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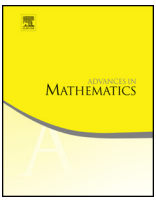

The localized slice spectral sequence, norms of Real bordism, and the Segal conjecture

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ABSTRACT

In this paper, we introduce the localized slice spectral sequence, a variant of the equivariant slice spectral sequence that computes geometric fixed points equipped with residue group actions. We prove convergence and recovery theorems for the localized slice spectral sequence and use it to analyze the norms of the Real bordism spectrum. As a consequence, we relate the Real bordism spectrum and its norms to a form of the C_2-Segal conjecture. We compute the localized slice spectral sequence of the C_4-norm of $BP_{\mathbb{R}}$ in a range and show that the Hill–Hopkins–Ravenel slice differentials are in one-to-one correspondence with a family of Tate differentials for $N_1^2 H\mathbb{F}_2$.

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1. Introduction

The complex conjugation action on the complex bordism spectrum MU defines a C_2-spectrum $MU_{\mathbb{R}}$, the Real bordism spectrum of Landweber, Fujii, and Araki [37,19,4]. Its norms

$$MU^{((C_{2^n}))} := N_{C_2}^{C_{2^n}} MU_{\mathbb{R}} = N_2^{2^n} MU_{\mathbb{R}}$$

have played a central role in the solution of the Kervaire invariant one problem [33]. After localizing at 2, the norm $MU^{((C_{2^n}))}$ splits as a wedge of suspensions of $BP^{((C_{2^n}))} := N_2^{2^n} BP_{\mathbb{R}}$, where $BP_{\mathbb{R}}$ is the Real Brown–Peterson spectrum.

The spectra $MU^{((C_{2^n}))}$ and $BP^{((C_{2^n}))}$ connect many fundamental objects and computations in non-equivariant stable homotopy theory to equivariant stable homotopy theory. The fixed points of these norms are ring spectra, and their Hurewicz images detect families of elements in the stable homotopy groups of spheres [33,30,40]. The Lubin–Tate spectra at prime 2 with finite group actions can also be built from these norms and their quotients [26,8]. They produce higher height analogues of topological K-theory and play a fundamental role in chromatic homotopy theory.

To compute the equivariant homotopy groups of $MU^{((C_{2^n}))}$ and $BP^{((C_{2^n}))}$, Hill, Hopkins, and Ravenel introduced the equivariant slice spectral sequence [33]. However, due to the complexity of the equivariant computations, besides $MU_{\mathbb{R}}$ and $BP_{\mathbb{R}}$, we still know relatively little about the behavior of their norms. For example, we are still far from a complete understanding of the equivariant homotopy groups of $BP^{((C_4))}$.

Our project arose from the desire to systematically understand the equivariant homotopy groups of $MU^{((C_{2^n}))}$ and $BP^{((C_{2^n}))}$. The goal of this paper is two-fold: first, we establish our main computational tool, the *localized slice spectral sequence*. This is a variant of the slice spectral sequence that is easier for computations while at the same time recovers the original slice differentials. Second, as an application of the localized slice spectral sequence, we focus on the C_4-norm $BP^{((C_4))}$. We compute its localized slice spectral sequence in a range and build a new connection to the Segal conjecture at C_2. As a consequence, we establish correspondences between families of slice differentials for $BP^{((C_4))}$ and families of differentials in the Tate spectral sequence for $N_1^2 H\mathbb{F}_2$.

1.1. Fixed points and geometric fixed points

It is well-known in equivariant stable homotopy theory that a map between G-spectra is a weak equivalence if and only if for all subgroups $H \subset G$, it induces (non-equivariant)

weak equivalences on all H-fixed points or H-geometric fixed points. Despite this fact, fixed points and geometric fixed points behave very differently.

The fixed points of a G-spectrum X can be difficult to understand. For a suspension spectrum, its fixed points can be described by using the tom Dieck splitting [39, Section V.11], but such a splitting does not exist in general. Nevertheless, by the Wirthmüller isomorphism, there are natural maps between fixed points of different subgroups of G. The induced maps on their homotopy groups can be assembled into an algebraic object $\underline{\pi}_* X$, called a Mackey functor. Organizing information in terms of Mackey functors is one of the most powerful ideas in equivariant stable homotopy theory, and this has produced new insights in both theory and computation (e.g. [24,33]).

As an important example, the C_2-fixed points of the Real bordism spectrum $MU_{\mathbb{R}}$ are computable but complicated [36,23]. For groups beyond C_2, we still don't know very much about the fixed points of the norms $MU^{((C_{2^n}))}$ aside from the computations in [33,34,30,31]. Nevertheless, these fixed points contain very rich information about the stable homotopy groups of spheres (such as the Kervaire invariant elements) and chromatic homotopy theory [33,40,26,8].

On the other hand, the geometric fixed points are easier to understand. The geometric fixed points functor $\Phi^H \colon \mathrm{Sp}_G \to \mathrm{Sp}$ is compatible with the suspension spectrum functor, commutes with all homotopy colimits, and is symmetric monoidal.

For the Real bordism spectrum $MU_{\mathbb{R}}$, a straightforward geometric argument, based on the fact that the fixed points of the C_2-Galois action on $\mathbb{C}$ are $\mathbb{R}$, shows that the C_2-geometric fixed points of $MU_{\mathbb{R}}$ and $BP_{\mathbb{R}}$ are MO (the unoriented bordism spectrum) and $H\mathbb{F}_2$, respectively. The geometric fixed points functor also behaves well with respect to the norm functor [33, Proposition 2.57]. This renders the geometric fixed points of the norms $MU^{((C_{2^n}))}$ easy to understand.

Although the homotopy groups of the geometric fixed points for various subgroups do not form a Mackey functor, there are reconstruction theorems which recovers a G-spectrum from structures on its geometric fixed points [1,21,5].

At this point, it is natural to ask the following questions:

(1) How do the fixed points and the geometric fixed points of an equivariant spectrum interact with each other?
(2) Computationally, how to recover the fixed points of equivariant spectra, such as norms of $MU_{\mathbb{R}}$, through their geometric fixed points, which are significantly easier to compute?

In order to attack these questions, the first observation is that it is necessary to consider the H-geometric fixed points not only as a non-equivariant spectrum, but as a $W_G(H)$-equivariant spectrum, where $W_G(H)$ is the Weyl group. In our examples of interest, H will be a normal subgroup of G, so that $W_G(H) \cong G/H$. When the G-spectrum is of the form $N_H^G X$, we prove the following theorem.

Theorem 1.1. *Let $H \subset G$ be a normal subgroup and X be an H-spectrum. Then we have an equivalence of G/H-spectra*

$$\Phi^H N_H^G X \simeq N_e^{G/H} \Phi^H(X).$$

If X is an H-commutative ring spectrum, then this equivalence is an equivalence of G/H-commutative ring spectra.

This theorem is by no means difficult to prove, and in fact it only marks the starting point of our analysis. To understand how the H-fixed points and H-geometric fixed points interact with each other, we introduce our main computational tool: the localized slice spectral sequence.

1.2. The localized slice spectral sequence

Let X be a G-spectrum and $H \subset G$ a normal subgroup. As a G/H-spectrum, $\Phi^H X$ can be constructed as $(\tilde{E}\mathcal{F}[H] \wedge X)^H$, where $\tilde{E}\mathcal{F}[H]$ is the universal space of the family $\mathcal{F}[H]$ consisting of all subgroups that do not contain H. In many cases, including G cyclic, smashing with $\tilde{E}\mathcal{F}[H]$ is equivalent to inverting an Euler class $a_V \in \pi_{-V}^G S^0$ for V a certain G-representation. In particular, the residue fixed points $(\Phi^H X)^{G/H}$ are equivalent to the fixed points $(a_V^{-1} X)^G$.

To define the localized slice spectral sequence, let $P^\bullet X$ be the regular slice tower of X [33][55]. The *a_V-localized slice spectral sequence* of X is, by definition, the spectral sequence corresponding to the localized tower $\{a_V^{-1} P^\bullet X\}$. It has E_2-page

$$E_2^{s,t} = \underline{\pi}_{t-s} a_V^{-1} P_t^t X.$$

Theorem 1.2. *Let X be a C_{2^n}-spectrum and V be an actual C_{2^n}-representation. Then the a_V-localized slice spectral sequence converges strongly to the homotopy groups $\underline{\pi}_{t-s} a_V^{-1} X$.*

The localized slice spectral sequence serves as a bridge between the fixed points X^G and the residue fixed points $(\Phi^H X)^{G/H}$. More precisely, even though the localized slice spectral sequence only computes the geometric fixed points, its E_2-page is closely related to the original slice spectral sequence, which computes the fixed points. From now on, we will denote the regular slice spectral sequence and the a_V-localized slice spectral sequence of X by $\mathrm{SliceSS}(X)$ and $a_V^{-1}\,\mathrm{SliceSS}(X)$, respectively. The following theorem directly follows from computations of the homotopy groups of $H\underline{\mathbb{Z}}$ [34, Section 3].

Theorem 1.3. *Let X be a (-1)-connected C_{2^n}-spectrum whose slices are wedges of the form $C_{2^n+} \wedge_{C_{2^k}} \Sigma^{i\rho_k} H\underline{\mathbb{Z}}$, and λ be the 2-dimensional real C_{2^n}-representation that is rotation by $\frac{\pi}{2^{n-1}}$. Then the localizing map*

$$\mathrm{SliceSS}(X) \longrightarrow a_\lambda^{-1}\,\mathrm{SliceSS}(X)$$

induces an isomorphism on the E_2-page for classes whose filtration is greater than 0*. On the* 0*-line, this map is surjective, with kernel consisting of elements in the image of the transfer $Tr_e^{C_{2^n}}$.*

By the slice theorem [33, Theorem 6.1], the C_{2^n}-norms of $MU_{\mathbb{R}}$ and $BP_{\mathbb{R}}$ both satisfy the conditions of Theorem 1.3.

An upshot of Theorem 1.3 is that despite the fact that the fixed points are harder to compute than the geometric fixed points, if we already know the differentials in the localized slice spectral sequence, then we can use the isomorphism on the E_2-page given by Theorem 1.3 to recover differentials in the original slice spectral sequence. This allows us to approach the computation of the fixed points X^G from the residue fixed points $(\Phi^H X)^{G/H}$.

A subtlety that arises from the localized slice spectral sequence is its compatibility with multiplicative structures. More precisely, let R be a connective G-commutative ring spectrum. Ullman [55] has shown that the slice tower of R is multiplicative. Therefore, the corresponding slice spectral sequence has all the desired multiplicative properties such as the Leibniz rule, the Frobenius relation [34, Definition 2.3], and most importantly, the norm [34, Corollary 4.8]. On the other hand, the localization $a_V^{-1}R$ can never be a G-commutative ring spectrum because its underlying spectrum is contractible.

To establish multiplicative properties for the localizations, we apply the theory of N_∞-operads from [10]. More precisely, in Section 2.5, we establish a criterion generalizing the results of [32] and [13]. As a consequence, we obtain the following theorem, which shows that a_V-localization preserves algebra structures over a certain N_∞-operad $\mathcal{O}$ that depends on the class a_V.

