtain an equivalent form of the above identity for the series in (4.3), rewriting the expression $\Gamma\left(n+\frac{1}{2}\right)$ using central binomial coefficients, according to the Legendre duplication formula. $\square$

The problem of evaluating the infinite series given below is difficult, in the sense that it does not seem to be feasible to apply the generating functions from [15, 37] to evaluate the series in (4.6).

Corollary 4.2.2. *The series*

$$\sum_{n=0}^{\infty} \left(-\frac{1}{16}\right)^n \frac{\binom{2n}{n}^2 H'_{2n}}{n+1} \tag{4.6}$$

is equal to

$$2+\frac{(4\ln(2)-\pi)\Gamma^2\left(\frac{1}{4}\right)}{4\sqrt{2}\pi^{3/2}}-\frac{\sqrt{2\pi}(4+\pi+4\ln(2))}{\Gamma^2\left(\frac{1}{4}\right)}.$$

Proof. Letting $f_n=(-1)^n$, by Lemma 4.2.1, we have that (4.6) is equal to

$$\frac{4}{\pi}\int_0^1 \sqrt{\frac{1-x^2}{1+x^2}}\ln(x)\,dx+2+\frac{\ln(2)\Gamma^2\left(\frac{1}{4}\right)}{\sqrt{2}\pi^{3/2}}-\frac{4\sqrt{2\pi}(2+\ln(2))}{\Gamma^2\left(\frac{1}{4}\right)},$$

and we may evaluate the above integral in the following manner. An antiderivative of the above integrand may be expressed as below:

$$\begin{aligned} &-x\,{}_3F_2\left[\begin{matrix}\frac{1}{4},\frac{1}{4},\frac{1}{2}\\ \frac{5}{4},\frac{5}{4}\end{matrix}\,\middle|\,x^4\right]+\frac{x^3}{9}\,{}_3F_2\left[\begin{matrix}\frac{1}{2},\frac{3}{4},\frac{3}{4}\\ \frac{7}{4},\frac{7}{4}\end{matrix}\,\middle|\,x^4\right]\\ &+x\,{}_2F_1\left[\begin{matrix}\frac{1}{4},\frac{1}{2}\\ \frac{5}{4}\end{matrix}\,\middle|\,x^4\right]\ln x-\frac{x^3}{3}\,{}_2F_1\left[\begin{matrix}\frac{1}{2},\frac{3}{4}\\ \frac{7}{4}\end{matrix}\,\middle|\,x^4\right]\ln x.\end{aligned}$$

Taking limits as $x\to 0$ and $x\to 1$, we may use evaluations for lemniscate-like constants (a term introduced in the subsequent paper [29]) given in the preceding Section. $\square$

Through the use of Lemma 4.2.1, we are also able to evaluate sums of the form

$$\sum_{n=0}^{\infty} \left(-\frac{1}{16}\right)^n \frac{\binom{2n}{n}^2 H'_{2n}}{n+z}$$

for natural numbers $z>1$. Applying partial fraction decomposition to the rational component of the series in (3.29), we obtain the expression

$$\frac{1}{3}\sum_{n=1}^{\infty}\left(\frac{1}{16}\right)^n \frac{(-1)^{n+1}\binom{2n}{n}^2 H'_{2n}}{n+1}+\frac{8}{3}\sum_{n=1}^{\infty}\left(\frac{1}{16}\right)^n \frac{(-1)^{n}\binom{2n}{n}^2 H'_{2n}}{2n-1},$$

and, in view of Theorem 4.2.2, this strongly motivates the evaluation of the series given in the following Theorem, since much of this Section is inspired by the open problem given in [18].

Corollary 4.2.3. *The equality*

$$\sum_{n=0}^{\infty} \frac{\left(-\frac{1}{16}\right)^n \binom{2n}{n}^2 H'_{2n}}{2n-1} = \frac{(\pi - 4\ln(2))\Gamma^2\left(\frac{1}{4}\right)}{8\sqrt{2}\pi^{3/2}} - \frac{\sqrt{\frac{\pi}{2}}(\pi + 4\ln(2) - 4)}{\Gamma^2\left(\frac{1}{4}\right)}$$

holds.

Proof. If we let $f_n = \frac{(-1)^n(n+1)}{2n-1}$, then by Lemma 4.2.1, we have that the series in Corollary 4.2.3 is equal to

$$\frac{4}{\pi}\int_0^1 -\frac{\sqrt{1-x^2}\left(3x^2+2\right)\ln(x)}{2\sqrt{x^2+1}}\,dx - \frac{\sqrt{2\pi}\ln(4)}{\Gamma^2\left(\frac{1}{4}\right)} - \frac{(2+\ln(8))\Gamma^2\left(\frac{1}{4}\right)}{6\sqrt{2}\pi^{3/2}},$$

and we thus obtain the desired result. $\square$

Lemma 4.2.1 may also be used to evaluate

$$\sum_{n=0}^{\infty} \left(-\frac{1}{16}\right)^n \frac{\binom{2n}{n}^2 H'_{2n}}{2n-2z-1}$$

for $z \in \mathbb{N}$. As a way of further demonstrating the utility of Lemma 4.2.1, we offer a simplified proof of the following result that had been introduced in [31].

Corollary 4.2.4. $\sum_{n=1}^{\infty} \left(\frac{1}{16}\right)^n \frac{\binom{2n}{n}^2 H_{2n}}{2n-1} = \frac{6\ln(2)-2}{\pi}$ [31].

Proof. Letting $f_n = \frac{n+1}{2n-1}$, by Lemma 4.2.1 we have that

$$\sum_{n=1}^{\infty} \left(\frac{1}{16}\right)^n \frac{\binom{2n}{n}^2 H'_{2n}}{2n-1} = \frac{4}{\pi}\int_0^1 \frac{1}{2}\left(3x^2-2\right)\ln(x)\,dx - \frac{2(2+\ln(8))}{3\pi},$$

and from the evaluation of $\sum_{n=1}^{\infty} \left(\frac{1}{16}\right)^n \frac{\binom{2n}{n}^2 H_n}{2n-1}$ given in [19] we obtain the desired result. $\square$

By letting $f_n = \frac{n+1}{2n-2z-1}$ for $z \in \mathbb{N}$ with regard to Lemma 4.2.1, we also obtain closed-form evaluations. The series in

$$\frac{4G - 12\ln 2 + 6}{\pi} = \sum_{n=0}^{\infty} \left(\frac{1}{16}\right)^n \binom{2n}{n}^2 \frac{H_{2n}}{(2n-1)^2} \tag{4.7}$$

from [32] is a natural extension of the Choi–Chen series for $\frac{1}{\pi}$ from [37, 38], and it is natural to explore new methods of deriving the formula in (4.7), in the hope of arriving at similar results.

Corollary 4.2.5. *The evaluation in* (4.7) *holds* [32].

Proof. By Lemma 4.2.1, if we let $f_n = \frac{n+1}{(2n-1)^2}$, then we see that the series from (4.7) equals

$$\int_0^1 \frac{1}{2} \ln(x) \left(-2x^2 + 3\sqrt{1-x^2} x \sin^{-1}(x) + 2\right) dx + \frac{4(4+9\ln 2)}{9\pi},$$

and using the symbolic form for $\sum_{n=1}^{\infty} \left(\frac{1}{16}\right)^n \frac{\binom{2n}{n}^2 H_n}{(2n-1)^2}$ given in [19], we obtain the desired result. $\square$

Using the technique given in the proof of Corollary 4.2.5, we are also able to evaluate series of the form $\sum_{n=1}^{\infty} \left(\frac{\binom{2n}{n}}{4^n(2n-2z-1)}\right)^2 H_{2n}$. for $z \in \mathbb{N}$. Again through an application of Lemma 4.2.1 together with Lemma 2.2.1 as in the proof of Theorem 4.2.5, we can evaluate series of the form $\sum_{n=1}^{\infty} \left(\frac{1}{16}\right)^n \frac{\binom{2n}{n}^2 H_{2n}}{n+z}$ for $z \in \mathbb{N}$.

Corollary 4.2.6. *The infinite series*

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2 H_{2n}}{16^n (n+1)^2} \tag{4.8}$$

is equal to $\frac{16G+24-48\ln(2)}{\pi} + 4 - 8\ln(2)$.

Proof. Through a direct application of Lemma 4.2.1 we find that the series

$$\sum_{n=1}^{\infty} \frac{(H_{2n} - H_n)\binom{2n}{n}^2}{16^n (n+1)^2}$$

is equal to $\frac{1}{2} {}_4F_3\left[\begin{matrix} \frac{1}{2}, \frac{1}{2}, 1, 1 \\ 2, 2, 2 \end{matrix} \middle| 1\right] - 4 + \frac{16\ln(2)}{\pi}$. So, from the evaluation

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2 H_n}{16^n (n+1)^2} = 16 + \frac{32G - 64\ln(2)}{\pi} - 16\ln(2)$$

introduced in [18], we find that the problem of evaluating binomial-harmonic series in the above Theorem reduces to the evaluation of a ${}_4F_3(1)$ series with half-integer parameters. As above, using the canonical integral formula

$$\frac{\binom{2i}{i}}{i+1} = \int_0^4 \frac{x^i \sqrt{\frac{4-x}{x}}}{2\pi} \, dx$$

for the Catalan sequence, we may evaluate (3.27). $\square$

Based on the evaluation provided in the above Theorem, through the use of iterative re-indexing, we are able to obtain new evaluations for sums of the form $\sum_{n=1}^{\infty} \left(\frac{\binom{2n}{n}}{4^n(n+z)}\right)^2 H_{2n}$ for natural numbers $z > 1$.

The main Lemma introduced in this Section can also be applied to series that do not involve squared central binomial coefficients. For example, a direct application of Lemma 4.2.1 shows that $\sum_{n=1}^{\infty} \left(\frac{1}{64}\right)^{n} \frac{\binom{2n}{n}\binom{4n}{2n}H_{2n}}{n+1}$ is equal to

$$\frac{8}{3} + \frac{8\sqrt{2} - 40\sqrt{2}\ln(2) + 8\ln\left(1+\sqrt{2}\right)}{3\pi},$$

and this can also be determined using a generating function from [37], together with a Wallis-type integral. The exploration of further applications of Lemma 4.2.1 seems like a worthwhile area to pursue.

Chapter 5

Background on and applications of Fourier–Legendre theory

5.1 Introduction

In the history of mathematical analysis, there are many strategies for computing infinite series in symbolic form and it remains a very active area of research. In our recent publication [31], we introduced a variety of new results on the closed-form evaluation of hypergeometric series and harmonic summations through the use of new techniques that are mainly based on the use of complete elliptic integrals and the theory of Fourier–Legendre (FL) expansions.

Inspired in part by our previous work on the evaluation of a ${}_3F_2(1)$-series related to the parbelos constant [30], which, in turn, came about through the discovery [19] of an integration technique for evaluating series involving squared central binomial coefficients and harmonic numbers in terms of $\frac{1}{\pi}$, in the article [31] we applied a related integration method to determine new identities for hypergeometric expressions, as well as new evaluations for binomial-harmonic series.

We recall the notation and definition for ${}_pF_q$-series, as indicated in (1.13). We are interested in evaluating integrals such as

$$\int_0^1 \mathbf{K}\left(\sqrt{x}\right) g(x)\, dx \tag{5.1}$$

for a suitable function g, by expanding $\mathbf{K}$ as a Maclaurin series, perhaps after a manipulation of the expression $\mathbf{K}$, and integrating term-by-term. However, by replacing $g(x)$ with its shifted Fourier–Legendre series expansion, integrating term-by-term and equating the two resulting series, we often obtain new closed-form evaluations. We illustrate this idea with the example described in Section 5.2 below that is taken from our publication [30], after a preliminary discussion concerning the basics of FL theory.

Legendre functions of order n are solutions to Legendre's differential equation

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$$

for $n > 0$ and $|x| < 1$. For $n \in \mathbb{N}_0$, *Legendre polynomials* $P_n(x)$ are examples of Legendre functions, and may be defined via the *Rodrigues formula*

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n. \tag{5.2}$$

The following equivalent definition for $P_n(x)$ will also be used:

$$P_n(x) = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}^2 (x-1)^{n-k}(x+1)^k. \tag{5.3}$$

The Legendre polynomials form an othogonal family on $(-1, 1)$, with

$$\int_{-1}^{1} P_n(x)P_m(x)\,dx = \frac{2}{2n+1}\delta_{m,n}, \tag{5.4}$$

which gives us the *Fourier–Legendre series* for a suitable function g:

$$g(x) = \sum_{n=0}^{\infty}\left[\frac{2n+1}{2}\int_{-1}^{1} g(t)P_n(t)\,dt\right]P_n(x). \tag{5.5}$$

Letting shifted Legendre polynomials be denoted as $\tilde{P}_n(x) = P_n(2x-1)$, polynomials of this form are orthogonal on $[0, 1]$, with

$$\int_{0}^{1} \tilde{P}_m(x)\tilde{P}_n(x)\,dx = \frac{1}{2n+1}\delta_{m,n}.$$

By analogy with the expansion from (5.5), for a reasonably well-behaved function f on $(0, 1)$, this function may be expressed in terms of shifted Legendre polynomials by writing

$$f(x) = \sum_{m=0}^{\infty} c_m P_m(2x-1).$$

We can determine the scalar coefficient c_m in a natural way using the orthogonality of the family of shifted Legendre polynomials. In particular, we see that if we integrate both sides of

$$\tilde{P}_m(x)f(x) = \sum_{n=0}^{\infty} c_n \tilde{P}_m(x)\tilde{P}_n(x)$$

over $[0, 1]$, we get

$$c_m = (2m+1)\int_0^1 P_m(2x-1)\, f(x)\, dx.$$

Brafman's formula states that

$$\sum_{n=0}^{\infty} \frac{(s)_n(1-s)_n}{(n!)^2} P_n(x)z^n = {}_2F_1\left[\begin{matrix} s, 1-s \\ 1 \end{matrix} \,\middle|\, \alpha\right] {}_2F_1\left[\begin{matrix} s, 1-s \\ 1 \end{matrix} \,\middle|\, \beta\right], \tag{5.6}$$

letting $\alpha = \frac{1-\rho-z}{2}$, $\beta = \frac{1-\rho+z}{2}$, and $\rho = \sqrt{1-2xz+z^2}$ [16]. The canonical generating function for Legendre polynomials is [44, 47]

$$\frac{1}{\sqrt{1-2xz+z^2}} = \sum_{n=0}^{\infty} P_n(x)z^n. \tag{5.7}$$

This gives the following result (see [40] and [47]) which we exploit heavily:

$$\mathbf{K}\left(\sqrt{k}\right) = \sum_{n\geq 0} \frac{2}{2n+1} P_n(2k-1). \tag{5.8}$$

If we make use of the standard moment formula

$$\int_0^1 x^i P_n(2x-1)\, dx = \frac{(i!)^2}{(i-n)!(i+n+1)!} \tag{5.9}$$

for shifted Legendre polynomials, then we can see why (5.8) holds. In particular, from the Maclaurin series for $\mathbf{K}$, we have that

$$\mathbf{K}\left(\sqrt{x}\right) P_n(2x-1) = \frac{\pi}{2} \sum_{i=0}^{\infty} \left(\frac{1}{16}\right)^i \binom{2i}{i}^2 x^i P_n(2x-1),$$

and by rewriting the right-hand side as

$$\frac{\pi}{2} \sum_{i=0}^{\infty} \left(\frac{(2i)!}{i!}\right)^2 \frac{1}{16^i (i-n)!(i+n+1)!} = \frac{2(\sin(\pi n)+1)}{(2n+1)^2}$$

using the moments for the family $\{P_n(2x-1)\}_{n\in\mathbb{N}_0}$, we obtain the desired result.

5.2 A motivating example

With regard to the proof of Lemma 2.2.1, since

$$\int_0^1 \frac{x^{4n}\ln\left(1-x^2\right)}{\sqrt{1-x^2}}\,dx = -\frac{\sqrt{\pi}\Gamma\left(2n+\frac{1}{2}\right)\left(H_{2n}+2\ln(2)\right)}{2\Gamma(2n+1)},$$

and since

$$\sum_{n=0}^{\infty}\frac{(-1)^n x^{4n}\binom{\frac{1}{2}}{n}\ln\left(1-x^2\right)}{\sqrt{1-x^2}} = \sqrt{x^2+1}\ln\left(1-x^2\right)$$

we have that the series

$$\sum_{n=0}^{\infty}\frac{\binom{2n}{n}\binom{4n}{2n}H_{2n}}{64^n(2n-1)}$$

may be evaluated in terms of the following ${}_3F_2(1)$-series:

$${}_3F_2\left[\begin{matrix}-\frac{1}{2},\frac{1}{4},\frac{3}{4}\\ \frac{1}{2},1\end{matrix}\,\middle|\,1\right].$$

Using the On-Line Encyclopedia of Integer Sequences, the author of this Thesis had experimentally discovered that the decimal expansion of the above series seems to agree with that for the the parbelos constant $\frac{\sqrt{2}+\ln(1+\sqrt{2})}{\pi}$ [30]. This conjecture is proved in a variety of different ways in [30], where the "palindromic" formula

$${}_3F_2\left[\begin{matrix}\frac{1}{4},\frac{1}{2},\frac{3}{4}\\ 1,\frac{3}{2}\end{matrix}\,\middle|\,1\right] = \frac{8}{\pi}\tanh^{-1}\tan\frac{\pi}{8} \tag{5.10}$$

is introduced and is used in one proof through an application of Fourier–Legendre theory that heavily makes use of the complete elliptic integral $\mathbf{K}$.

Adopting notation from (5.1), we let $g(x) = \frac{1}{(2-x)^{3/2}}$; the main integral under investigation is then

$$\int_0^1 \mathbf{K}\left(\sqrt{x}\right)\frac{1}{(2-x)^{3/2}}\,dx. \tag{5.11}$$

Letting $(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)}$ denote the Pochhammer symbol, by writing the series in (5.10) as

$${}_3F_2\left[\begin{matrix}\frac{1}{4},\frac{1}{2},\frac{3}{4}\\ 1,\frac{3}{2}\end{matrix}\,\middle|\,1\right] = \sum_{n=0}^{\infty}\frac{\left(\frac{1}{4}\right)_n\left(\frac{1}{2}\right)_n\left(\frac{3}{4}\right)_n}{(1)_n\left(\frac{3}{2}\right)_n n!}$$

$$= \sum_{n=0}^{\infty} \frac{\binom{2n}{n}\binom{4n}{2n}}{(2n+1)64^n}$$

and by noting that

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}\binom{4n}{2n}}{64^n} t^{2n} = \frac{2\mathbf{K}\left(\sqrt{\frac{2t}{t+1}}\right)}{\pi\sqrt{t+1}},$$

we thus find that

$$ {}_3F_2\left[\begin{matrix} \frac{1}{4}, \frac{1}{2}, \frac{3}{4} \\ 1, \frac{3}{2} \end{matrix} \middle| \, 1\right] = \frac{2\sqrt{2}}{\pi} \int_0^1 \frac{\mathbf{K}\left(\sqrt{r}\right)}{(2-r)^{3/2}}\, dr. \tag{5.12}$$

Manipulating (5.7) so as to obtain the generating function for shifted Legendre polynomials, we find that

$$\frac{1}{(2-x)^{\frac{3}{2}}} = \sum_{n=0}^{\infty} (2n+1)\sqrt{2}\left(\sqrt{2}-1\right)^{2n+1} \tilde{P}_n(x),$$

so that

$$\int_{x\in[0,1]} \tilde{P}_m(x) \frac{1}{(2-x)^{\frac{3}{2}}}\, dx = \sqrt{2}\left(\sqrt{2}-1\right)^{2m+1}$$

for all $m \in \mathbb{N}_0$, from the orthogonality relations for shifted FL polynomials. So, from the important expansion in (5.8), we have that the integral in (5.11) equals

$$\begin{aligned}\int_0^1 \sum_{n\geq 0} \frac{2}{2n+1} \tilde{P}_n(x) \frac{1}{(2-x)^{\frac{3}{2}}}\, dx &= \sum_{n\geq 0} \frac{2}{2n+1} \sqrt{2}\left(\sqrt{2}-1\right)^{2n+1} \\ &= \sqrt{2}\ln\left(1+\sqrt{2}\right).\end{aligned}$$

This, together with equation (5.12), gives us the desired evaluation in (5.10).

5.3 An evaluation method due to Campbell, D'Aurizio, and Sondow

Generalized harmonic functions are of the form

$$H_a^{(b)} = \zeta(b) - \zeta(b, a+1), \tag{5.13}$$

where $\zeta(b)$ denotes the Riemann zeta function evaluated at b, and

$$\zeta\left(b, a+1\right) = \sum_{i=0}^{\infty} \frac{1}{\left(i+a+1\right)^{b}}$$

denotes the Hurwitz zeta function with parameters b and $a+1$. In the case $b = 1$, we often omit the superscript on the left-hand side of (5.13). We also adopt the standard convention whereby $H_a^{(0)} = a$. If we let c_n, c'_n, …, $c_n^{(m)}$ be hypergeometric expressions, then we define a *twisted hypergeometric series* to be one of the form

$$\sum_{n=0}^{\infty} \left(c_n H_{\alpha n+\beta}^{(\gamma)} + c'_n H_{\alpha' n+\beta'}^{(\gamma')} + \cdots + c_n^{(m)} H_{\alpha^{(m)} n+\beta^{(m)}}^{(\gamma^{(m)})} \right).$$

The evaluation of such series using our main technique is of central importance in our work.

We have shown in [31] how this technique may be used to prove that

$$\frac{\pi}{4} = \frac{{}_3F_2\left[\begin{matrix} -\eta, \frac{1}{2}, 1 \\ \frac{3}{2}, 2+\eta \end{matrix} \,\middle|\, -1\right]}{{}_3F_2\left[\begin{matrix} \frac{1}{2}, \frac{1}{2}, 1+\eta \\ 1, 2+\eta \end{matrix} \,\middle|\, 1\right]} \tag{5.14}$$

for $\eta > -1$, and that we may obtain the following identity on the moments of the elliptic-type function $\mathbf{E}(\sqrt{x})$:

$$\begin{aligned} \int_0^1 \mathbf{E}(\sqrt{x})\, x^{\eta}\, dx &= \frac{\pi}{2(1+\eta)} \cdot {}_3F_2\left[\begin{matrix} -\frac{1}{2}, \frac{1}{2}, 1+\eta \\ 1, 2+\eta \end{matrix} \,\middle|\, 1\right] \\ &= \frac{4}{3(1+\eta)} \cdot {}_3F_2\left[\begin{matrix} -\frac{1}{2}, 1, -\eta \\ \frac{5}{2}, 2+\eta \end{matrix} \,\middle|\, -1\right]. \end{aligned}$$

In [31], we have proved the equality

$$\sum_{n\geq 0} \binom{4n}{2n}\binom{2n}{n} \frac{H_n - H_{n-1/2}}{64^n} = \frac{\pi}{\sqrt{2}} - \frac{2\sqrt{2}}{\pi} \ln^2\left(\sqrt{2}+1\right)$$

and we also have provided a closed-form evaluation for the series

$$\sum_{n=0}^{\infty} \binom{2n}{n}\binom{4n}{2n} \frac{\frac{1}{2} + nH_{n-\frac{1}{2}} - nH_n}{64^n}.$$