Theorem 1.4. *Let V be a G-representation. Assume that $\mathrm{Ind}_K^H \mathrm{Res}_K^G V$ is a summand of a multiple of $\mathrm{Res}_H^G V$ for every $K \subset H \subset G$ such that H/K is an admissible H-set. Then localization at a_V preserves $\mathcal{O}$-algebras.*

Therefore, the homotopy of the a_V-localization of an equivariant commutative ring spectrum such as $MU^{((C_{2^n}))}$ forms an incomplete Tambara functor [11], and the norm maps essential to our computation are still available. In Section 3.4, we draw consequences of the behavior of norms in the localized slice spectral sequence.

Aside from the localized slice spectral sequence $a_\lambda^{-1}\,\mathrm{SliceSS}(X)$, the G/H-slice spectral sequence of $\Phi^H X$ also computes the residue fixed points $(\Phi^H X)^{G/H}$. Even though both spectral sequences compute the same homotopy groups, their behaviors can be very different. Surprisingly, we have the following theorem, which shows that after a modification of filtrations, there is map between the two spectral sequences.

Theorem 1.5. *Let X be a C_{2^n}-spectrum, then there is a canonical map of spectral sequences*

$$a_\lambda^{-1}\,\mathrm{SliceSS}^{C_{2^n}}(X) \to P^*_{C_{2^n}/C_2}\mathscr{D}\mathrm{SliceSS}^{C_{2^n}/C_2}(\Phi^{C_2}X)$$

that converges to an isomorphism in homotopy groups. Here $\mathscr{D}$ is the doubling operation defined in Section 3.5, which slows down a tower by a factor of 2, and $P^*_{C_{2^n}/C_2}$ is the pullback functor from [29], which is recalled in Section 2.2.

In the second half of the paper, as an application of all the tools that we have developed, we will use the localized slice spectral sequence to analyze the norms of $MU_{\mathbb{R}}$.

1.3. Norms of Real bordism and the Segal conjecture

The Segal conjecture is a deep result in equivariant homotopy theory. In its original formulation, it was proven by Lin [41] for the group C_2 and by Carlsson [16] for all finite groups, building on the works of May–McClure [46] and Adams–Gunawardena–Miller [2]. When the group is C_2, the most general formulation can be found in Lunøe-Nielsen–Rognes [42] and Nikolaus–Scholze [54]: for every bounded below spectrum X, the Tate diagonal map $X \to (N_1^2 X)^{tC_2}$ is a 2-adic equivalence.

We are interested in the case when $X = H\mathbb{F}_2$, the mod 2 Eilenberg–Mac Lane spectrum. This case is intriguing for at least two reasons: first, Nikolaus–Scholze [54] show that the general formulation follows formally from this case. Second, even though the Segal conjecture implies the equivalence $H\mathbb{F}_2 \simeq (N_1^2 H\mathbb{F}_2)^{tC_2}$, this is still a mystery from a computational perspective.

More precisely, the Tate spectral sequence computing $(N_1^2 H\mathbb{F}_2)^{tC_2}$ has E_2-page $\hat{H}^*(C_2; \mathcal{A}_*)$, the Tate cohomology of the dual Steenrod algebra $\mathcal{A}_*$ with the conjugate C_2-action. This cohomology is highly nontrivial and we currently don't even have a closed formula [15]. However, because of the equivalence $H\mathbb{F}_2 \simeq (N_1^2 H\mathbb{F}_2)^{tC_2}$ given by the Segal conjecture, every element besides $1 \in \mathbb{F}_2 \cong \hat{H}^0(C_2; (\mathcal{A}_*)_0)$ must either support or receive a differential.

Understanding equivariant equivalences from a computational perspective can be extremely useful. For example, in the case of $BP_{\mathbb{R}}$ and its norms, it is relatively straightforward to establish the equivalence $\Phi^{C_{2^n}} BP^{((C_{2^n}))} \simeq \Phi^{C_2} BP_{\mathbb{R}} \simeq H\mathbb{F}_2$. By working backwards, Hill–Hopkins–Ravenel used this equivalence to prove a family of differentials in the slice spectral sequence of $BP^{((C_{2^n}))}$ [33, Theorem 9.9], from which their Periodicity Theorem and eventually the nonexistence of the Kervaire invariant elements followed.

By Theorem 1.1, we have a C_2-equivalence

$$\Phi^{C_2} BP^{((C_4))} \simeq N_1^2 H\mathbb{F}_2.$$

For the left hand side, we can use the localized slice spectral sequence to compute the C_2-fixed points of $\Phi^{C_2} BP^{((C_4))}$. We demonstrate this computation in a range (Theorem 4.4). Note that we can actually compute much further than the range we have shown, but the point is to give the readers a taste of the computations involved and to draw comparisons to the slice spectral sequence computations in [33,34,31].

After demonstrating these computations, we use Theorem 1.5 to establish a map between the slice spectral sequence of $BP^{((C_4))}$ and the Tate spectral sequence of $N_1^2 H\mathbb{F}_2$.

We prove that this map establishes a correspondence between families of differentials in the two spectral sequences.

Theorem 1.6. *After the E_2-page, the Hill–Hopkins–Ravenel slice differentials [33, Theorem 9.9] are in one-to-one correspondence to a family of differentials on the first diagonal of slope (-1) in the Tate spectral sequence of $N_1^2 H\mathbb{F}_2$. This completely determines all the differentials in the Tate spectral sequence that originate from the first diagonal of slope (-1).*

In the future, we wish to reverse the flow of information: to prove slice differentials from spectral sequences associated to $N_1^2 H\mathbb{F}_2$. Computations along this line appear in [7] and will be refined in an upcoming article by the same authors. There are various methods to study the norms of $H\mathbb{F}_2$ and their modules, such as the modified Adams spectral sequence [49,14] and the descent spectral sequence [27]. These methods allow one to understand modules over norms of $H\mathbb{F}_2$ and $BP_{\mathbb{R}}$ from different perspectives.

It is worth noting that in another direction, one can prove the C_2-Segal conjecture by showing that $N_2^4 MU_{\mathbb{R}}$ is cofree and using Theorem 1.1. This approach is taken by Carrick in [17].

Theorem 1.6 has an unexpected consequence. Let R be an arbitrary non-equivariant (-1)-connected homotopy ring spectrum with $\pi_0 R \cong \mathbb{Z}$ (or a localization thereof not containing $\frac{1}{2}$). We can use the (stable) EHP spectral sequence and the Tate spectral sequence of $N_1^2 H\mathbb{F}_2$ to bound the length of differentials on powers of the Tate generator in the Tate spectral sequence of $N_1^2 R$.

Theorem 1.7. *Let $v \in \hat{H}^2(C_2; \pi_0 N_1^2 R)$ be the generator of the Tate cohomology, and l_k be the length of differential that v^{2^k} supports in the Tate spectral sequence of $N_1^2 R$. Then*

$$\rho(2^{k+1}) \leq l_k \leq 2^{k+2} - 1,$$

where $\rho(n)$ is the Radon–Hurwitz number.

1.4. Outline of paper

In Section 2, we recall a few basics of equivariant homotopy theory. In particular, we discuss the interplay between the norm functor, the geometric fixed points functor, and the pull back functor. We prove Theorem 1.1. We also investigate the multiplicative structure of localizations and give a criterion for a localization at an element to preserve multiplicative structures, thus proving Theorem 1.4.

In Section 3, we recall the spectra $MU^{((G))}$ and $BP^{((G))}$ and their slice spectral sequences. We then introduce the main computational tool for this paper, the localized slice spectral sequence. We prove Theorem 1.2, the strong convergence of the localized slice spectral sequence (Theorem 3.3). We also discuss exotic extensions and norms.

Sections 4 and 5 are dedicated to the computation of the localized slice spectral sequence of $a_\lambda^{-1}BP^{((C_4))}$. In Section 4, we give an outline of the computation and list our main results (Theorem 4.1 and Theorem 4.4). The detailed computations are in Section 5. While computing differentials, we make full use of the Mackey functor structure of the spectral sequence. Certain differentials are proven using exotic extensions and norms by methods established in Section 3.

In Section 6, we turn our attention to the Tate spectral sequence of $N_1^2 H\mathbb{F}_2$. We use the computation of the localized slice spectral sequence of $BP^{((C_4))}$ to prove families of differentials and compute the Tate spectral sequence in a range. In particular, Theorem 1.6 is proven as Theorem 6.6, which describes the first infinite family of differentials in the Tate spectral sequence.

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Conventions

(1) Given a finite group G, all representations will be finite-dimensional and orthogonal. Per default actions will be from the left.
(2) We denote by ρ_G the real regular representation of a finite group G and we abbreviate ρ_{C_2} to ρ_2.
(3) We will often use the abbreviation $BP^{((C_4))}$ for $N_2^4 BP_{\mathbb{R}}$ and more generally $BP^{((G))}$ for $N_{C_2}^G BP_{\mathbb{R}}$.
(4) All spectral sequences use the Adams grading.
(5) We use the regular slice filtration and its corresponding tower and spectral sequence defined in [55] throughout the paper, often omitting "regular".

2. Equivariant stable homotopy theory

2.1. A few basics

We work in the category of genuine G-spectra for a finite group G, and our particular model will be the category of orthogonal G-spectra Sp_G. For us these will be simply G-objects in orthogonal spectra as in [52], which will often be just called *G-spectra*. This

category is equivalent to the categories of orthogonal G-spectra considered in [45] and [33]. In particular, we are able to evaluate a G-spectrum at an arbitrary G-representation to obtain a G-space. We refer to the three cited sources for general background on G-equivariant stable homotopy theory, of which we will recall some for the convenience of the reader.

For each G-representation V, we denote by S^V its one-point compactification. Denoting further by ρ_G the regular representation, we obtain for each subgroup $H \subset G$ and each G-spectrum its homotopy groups

$$\pi_n^H(X) = \operatorname{colim}_k [S^{k\rho_G+n}, X(k\rho_G)]^H.$$

These assemble into a Mackey functor $\underline{\pi}_n(X)$. A map of G-spectra is an *equivalence* if it induces an isomorphism on all $\underline{\pi}_n$. Inverting the equivalences of G-spectra in the 1-categorical sense yields the genuine equivariant stable homotopy category $\mathrm{Ho}(\mathrm{Sp}_G)$ and inverting them in the ∞-categorical sense the ∞-category of G-spectra Sp_G^∞. These constructions are well-behaved as there is a stable model structure on Sp_G with the weak equivalences we just described [45, Theorem III.4.2]. The fibrant objects are precisely the Ω-G-spectra. In the main body of the paper we will implicitly work in $\mathrm{Ho}(\mathrm{Sp}_G)$ or Sp_G^∞; in particular, commutative squares are meant to be only commutative up to (possibly specified) homotopy.

By [45, Proposition V.3.4], the categorical fixed point construction $\mathrm{Sp}_G \to \mathrm{Sp}$ is a right Quillen functor. We call the right derived functor $(-)^G \colon \mathrm{Sp}_G^\infty \to \mathrm{Sp}^\infty$ the *(genuine) fixed points*. We can define fixed point functors for subgroups $H \subset G$ by applying first the restriction functor $\mathrm{Sp}_G \to \mathrm{Sp}_H$ and then the H-fixed point functor. One easily shows that $\pi_n X^H \cong \pi_n^H X$. Thus, a map is an equivalence if it is an equivalence on all fixed points.