In [31], we have offered a new FL-based proof of the $\frac{1}{\pi}$ formula

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2 H_n}{16^n(2n-1)} = \frac{8\ln(2)-4}{\pi} \tag{5.15}$$

introduced in [19], along with a new proof of the formula

$$\sum_{m,n\geq 0} \left(\frac{1}{16}\right)^m \frac{\binom{2m}{m}^2 H_n}{(n+1)(m+n+2)} = \frac{48+32(\ln(2)-2)\ln(2)}{\pi} - \frac{4\pi}{3}$$

given in [19]. By applying a moment formula that is used to prove the identity in (5.14), we have proved [31] the following double series result:

$$\sum_{n,m\in\mathbb{N}_0} \frac{(n+1)!n!H_n}{(2m+1)(n-m+1)!(m+n+2)!} = 12 - \frac{\pi^2}{3} + 8\ln^2(2) - 16\ln(2).$$

Inspired in part by our above integration method for evaluating series containing factors of the form $H_n^2 + H_n^{(2)}$, we have offered [31] a new proof of the formula

$$\sum_{n=0}^{\infty} \left(\frac{1}{16}\right)^n \binom{2n}{n}^2 \frac{H_n^2 + H_n^{(2)}}{n+1} = \frac{64\ln^2(2)}{\pi} - \frac{8\pi}{3}$$

using the machinery of Fourier–Legendre expansions. We have also proved [31] the formula

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2 H_{2n}}{16^n(2n-1)} = \frac{6\ln(2)-2}{\pi},$$

which extends our proof of (5.15). In [31], we have proved the equality

$$\sum_{n=0}^{\infty} \binom{2n}{n}^2 \frac{H_{n+\frac{1}{4}} - H_{n-\frac{1}{4}}}{16^n} = \frac{\Gamma^4\left(\frac{1}{4}\right)}{8\pi^2} - \frac{4G}{\pi},$$

and have offered a new proof of the equation

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2 H_{2n}}{16^n(2n-1)^2} = \frac{4G+6-12\ln(2)}{\pi}$$

introduced in [32]. Using FL theory, we have, as in [31], proved the formula

$$\sum_{m,n\geq 0} \frac{\binom{2m}{m}^2\binom{2n}{n}^2}{16^{m+n}(m+n+1)(2m+3)} = \frac{7\zeta(3)-4G}{\pi^2},$$

strongly motivating further explorations on our main techniques from [31].

Recall that the polylogarithm function $\mathrm{Li}_n(z)$ is defined so that

$$\mathrm{Li}_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n}$$

for $|z| \leq 1$, and in the case $n = 2$ we obtain the dilogarithm mapping. In [31], we have applied our integration methods to explore connections between generalized hypergeometric functions and polylogarithmic functions, providing an evaluation of

$$\sum_{n\geq 1} \frac{\binom{4n}{2n}\binom{2n}{n}}{n\,64^n} = \frac{3}{16}\cdot {}_4F_3\left[\begin{matrix}1,1,\frac{5}{4},\frac{7}{4}\\ 2,2,2\end{matrix}\,\middle|\,1\right]$$

in terms of a dilogarithmic expression, with a similar evaluation being given for

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2(2n+1)}{16^n(n+1)^4}.$$

In [31], we have proved a variety of new results on generalized complete elliptic integrals of the form

$$\mathfrak{I}(x) = \int_0^{\pi/2} \left(\sqrt{1-x\sin^2\theta}\right)^3\, d\theta.$$

Using the moments of this function, we have proved [31] that the identity

$$\frac{15\pi}{32} = \frac{{}_3F_2\left[\begin{matrix}-\frac{3}{2},1,-\eta\\ \frac{7}{2},2+\eta\end{matrix}\,\middle|\,-1\right]}{{}_3F_2\left[\begin{matrix}-\frac{3}{2},\frac{1}{2},1+\eta\\ 1,2+\eta\end{matrix}\,\middle|\,1\right]}$$

holds for $\eta > -1$. A generalization $\mathfrak{I}_m(x)$ of $\mathfrak{I}(x)$ is also introduced in [31], in which we introduced formulas for evaluating the moments of $\mathfrak{I}_m(x)$.

5.4 Related mathematical literature

Some of our main results from [31] are given by the construction of new formulas for $\frac{1}{\pi}$ using Fourier–Legendre expansions. So, it is natural to consider FL-based techniques that have previously been applied to derive new infinite series formulas for $\frac{1}{\pi}$. One of Ramanujan's most famous formulas for $\frac{1}{\pi}$ is

$$\frac{2}{\pi} = \sum_{n=0}^{\infty} \left(-\frac{1}{64}\right)^n (4n+1)\binom{2n}{n}^3,$$

which was proved by Bauer in 1859 in [6] using a Fourier–Legendre expansion. Levrie applied Bauer's classical FL-based method to functions of the form $(\sqrt{1-a^2x^2})^{2k-1}$ to derive new infinite sums for $\frac{1}{\pi}$, such as the Ramanujan-like equation [61]

$$\frac{8}{9\pi} = \sum_{n=0}^{\infty} \left(-\frac{1}{64}\right)^n \frac{(4n+1)\binom{2n}{n}^3}{(n+1)(n+2)(2n-3)(2n-1)}.$$

Much of the subject matter in Wan's Thesis [82] is closely related to some of our main techniques from [31]. One of our key methods from [31] is the manipulation of generating functions for Legendre polynomials to construct new rational approximations for $\frac{1}{\pi}$, as is the case with [82]. The Section in [82] on Legendre polynomials and series for $\frac{1}{\pi}$ makes use of Brafman's formula (5.6), proving many series for $\frac{1}{\pi}$ typically involving summands with irrational powers.

Wan also explored the use of Legendre polynomials to construct new series for $\frac{1}{\pi}$ in [83] and new results in this area were also introduced by Chan, Wan, and Zudilin in [36]. Brafman's formula is also applied in [83] to produce new results on $\frac{1}{\pi}$ series, whereas the FL-based methods in [31] mainly make use of Fourier–Legendre expansions for elliptic-type expressions such as $\mathbf{K}(\sqrt{x})$. New series for $\frac{1}{\pi}$ are given in [84] through a generalization of Bailey's identity for generating functions given by componentwise products of Apéry-type sequences and the sequence of Legendre polynomials. The construction of hypergeometric series identities using expansions in terms of Legendre polynomials has practical applications in mathematical physics [56] and related areas; a variety of binomial sum identities given in terms of generalized hypergeometric functions are proved in [43] through the use of the family $\{P_n(x) : n \in \mathbb{N}_0\}$.

Chapter 6

Families of double hypergeometric series for constants involving $\frac{1}{\pi^2}$

6.1 Introduction and motivation

The 17 hypergeometric series for $\frac{1}{\pi}$ introduced by Ramanujan are among the most celebrated out of the myriad of groundbreaking mathematical results discovered by Ramanujan. Within the many new areas of research that are still directly inspired by the $\frac{1}{\pi}$ series that Ramanujan discovered over 100 years ago, of particular note are the techniques introduced by Guillera for constructing Ramanujan-like hypergeometric series for $\frac{1}{\pi^2}$; see [2, 48, 49, 50, 52, 53, 54]. This is a main source of inspiration behind the results given in this Section, in which we make use of recent developments in Fourier–Legendre (FL) theory to formulate a powerful method for constructing families of rational double hypergeometric series for expressions containing $\frac{1}{\pi^2}$ as a factor, especially the constant $\frac{\zeta(3)}{\pi^2}$, letting

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}$$

denote the Riemann zeta function. We adopt the standard definition of the term *double hypergeometric series*, which traces back to the 1889 article [57] of Jacob Horn [35], in which this expression is defined as a double power series

$$\sum_{m,n \geq 0} a_{m,n}\, x^m y^n$$

such that the quotients $\frac{a_{m+1,n}}{a_{m,n}}$ and $\frac{a_{m,n+1}}{a_{m,n}}$ are both rational functions in m and n. In this Section, we show how the integration methods introduced in the recent article [31] may be applied to transform double hypergeometric sums that cannot be evaluated with elementary or previously known methods into strikingly simple expressions that provide us with evaluations involving $\frac{1}{\pi^2}$ for the original double series. We prove, via the main FL-based technique from [31], and often in conjunction with Bonnet's recursion formula [42, p. 124], many new double series transformation formulas that may be successfully applied to evaluate infinite families of Guillera-inspired double series for expressions involving $\frac{1}{\pi^2}$.

The families of rational double hypergeometric series that we introduce for constants containing $\frac{1}{\pi^2}$ as a factor are of interest for a variety of reasons. These double sums, despite having very simple summands, are such that the single sums that we obtain by summing over a fixed index also have no known symbolic evaluation, and it appears that the existing literature on double hypergeometric transforms does not apply to the infinite families of double hypergeometric series under consideration, which cannot be expressed with classical families of double hypergeometric sums, as in with Appell functions, Humbert series, etc. The techniques from FL theory that we apply in this Section give us an efficient way of symbolically computing many different kinds of families of double hypergeometric series involving products of binomial coefficients.

6.1.1 A motivating example

Let us highlight the formula

$$\frac{14\zeta(3)}{\pi^2} = \sum_{m,n\geq 0} \left(\frac{1}{16}\right)^{m+n} \frac{\binom{2m}{m}^2\binom{2n}{n}^2}{m+n+1} \tag{6.1}$$

introduced in this Section, in which we construct infinite families of generalizations of the closed-form evaluation in (6.1). The techniques that we introduce in this Section are especially useful for the construction of new families of rational approximations to $\frac{\zeta(3)}{\pi^2}$, and hence our preliminary discussions on the evaluation in (6.1) as a motivating example. However, the main purpose of the research in this Section is to introduce new rational hypergeometric series for constants involving $\frac{1}{\pi^2}$ more generally, as in the series

$$\frac{16\sqrt{2}\ln(2)}{\pi^2} = \sum_{m,n\geq 0} \left(\frac{1}{4}\right)^{2m+3n} \frac{\binom{2m}{m}^2\binom{2n}{n}\binom{4n}{2n}}{m+n+1} \tag{6.2}$$

that is introduced via the main technique in this Section.

Summing with respect to either of the indices involved in the double sums that we introduce produces expressions with no symbolic form involving elementary functions or "established" special functions such as the Riemann zeta function. In particular, if we take the summand in (6.1) and sum over $m \in \mathbb{N}_0$, we obtain

$$\left(\frac{1}{16}\right)^n \frac{\binom{2n}{n}^2}{n+1} \cdot {}_3F_2\left[\begin{matrix} \frac{1}{2}, \frac{1}{2}, n+1 \\ 1, n+2 \end{matrix} \,\middle|\, 1\right].$$

As Adamchik describes in [1], there is a rich mathematical history associated with hypergeometric expressions of the form

$$S(r) = \sum_{k=0}^{\infty} \left(\frac{1}{16}\right)^k \frac{\binom{2k}{k}^2}{k+r} = \frac{1}{r} {}_3F_2\left[\begin{matrix} \frac{1}{2}, \frac{1}{2}, r \\ 1, r+1 \end{matrix} \,\middle|\, 1\right], \tag{6.3}$$

which were of considerable interest to Ramanujan [1], letting $r \in \mathbb{N}/2$. Generalizations of such families of hypergeometric series were also explored in [12]. However, the ${}_3F_2(1)$ series in (6.3) does not admit any closed-form evaluation, despite Ramanujan's identity whereby

$$S(r) = \frac{16^r}{\pi r^2 \binom{2r}{r}^2} \sum_{k=0}^{r-1} \left(\frac{1}{16}\right)^k \binom{2k}{k}^2$$

in the case whereby $r \in \mathbb{N}$. Recalling that the complete elliptic integral of the first kind may be defined so that

$$\mathbf{K}(x) = \frac{\pi}{2}\, {}_2F_1\left[\begin{matrix} \frac{1}{2}, \frac{1}{2} \\ 1 \end{matrix} \,\middle|\, x^2\right],$$

we find that: From the generating function for the sequence of squared central binomial coefficients, we obtain the moment formula

$$S(r) = \frac{2}{\pi} \int_0^1 z^{r-1} \mathbf{K}\left(\sqrt{z}\right)\, dz. \tag{6.4}$$

We also recall the above given Maclaurin series expansion for $\mathbf{E}$. The moments of Ramanujan's generalizations of $\mathbf{K}$ and $\mathbf{E}$ were recently applied in [12] to prove some hypergeometric identities, and to evaluate some ${}_3F_2(1)$-series. However, the evaluation of series involving expressions as in (6.4) so as to form new double series evaluations seems to be a new area of research. Elementary methods of series evaluation, as in with the manipulation of generating functions and the like, cannot be applied to evaluate the series that we introduce.