Note that if $H \subset G$ is normal, the categorical fixed points carry a residual G/H-action. The resulting functor $\mathrm{Sp}_G \to \mathrm{Sp}_{G/H}$ is a right Quillen functor as well [45, p. 81] and thus H-fixed points actually define a functor $\mathrm{Sp}_G^\infty \to \mathrm{Sp}_{G/H}^\infty$. The left adjoint of this is the inflation functor p^* associated to the projection $p \colon G \to G/H$.

As π_n^H translates filtered homotopy colimits into colimits, we see that fixed points $\mathrm{Sp}_G^\infty \to \mathrm{Sp}^\infty$ preserve filtered homotopy colimits. As they preserve homotopy limits as well (as they are induced by a Quillen right adjoint) and are a functor between stable ∞-categories, they preserve all finite homotopy colimits [44, Proposition 1.1.4.1] and hence all homotopy colimits [43, Proposition 4.4.2.7]. By the associativity of fixed points, the same is true for $(-)^H \colon \mathrm{Sp}_G^\infty \to \mathrm{Sp}_{G/H}^\infty$ for a normal subgroup $H \subset G$.

2.2. Geometric fixed points and pullbacks

To define other versions of fixed points, we need the notion of a universal space for a given family $\mathcal{F}$ of subgroups of G, i.e. a collection of subgroups closed under taking subgroups and conjugation. For every such family there exists a *universal space*, i.e. a

G-CW complex $E\mathcal{F}$, which is up to G-homotopy equivalence characterized by its fixed points:

$$(E\mathcal{F})^H \simeq \begin{cases} * & \text{if } H \in \mathcal{F} \\ \varnothing & \text{if } H \notin \mathcal{F} \end{cases}$$

Examples of such families include the case $\mathcal{F} = \{e\}$ of just the trivial group, where we denote $E\mathcal{F}$ by EG, and the case $\mathcal{F} = \mathcal{P}$ of all proper subgroups. To each family, we can associate furthermore the cofiber $\tilde{E}\mathcal{F}$ of $E\mathcal{F}_+ \to S^0$, which is again characterized by its fixed points

$$(\tilde{E}\mathcal{F})^H \simeq \begin{cases} * & \text{if } H \in \mathcal{F} \\ S^0 & \text{if } H \notin \mathcal{F} \end{cases}$$

For each family $\mathcal{F}$ and every G-spectrum we have an associated isotropy separation diagram, whose rows are parts of cofiber sequences:

$$\begin{array}{ccccc} X \wedge E\mathcal{F}_+ & \longrightarrow & X & \longrightarrow & X \wedge \tilde{E}\mathcal{F} \\ \downarrow \simeq & & \downarrow & & \downarrow \\ X \wedge E\mathcal{F}_+ & \longrightarrow & X^{E\mathcal{F}_+} & \longrightarrow & X^{E\mathcal{F}_+} \wedge \tilde{E}\mathcal{F} \end{array}$$

Upon taking fixed points, we can identify some of the entries with well-known constructions. If $E\mathcal{F} = EG$, then $(X^{E\mathcal{F}_+})^G$ is the spectrum of *homotopy fixed points* X^{hG} and $(X \wedge E\mathcal{F}_+)^G$ is (by the Adams isomorphism) the spectrum of *homotopy orbits* X_{hG}. Moreover, one calls in this case $(X^{E\mathcal{F}_+} \wedge \tilde{E}\mathcal{F})^G$ the *Tate construction* and denotes it by X^{tG}. If $\mathcal{F} = \mathcal{P}$, then $(X \wedge \tilde{E}\mathcal{P})^G$ is called the *geometric fixed points* and denoted by $\Phi^G X$. One can show that $\Phi^G(\Sigma^\infty X) \simeq \Sigma^\infty X^G$ for pointed G-spaces X.

Let $H \subset G$ be normal. As mentioned above, H-fixed points define a functor $\mathrm{Sp}_G^\infty \to \mathrm{Sp}_{G/H}^\infty$. We want to define a similar version for geometric fixed points. Let $\mathcal{F}[H]$ be the family of all subgroups of G not containing H. We consider the functor

$$\Phi^H \colon \mathrm{Sp}_G^\infty \to \mathrm{Sp}_{G/H}^\infty, \qquad X \mapsto (\tilde{E}\mathcal{F}[H] \wedge X)^H.$$

This agrees with our previous definition when $H = G$ since $\mathcal{F}[G] = \mathcal{P}$. Another important special case is $G = C_{2^n}$ and $H = C_2$; then $\tilde{E}\mathcal{F}[H] = \tilde{E}G$.

As the geometric fixed points functor $\Phi^H \colon \mathrm{Sp}_G^\infty \to \mathrm{Sp}_{G/H}^\infty$ is the composition of smashing with a space and taking fixed points, it preserves all homotopy colimits as well.

This property implies that Φ^H must possess a right adjoint, which was constructed in [29, Definition 4.1] as the *pullback functor*

$$P^*_{G/H}\colon \mathrm{Sp}^\infty_{G/H} \to \mathrm{Sp}^\infty_G, \qquad X \mapsto \tilde{E}\mathcal{F}[H] \wedge p^*X,$$

where p^* is the functor induced by the projection $p\colon G \to G/H$, as defined in the previous subsection. For the adjointness see [29, Proposition 4.4], at least on the level of homotopy categories. Several pleasant properties of $P^*_{G/H}$ are shown in [29, Section 4.1], in particular that $P^*_{G/H}$ defines a fully-faithful embedding of G/H-spectra into G-spectra (with image agreeing with that of $-\wedge\tilde{E}\mathcal{F}[H]$). Equivalently, the unit map $\mathrm{id} \to \Phi^H P^*_{G/H}$ is an equivalence. This also implies that $P^*_{G/H}$ is (strong) symmetric monoidal (since the image of $P^*_{G/H}$ is closed under $\wedge$). Moreover, it follows that $P^*_{G/H}\Phi^H$ is equivalent to $-\wedge\tilde{E}\mathcal{F}[H]$.

We furthermore note:

Lemma 2.1. *For every G/H-spectrum X, every $H \subset K \subset G$ and every $V \in RO(G/H)$, there is a canonical isomorphism*

$$\pi^K_V(P^*_{G/H}X) \cong \pi^{K/H}_V(X).$$

Here we view V also as an element of $RO(G)$ by pullback along $G \to G/H$.

Proof. Essentially by definition, $\pi^K_V(P^*_{G/H}X) \cong [\Sigma^V G/K_+, P^*_{G/H}X]^G$. By the containment $H \subset K$, all points in G/K are H-fixed and moreover $V^H = V$. Hence we get $\Phi^H\Sigma^V G/K_+ \simeq \Sigma^V (G/H)/(K/H)_+$. By the adjointness of Φ^H and $P^*_{G/H}$ we thus obtain the result. □

2.3. Universal properties of G-spectra

In [20, Corollary C.7], Gepner and the first-named author established a universal property for symmetric monoidal colimit-preserving functors out of Sp^∞_G. We will need a variant of this for functors just preserving *filtered* colimits.

Localizing the 1-category of pointed finite G-CW-complexes at G-homotopy equivalences yields an ∞-category $\mathcal{S}^{\mathrm{fin},G}_*$. This ∞-category is essentially small. For every essentially small ∞-category $\mathcal{C}$, we can freely adjoin filtered colimits to obtain an ∞-category $\mathrm{Ind}(\mathcal{C})$ [43, Section 5.3]. The inclusion $\mathcal{S}^{\mathrm{fin},G}_* \to \mathcal{S}^G_*$ into the ∞-category of pointed G-spaces induces a functor $\mathrm{Ind}(\mathcal{S}^{\mathrm{fin},G}_*) \to \mathcal{S}^G_*$. Since $\mathcal{S}^{\mathrm{fin},G}_*$ consists of compact objects inside $\mathcal{S}^G_*$ and generates $\mathcal{S}^G_*$ under filtered colimits, the functor is an equivalence.

Let us explain to obtain G-spectra and finite G-spectra as stabilization of $\mathcal{S}^G_*$ and $\mathcal{S}^{\mathrm{fin},G}_*$ respectively. Let $\mathcal{U}$ be a complete G-universe and denote by $\mathrm{Sub}_\mathcal{U}$ the poset of finite-dimensional sub-representations. Following [20, Appendix C], we can consider functors $\mathcal{T}$ and $\mathcal{T}^{\mathrm{fin}}$ from $\mathrm{Sub}_\mathcal{U}$ to $\mathrm{Cat}^\omega_\infty$ (resp. Cat_∞), sending each $V \in \mathrm{Sub}_\mathcal{U}$ to $\mathcal{S}^G_*$ (resp. $\mathcal{S}^{\mathrm{fin},G}_*$) and each inclusion $V \subset W$ to smashing with S^{W-V}. Here, $\mathrm{Cat}^\omega_\infty$ is the ∞-category of compactly generated ∞-categories with compact object preserving left adjoints as morphisms, and $W - V$ is the orthogonal complement of V in W. As explained in [20,

Appendix C], $\operatorname{colim}_{\mathrm{Sub}_{\mathcal{U}}} \mathcal{T}$ carries a canonical symmetric monoidal structure, which is as a symmetric monoidal ∞-category canonically equivalent to Sp^G_∞. Denote $\operatorname{colim}_{\mathrm{Sub}_{\mathcal{U}}} \mathcal{T}^{\mathrm{fin}}$ by $\mathrm{Sp}^{G,\mathrm{fin}}_\infty$. General properties of colimits in $\mathrm{Cat}^\omega_\infty$ ([43, Proposition 5.5.7.10]) imply that the functor $\mathrm{Sp}^{G,\mathrm{fin}}_\infty \to \mathrm{Sp}^G_\infty$ extends to an equivalence $\mathrm{Ind}(\mathrm{Sp}^{G,\mathrm{fin}}_\infty) \simeq \mathrm{Sp}^G_\infty$. This yields directly:

Lemma 2.2. *Let $\mathcal{D}$ be an ∞-category with filtered colimits. The space of functors $\mathrm{Sp}^G_\infty \to \mathcal{D}$ preserving filtered colimits is equivalent to that of functors $\mathrm{Sp}^{\mathrm{fin},G}_\infty \to \mathcal{D}$.*

Remark 2.3. With our convention that G is always finite, we could simplify the colimit $\operatorname{colim}_{\mathrm{Sub}_{\mathcal{U}}} \mathcal{T}$ to the colimit of the directed system

$$\mathcal{S}^G_* \xrightarrow{S^{\rho G}} \mathcal{S}^G_* \xrightarrow{S^{\rho G}} \mathcal{S}^G_* \xrightarrow{S^{\rho G}} \cdots$$

and similarly for $\mathcal{S}^{\mathrm{fin},G}_*$. For possible future applications, we chose however to present the proofs in this section in a way that applies to all compact Lie groups.