In consideration of the amount of interest in ${}_3F_2(1)$-series of the form indicated in (6.3), as well as the study of the moments of elliptic-like integrals, more generally, this motivates researching summations involving these kinds of expressions. We again examine the summand of the series in (6.1), noting its symmetry, i.e., its forming a symmetric function. As noted above, the problem of evaluating the series

$$\sum_{m,n\geq 0}\left(\frac{1}{16}\right)^{m+n}\frac{\binom{2m}{m}^2\binom{2n}{n}^2}{m+n+1}=\,? \tag{6.5}$$

is equivalent to the problem of evaluating the following series:

$$\sum_{m=0}^{\infty}\left(\frac{1}{16}\right)^{m}\binom{2m}{m}^2 S(m+1). \tag{6.6}$$

Of course, by the symmetry of the summand in (6.5), we obtain the same series as in (6.6) by summing over either of the indices of the above double sum. However, it is unclear as to what kinds of known results on the hypergeometric S-function defined in (6.3) could be applied successfully to evaluate the difficult double sum in (6.5). Moreover, it appears that it is not possible to use Wallis-type integral formulas for central binomial coefficients to evaluate this sum. The key idea behind our proof for (6.1) is given by using the main method from [31] in conjunction with the moment formula for shifted Legendre polynomials.

6.1.2 Preliminaries

We refer to previous sections on preliminaries on the special functions involved in this Section. However, we find it appropriate to highlight the recurrence

$$(2n+1)xP_n(x)=(n+1)P_{n+1}(x)+nP_{n-1}(x), \tag{6.7}$$

with $P_0(x)=1$ and $P_1(x)=x$. Using this recurrence, we can show that the FL expansion for $x\cdot g(x)$ is given as

$$x\cdot g(x)=\sum_{n=0}^{\infty}\left(\frac{2n+1}{4n+1}A_{2n}+\frac{2n+2}{4n+5}A_{2n+2}\right)P_{2n+1}(x),$$

in the case whereby g is continuous on $[-1,1]$ [61]. The recursion in (6.7) is often referred to as *Bonnet's recursion formula*. We make note of the corresponding recursion $(2n+1)(2x-1)P_n(2x-1)=(n+1)P_{n+1}(2x-1)+n\,P_{n-1}(2x-1)$ for *shifted Legendre polynomials* [33], i.e., polynomials of the form $P_m(2x-1)$ for $m\in\mathbb{N}_0$. Of particular importance in this Section is the moment formula for shifted Legendre polynomials.

We adopt the notational convention concerning the Γ-function indicated below:

$$\Gamma\begin{bmatrix} \alpha, \beta, \ldots, \gamma \\ \\ A, B, \ldots, C \end{bmatrix} = \frac{\Gamma(\alpha)\Gamma(\beta)\cdots\Gamma(\gamma)}{\Gamma(A)\Gamma(B)\cdots\Gamma(C)}.$$

Recalling the moment formula in (5.9), we record the following more general form of this identity:

$$\int_{-a}^{a} (x+a)^{\alpha-1} P_{\nu}\left(\frac{x}{a}\right) dx = (2a)^{\alpha}\,\Gamma\begin{bmatrix} \alpha, \alpha \\ \\ \alpha+\nu+1, \alpha-\nu \end{bmatrix}$$

for $a > 0$ and $\Re(\alpha) > 0$, as given in [68, p. 195].

We offer a full proof for our motivating example in (6.1) after we prove a much more powerful result, as in the transformation formula given as Theorem 6.2.1 below.

6.2 Transformation methods based on the work of Campbell et al.

The study of hypergeometric transforms forms an important aspect about the field of classical analysis. There is much mathematical literature on transformation identities for double hypergeometric series [35, 59, 63, 71, 72, 81], but it seems that previously known results in this area cannot be applied to prove the results that we introduce.

Theorem 6.2.1. *Let* $(f_n : n \in \mathbb{N}_0)$ *be such that the function* $g(x)$ *given by the ordinary generating function for this sequence is well-defined on* $(0, 1)$*, and such that that the integral*

$$\int_0^1 \mathbf{K}(\sqrt{x})g(x)\, dx \tag{6.8}$$

is well-defined and is such that the following holds: If we replace $g(x)$ *in the above integrand with the series* $\sum_{n=0}^{\infty} f_n x^n$*, and replace* $\mathbf{K}(\sqrt{x})$ *by either its Maclaurin series or its shifted FL series, summing over* $m \in \mathbb{N}_0$*, and if the operators* $\int_0^1 \cdot\, dx$*,* $\sum_{n=0}^{\infty} \cdot$*, and* $\sum_{m=0}^{\infty} \cdot$ *commute in either case and are such that the following series are convergent, then*

$$\frac{\pi}{2} \sum_{m,n \geq 0} \left(\frac{1}{16}\right)^m \frac{\binom{2m}{m}^2}{m+n+1} \cdot f_n \tag{6.9}$$

equals

$$2\sum_{\substack{m,n\in\mathbb{N}_0\\ n\geq m}} \frac{1}{2m+1}\frac{(n!)^2}{(n-m)!(n+m+1)!}\cdot f_n. \tag{6.10}$$

Proof. Let f and g be as above, satisfying the above conditions. Starting with the integral in (6.8), let us rewrite this expression as

$$\int_0^1 \left(\sum_{n=0}^{\infty} f_n x^n \mathbf{K}(\sqrt{x})\right) dx. \tag{6.11}$$

We replace the factor $\mathbf{K}(\sqrt{x})$ with its Maclaurin series:

$$\frac{\pi}{2}\int_0^1 \sum_{n=0}^{\infty}\left(f_n x^n \sum_{m=0}^{\infty}\left(\frac{1}{16}\right)^m \binom{2m}{m}^2 x^m\right) dx.$$

So, under the commutativity assumptions of the above Theorem, we obtain the series in (6.9). Now, rewrite (6.11) by replacing $\mathbf{K}(\sqrt{x})$ with the shifted FL series for this expression, so as to obtain:

$$2\int_0^1 \left(\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{f_n}{2m+1} x^n P_m(2x-1)\right) dx. \tag{6.12}$$

From the moment formula for shifted Legendre polynomials, as in (5.9) above, together with the commutativity assumptions given above, we may rewrite (6.12) to yield the expression in (6.10). □

Remark 6.2.2. *With regard to the double sum in* (6.10)*, it is actually not necessary to enforce the condition whereby $n \geq m$, since $\frac{1}{(n-m)!}$ vanishes if $n-m$ is a negative integer, by convention.*

As an application of the above Theorem, we apply this identity to prove the motivating example from (6.1). Letting $f_n := \left(\frac{1}{16}\right)^n \binom{2n}{n}^2$ for $n \in \mathbb{N}_0$, and writing $g(x)$ in place of the power series expansion $\sum_{n=0}^{\infty} x^n f_n$, from the above Theorem, we obtain that

$$\frac{\pi}{2}\sum_{m,n\geq 0}\left(\frac{1}{16}\right)^{m+n} \frac{\binom{2m}{m}^2\binom{2n}{n}^2}{n+m+1} \tag{6.13}$$

must equal

$$2\sum_{\substack{m,n\in\mathbb{N}_0\\ n\geq m}}\left(\frac{1}{16}\right)^n \frac{\binom{2n}{n}^2}{2m+1}\frac{(n!)^2}{(n-m)!(n+m+1)!}.$$

We claim that we may evaluate the finite sum

$$\sum_{n=0}^{w} \left(\frac{1}{16}\right)^{n} \frac{\binom{2n}{n}^2}{2m+1} \frac{(n!)^2}{(n-m)!(n+m+1)!} \tag{6.14}$$

as

$$\left(\frac{1}{16}\right)^{w} \frac{(2w+1)^2\binom{2w}{w}^2}{(2m+1)^3} \frac{(w!)^2}{(w-m)!(w+m+1)!}. \tag{6.15}$$

Let us show this inductively, for $w \in \mathbb{N}_0$. To begin with, we have that

$$\sum_{n=0}^{0} \left(\frac{1}{16}\right)^{n} \frac{\binom{2n}{n}^2}{2m+1} \frac{(n!)^2}{(n-m)!(n+m+1)!} = \delta_{m,0}$$

for $m \in \mathbb{N}_0$, letting δ denote the Kronecker delta function, i.e., so that $\delta_{0,0} = 1$ and $\delta_{m,0}$ vanishes for nonzero m. Setting $w = 0$ in (6.15), the resultant expression also must be equal to $\delta_{m,0}$ for $m \in \mathbb{N}_0$. So, the base case for our inductive proof holds. Now, under the assumption that (6.14) equals (6.15) for a given $w \in \mathbb{N}_0$, this gives us that

$$\sum_{n=0}^{w+1} \left(\frac{1}{16}\right)^{n} \frac{\binom{2n}{n}^2}{2m+1} \frac{(n!)^2}{(n-m)!(n+m+1)!} \tag{6.16}$$

must equal the following:

$$\left(\frac{1}{16}\right)^{w} \frac{(2w+1)^2\binom{2w}{w}^2}{(2m+1)^3} \frac{(w!)^2}{(w-m)!(w+m+1)!} +$$
$$\left(\frac{1}{16}\right)^{w+1} \frac{\binom{2(w+1)}{w+1}^2}{2m+1} \frac{((w+1)!)^2}{(w+1-m)!(w+1+m+1)!}.$$

So, it remains to be seen that the above expression is equal to the following, recalling (6.15):

$$\left(\frac{1}{16}\right)^{w+1} \frac{\binom{2(w+1)}{(w+1)}^2}{(2m+1)^3} \frac{(2(w+1)+1)^2((w+1)!)^2}{((w+1)-m)!((w+1)+m+1)!}.$$

This may be shown by rewriting $\binom{2(w+1)}{w+1}$ as $\frac{2(2w+1)}{w+1}\binom{2w}{w}$ and then simplifying our evaluation for (6.16).

So, now that we have proved our closed-form evaluation for the finite sum in (6.14), let us consider the problem of evaluating the limit of this evaluation as $w \to \infty$. We claim that

$$\lim_{w\to\infty} \left(\frac{1}{16}\right)^{w} \frac{(2w+1)^2\binom{2w}{w}^2}{(2m+1)^3} \frac{(w!)^2}{(w-m)!(w+m+1)!}$$

must equal $\frac{4}{\pi(2m+1)^3}$. To prove our closed form for the above limit, one way of going about with this would be to use *Stirling's approximation*, i.e., so that by repeated applications of the asymptotic equivalence whereby $n! \sim \sqrt{2\pi n}\left(\frac{n}{2}\right)^n$, we obtain that

$$\left(\frac{1}{16}\right)^w \binom{2w}{w}^2 (2w+1)^2 \frac{(w!)^2}{(w-m)!(w+m+1)!} \sim$$
$$\frac{e}{\pi} w^{2w}(w-m)^{-w+m-\frac{1}{2}}(w+m+1)^{-w-m-\frac{3}{2}}(2w+1)^2$$

for fixed m. Taking the limit of this right-hand expression as w approaches infinity, we obtain $\frac{4}{\pi}$.