We want to discuss a universal property of $\mathrm{Sp}^{\mathrm{fin},G}_\infty$ using symmetric monoidal structures. For this, we need the following result of Robalo. Recall here that an object X in a symmetric monoidal ∞-category is *symmetric* if the cyclic permutation of $X \otimes X \otimes X$ is homotopic to the identity.

Proposition 2.4. *Let $\mathcal{C}$ be a small symmetric monoidal ∞-category and $X \in \mathcal{C}$ symmetric. Then $\mathcal{C}[X^{-1}] := \operatorname{colim} \mathcal{C} \xrightarrow{X\otimes} \mathcal{C} \xrightarrow{X\otimes} \cdots$ has a symmetric monoidal structure such that $\mathcal{C} \to \mathcal{C}[X^{-1}]$ refines to a symmetric monoidal functor, which is initial among all those that send X to an invertible object.*

Proof. The proof is the same as that of [50, Corollary 2.22]; all necessary previous results are actually proven for small ∞-categories and not just for presentable ones. □

Corollary 2.5. *Let $\mathcal{D}$ be a symmetric monoidal ∞-category. Then taking the suspension spectrum defines an equivalence between the space of symmetric monoidal functors $\mathrm{Sp}^{\mathrm{fin},G}_\infty \to \mathcal{D}$ and the space of symmetric monoidal functors $\mathcal{S}^{\mathrm{fin},G}_* \to \mathcal{D}$ sending S^V for any G-representation to an invertible object.*

Proof. This can be deduced from the previous proposition as in [20, Corollary C.7] □

Corollary 2.6. *Let $\mathcal{D}$ be a symmetric monoidal ∞-category with filtered colimits. Then any symmetric monoidal functor $F\colon \mathrm{Sp}^G_\infty \to \mathcal{D}$ which preserves filtered colimits is uniquely (up to equivalence) determined by its restriction $F\Sigma^\infty\colon \mathcal{S}^{\mathrm{fin},G}_* \to \mathcal{D}$ (as a symmetric monoidal functor).*

Remark 2.7. The idea behind the preceding corollary is that we can write every G-spectrum canonically as a filtered homotopy colimit of $S^{-V} \wedge \Sigma^\infty X$. We chose the above treatment to give a precise meaning to how canonical this colimit actually is.

2.4. Norms and pullbacks

In this section, we will identify certain localizations of norm functors with pullbacks of norms from quotient groups. In the case of $BP^{((G))}$ this is a central ingredient of this paper.

First, we will recall the norm construction. For a group G, let $\mathcal{B}G$ denote the category with one object and having G as morphisms. Given an arbitrary symmetric monoidal category $(\mathcal{C}, \otimes, \mathbb{1})$, there is for a subgroup $H \subset G$ a norm functor

$$\mathcal{C}^{\mathcal{B}H} \to \mathcal{C}^{\mathcal{B}G}, \qquad X \mapsto X^{\otimes_H G}$$

from H-objects to G-objects, where the G-action is induced by the right G-action on G. In the case of spaces or sets, one can identify $X^{\times_H G}$ with $\mathrm{Map}_H(G, X)$ and for based spaces or sets, one can likewise identify $X^{\wedge_H G}$ with $\mathrm{Map}^*_H(G, X)$. In the case of orthogonal spectra, one can by [33, Proposition B.105] left derive the functor $(-)^{\wedge_H G}$ to obtain a functor N_H^G. (Often, N_H^G is also used for the corresponding underived functor, but the derived functor will be more important for us.) The functor N_H^G commutes with filtered (homotopy) colimits by [33, Propositions A.53, B.89]. Note moreover that $N_H^G \Sigma^\infty X \simeq \Sigma^\infty \mathrm{Map}^*_H(G, X)$ (if X is cofibrant or at least well-pointed) as Σ^∞ is symmetric monoidal.

Lemma 2.8. *Let G be a finite group, $K, H \subset G$ be two subgroups and X be a (based) topological H-space. Let $H\backslash G/K = \{Hg_1K, \dots, Hg_lK\}$. Then there are natural (based) homeomorphisms*

$$\mathrm{Map}_H(G, X)^K \cong X^{g_1 K g_1^{-1} \cap H} \times \cdots \times X^{g_l K g_l^{-1} \cap H}$$

and

$$\mathrm{Map}^*_H(G, X)^K \cong X^{g_1 K g_1^{-1} \cap H} \wedge \cdots \wedge X^{g_l K g_l^{-1} \cap H}, \tag{1}$$

where the K-action on the mapping spaces is induced by the right K-action on G. In particular, if $H = K$ is normal, we obtain a natural G/H-equivariant homeomorphism

$$\mathrm{Map}^*_H(G, X)^H \cong \mathrm{Map}^*(G/H, X^H).$$

Proof. The first two statements follow from the H-K-equivariant decomposition of G into $\coprod_{i=1}^l Hg_iK$. For the last one observe that if $H = K$ is normal, $H\backslash G/K = G/H$ and G/H permutes the factors of the decomposition in (1). □

To put the following theorem and its corollary into context, recall from [33, Proposition B.213] that $\Phi^G N_K^G X \simeq \Phi^K X$. We show more generally that $\Phi^H N_K^G X \simeq N_e^{G/H} \Phi^K X$ if $K \subset H \subset G$ and H is normal. Before we do so in Corollary 2.11, we provide a version that gives an equivalence on the level of G-spectra, i.e. before taking fixed points.

Theorem 2.9. *Let $H \subset G$ be a normal subgroup and X be an H-spectrum. Then we have an equivalence of G-spectra*

$$\tilde{E}\mathcal{F}[H] \wedge N_H^G X \simeq P_{G/H}^*(N_e^{G/H} \Phi^H(X)).$$

Proof. We have

$$\Phi^H N_H^G X \simeq N_e^{G/H} \Phi^H X \tag{2}$$

for all H-spectra X. Indeed: If X is a suspension spectrum, this reduces to the space-level statement $\mathrm{Map}_H^*(G, X)^H \simeq \mathrm{Map}^*(G/H, X^H)$, which is part of Lemma 2.8. Both sides of (2) are symmetric monoidal and commute with filtered homotopy colimits. Thus Corollary 2.6 implies the claim.

Applying $P_{G/H}^*$ to (2), it suffices to check that $P_{G/H}^* \Phi^H N_H^G X$ is equivalent to $\tilde{E}\mathcal{F}[H] \wedge N_H^G X$. But the equivalence of $P_{G/H}^* \Phi^H$ with $\tilde{E}\mathcal{F}[H] \wedge -$ was already noted above. □

Corollary 2.10. *Let $K \subset H \subset G$ be subgroups and assume that $H \subset G$ is normal. Let moreover X be a K-spectrum. Then there is an equivalence of G-spectra*

$$\tilde{E}\mathcal{F}[H] \wedge N_K^G X \simeq P_{G/H}^*(N_e^{G/H} \Phi^K(X)).$$

Proof. This follows from Theorem 2.9 by applying it to $N_K^H X$. Here, we use $N_K^G X \simeq N_H^G N_K^H X$ and $\Phi^H N_K^H X \simeq \Phi^K X$. □

Taking H-fixed points we obtain a strengthened form of Theorem 1.1:

Corollary 2.11. *Let $K \subset H \subset G$ be subgroups and assume that $H \subset G$ is normal. Let moreover X be a K-spectrum. Then there is an equivalence of G/H-spectra*

$$\Phi^H N_K^G X \simeq N_e^{G/H} \Phi^K(X).$$

Remark 2.12. An alternative proof of this result is possible using [56, Theorem 2.7].

As we will recall below, there is a C_2-spectrum $BP_{\mathbb{R}}$ with geometric fixed points $H\mathbb{F}_2$. For $G = C_4$ and $H = C_2$, we can express $\tilde{E}\mathcal{F}[H]$ as $S^{\infty\lambda}$, where λ is the 2-dimension representation of C_4 corresponding to rotation by an angle of $\frac{\pi}{2}$. Denoting the norm $N_{C_2}^{C_4} BP_{\mathbb{R}}$ by $BP^{((C_4))}$, we obtain our main example for Theorem 2.9.

Corollary 2.13. *There is an equivalence*

$$BP^{((C_4))} \wedge S^{\infty\lambda} \simeq P^*_{C_4/C_2} N_1^2(H\mathbb{F}_2).$$

We end this section with a different kind of compatibility of norms and pullbacks.

Proposition 2.14. *Let $K \subset H \subset G$ be subgroups such that K is normal in G. Then there is a natural equivalence $N_H^G P^*_{H/K} \simeq P^*_{G/K} N^{G/K}_{H/K}$ of functors $\mathrm{Sp}^\infty_{H/K} \to \mathrm{Sp}^\infty_G$.*

Proof. Since both $N_H^G P^*_{H/K}$ and $P^*_{G/K} N^{G/K}_{H/K}$ commute with filtered colimits and are symmetric monoidal, it suffices (as in the proof of Theorem 2.9) to provide a natural equivalence of their restriction to suspension spectra. We compute

$$N_H^G P^*_{H/K} \Sigma^\infty X \simeq N_H^G \Sigma^\infty \tilde{E}\mathcal{F}[H] \wedge X \simeq \Sigma^\infty \mathrm{Map}^*_H(G, \tilde{E}\mathcal{F}_H[K]) \wedge \mathrm{Map}^*_H(G, X)$$

and

$$P^*_{G/K} N^{G/K}_{H/K} \Sigma^\infty X \simeq P^*_{G/K} \Sigma^\infty \mathrm{Map}^*_{H/K}(G/K, X) \simeq \Sigma^\infty \tilde{E}\mathcal{F}_G[K] \wedge \mathrm{Map}^*_H(G, X),$$

where we used a subscript at $\tilde{E}\mathcal{F}$ to indicate whether it is a G-space or an H-space. Using Lemma 2.8, one can check that $\mathrm{Map}^*_H(G, \tilde{E}\mathcal{F}_H[K])^L \simeq S^0$ if $L \subset K$ and is contractible otherwise; thus indeed $\mathrm{Map}^*_H(G, \tilde{E}\mathcal{F}_H[K]) \simeq \tilde{E}\mathcal{F}_G[K]$. □

2.5. Multiplicative structures of localizations

In many cases, smashing with $\tilde{E}\mathcal{F}[H]$ is equivalent to localizing at a certain element in $\pi^G_\bigstar \mathbb{S}$ (for example if G is cyclic). The goal of this section is to investigate which kind of multiplicative structure localization at such an element preserves. More specifically let us fix an N_∞-operad $\mathcal{O}$, i.e. an operad $\mathcal{O}$ in (unbased) G-spaces such that each $\mathcal{O}(n)$ is a universal space for a family $\mathcal{F}_n$ of graph subgroups of $G \times \Sigma_n$, containing all $H \times \{e\}$ for subgroups $H \subset G$. This notion was introduced in [10]. In the maximal case, we speak of a *G-E_∞-operad* and by [10, Theorem A.6] every algebra over such an operad can be strictified to a commutative G-spectrum. In the minimal case, we speak of a *(naive) E_∞-operad*.

Essentially, the different versions of N_∞-operads encode which norms we see in the homotopy groups of an $\mathcal{O}$-algebra. To be more precise, call an H-set T *admissible* if the graph of the H-action on T lies in $\mathcal{F}_{|T|}$. By [3, Remark 5.15] an $\mathcal{O}$-algebra R admits norms $N_K^H \colon \pi_V^K R \to \pi^H_{\mathrm{Ind}_K^H V} R$ if H/K is admissible, and the groups $\pi^H_\bigstar R$ assemble into an $RO(G)$-graded incomplete Tambara functor.

As already observed in [47], localizations only need to preserve naive E_∞-structures, but not G-E_∞-structures. Later, [32] gave a criterion when localizations indeed preserve

G-E_∞-structures and this was extended in [13] to N_∞-algebras, albeit only for localizations of elements in degree 0. In this section, we will extend this work to elements in non-trivial degree and follow the proof strategy of [13, Proposition 2.30].

Let us first recall what localizing at some $x \in \pi_V^G \mathbb{S}$ means. We say that a G-spectrum E is *x-local* if x acts invertibly on E or, equivalently, on $\pi_\star^G E$. Given a G-spectrum E, we construct its *x-localization* as

$$x^{-1}E = \operatorname{hocolim}\left(E \xrightarrow{x} \Sigma^{-V}E \xrightarrow{x} \Sigma^{-2V}E \xrightarrow{x} \cdots\right).$$

Note that $x^{-1}E \simeq E \wedge x^{-1}\mathbb{S}$.

Example 2.15. Given a G-representation V, let $a_V \colon S^0 \to S^V$ be the Euler class. Then $a_V^{-1}\mathbb{S} \simeq S^{\infty V}$ and hence in general $a_V^{-1}E \simeq S^{\infty V} \wedge E$. In particular, we can reformulate Corollary 2.13 as

$$a_\lambda^{-1} BP^{((C_4))} \simeq P^*_{C_4/C_2} N_1^2(H\mathbb{F}_2).$$

A map $f \colon E \to F$ is an *x-local equivalence* if $f \wedge x^{-1}\mathbb{S}$ is an equivalence; by abuse of notation, we call for $H \subset G$ a map of H-spectra an x-equivalence if it is a $\operatorname{Res}^G_H(x)$-equivalence.

Definition 2.16. Localization at x *preserves $\mathcal{O}$-algebras* if for every $\mathcal{O}$-algebra R, we can lift the morphism $R \to x^{-1}R$ in $\mathrm{Ho}(\mathrm{Sp}^G)$ (up to isomorphism) to a morphism in $\mathrm{Ho}(\mathcal{O}-\mathrm{Alg})$.

We will use the following specialization of a criterion of [25, Corollary 7.10]:

Proposition 2.17. *Localization at x preserves $\mathcal{O}$-algebras if and only if*

$$N_K^H \operatorname{Res}_K^G \colon \mathrm{Sp}^G_\infty \to \mathrm{Sp}^H_\infty$$

preserves x-equivalences for every $K \subset H \subset G$ such that H/K is admissible as an H-set.

To reformulate this criterion, we need the following lemma.

Lemma 2.18. *There is an equivalence $N_K^H \operatorname{Res}_K^G(x^{-1}\mathbb{S}) \simeq (N_K^H \operatorname{Res}_K^G(x))^{-1}(\mathbb{S}_H)$ for $\mathbb{S}_H$ the H-equivariant sphere spectrum.*

Proof. Applying $N_K^H \operatorname{Res}_K^G$ to

$$\mathbb{S} \xrightarrow{x} \Sigma^{-V}\mathbb{S} \xrightarrow{x} \Sigma^{-2V}\mathbb{S} \xrightarrow{x} \cdots,$$

we obtain precisely

$$\mathbb{S}_H \xrightarrow{N_K^H \operatorname{Res}_K^G(x)} \Sigma^{-} \operatorname{Ind}_K^H \operatorname{Res}_K^G V \mathbb{S}_H \xrightarrow{N_K^H \operatorname{Res}_K^G(x)} \Sigma^{-2} \operatorname{Ind}_K^H \operatorname{Res}_K^G V \mathbb{S}_H \xrightarrow{N_K^H \operatorname{Res}_K^G(x)} \dots$$

Here we have used that the norm of a representation sphere is computed by induction. As both N_K^H and Res_K^G preserve filtered homotopy colimits, the result follows. □

Proposition 2.19. *Localization at x preserves $\mathcal{O}$-algebras if and only if $N_K^H \operatorname{Res}_K^G(x)$ divides a power of $\operatorname{Res}_H^G(x)$ for every $K \subset H \subset G$ such that H/K is admissible as an H-set.*

Proof. Let $K \subset H \subset G$ be subgroups such that H/K is admissible as an H-set. By Proposition 2.17, we have to show that $N_K^H \operatorname{Res}_K^G(x)$ divides a power of $\operatorname{Res}_H^G(x)$ if and only if

$$N_K^H \operatorname{Res}_K^G \colon \operatorname{Sp}_\infty^G \to \operatorname{Sp}_\infty^H$$

preserves x-equivalences.

Assume first that $N_K^H \operatorname{Res}_K^G$ preserves x-equivalences. By the preceding lemma, we see in particular that $\mathbb{S}_H \to (N_K^H \operatorname{Res}_K^G(x))^{-1}\mathbb{S}_H$ is an x-equivalence, i.e. $N_K^H \operatorname{Res}_K^G(x)$ becomes a unit after inverting $\operatorname{Res}_H^G(x)$ and just must divide a power of it.

Assume now that $N_K^H \operatorname{Res}_K^G(x)$ divides a power of $\operatorname{Res}_H^G(x)$. Then the map $\mathbb{S}_H \to \operatorname{Res}_H^G(x)^{-1}\mathbb{S}_H$ factors over the standard map $\mathbb{S}_H \to (N_K^H \operatorname{Res}_K^G(x))^{-1}\mathbb{S}_H$.

Let now $f \colon E \to F$ be an x-equivalence of G-spectra, i.e. we assume that $f \wedge x^{-1}\mathbb{S}$ is an equivalence. As N_K^H and Res_H^G are symmetric monoidal, we see that $N_K^H \operatorname{Res}_H^G(f \wedge x^{-1}\mathbb{S})$ is equivalent to $N_K^H \operatorname{Res}_H^G(f) \wedge (N_K^H \operatorname{Res}_K^G(x))^{-1}\mathbb{S}_H$, which is thus an equivalence. Tensoring with $\operatorname{Res}_H^G(x)^{-1}\mathbb{S}_H$ over $(N_K^H \operatorname{Res}_K^G(x))^{-1}\mathbb{S}_H$ yields the result. □

We specialize now to the case that x is the Euler class $a_V \colon S^0 \to S^V$. In this case we have $N_K^G \operatorname{Res}_H^G a_V = a_{\operatorname{Ind}_K^G \operatorname{Res}_H^G V}$. Thus to see which multiplicative structure localization at a_V preserves, we only have to understand divisibility relations between Euler classes. In particular, we obtain the following corollary:

Corollary 2.20. *Let V be a G-representation. Assume that $\operatorname{Ind}_K^H \operatorname{Res}_K^G V$ is a summand of a multiple of $\operatorname{Res}_H^G V$ for every $K \subset H \subset G$ such that H/K is an admissible H-set. Then localizing at a_V preserves $\mathcal{O}$-algebras.*

Remark 2.21. While this corollary is everything we need, one can be more precise. For a H-representation V, let $\widetilde{\mathcal{F}_V^{\mathrm{fix}}}$ be the family of subgroups $K \subset H$ such that $V^K \neq 0$. Thus, $a_V^{-1}\mathbb{S}_H \simeq S^{\infty V} \simeq \widetilde{E\mathcal{F}_V^{\mathrm{fix}}}$. In general, a_W divides a power of a_V if and only if $a_W^{-1}a_V^{-1}\mathbb{S}_H \simeq a_V^{-1}\mathbb{S}_H$, i.e. if $\mathcal{F}_W^{\mathrm{fix}} \subset \mathcal{F}_V^{\mathrm{fix}}$. (This is a weaker condition than W being contained in a multiple of V: for example, take $G = C_8$ and W and V be the two-dimensional real representation corresponding to rotation by $\frac{1}{8}\cdot 2\pi$ and $\frac{3}{8}\cdot 2\pi$, respectively, which both have trivial fixed point family.)

Specializing Proposition 2.19 thus yields: For a G-representation V, localization at a_V preserves $\mathcal{O}$-algebras if and only if $\mathcal{F}^{\mathrm{fix}}_{\mathrm{Ind}_K^H \mathrm{Res}_K^G V} \subset \mathcal{F}^{\mathrm{fix}}_{\mathrm{Res}_H^G V}$ for all $K \subset H \subset G$ such that H/K is an admissible H-set.

Example 2.22. Let $G = C_{2^n}$ and $\lambda = \lambda^n$ be the two-dimensional representation of C_{2^n} given by rotation by an angle of $\frac{2\pi}{2^n}$. We observe that $\mathrm{Res}^{C_2^n}_{C_{2^k}} \lambda^n = \lambda^k$ and $\mathrm{Ind}^{C_2^m}_{C_{2^k}} \lambda^k = 2^{m-k}\lambda^m$ unless $k = 1$. Thus localizing at a_λ preserves $\mathcal{O}$-algebras if the following holds: H/K is H-admissible if and only if $K \neq e$. In particular, we see that for any commutative C_{2^n}-spectrum R, the localization $a_\lambda^{-1}R$ admits norms from $\pi_*^{C_{2^k}}$ to $\pi_*^{C_{2^n}}$ for $0 < k < n$, but will not admit norms from π_*^e unless the target is zero. The example we care most about is $a_\lambda^{-1}MU^{((C_{2^n}))}$.

These considerations have consequences for the multiplicative behavior of the pullback functor $P^*_{C_{2^n}/C_2}$. Let R be an algebra over a C_{2^n}/C_2-E_∞-operad $\mathcal{O}$ in C_{2^n}/C_2-spectra. Denoting the projection $C_{2^n} \to C_{2^n}/C_2$ by p, we see that $p^*\mathcal{O}$ is an N_∞-operad for which $\Gamma \subset C_{2^n} \times \Sigma_n$ is in $\mathcal{F}_n$ if and only if $\Gamma/(C_2 \times e)$ is a graph subgroup of $(C_{2^n}/C_2) \times \Sigma_n$. This means that H/K is H-admissible if and only if $K \neq e$. Note further that

$$P^*_{C_{2^n}/C_2} R = p^*R \wedge \widetilde{E}\mathcal{F}[C_2] \simeq p^*R[a_\lambda^{-1}]$$

since $\lambda^K = 0$ unless K is trivial. Using the paragraph above we see that $P^*_{C_{2^n}/C_2} R$ retains the structure of a $p^*\mathcal{O}$-algebra.

Likewise we can apply our considerations to the geometric fixed point functor. With $p^*\mathcal{O}$ as above, we see that for a G-commutative ring spectrum R, the localization $a_\lambda^{-1}R$ retains an action of $p^*\mathcal{O}$ and thus $\Phi^{C_2}R \simeq (a_\lambda^{-1}R)^{C_2}$ has the structure of a $\mathcal{O}$-algebra. Thus $\Phi^{C_2}R$ is equivalent to a G/C_2-commutative ring spectrum.