So, from the foregoing discussion, we have that the identity whereby

$$\frac{4}{\pi(2m+1)^3} = \sum_{n=0}^{\infty} \left(\frac{1}{16}\right)^n \frac{\binom{2n}{n}^2}{2m+1} \frac{(n!)^2}{(n-m)!(n+m+1)!}$$

holds for $m \in \mathbb{N}_0$, recalling that we are letting $\frac{1}{z!}$ vanish for $z \in \mathbb{Z}_{<0}$, so that we may instead let the initial index for the above infinite series be equal to m instead of $n = 0$. Summing over $m \in \mathbb{N}_0$ on both sides of this equation, we can see that the expression in (6.13) must be equal to

$$\frac{8}{\pi} \sum_{m=0}^{\infty} \frac{1}{(2m+1)^3} = \frac{7\zeta(3)}{\pi},$$

giving us the desired result.

We are interested in mimicking the above proof technique, but with the use of variants and generalizations of the integrand factor $\mathbf{K}(\sqrt{x})$, as we later explore.

6.2.1 New results

The above Theorem, on its own, is powerful enough to be able to provide $\frac{1}{\pi^2}$ evaluations for both of the infinite families of rational hypergeometric series suggested below. However, actually applying this Theorem to prove these evaluations requires some work, as we discuss below.

$$-\frac{7\zeta(3)+2}{\pi^2} = \sum_{m,n\geq 0} \left(\frac{1}{16}\right)^{m+n} \frac{\binom{2m}{m}^2\binom{2n}{n}^2}{(m+n+1)(2n-1)} \tag{6.17}$$

$$-\frac{63\zeta(3)+14}{18\pi^2} = \sum_{m,n\geq 0} \left(\frac{1}{16}\right)^{m+n} \frac{\binom{2m}{m}^2\binom{2n}{n}^2}{(m+n+1)(2n-3)} \tag{6.18}$$

$$\cdots$$

$$\frac{7\zeta(3)+6}{\pi^2} = \sum_{m,n\geq 0}\left(\frac{1}{16}\right)^{m+n}\frac{\binom{2m}{m}^2\binom{2n}{n}^2}{(m+n+1)(2n-1)^2}$$

$$\frac{189\zeta(3)+194}{108\pi^2} = \sum_{m,n\geq 0}\left(\frac{1}{16}\right)^{m+n}\frac{\binom{2m}{m}^2\binom{2n}{n}^2}{(m+n+1)(2n-3)^2}$$

$$\cdots$$

We offer a detailed explanation as to how the first out of the above series for $\frac{\zeta(3)}{\pi^2}$ may be obtained from Theorem 6.2.1; the remaining members of the infinite families of evaluations indicated above may be proved in essentially the same way. How can we find more explicit identities for

$$\sum_{m,n\geq 0}\left(\frac{1}{16}\right)^{m+n}\frac{\binom{2m}{m}^2\binom{2n}{n}^2}{(m+n+1)(2n-2z+1)} \tag{6.19}$$

and

$$\sum_{m,n\geq 0}\left(\frac{1}{16}\right)^{m+n}\frac{\binom{2m}{m}^2\binom{2n}{n}^2}{(m+n+1)(2n-2z+1)^2} \tag{6.20}$$

for arbitrary $z \in \mathbb{N}$?

Example 6.2.3. *Via a direct application of Theorem 6.2.1, by letting*

$$f_n = \left(\frac{1}{16}\right)^{n}\frac{\binom{2n}{n}^2}{2n-1}, \tag{6.21}$$

we have that

$$\frac{\pi}{2}\sum_{m,n\geq 0}\left(\frac{1}{16}\right)^{m+n}\frac{\binom{2m}{m}^2\binom{2n}{n}^2}{(n+m+1)(2n-1)} \tag{6.22}$$

may be rewritten as

$$2\sum_{\substack{m,n\in\mathbb{N}_0\\ n\geq m}} f_n\cdot\frac{(n!)^2}{(2m+1)(n-m)!(n+m+1)!}.$$

We claim that the single sum

$$\sum_{n=m}^{\infty} f_n\cdot\frac{(n!)^2}{(2m+1)(n-m)!(n+m+1)!}$$

may be evaluated as $\frac{8}{\pi}\cdot\frac{1}{(2m-1)(2m+1)^3(2m+3)}$*. To show this, we begin by evaluating the finite sum*

$$\sum_{n=m}^{w}\left(\frac{1}{16}\right)^{n}\frac{\binom{2n}{n}^2}{(2n-1)(2m+1)}\frac{(n!)^2}{(n-m)!(n+m+1)!} \tag{6.23}$$

in closed form as follows:

$$\frac{4m^2+4m+4w+3}{4^{2w+1}(2m-1)(2m+3)(2m+1)^3(2w+1)} \times$$
$$\frac{\binom{2w+2}{w+1}^2((w+1)!)^2}{(w-m)!(w+m+1)!}.$$

This can be shown using induction, and again with the usual recursion for central binomial coefficients. More specifically, we have that

$$\sum_{n=m}^{0}\left(\frac{1}{16}\right)^n \frac{\binom{2n}{n}^2}{(2n-1)(2m+1)} \frac{(n!)^2}{(n-m)!(n+m+1)!} = -\delta_{m,0},$$

and by rewriting the above expression involving $\binom{2w+2}{w+1}^2$ *according to the relation whereby*

$$\binom{2w+4}{w+2} = \frac{2(2w+3)\binom{2w+2}{w+1}}{w+2},$$

we obtain our evaluation of (6.23)*. Again by using the asymptotic equivalence given by Stirling's approximation, we may evaluate the limit of our evaluation for* (6.23) *as* $w \to \infty$*, in much the same way as in with our proof of the motivating example highlighted in Section 6.1.1. So, from the identity whereby*

$$\sum_{n=m}^{\infty} f_n \cdot \frac{(n!)^2}{(2m+1)(n-m)!(m+n+1)!} = \frac{8}{\pi} \cdot \frac{1}{(2m-1)(2m+1)^3(2m+3)}$$

that we obtain, we apply $\sum_{m=0}^{\infty} \cdot$ *to both sides of this identity, and, according to Theorem 6.2.1, this gives us that the expression in* (6.22) *equals*

$$\frac{16}{\pi} \sum_{m \geq 0} \frac{1}{(2m-1)(2m+1)^3(2m+3)},$$

and by applying partial fraction decomposition to the above summand, we may obtain the desired closed form for (6.22)*.*

6.2.2 Further applications of the above hypergeometric transform

From the infinite families of series for $\frac{\zeta(3)}{\pi^2}$ displayed above, we are curious as to how the series transformation method given by Theorem 6.2.1 may be applied more generally. What kinds of series for constants involving $\frac{1}{\pi^2}$ can we determine more generally using this result? Adopting notation from Theorem 6.2.1, if we define f_n

with more general products of binomial coefficients, compared to (6.21), then this leads us toward interesting evaluations, as we consider below.

Example 6.2.4. *We claim that the evaluation*

$$\frac{16\sqrt{2}\ln(2)}{\pi^2} = \sum_{m,n\geq 0} \left(\frac{1}{4}\right)^{2m+3n} \frac{\binom{2m}{m}^2\binom{2n}{n}\binom{4n}{2n}}{m+n+1} \tag{6.24}$$

may be proved through a direct application of Theorem 6.2.1, in much the same as in with the steps that we had taken to obtain the rational $\frac{\zeta(3)}{\pi^2}$ series in Example 6.2.3. By setting $f_n := \left(\frac{1}{4}\right)^{3n}\binom{2n}{n}\binom{4n}{2n}$ in Theorem 6.2.1, this Theorem gives us that

$$\frac{\pi}{2}\sum_{m,n\geq 0} \left(\frac{1}{4}\right)^{2m+3n} \frac{\binom{2m}{m}^2\binom{2n}{n}\binom{4n}{2n}}{m+n+1} \tag{6.25}$$

must be equal to

$$2\sum_{\substack{m,n\in\mathbb{N}_0\\ n\geq m}} \left(\frac{1}{2}\right)^{6n} \frac{\binom{2n}{n}\binom{4n}{2n}}{(2m+1)} \frac{(n!)^2}{(n-m)!(n+m+1)!}. \tag{6.26}$$

We claim that the sum

$$\sum_{n=m}^{\infty} \left(\frac{1}{2}\right)^{6n} \frac{\binom{2n}{n}\binom{4n}{2n}}{2m+1} \frac{(n!)^2}{(n-m)!(n+m+1)!}$$

may be evaluated as $\frac{8\sqrt{2}}{\pi}\cdot\frac{1}{(2m+1)(4m+1)(4m+3)}$. This can be shown to be true by evaluating the partial sum

$$\sum_{n=m}^{w} \left(\frac{1}{2}\right)^{6n} \frac{\binom{2n}{n}\binom{4n}{2n}}{2m+1} \frac{(n!)^2}{(n-m)!(n+m+1)!}$$

as

$$\frac{4}{\sqrt{\pi}}\left(\frac{1}{4}\right)^{w} \frac{\Gamma\left(2w+\frac{5}{2}\right)}{(2m+1)(4m+1)(4m+3)\Gamma(w-m+1)\Gamma(w+m+2)},$$

as may be shown inductively. This gives us that the identity whereby

$$\sum_{n=m}^{\infty} \left(\frac{1}{2}\right)^{6n} \frac{\binom{2n}{n}\binom{4n}{2n}}{2m+1} \frac{(n!)^2}{(n-m)!(n+m+1)!} =$$
$$\frac{8\sqrt{2}}{\pi}\cdot\frac{1}{(2m+1)(4m+1)(4m+3)}$$

must hold for $m \in \mathbb{N}_0$. *So, by applying the operator* $\sum_{m=0}^{\infty} \cdot$ *to both sides of this equality, i.e., so that*

$$\sum_{\substack{m,n\in\mathbb{N}_0\\ n\geq m}} \left(\frac{1}{2}\right)^{6n} \frac{\binom{2n}{n}\binom{4n}{2n}}{2m+1} \frac{(n!)^2}{(n-m)!(n+m+1)!} =$$
$$\frac{8\sqrt{2}}{\pi} \sum_{m=0}^{\infty} \frac{1}{(2m+1)(4m+1)(4m+3)},$$

we find that we may evaluate this latter sum, i.e., the single hypergeometric sum given above. Proving that the right-hand side of the above equality equals $\frac{4\sqrt{2}\ln(2)}{\pi}$ *is straightforward, so this evaluation must give us the desired evaluation for the double sum in* (6.24)*, thanks to our application of Theorem 6.2.1, as given by the equality of* (6.25) *and* (6.26).

Example 6.2.5. *The above transformation Theorem also may be used to give us new rational series for constants involving* $\frac{1}{\pi^2}$ *that involve non-central binomial coefficients. For example, a direct application of this Theorem gives us that*

$$\frac{18\sqrt{3}\ln\left(\frac{27}{16}\right)}{\pi^2} = \sum_{m,n\geq 0} \left(\frac{1}{16}\right)^m \left(\frac{1}{27}\right)^n \frac{\binom{2m}{m}^2\binom{2n}{n}\binom{3n}{n}}{m+n+1},$$

as may be proved in much the same way as above. Explicitly, and again adopting our notation from Theorem 6.2.1, we set

$$f_n := \left(\frac{1}{27}\right)^n \binom{2n}{n}\binom{3n}{n},$$

so that Theorem 6.2.1 directly gives us the equality of

$$\frac{\pi}{2} \sum_{m,n\geq 0} \left(\frac{1}{16}\right)^m \left(\frac{1}{27}\right)^n \frac{\binom{2m}{m}^2\binom{2n}{n}\binom{3n}{n}}{m+n+1}$$

and

$$2 \sum_{\substack{m,n\in\mathbb{N}_0\\ n\geq m}} \left(\frac{1}{27}\right)^n \frac{\binom{2n}{n}\binom{3n}{n}}{2m+1} \frac{(n!)^2}{(n-m)!(n+m+1)!}.$$

So, by evaluating

$$\sum_{n=m}^{\infty} \left(\frac{1}{27}\right)^n \frac{\binom{2n}{n}\binom{3n}{n}}{2m+1} \frac{(n!)^2}{(n-m)!(n+m+1)!}$$

in closed form as

$$\frac{9\sqrt{3}}{2\pi} \cdot \frac{1}{(2m+1)(3m+1)(3m+2)}$$

and by applying the operator $\sum_{m=0}^{\infty}$ to both sides of this equality, we may obtain the desired result.

6.2.3 Applying the shifted FL expansion for the complete elliptic integral of the second kind

Many of the main results from [31] are based on the use of integrals given by replacing $\mathbf{K}\left(\sqrt{x}\right)$ with $\mathbf{E}\left(\sqrt{x}\right)$ in (5.1). So, it is natural to construct an analogue of Theorem 6.2.1 as applied to integrals as in (6.27) below, subject to the conditions specified below.

Theorem 6.2.6. *Let the sequence $(f_n : n \in \mathbb{N}_0)$ satisfy the following conditions. We let $g(x)$ denote the ordinary generating function for this sequence, and we suppose that g is well-defined on $(0, 1)$, and is such that that the integral*

$$\int_0^1 \mathbf{E}(\sqrt{x}) g(x)\, dx \tag{6.27}$$

is well-defined and is such that the following holds: If we replace $g(x)$ in the above integrand with the series $\sum_{n=0}^{\infty} f_n x^n$, and replace $\mathbf{E}(\sqrt{x})$ by either its Maclaurin series or its shifted FL series, summing over $m \in \mathbb{N}_0$, and if the operators $\int_0^1 \cdot\, dx$, $\sum_{n=0}^{\infty} \cdot$, and $\sum_{m=0}^{\infty} \cdot$ commute in either case and are such that the following series are convergent, then the expression

$$-\frac{\pi}{2} \sum_{m,n \geq 0} \left(\frac{1}{16}\right)^m \frac{\binom{2m}{m}^2}{(2m-1)(m+n+1)} \cdot f_n \tag{6.28}$$

equals

$$-4 \sum_{\substack{m,n \in \mathbb{N}_0 \\ n \geq m}} \frac{1}{(2m-1)(2m+1)(2m+3)} \frac{(n!)^2}{(n-m)!(n+m+1)!} \cdot f_n. \tag{6.29}$$

Proof. Working under the assumptions of the above Theorem, we rewrite the integral in (6.27) as

$$\int_0^1 \left(\sum_{n=0}^{\infty} x^n f(n) \right) \mathbf{E}\left(\sqrt{x}\right)\, dx, \tag{6.30}$$

and by expanding the factor $\mathbf{E}\left(\sqrt{x}\right)$ with its Maclaurin series, we obtain that (6.30) equals (6.28), by moving the integration operator into the summand of the double series that we obtain. Replacing the integrand factor $\mathbf{E}\left(\sqrt{x}\right)$ with its shifted FL expansion in (6.30), we obtain the expression in (6.29), again under the assumption that we may move the $\int_0^1 \cdot\, dx$ operator inside the double sum that we obtain. $\square$

Through a direct application of the above Theorem, we obtain the infinite families of series for constants involving $\frac{1}{\pi^2}$ indicated below, as may be verified.

$$\frac{7\zeta(3)+6}{2\pi^2} = \sum_{m,n\geq 0}\left(\frac{1}{16}\right)^{m+n}\frac{\binom{2m}{m}^2\binom{2n}{n}^2}{(m+n+1)(2m-1)(2n-1)} \tag{6.31}$$

$$\frac{189\zeta(3)+130}{108\pi^2} = \sum_{m,n\geq 0}\left(\frac{1}{16}\right)^{m+n}\frac{\binom{2m}{m}^2\binom{2n}{n}^2}{(m+n+1)(2m-1)(2n-3)}$$

$$\cdots$$

$$\frac{-21\zeta(3)-50}{8\pi^2} = \sum_{m,n\geq 0}\left(\frac{1}{16}\right)^{m+n}\frac{\binom{2m}{m}^2\binom{2n}{n}^2}{(2m-1)(m+n+1)(2n-1)^2}$$

$$-\frac{189\zeta(3)+514}{324\pi^2} = \sum_{m,n\geq 0}\left(\frac{1}{16}\right)^{m+n}\frac{\binom{2m}{m}^2\binom{2n}{n}^2}{(2m-1)(m+n+1)(2n-3)^2}$$

$$\cdots$$

For the sake of clarity let us offer a proof for the first out of the evaluations listed above. Setting

$$f_n := \left(\frac{1}{16}\right)^n \binom{2n}{n}^2 \frac{1}{2n-1}$$

in Theorem 6.2.6, from this Theorem, we immediately have that

$$-\frac{\pi}{2}\sum_{m,n\geq 0}\left(\frac{1}{16}\right)^{m+n}\frac{\binom{2m}{m}^2\binom{2n}{n}^2}{(2m-1)(2n-1)(m+n+1)}$$

and

$$-4\sum_{\substack{m,n\in\mathbb{N}_0\\ n\geq m}}\left(\frac{1}{16}\right)^n\frac{\binom{2n}{n}^2}{(2m-1)(2m+1)(2m+3)(2n-1)}\frac{(n!)^2}{(n-m)!(n+m+1)!}$$

must be equal. With regard to the latter summand displayed above, we may verify the closed-form identity whereby the single sum

$$\sum_{n=m}^{\infty}\left(\frac{1}{16}\right)^n\frac{\binom{2n}{n}^2}{(2m-1)(2m+1)(2m+3)(2n-1)}\frac{(n!)^2}{(n-m)!(n+m+1)!}$$

must equal

$$\frac{8}{\pi}\cdot\frac{1}{(2m-1)^2(2m+1)^3(2m+3)^2},$$

and by applying $\sum_{m=0}^{\infty}\cdot$ to both sides of this equality, we may obtain the closed-form evaluation displayed in (6.31).

We may also apply Theorem 6.2.6 to obtain rational double hypergeometric series for constants involving expressions as in $\frac{\sqrt{2}\ln(2)}{\pi^2}$, as suggested below:

$$\frac{16\sqrt{2}(-1-8\ln(2))}{15\pi^2} = \sum_{m,n\geq 0} \left(\frac{1}{2}\right)^{4m+6n} \frac{\binom{2m}{m}^2\binom{2n}{n}\binom{4n}{2n}}{(m+n+1)(2m-1)}.$$

6.3 Applications of Bonnet's recursion formula

How can we generalize the transformation methods given in Theorem 6.2.1 and Theorem 6.2.6 so as to be applicable to more general integrals of the form

$$\int_0^1 e(x)g(x)\,dx \tag{6.32}$$

for an elliptic-type expression $e(x)$? For example, let us consider making use of the following Maclaurin series:

$$\frac{\pi}{4}\sum_{m=0}^{\infty} \frac{x^{m+1}\binom{2m}{m}^2}{16^m(m+1)} = \mathbf{E}\left(\sqrt{x}\right) - \mathbf{K}\left(\sqrt{x}\right) + x\,\mathbf{K}\left(\sqrt{x}\right). \tag{6.33}$$

To mimic our proofs of Theorem 6.2.1 and Theorem 6.2.6, we need to compute the shifted FL expansion for the right-hand side of the above equality. To determine this expansion, we make use of Bonnet's recursion formula.

In general, we obtain much more complicated coefficients in the shifted FL expansions for expressions such as

$$\frac{\mathbf{E}(\sqrt{x}) - \mathbf{K}(\sqrt{x})}{x}, \tag{6.34}$$

i.e., expressions involving negative powers of x times elliptic-type integral expressions. For example, we have that

$$\int_0^1 \frac{\mathbf{E}\left(\sqrt{x}\right) - \mathbf{K}\left(\sqrt{x}\right)}{x} P_n(2x-1)\,dx = 4(-1)^n\overline{O}_{n+1} - \frac{2}{2n+1} + (-1)^{n+1}\pi,$$

where $\overline{O}_n = \sum_{k=0}^{n-1}\frac{(-1)^k}{2k+1}$. These more complicated kinds of FL expansions make it difficult to apply analogues of the transformation identities as in Theorem 6.2.1 and Theorem 6.2.6 in the case whereby we use the Maclaurin/FL series for expressions involving negative powers of x and elliptic-type functions, as in (6.34). On the other hand, using a direct analogue of Theorems 6.2.1 and 6.2.6 based on the identity in (6.33), together with an application of Bonnet's recursion formula to give us the shifted FL identity whereby

$$\mathbf{E}\left(\sqrt{x}\right) - \mathbf{K}\left(\sqrt{x}\right) + x\mathbf{K}\left(\sqrt{x}\right) = 4\sum_{m=0}^{\infty} \frac{(2m+1)}{(2m-1)^2(2m+3)^2} P_m(2x-1),$$

we may obtain the infinite families of double series for constants involving $\frac{1}{\pi^2}$ indicated below, letting f_n be equal to

$$\frac{\binom{2n}{n}^2}{16^n(2n-2z+1)} \quad \text{or} \quad \frac{\binom{2n}{n}^2}{16^n(2n-2z+1)^2}$$

for a fixed parameter $z \in \mathbb{N}$.

$$\begin{aligned} \frac{7\zeta(3)-26}{4\pi^2} &= \sum_{m,n\geq 0} \left(\frac{1}{16}\right)^{m+n} \frac{\binom{2m}{m}^2\binom{2n}{n}^2}{(m+1)(m+n+2)(2n-1)} \\ \frac{189\zeta(3)-958}{324\pi^2} &= \sum_{m,n\geq 0} \left(\frac{1}{16}\right)^{m+n} \frac{\binom{2m}{m}^2\binom{2n}{n}^2}{(m+1)(m+n+2)(2n-3)} \\ &\cdots \end{aligned} \tag{6.35}$$

$$\begin{aligned} \frac{78-21\zeta(3)}{8\pi^2} &= \sum_{m,n\geq 0} \left(\frac{1}{16}\right)^{m+n} \frac{\binom{2m}{m}^2\binom{2n}{n}^2}{(m+1)(m+n+2)(2n-1)^2} \\ \frac{1682-315\zeta(3)}{648\pi^2} &= \sum_{m,n\geq 0} \left(\frac{1}{16}\right)^{m+n} \frac{\binom{2m}{m}^2\binom{2n}{n}^2}{(m+1)(m+n+2)(2n-3)^2} \\ &\cdots \end{aligned} \tag{6.36}$$

We offer a proof for the first out of the formulas listed above; the remaining evaluations may be proved in the same kind of way. We begin by defining

$$f_n := \left(\frac{1}{16}\right)^n \binom{2n}{n}^2 \frac{1}{2n-1},$$

and with regard to (6.32), we also set

$$e(x) := \mathbf{E}(\sqrt{x}) - \mathbf{K}(\sqrt{x}) + x\,\mathbf{K}(\sqrt{x}),$$

and we define $g(x)$ so that:

$$g(x) := \sum_{n=0}^{\infty} f_n\, x^n.$$

Mimicking our proofs of Theorems 6.2.1 and 6.2.6, we can show that

$$\frac{\pi}{4} \sum_{m,n \geq 0} \left(\frac{1}{16}\right)^m \frac{\binom{2m}{m}^2}{(m+1)(m+n+2)} \cdot f_n$$

must be equal to:

$$4 \sum_{\substack{m,n \in \mathbb{N}_0 \\ n \geq m}} \frac{2m+1}{(2m-1)^2(2m+3)^2} \frac{(n!)^2}{(n-m)!(n+m+1)!} \cdot f_n.$$

With regard to this latter summand, applying $\sum_{n=m}^{\infty} \cdot$ to this expression and evaluating this resultant series, and then summing over $m \in \mathbb{N}_0$, we are led to the explicit evaluation in (6.35).