3. The slice spectral sequence and the localized slice spectral sequence

3.1. The slice spectral sequence of $MU^{((C_{2^n}))}$ and $BP^{((C_{2^n}))}$

Our main computational tool in this paper is a modification of the equivariant slice spectral sequence of Hill–Hopkins–Ravenel. In this subsection, we list some important facts about the slice filtration for norms of $MU_{\mathbb{R}}$ and $BP_{\mathbb{R}}$, which we will need for the rest of the paper. For a detailed construction of the slice spectral sequence and its properties, see [33, Section 4] and [34].

Let $G = C_{2^n}$ be the cyclic group of order 2^n, with generator γ. The spectrum $MU^{((G))}$ is defined as

$$MU^{((G))} := N^G_{C_2} MU_{\mathbb{R}}.$$

The underlying spectrum of $MU^{((G))}$ is the smash product of 2^{n-1}-copies of MU.

Hill, Hopkins, and Ravenel [33, Section 5] constructed elements

$$\overline{r}_i \in \pi^{C_2}_{i\rho_2} MU^{((G))}$$

such that

$$\pi^{C_2}_{*\rho_2} MU^{((G))} \cong \mathbb{Z}[G \cdot \overline{r}_1, G \cdot \overline{r}_2, \ldots].$$

Here, $G \cdot x$ denotes the set $\{x, \gamma x, \gamma^2 x, \ldots, \gamma^{2^{n-1}} x\}$, and the Weyl action is given by

$$\gamma \cdot \gamma^j \overline{r}_i = \begin{cases} \gamma^{j+1} \overline{r}_i & 0 \leq j \leq 2^{n-1} - 2 \\ (-1)^i \overline{r}_i & j = 2^{n-1} - 1. \end{cases}$$

Adjoint to each map

$$\overline{r}_i : S^{i\rho_2} \longrightarrow i^*_{C_2} MU^{((G))}$$

is an associative algebra map from the free associative algebra

$$S^0[\overline{r}_i] = \bigvee_{j \geq 0} (S^{i\rho_2})^{\wedge j} \longrightarrow i^*_{C_2} MU^{((G))}.$$

Applying the norm and using the norm-restriction adjunction, this gives a G-equivariant associative algebra map

$$S^0[G \cdot \overline{r}_i] = N^G_{C_2} S^0[\overline{r}_i] \longrightarrow MU^{((G))}.$$

Smashing these maps together produces an associative algebra map

$$A := S^0[G \cdot \overline{r}_1, G \cdot \overline{r}_2, \ldots] = \bigwedge_{i=1}^{\infty} S^0[G \cdot \overline{r}_i] \longrightarrow MU^{((G))}.$$

Note that by construction, A is a wedge of representation spheres, indexed by monomials in the $\overline{r}_i$s. By the Slice Theorem [33, Theorem 6.1], the slice filtration of $MU^{((G))}$ is the filtration associated with the powers of the augmentation ideal of A. The slice associated graded for $MU^{((G))}$ is the graded spectrum

$$S^0[G \cdot \overline{r}_1, G \cdot \overline{r}_2, \ldots] \wedge H\underline{\mathbb{Z}},$$

where the degree of a summand corresponding to a monomial in the $\overline{r}_i$ generators and their conjugates is the underlying degree.

As a consequence of the slice theorem, the slice spectral sequence for the $RO(G)$-graded homotopy groups of $MU^{((G))}$ has E_2-term the $RO(G)$-graded homology of $S^0[G \cdot$

$\overline{r}_1, G\cdot\overline{r}_2, \ldots]$ with coefficients in the constant Mackey functor $\underline{\mathbb{Z}}$. To compute this, note that $S^0[G\cdot\overline{r}_1, G\cdot\overline{r}_2, \ldots]$ can be decomposed into a wedge sum of slice cells of the form

$$G_+ \wedge_{H_p} S^{\frac{|p|}{|H_p|}\rho_{H_p}},$$

where p ranges over a set of representatives for the orbits of monomials in the $\gamma^j\overline{r}_i$ generators, and $H_p \subset G$ is the stabilizer of p (mod 2). Therefore, the E_2-page of the integer graded slice spectral sequence can be computed completely by writing down explicit equivariant chain complexes for the representation spheres $S^{\frac{|p|}{|H_p|}\rho_{H_p}}$.

The exact same story holds for norms of $BP_{\mathbb{R}}$ as well. By [36, Theorems 2.25, 2.33], the classical Quillen idempotent $MU \longrightarrow MU$ lifts to a multiplicative idempotent $MU_{\mathbb{R}} \to MU_{\mathbb{R}}$ with image $BP_{\mathbb{R}}$, resulting in particular in a multiplicative C_2-equivariant map

$$MU_{\mathbb{R}} \longrightarrow BP_{\mathbb{R}}.$$

Taking the norm $N_{C_2}^G(-)$ of this map produces a multiplicative G-equivariant map

$$MU^{((G))} \longrightarrow BP^{((G))} =: N_{C_2}^G BP_{\mathbb{R}}.$$

The exact same technique as the one used in [33, Section 5] shows that there are generators

$$\overline{t}_i \in \pi^{C_2}_{(2^i-1)\rho_2} BP^{((G))}$$

such that

$$\pi^{C_2}_{*\rho_2} BP^{((G))} \cong \mathbb{Z}_{(2)}[G\cdot\overline{t}_1, G\cdot\overline{t}_2, \ldots].$$

Throughout the paper, the generators $\overline{t}_i$ are chosen to be the coefficients of the canonical isomorphism from the formal group law of the first $BP_{\mathbb{R}}$ component to the formal group law of the second $BP_{\mathbb{R}}$-component. In the case when $G = C_4$, it is the canonical isomorphism from the formal group law F_L to F_R, where F_L is induced by the map

$$BP_{\mathbb{R}} \simeq BP_{\mathbb{R}} \wedge S^0 \longrightarrow BP_{\mathbb{R}} \wedge BP_{\mathbb{R}},$$

and F_R is induced by the map

$$BP_{\mathbb{R}} \simeq S^0 \wedge BP_{\mathbb{R}} \longrightarrow BP_{\mathbb{R}} \wedge BP_{\mathbb{R}}.$$

Remark 3.1. Our specific choice of the formal group law and the generators $\overline{t}_i$ is because we would like to control their geometric fixed points (see Proposition 6.2). Nevertheless, we would like to remark that the proofs and formulas in both [33] and [8] work for any choice of formal group law and the corresponding $\overline{t}_i$ generators we get for $\pi^{C_2}_{*\rho_2} BP^{((G))}$, as long as the conditions in [33, Proposition 5.45] are satisfied.

Just like $MU^{((G))}$, we can build an equivariant refinement

$$S^0[G \cdot \overline{t}_1, G \cdot \overline{t}_2, \ldots] \longrightarrow BP^{((G))}$$

from which the Slice Theorem implies that the slice associated graded for $BP^{((G))}$ is the graded spectrum $S^0[G \cdot \overline{t}_1, G \cdot \overline{t}_2, \ldots] \wedge H\underline{\mathbb{Z}}_{(2)}$.

Remark 3.2. The regular slice towers of $MU^{((G))}$ and $BP^{((G))}$ are isomorphic to their slice towers in [33], since all the HHR-slices of them are regular slices. For a proof of the slice theorem in terms of the regular slice filtration, see [35, Chapter 12.4].

Since the slice filtration is an equivariant filtration, the slice spectral sequence is a spectral sequence of $RO(G)$-graded Mackey functors. Moreover, the slice spectral sequences for $MU^{((G))}$ and $BP^{((G))}$ are multiplicative spectral sequences and the natural maps between them are multiplicative as well (see [33, Section 4.7]), and the slice spectral sequence for $BP^{((G))}$ is a spectral sequence of modules over the spectral sequence of $MU^{((G))}$ in Mackey functors.

3.2. The localized spectral sequence

In this subsection, we introduce a variant of the slice spectral sequence which we call the localized slice spectral sequence. This will be our main computational tool to compute $a_\lambda^{-1}BP^{((C_4))}$ in the later sections.

Let $\lambda_{2^{n-i}}$ denote the 2-dimensional real C_{2^n}-representation corresponding to rotation by $\left(\frac{\pi}{2^{n-i}}\right)$ and σ denote the real sign representation of C_{2^n}. Given a C_{2^n}-spectrum X, we have an equivalence

$$\tilde{E}\mathcal{F}[C_{2^i}] \wedge X \simeq S^{\infty\lambda_{2^{n-i}}} \wedge X \simeq a^{-1}_{\lambda_{2^{n-i}}} X$$

for all $1 \leq i \leq n$. For example, there are equivalences

$$\begin{aligned}
\widetilde{E}\mathcal{F}[C_{2^n}] \wedge X &\simeq a_{\lambda_1}^{-1} X = a_{2\sigma}^{-1} X = a_\sigma^{-1} X, \\
\widetilde{E}\mathcal{F}[C_{2^{n-1}}] \wedge X &\simeq a_{\lambda_2}^{-1} X, \\
\widetilde{E}\mathcal{F}[C_{2^{n-2}}] \wedge X &\simeq a_{\lambda_4}^{-1} X.
\end{aligned}$$

The following theorem shows that one can compute the homotopy groups of $\tilde{E}\mathcal{F}[C_{2^i}] \wedge X = a^{-1}_{\lambda_{2^{n-i}}} X$ by smashing the slice tower of X with $\tilde{E}\mathcal{F}[C_{2^i}]$. The resulting localized slice spectral sequence will converge to the homotopy groups of $a^{-1}_{\lambda_{2^{n-i}}} X$.

Theorem 3.3. *Let X be a C_{2^n}-spectrum, and let $\{P^\bullet\}$ denote the (regular) slice tower for X. Consider the tower*

$$\{Q^\bullet\} := \{\tilde{E}\mathcal{F}[C_{2^i}] \wedge P^\bullet\}$$

obtained by smashing $\{P^\bullet\}$ *with* $\tilde{E}\mathcal{F}[C_{2^i}]$*. The spectral sequence associated to* $\{Q^\bullet\}$ *converges strongly to the homotopy groups of* $\tilde{E}\mathcal{F}[C_{2^i}] \wedge X$*.*

Proof. Let $\lambda := \lambda_{2^{n-i}}$. Consider the tower

$$\begin{array}{ccc}
S^{\infty\lambda} \wedge X & \longrightarrow & \varprojlim(S^{\infty\lambda} \wedge P^\bullet X) \\
 & & \downarrow \\
 & & \vdots \\
 & & \downarrow \\
 & & S^{\infty\lambda} \wedge P^n X \\
 & & \downarrow \\
 & & S^{\infty\lambda} \wedge P^{n-1} X \\
 & & \downarrow \\
 & & \vdots
\end{array}
\qquad
\begin{array}{c}
\\ \\ \\ \\ \longleftarrow \\ \\ \longleftarrow \\ \\ \\
\end{array}
\qquad
\begin{array}{c}
\\ \\ \\ \\ S^{\infty\lambda} \wedge P^n_n X \\ \\ S^{\infty\lambda} \wedge P^{n-1}_{n-1} X \\ \\ \\
\end{array}$$

We will first show that the spectral sequence converges to the limit, $\varprojlim(S^{\infty\lambda} \wedge P^\bullet X)$. Since smash products commute with colimits, we have the equivalence

$$\varinjlim(S^{\infty\lambda} \wedge P^\bullet X) \simeq *$$

and so the colimit of the tower is contractible. The slices $P^n_n X$ satisfy $P^n_n X \geq n$ for all n. Furthermore, since $S^{\infty\lambda} \geq 0$, we also have

$$S^{\infty\lambda} \wedge P^n_n X \geq n$$

by [33, Proposition 4.26].[1] Applying Proposition 4.40 in [33] to $S^{\infty\lambda} \wedge P^n_n X$ shows that the homotopy groups

$$\underline{\pi}_k(S^{\infty\lambda} \wedge P^n_n X) = 0 \text{ if } \begin{cases} n \geq 0 \text{ and } k < \lfloor \frac{n}{|G|} \rfloor, \\ n < 0 \text{ and } k < n. \end{cases}$$

This gives a vanishing line on the E_2-page of the spectral sequence. It follows that the spectral sequence converges strongly to the homotopy groups of the limit, $\underline{\pi}_k \varprojlim(S^{\infty\lambda} \wedge P^\bullet X)$ [12, Section 5-6].