Again applying our main technique in the case whereby we set the elliptic-type factor $e(x)$ in the integrand in (6.32) to be equal to $e(x) = \mathbf{E}\left(\sqrt{x}\right) - \mathbf{K}\left(\sqrt{x}\right) + x\mathbf{K}\left(\sqrt{x}\right)$ we may obtain families of rational double hypergeometric series for constants involving $\frac{1}{\pi^2}$, with non-central binomial coefficients involved in the summand, as below:

$$\frac{72\sqrt{3}(71 - 36\ln(2) + 27\ln(3))}{1225\pi^2} = \sum_{m,n \geq 0} \frac{\binom{2m}{m}^2 \binom{2n}{n} \binom{3n}{n}}{16^m 27^n (m+1)(m+n+2)}.$$

6.3.1 The generating function for squares of Catalan numbers

From the Maclaurin series for $\mathbf{K}$ and $\mathbf{E}$, we obtain the following identity:

$$\sum_{m=0}^{\infty} \left(\frac{1}{16}\right)^m \binom{2m}{m}^2 \frac{x^{m+1}}{(m+1)^2} = \frac{16\,\mathbf{E}\left(\sqrt{x}\right)}{\pi} - \frac{8\mathbf{K}\left(\sqrt{x}\right)}{\pi} + \frac{8x\,\mathbf{K}\left(\sqrt{x}\right)}{\pi} - 4.$$

Through a direct application of Bonnet's recursion formula, we may express the right-hand side of the above identity with the shifted FL expansion given as below:

$$\frac{128}{\pi} \sum_{m=0}^{\infty} \frac{P_m(2x-1)}{(2m-1)^2(2m+1)(2m+3)^2} - 4.$$

Mimicking our proofs for Theorems 6.2.1 and 6.2.6, and exploiting the above shifted FL expansion, we can show, for sequences $(f_n : n \in \mathbb{N}_0)$ satisfying the appropriate analogues of the conditions in Theorems 6.2.1 and 6.2.6, that the identity whereby

$$\sum_{m,n \geq 0} \left(\frac{1}{16}\right)^m \frac{\binom{2m}{m}^2}{(m+1)^2(m+n+2)} \cdot f_n$$

equals $-4\sum_{n=0}^{\infty}\frac{1}{n+1}\cdot f_n$ plus

$$\frac{128}{\pi}\sum_{\substack{m,n\in\mathbb{N}_0\\ n\geq m}}\frac{(n!)^2}{(2m-1)^2(2m+1)(2m+3)^2(n-m)!(n+m+1)!}\cdot f_n$$

holds. This may be used to prove that

$$\frac{24+28\zeta(3)}{\pi^2}-\frac{16}{\pi}=\sum_{m,n\geq 0}\left(\frac{1}{16}\right)^{m+n}\frac{\binom{2m}{m}^2\binom{2n}{n}^2}{(m+1)^2(m+n+2)},$$

which is particularly interesting, providing a rational double hypergeometric series for a closed form involving the constants $\frac{1}{\pi}$, $\frac{1}{\pi^2}$, and $\frac{\zeta(3)}{\pi^2}$. We obtain the infinite generalizations of the above evaluation suggested below.

$$\frac{32}{3\pi}-\frac{21\zeta(3)+50}{2\pi^2}=\sum_{m,n\geq 0}\left(\frac{1}{16}\right)^{m+n}\frac{\binom{2m}{m}^2\binom{2n}{n}^2}{(m+1)^2(m+n+2)(2n-1)}$$

$$\frac{224}{45\pi}-\frac{945\zeta(3)+1738}{162\pi^2}=\sum_{m,n\geq 0}\left(\frac{1}{16}\right)^{m+n}\frac{\binom{2m}{m}^2\binom{2n}{n}^2}{(m+1)^2(m+n+2)(2n-3)}$$

$$\cdots$$

$$\frac{21\zeta(3)+178}{4\pi^2}-\frac{128}{9\pi}=\sum_{m,n\geq 0}\left(\frac{1}{16}\right)^{m+n}\frac{\binom{2m}{m}^2\binom{2n}{n}^2}{(m+1)^2(m+n+2)(2n-1)^2}$$

$$\frac{1246+147\zeta(3)}{108\pi^2}-\frac{2432}{675\pi}=\sum_{m,n\geq 0}\left(\frac{1}{16}\right)^{m+n}\frac{\binom{2m}{m}^2\binom{2n}{n}^2}{(m+1)^2(m+n+2)(2n-3)^2}$$

$$\cdots$$

For example, setting

$$f_n:=\left(\frac{1}{16}\right)^n\binom{2n}{n}^2\frac{1}{2n-1},$$

we find that the problem of evaluating the first out of the double series listed above is equivalent to that for:

$$\sum_{\substack{m,n\in\mathbb{N}_0\\ n\geq m}}\frac{(n!)^2f_n}{(2m-1)^2(2m+1)(2m+3)^2(n-m)!(n+m+1)!},$$

and by applying $\sum_{n=m}^{\infty}\cdot$ to the expression given by the above summand, we obtain

$$\frac{8}{\pi}\cdot\frac{1}{(2m-1)^3(2m+1)^3(2m+3)^3},$$

and by summing over $m \in \mathbb{N}_0$, this allows us to determine the desired closed form, in this case.

We may greatly generalize the double hypergeometric transforms given in this Section, through the use of Bonnet's recursion formula, by mimicking the above techniques and by letting the elliptic-type integral function $e(x)$ in (6.32) be equal to power series expansions as below, letting $z_1 \in \mathbb{N}$ and $z_2 \in \mathbb{N}_0$:

$$\sum_{m=0}^{\infty} \left(\frac{1}{16}\right)^m \frac{\binom{2m}{m}^2}{m+z_1} x^{m+z_1+z_2}, \sum_{m=0}^{\infty} \left(\frac{1}{16}\right)^m \frac{\binom{2m}{m}^2}{(m+z_1)^2} x^{m+z_1+z_2},$$
$$\sum_{m=0}^{\infty} \left(\frac{1}{16}\right)^m \frac{\binom{2m}{m}^2}{2m-2z_1+1} x^{m+z_2}, \sum_{m=0}^{\infty} \left(\frac{1}{16}\right)^m \frac{\binom{2m}{m}^2}{(2m-2z_1+1)^2} x^{m+z_2}.$$

We strongly encourage the pursuit of new research areas based on these kinds of applications of the techniques introduced in this Section.

Chapter 7

Further explorations

This concluding section is partly based on feedback on this Thesis from members of this author's supervisory committee regarding further areas of research regarding the material in this Thesis.

To begin with, Nantel Bergeron [7] suggested that the material in Sections 5 and 6 may be generalized using families of orthogonal polynomials apart from the Legendre polynomials. In this regard, it may be worthwhile to explore the use of Hermite polynomials, but a full exploration of this kind of subject is beyond the scope of our current considerations. Incidentally, the use of orthogonal polynomials in what is referred to as the *Askey scheme* was suggested in the context of the author's recent research with Wenchang Chu related to the double series introduced in [28]. Systematically applying polynomials in the Askey scheme using direct analogues or variants of main identities from Sections 5 and 6 may serve as a basis for a suitable follow-up to this Thesis.

The recent publication [28] indicated above was based on further applications of the double series transforms from Section 6, using special values for expressions involving the dilogarithm function. In the author's forthcoming paper for the *Bulletin of the Irish Mathematical Society* [22], it is noted that previously known two-term dilogarithm relations as in [21] may be applied in conjunction with identities from [28] in order to obtain interesting formulas as below [22]:

$$\sum_{m,n\geq 0} \left(\frac{1}{16}\right)^m \left(\frac{1}{20}\right)^n \frac{\binom{2m}{m}^2\binom{2n}{n}}{m+n+1} = \frac{\sqrt{5}\pi}{3} - \frac{6\sqrt{5}\ln^2(\phi)}{\pi},$$

$$\sum_{m,n\geq 0} \left(\frac{1}{16}\right)^m \left(-\frac{1}{12}\right)^n \frac{\binom{2m}{m}^2\binom{2n}{n}}{m+n+1} = \frac{16G}{\sqrt{3}\pi} - \frac{2\ln\left(2+\sqrt{3}\right)}{\sqrt{3}}.$$

As in [22], we encourage a full exploration of this subject.

References citing the journal article reproduced in Section 2 include [33, 86, 87, 88]. We encourage the development of the methods given in this Thesis in conjunction with the material from [33, 86, 87, 88]. In this regard, the coefficient-extraction methods due to Wang and Chu [87] are especially noteworthy, since these methods introduced in [87] yielded explicit identities for infinite families of generalizations of series introduced in [19]. The bijective result known as the modified Abel lemma on summation by parts was recently employed in [27] to build on the material from both [19] and [87]. The integration-based methods from Section 2, the hypergeometric-based methods from [87], and the combinatorial methods from [27] are remarkably and fundamentally different, but since these disparate methods have been successfully applied to achieve similar results, this suggests that it may be worthwhile to pursue interdisciplinary explorations based on "combinations" of the methods from [19, 27, 87].

With reference to the list of the author's publications given above, prior to Section 1, the author has explored the use of a variety of different techniques in relation to the contents of Sections 2–6. In this regard, the Wilf–Zeilberger method [67] has been applied by the author in [23, 24, 25], and fractional calculus-based methods have been applied in [26]. The author is actively involved in research based on WZ-type telescoping methods relevant to the contents of Sections 2–6, as well as research based on the "semi-integration by parts" method formulated in [26].

Bibliography

[1] V. S. Adamchik, *A certain series associated with Catalan's constant*, Z. Anal. Anwendungen **21** (2002), 817–826.

[2] G. Almkvist and J. Guillera, *Ramanujan-like series for* $1/\pi^2$ *and string theory*, Exp. Math. **21** (2012), 223–234.

[3] H. Alzer and A. Sofo, *New series representations for Apéry's and other classical constants*, Anal. Math. **44** (2018), 287–297.

[4] R. Apéry, *Irrationalité de* $\zeta 2$ *et* $\zeta 3$, Astérisque, (1979), 11–13.

[5] N. D. Baruah, B. C. Berndt, and H. H. Chan, *Ramanujan's series for* $1/\pi$*: a survey*, Amer. Math. Monthly **116** (2009), 567–587.

[6] G. Bauer, *Von den Coefficienten der Reihen von Kugelfunctionen einer Variablen*, J. Reine Angew. Math. **56** (1859), 101–121.

[7] N. Bergeron, Personal communication, 2022.

[8] B. C. Berndt, *Ramanujan's notebooks. Part I*, Springer-Verlag, New York, 1985.

[9] B. C. Berndt, *Ramanujan's Notebooks, Part IV*, Springer–Verlag, New York, 1994.

[10] J. M. Borwein and P. B. Borwein, *Pi and the AGM*, John Wiley & Sons, Inc., New York, 1987.

[11] J. M. Borwein, P. B. Borwein, and D. H. Bailey, *Ramanujan, modular equations, and approximations to pi, or How to compute one billion digits of pi*, Amer. Math. Monthly **96** (1989), 201–219.

[12] D. Borwein, J. M. Borwein, M. L. Glasser, and J. G. Wan, *Moments of Ramanujan's generalized elliptic integrals and extensions of Catalan's constant*, J. Math. Anal. Appl. **384** (2011), 478–496.

[13] J. M. Borwein and M. Chamberland, *Integer powers of arcsin*, Int. J. Math. Math. Sci. (2007), Art. ID 19381, 10.

[14] J. M. Borwein and R. E. Crandall, *Closed forms: what they are and why we care*, Notices Amer. Math. Soc. **60** (2013), 50–65.

[15] K. N. Boyadzhiev, *Series with central binomial coefficients, Catalan numbers, and harmonic numbers*, J. Integer Seq. **15** (2012), Article 12.1.7, 11.

[16] F. Brafman, *Generating functions of Jacobi and related polynomials*, Proc. Amer. Math. Soc. **2** (1951), 942–949.

[17] J. M. Campbell, *New families of double hypergeometric series for constants involving* $1/\pi^2$, Ann. Polon. Math. **126** (2021), 1–20.