[1] The proof of this result and of the part of [33, Proposition 4.40] we need are still valid for the regular slice filtration instead of the slice filtration as used in [33].

To finish our proof, it suffices to show that the map

$$S^{\infty\lambda} \wedge X \longrightarrow \varprojlim (S^{\infty\lambda} \wedge P^\bullet X)$$

is an equivalence.

Consider the cofiber sequence

$$P_{n+1}X \longrightarrow X \longrightarrow P^n X$$

used in the definition of the slice tower. In the cofiber sequence, $P_{n+1}X \geq n+1$ and $P^n X \leq n$. Smashing this cofiber sequence with $S^{\infty\lambda}$ produces a new cofiber sequence

$$S^{\infty\lambda} \wedge P_{n+1}X \longrightarrow S^{\infty\lambda} \wedge X \longrightarrow S^{\infty\lambda} \wedge P^n X.$$

Since $S^{\infty\lambda} \geq 0$, [33, Proposition 4.26] implies that

$$S^{\infty\lambda} \wedge P_{n+1}X \geq n+1.$$

Applying [33, Proposition 4.40] to $S^{\infty\lambda} \wedge P_{n+1}X$ shows that

$$\underline{\pi}_k(S^{\infty\lambda} \wedge P_{n+1}X) = 0 \text{ if } \begin{cases} n+1 \geq 0 \text{ and } k < \lfloor \frac{n+1}{|G|} \rfloor, \\ n+1 < 0 \text{ and } k < n+1. \end{cases}$$

The cofiber sequence above induces the following long exact sequence in homotopy groups:

$$\underline{\pi}_k(S^{\infty\lambda} \wedge P_{n+1}X) \longrightarrow \underline{\pi}_k(S^{\infty\lambda} \wedge X) \longrightarrow \underline{\pi}_k(S^{\infty\lambda} \wedge P^n X) \longrightarrow \underline{\pi}_{k-1}(S^{\infty\lambda} \wedge P_{n+1}X) \longrightarrow \cdots$$

It follows from this long exact sequence and the discussion above that

$$\underline{\pi}_k(S^{\infty\lambda} \wedge X) \cong \underline{\pi}_k(S^{\infty\lambda} \wedge P^n X) \text{ if } \begin{cases} n+1 \geq 0 \text{ and } k < \lfloor \frac{n+1}{|G|} \rfloor, \\ n+1 < 0 \text{ and } k < n+1. \end{cases}$$

This means that for any k, the kth homotopy groups of $S^{\infty\lambda} \wedge X$ and $S^{\infty\lambda} \wedge P^n X$ will be isomorphic when n is large enough. In particular, the map $S^{\infty\lambda} \wedge P^{n+1}X \to S^{\infty\lambda} \wedge P^n X$ will induce an isomorphism on $\underline{\pi}_k$. It is then immediate that the system $\underline{\pi}_k(S^{\infty\lambda} \wedge P^\bullet X)$ satisfies the Mittag–Leffler condition and therefore

$$\underline{\pi}_k \varprojlim (S^{\infty\lambda} \wedge P^\bullet X) \cong \varprojlim \underline{\pi}_k(S^{\infty\lambda} \wedge P^\bullet X) \cong \underline{\pi}_k(S^{\infty\lambda} \wedge P^n X)$$

for n large.

Another way to observe this is by using the localized slice spectral sequence. As we have shown, the spectral sequence associated to the tower $\{Q^\bullet\} := \{S^{\infty\lambda} \wedge P^\bullet\}$ converges to the homotopy groups of $\varprojlim (S^{\infty\lambda} \wedge P^\bullet X)$. It takes the form

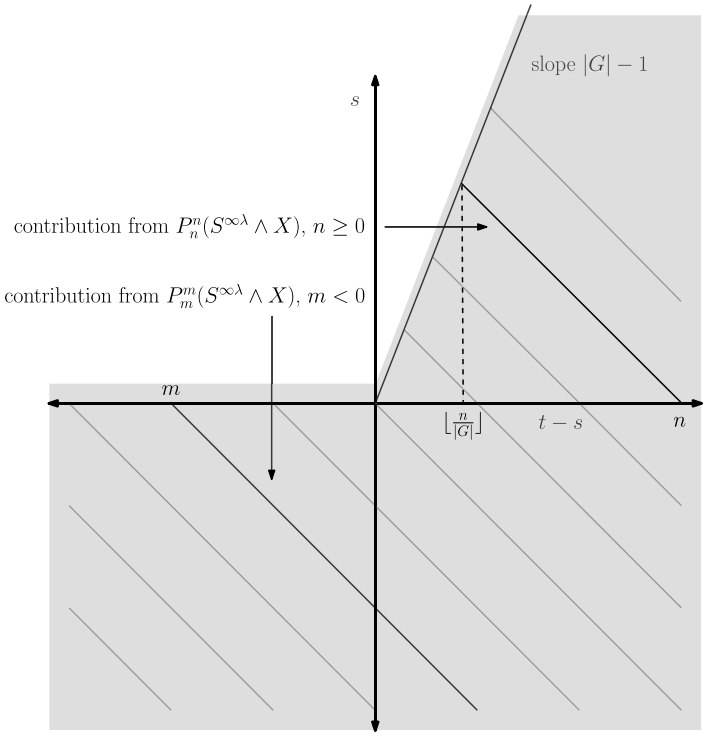


Fig. 1. Spectral sequence associated to the tower $\{\tilde{E}\mathcal{F}[C_{2^i}]\wedge P^\bullet\}$.

$$E_2^{s,n}=\underline{\pi}_{n-s}(S^{\infty\lambda}\wedge P^n_n X)\Longrightarrow \underline{\pi}_{n-s}\varprojlim(S^{\infty\lambda}\wedge P^\bullet X).$$

By [33, Proposition 4.40], the homotopy groups

$$\underline{\pi}_{n-s}(S^{\infty\lambda}\wedge P^n_n X)$$

do not contribute to $\underline{\pi}_k\varprojlim(S^{\infty\lambda}\wedge P^\bullet X)$ when $n\geq 0$ and $k<\lfloor\frac{n}{|G|}\rfloor$, or when $n<0$ and $k<n$ (see Fig. 1). Therefore,

$$\underline{\pi}_k\varprojlim(S^{\infty\lambda}\wedge P^\bullet X)\cong\underline{\pi}_k(S^{\infty\lambda}\wedge P^n X)\ \text{ if }\begin{cases} n\geq 0 \text{ and } k<\lfloor\frac{n}{|G|}\rfloor,\\ n<0 \text{ and } k<n.\end{cases}$$

For any k, consider the diagram

$$\begin{array}{ccc}\underline{\pi}_k(S^{\infty\lambda}\wedge X) & \longrightarrow & \underline{\pi}_k\varprojlim(S^{\infty\lambda}\wedge P^\bullet X)\\ & \searrow^{\cong} & \downarrow\cong\\ & & \underline{\pi}_k(S^{\infty\lambda}\wedge P^n X)\end{array}$$

We have proven that when n is large enough ($n > k$), the vertical arrow and the diagonal arrow are isomorphisms. Therefore, the horizontal arrow induces an isomorphism

$$\underline{\pi}_k(S^{\infty\lambda} \wedge X) \cong \underline{\pi}_k \varprojlim (S^{\infty\lambda} \wedge P^\bullet X)$$

for all k. It follows that $S^{\infty\lambda} \wedge X \simeq \varprojlim (S^{\infty\lambda} \wedge P^\bullet X)$, as desired. □

From the discussion in [55, Section I.4] it follows that the localized slice spectral sequences of $MU^{((G))}$ and $BP^{((G))}$ (and more generally of G-ring spectra) are multiplicative spectral sequences.

3.3. Exotic transfers

If the transfer of a given class in the slice spectral sequence is zero, it might still support a non-trivial *exotic transfer* in a higher filtration. Understanding these is both crucial for understanding the Mackey functor structure of the spectral sequence and helpful to deduce differentials and extensions inside the spectral sequence. While the concept of exotic transfers is pretty transparent for permanent cycles, it is slightly more subtle for exotic transfers just happening on finite pages. Following the lead of [6] (in the case of the Picard spectral sequence), we will give a precise definition of this phenomenon and show how it behaves with respect to differentials. It turns out that it is no more difficult to treat a more general setting, which specializes to several different known spectral sequences and allows also for more general operations than just transfers.

In this subsection, we will first state a general definition of exotic w-operations and prove some general results. Then, we will specialize to the case of cyclic 2-groups and prove a variant of [34, Theorem 4.4] that also work for exotic transfers and restrictions on finite pages.

We consider a tower

$$\cdots \to X^{i+1} \to X^i \to X^{i-1} \to \cdots$$

of G-spectra. Recall that to this we can associate a spectral sequence as follows: Let $X^m_n = \mathrm{fib}(X^m \to X^{n-1})$. For V a virtual G-representation of dimension t, we set $E_2^{s,V} = \underline{\pi}_{V-s}(X^t_t)$ and more generally

$$E_r^{s,V} = \mathrm{im}(\underline{\pi}_{V-s} X^{t+(r-2)}_t \to \underline{\pi}_{V-s} X^t_{t-(r-2)}).$$

The differentials $d_r \colon E_r^{s,V} \to E_r^{s+r,V+r-1}$ are defined as the restrictions of the boundary maps $\delta \colon \underline{\pi}_{V-s} X^t_{t-(r-2)} \to \underline{\pi}_{V-s-1} X^{t+r-1}_{t+1}$ (coming from the cofiber sequence $X^{t+(r-1)}_{t+1} \to X^{t+(r-1)}_{t-(r-2)} \to X^t_{t-(r-2)}$). See e.g. [44, Section 1.2.2] for some details in the setting of an ascending filtration. Our setting specializes in particular to the following spectral sequences:

(1) Given a spectrum Z with a G-action, set $X^i = (\tau_{\leq i}Z)^{EG_+}$. We recover the homotopy fixed point spectral sequence.
(2) Given a spectrum Z with a G-action, set $X^i = (\tau_{\leq i}Z \wedge \tilde{E}G)^{EG_+}$. We recover the Tate spectral sequence.
(3) Given a G-spectrum Z, set $X^i = P^iZ$, the slice tower. We obtain the slice spectral sequence.
(4) Given a C_{2^n}-spectrum Z and $1 \leq j \leq n$, set $X^i = \tilde{E}\mathcal{F}[C_{2^j}] \wedge P^iZ$. We obtain the localized slice spectral sequence. This will be the main example of relevance for us.