[18] J. M. Campbell, *New series involving harmonic numbers and squared central binomial coefficients*, Rocky Mountain J. Math. **49** (2019), 2513–2544.

[19] J. M. Campbell, *Ramanujan-like series for* $\frac{1}{\pi}$ *involving harmonic numbers*, Ramanujan J. **46** (2018), 373–387.

[20] J. M. Campbell, *Series containing squared central binomial coefficients and alternating harmonic numbers*, Mediterr. J. Math. **16** (2019), Paper No. 37, 7.

[21] J. M. Campbell, *Some nontrivial two-term dilogarithm identities*, Irish Math. Soc. Bull. (2021), 31–37.

[22] J. M. Campbell, *Special values of Legendre's chi-function and the inverse tangent integral*, to appear in Irish Math. Soc. Bull. (2022).

[23] J. M. Campbell, *A Wilf-Zeilberger-based solution to the Basel problem with applications*, Discrete Math. Lett. **10** (2022), 21–27.

[24] J. M. Campbell, *A WZ proof for a Ramanujan-like series involving cubed binomial coefficients*, J. Difference Equ. Appl. **27** (2021), 1507–1511.

[25] J. M. Campbell, *WZ proofs for lemniscate-like constant evaluations*, Integers **21** (2021), Paper No. A107, 15.

[26] J. M. Campbell, M. Cantarini, and J. D'Aurizio, *Symbolic computations via Fourier-Legendre expansions and fractional operators*, Integral Transforms Spec. Funct. **33** (2022), 157–175.

[27] J. M. Campbell and K.-W. Chen, *Explicit identities for infinite families of series involving squared binomial coefficients*, J. Math. Anal. Appl. **513** (2022), Paper No. 126219, 23.

[28] J. M. Campbell and W. Chu, *Double series transforms derived from Fourier-Legendre theory*, Commun. Korean Math. Soc. **37** (2022), 551–566.

[29] J. M. Campbell and W. Chu, *Lemniscate-like constants and infinite series*, Math. Slovaca **71** (2021), 845–858.

[30] J. M. Campbell, J. D'Aurizio, and J. Sondow, *Hypergeometry of the parbelos*, Amer. Math. Monthly **127** (2020), 23–32.

[31] J. M. Campbell, J. D'Aurizio and J. Sondow, *On the interplay among hypergeometric functions, complete elliptic integrals, and Fourier-Legendre expansions*, J. Math. Anal. Appl. **479** (2019), 90–121.

[32] J. M. Campbell and A. Sofo, *An integral transform related to series involving alternating harmonic numbers*, Integr. Transf. Spec. F. **28** (2017), 547–559.

[33] M. Cantarini and J. D'Aurizio, *On the interplay between hypergeometric series, Fourier-Legendre expansions and Euler sums*, Boll. Unione Mat. Ital. **12** (2019), 623–656.

[34] B. C. Carlson, *Elliptic integrals*, NIST handbook of mathematical functions, U.S. Dept. Commerce, Washington, DC, 2010, pp. 485–522.

[35] B. C. Carlson, *The need for a new classification of double hypergeometric series*, Proc. Amer. Math. Soc. **56** (1976), 221–224.

[36] H. H. Chan, J. Wan, and W. Zudilin, *Legendre polynomials and Ramanujan-type series for* $1/\pi$, Israel J. Math. **194** (2013), 183–207.

[37] H. Chen, *Interesting series associated with central binomial coefficients, Catalan numbers and harmonic numbers*, J. Integer Seq. **19** (2016), Article 16.1.5, 11.

[38] J. Choi, *Summation formulas involving binomial coefficients, harmonic numbers, and generalized harmonic numbers*, Abstr. Appl. Anal. (2014), Art. ID 501906, 10.

[39] W. Chu and D. Zheng, *Infinite series with harmonic numbers and central binomial coefficients*, Int. J. Number Theory **5** (2009), 429–448.

[40] H. S. Cohl and C. MacKenzie, *Generalizations and specializations of generating functions for Jacobi, Gegenbauer, Chebyshev and Legendre polynomials with definite integrals*, J. Class. Anal. **3** (2013), 17–33.

[41] P. Duren, *The Legendre relation for elliptic integrals*, Paul Halmos, Springer, New York, 1991, pp. 305–315.

[42] R. Earl, *Towards higher mathematics: a companion*, Cambridge University Press, Cambridge, 2017.

[43] M. E. A. El-Mikkawy and G.-S. Cheon, *Combinatorial and hypergeometric identities via the Legendre polynomials—a computational approach*, Appl. Math. Comput. **166** (2005), 181–195.

[44] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Higher transcendental functions. Vols. I, II*, McGraw-Hill Book Company, Inc., New York-Toronto-London, 1953.

[45] S. R. Finch, *Mathematical constants*, Cambridge University Press, Cambridge, 2003.

[46] M. Genčev, *Binomial sums involving harmonic numbers*, Math. Slovaca **61** (2011), 215–226.

[47] M. O. González, *Elliptic integrals in terms of Legendre polynomials*, Proc. Glasgow Math. Assoc. **2** (1954), 97–99.

[48] J. Guillera, *About a new kind of Ramanujan-type series*, Experiment. Math. **12** (2003), 507–510.

[49] J. Guillera, *Generators of some Ramanujan formulas*, Ramanujan J. **11** (2006), 41–48.

[50] J. Guillera, *A matrix form of Ramanujan-type series for* $1/\pi$, Gems in experimental mathematics, Amer. Math. Soc., Providence, RI, 2010, 189–206.

[51] J. Guillera, *More hypergeometric identities related to Ramanujan-type series*, Ramanujan J. **32** (2013), 5–22.

[52] J. Guillera, *A new Ramanujan-like series for* $1/\pi^2$, Ramanujan J. **26** (2011), 369–374.

[53] J. Guillera, *Series de Ramanujan: Generalizaciones y Conjeturas*, Thesis (Ph.D.)–University of Zaragoza (2007).

[54] J. Guillera, *Some binomial series obtained by the WZ-method*, Adv. in Appl. Math. **29** (2002), 599–603.

[55] J. P. Hannah, *Identities for the gamma and hypergeometric functions: an overview from Euler to the present*, University of the Witwatersrand, (2013).

[56] J. T. Holdeman, Jr., *Legendre polynomial expansions of hypergeometric functions with applications*, J. Mathematical Phys. **11** (1970), 114–117.

[57] J. Horn, *Ueber die Convergenz der hypergeometrischen Reihen zweier und dreier Veränderlichen*, Math. Ann. **34** (1889), 544–600.

[58] E. L. Kaplan, *Multiple elliptic integrals*, Stud. Appl. Math. **29** (1950), 69–75.

[59] Y. S. Kim, A. K. Rathie and R. B. Paris, *On a new class of summation formulae involving the Laguerre polynomial*, Integral Transforms Spec. Funct. **23** (2012), 435–444.

[60] F. Klein, "Development of Mathematics in the 19th Century," 1928, *Trans. Math. Sci. Press*, R. Hermann Ed. (Brookline, MA 1979).

[61] P. Levrie, *Using Fourier-Legendre expansions to derive series for* $\frac{1}{\pi}$ *and* $\frac{1}{\pi^2}$, Ramanujan J. **22** (2010), 221–230.

[62] H. Liu, W. Zhou, and S. Ding, *Generalized harmonic number summation formulae via hypergeometric series and digamma functions*, J. Differ. Equations Appl. **23** (2017), 1204–1218.

[63] V. V. Manako, *A connection formula between double hypergeometric series* Ψ_2 *and* Φ_3, Integral Transforms Spec. Funct. **23** (2012), 503–508.

[64] G.-S. Mao, C. Wang, and J. Wang, *Symbolic summation methods and congruences involving harmonic numbers*, C. R. Math. Acad. Sci. Paris **357** (2019), 756–765.

[65] S. Mattarei and R. Tauraso, *Congruences for central binomial sums and finite polylogarithms*, J. Number Theory **133** (2013), 131–157.

[66] M. Nicholson, *Quadratic Transformations of Hypergeometric Function and Series with Harmonic Numbers*, `arXiv:1801.02428`

[67] M. Petkovšek, H. S. Wilf, and D. Zeilberger, $A = B$, A K Peters, Ltd., Wellesley, MA, 1996.

[68] A. P. Prudnikov, Yu. A. Brychkov and O. I. Marichev, *Integrals and series.* Vol. 3, Gordon and Breach Science Publishers, New York, 1990.

[69] E. D. Rainville, *Special Functions*, Macmillan, New York, 1960.

[70] S. Ramanujan, *Modular equations and approximations to* π, Q. J. Math. **45** (1914), 350–372.

[71] A. K. Rathie and A. Kılıçman, *On a new class of summation formulas involving the generalized hypergeometric $_2F_2$ polynomial*, Adv. Difference Equ. (2014), 2014:43, 10.

[72] R. P. Singal, *A transformation formula for double hypergeometric series*, Rocky Mountain J. Math. **3** (1973), 377–381.

[73] A. Sofo, *General order Euler sums with multiple argument*, J. Number Theory **189** (2018), 255–271.

[74] A. Sofo, *Integrals of logarithmic and hypergeometric functions*, Comm. Math. **24** (2016), 7–22.

[75] P. K. Suetin, *On the representation of continuous and differentiable functions by Fourier series in Legendre polynomials*, Dokl. Akad. Nauk SSSR **158** (1964), 1275–1277.

[76] Z.-W. Sun, *p-adic congruences motivated by series*, J. Number Theory **134** (2014), 181–196.

[77] Z.-W. Sun, *A new series for π^3 and related congruences*, Internat. J. Math. **26** (2015), 1550055, 23.

[78] Z.-W. Sun, *New series for some special values of L-functions*, Nanjing Univ J Math. **32** (2015), 189–218.

[79] R. Tauraso, *Supercongruences related to $_3F_2(1)$ involving harmonic numbers*, Int. J. Number Theory **14** (2018), 1093–1109.

[80] J. Todd, *The Lemniscate Constants*, Pi: A Source Book, Springer New York (2004), 412–417.

[81] J. Van der Jeugt, S. N. Pitre, V. V. Manako and K. Srinivasa Rao, *Transformation and summation formulas for double hypergeometric series*, J. Comput. Appl. Math. **83** (1997), 185–193.

[82] J. G. F. Wan, *Random walks, elliptic integrals and related constants*, PhD Thesis, University of Newcastle, Newcastle, 2013.

[83] J. G. Wan, *Series for $1/\pi$ using Legendre's relation*, Integral Transforms Spec. Funct. **25** (2014), 1–14.

[84] J. Wan and W. Zudilin, *Generating functions of Legendre polynomials: a tribute to Fred Brafman*, J. Approx. Theory **164** (2012), 488–503.

[85] H. Wang and S. Xiang, *On the convergence rates of Legendre approximation*, Math. Comp. **81** (2012), 861–877.

[86] W. Wang and Y. Chen, *Explicit formulas of sums involving harmonic numbers and Stirling numbers*, J. Difference Equ. Appl. **26** (2020), 1369–1397.

[87] X. Wang and W. Chu, *Further Ramanujan-like series containing harmonic numbers and squared binomial coefficients*, Ramanujan J. **52** (2020), 641–668.

[88] W. Wang and C. Xu, *Alternating multiple zeta values, and explicit formulas of some Euler-Apéry-type series*, European J. Combin. **93** (2021), 103283, 25.

[89] E. W. Weisstein, *Arithmetic-Geometric Mean*. From *MathWorld*–A Wolfram Web Resource.

[90] E. W. Weisstein, *Elliptic Integral*. From MathWorld–A Wolfram Web Resource.

[91] R. L. Wheeden and A. Zygmund, *Measure and integral*, Marcel Dekker, Inc., New York-Basel, 1977.

[92] W. Zudilin, *Ramanujan-type formulae for* $1/\pi$*: a second wind?*, Modular forms and string duality (Fields Inst. Commun.), Amer. Math. Soc., Providence, RI, vol. 54, 2008, pp. 179–188.