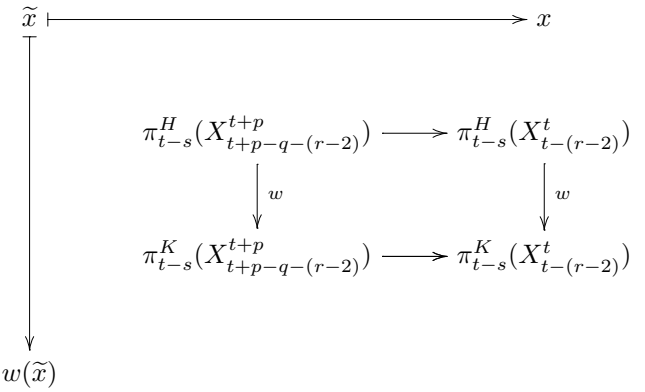
We fix an arbitrary map $\Sigma^\infty G/K \to \Sigma^\infty G/H$ and denote the resulting operation $\pi_n^H \to \pi_n^K$ by w. The most important case for us will be $H \subset K$ and $w = \mathrm{Tr}_H^K$. But equally well w could be a restriction map, multiplication by a fixed element such as 2, or any combination of these.

For notational simplicity, we will restrict for our treatment of exotic w-operations to integer degrees. By suspending by a representation sphere, one can easily translate our definitions and results to the $RO(G)$-grading.

Definition 3.4. Let $x \in E_r^{s,t}(G/H)$, and let $0 \leq p \leq r-2$ and $0 \leq q \leq p$. We may lift the corresponding element in $\pi_{t-s}^H X_{t-r+2}^t$ to an element $\widetilde{x} \in \pi_{t-s}^H X_{t+p-q-(r-2)}^{t+p}$ since by definition, we can actually lift it to an element in $\pi_{t-s}^H X_t^{t+(r-2)}$. If $w(\widetilde{x}) \in \pi_{t-s}^K X_{t+p-q-(r-2)}^{t+p}$ lies in $E_{r+q}^{s+p,t+p}(G/K)$, we call it a *w-operation of x of filtration jump p and page jump q*.

If $p > 0$, we speak of an *exotic w-operation*, which, depending on w, might be an *exotic transfer*, *exotic restriction* etc.[2] If the page jump is zero, we omit the mention of it.

This definition can be illustrated with the following diagram:



[2] If w is multiplication by an integer, then the existence of exotic w-operations corresponds essentially to hidden extensions. The basic issue of dependence on choices is already present in this more classical case.

Remark 3.5.

(1) The w-operations of filtration jump 0 are just the algebraic w-operations on the E_r-page as inherited from the E_2-page. This is why we call the w-operations of higher filtration jump *exotic*.
(2) The most classical case of exotic w-operations is the limiting case when $r = \infty$. If $x \in E_\infty^{s,t}(G/H)$ and X denotes $\lim_t X^t$, we can actually lift x to $\widetilde{x} \in \pi^H_{t-s}X$ (which is further than to $\pi^H_{t-s}X^{t+p}$ as required by the previous definition). If $w(\widetilde{x}) \neq 0$, it must be detected in some $E_\infty^{s+p,t+p}(G/K)$ and the resulting element is an example of a w-operation of filtration jump p on x. In the case when x is just on a finite page, we can suitably truncate the original spectral sequence to force x to be a permanent cycle that survives to the E_∞-page. We will do this in the proof of Lemma 3.7.
(3) Even in the classical situation of the last item, exotic w-operations are in general not unique; in other words, $w(\widetilde{x})$ will depend on the choice of lift $\widetilde{x}$. With notation as in the last bullet point, suppose for example that there exists $z \in E_\infty^{s+i,t+i}(G/H)$ for $0 < i < p$ such that z supports a non-exotic w-operation. If we lift z to $\widetilde{z} \in \pi^H_{t-s}X$, then $w(\widetilde{x} + \widetilde{z})$ will be detected by $w(z) \in E_\infty^{s+i,t+i}(G/K)$, while $\widetilde{x} + \widetilde{z}$ lifts x. In the extreme case, x might even be zero. In Lemma 3.7, we will prove a criterion that ensures the uniqueness of exotic w-operations. This criterion is often fulfilled in practice.
(4) A w-operation $z = w(\widetilde{x})$ of filtration jump p and page jump 0 defines a w-operation of filtration jump p and page jump q if $d_r(z) = \cdots = d_{r+q-1}(z) = 0$ by just mapping $z \in \pi^K_{t-s}X^{t+p}_{t+p-(r-2)}$ down to $\pi^K_{t-s}X^{t+p}_{t+p-q-(r-2)}$. All w-operations of page jump q are of this form.
(5) With x and $\widetilde{x}$ fixed, a w-operation of filtration jump p can only exist if all w-operations of lower filtration jump vanish. Indeed, if the image of $w(\widetilde{x})$ in $\pi^K_{t-s}X^{t+p}_{t+p-q-(r-2)}$ lies in $E_{r+q}^{s+p,t+p}(G/K)$, it is in the image of $\pi^K_{t-s}X^{t+p+q+(r-2)}_{t+p}$. The map from this group to $\pi^K_{t-s}X^{t+p-1}_{t+p-q-(r-2)-1}$ factors through $\pi^K_{t-s}X^{t+p-1}_{t+p} = 0$.

The following lemma holds by definition.

Lemma 3.6. *Let $x \in E_r^{s,t}(G/H)$ be a d_r-cycle and denote by $\overline{x}$ its image in $E_{r+1}^{s,t}(G/H)$. Let $z \in E_{r+q}^{s+p,t+p}$ be a w-operation on x of filtration jump p and page jump $q \geq 1$. Then z is a w-operation on $\overline{x}$ of filtration jump p and page jump $q-1$.*

The following is the uniqueness result for exotic w-operations that we will use.

Lemma 3.7. *Let $x \in E_r^{s,t}(G/H)$ and $0 < p \leq r-2$. Suppose every class in $E_2^{s+k,t+k}(G/H)$ for $0 < k < p$ is either hit by a differential of length at most $r+k-1$ or supports a differential of length at most $p-k+1$. Denoting by I the image of all $(r+p)$-cycles in $w(E_2^{s+p,t+p}(G/H))$ in $E_{r+p}^{s+p,t+p}(G/K)$, then there is at most one class in $E_{r+p}^{s+p,t+p}(G/K)/I$ that is a w-operation of x of filtration jump p and page jump p.*

Proof. Consider the towers $\overline{X}^\bullet$ and $\widetilde{X}^\bullet$ with

$$\overline{X}^i = \begin{cases} X^{t+p} & \text{if } i \geq t+p \\ X^i & \text{else} \end{cases}$$

and

$$\widetilde{X}^i = \begin{cases} X^{t+p}_{t-(r-2)} & \text{if } i \geq t+p \\ X^i_{t-(r-2)} & \text{if } t-(r-2) \leq i \leq t+p \\ 0 & \text{else} \end{cases}.$$

The maps $X^\bullet \to \overline{X}^\bullet \leftarrow \widetilde{X}^\bullet$ induce maps of spectral sequences $E^{\bullet,\bullet}_\bullet \to \overline{E}^{\bullet,\bullet}_\bullet \leftarrow \widetilde{E}^{\bullet,\bullet}_\bullet$. The first induces isomorphisms $E_2^{s',t'} \to \overline{E}_2^{s',t'}$ for $t' \leq t+p$ and the second isomorphisms $\widetilde{E}_2^{s',t'} \to \overline{E}_2^{s',t'}$ for $t' \geq t-(r-2)$. Via the maps of spectral sequences, differentials in the original spectral sequence enforce corresponding differentials in the $\widetilde{E}$-spectral sequence in the range $t-(r-2) \leq t' \leq t+p$. In particular, $E_r^{s,t}$ injects into $\widetilde{E}_r^{s,t}$. Note moreover that the $\widetilde{E}$-spectral sequence converges to $\underline{\pi}_* X^{t+p}_{t-(r-2)}$.

Our assumptions imply that $\widetilde{E}_\infty^{s+k,t+k}(G/H) = 0$ for $0 < k < p$ and moreover $\widetilde{E}_r^{s,t} = \widetilde{E}_\infty^{s,t}$. Thus, we can lift the image of x in $\widetilde{E}_r^{s,t}$ uniquely to $\pi^H_{t-s} X^{t+p}_{t-(r-2)}$ modulo $\widetilde{E}_\infty^{s+p,t+p}$. The latter term is a quotient of $\widetilde{E}_2^{s+p,t+p} = E_2^{s+p,t+p}$.

In summary, we have shown that we can lift x uniquely to $\widetilde{x} \in \pi^H_{t-s} X^{t+p}_{t-(r-2)}$ modulo the image from $\pi^H_{t-s} X^{t+p}_{t+p}$. Thus, $w(\widetilde{x}) \in \pi^K_{t-s} X^{t+p}_{t-(r-2)}$ is indeed well-defined modulo the image of $w(\pi^H_{t-s} X^{t+p}_{t+p}) = w(E_2^{s+p,t+p})$. □

Remark 3.8. One can probably formulate a sharper criterion for the uniqueness of exotic w-operations, without requiring that all classes between x and its target vanish. The essential point is to require that there are no interleaving w-operations such as classes in $E_r^{s+k,t+k}$ with $0 < k < p$ that admit nonzero w-operations of filtration jump smaller than $p-k$. Moreover, one would have to enlarge I to include exotic w-operations as well. We refrain from making this precise.

Proposition 3.9. *Let $x \in E_r^{s,t}(G/H)$ and z a class with $d_r(z) = x$. Suppose $d_{r+q}(w(z))$ is zero for $q < p$. Then $d_{r+p}(w(z))$ is a w-operation of x of filtration jump p and page jump p.*

Proof. We choose a lift of $z \in \pi^H_{t-s+1} X^{t-r+1}_{t-2r+3}$ to $\widetilde{z} \in \pi^H_{t-s+1} X^{t-1}_{t-r+1}$. As $\delta(\widetilde{z})$ in the diagram below is a lift of x, contemplating the fate of $w(\widetilde{z})$ passing along the two different travel paths from the upper left corner to the lower right corner proves the proposition.

$$\begin{array}{ccc}
\pi^K_{t-s+1}X^{t-1}_{t-r+1} & \xrightarrow{\delta} & \pi^K_{t-s}X^{t+r-2}_{t} \\
\downarrow & & \downarrow \\
E_r^{s-r,t-r+1} \subseteq \pi^K_{t-s+1}X^{t-r+1}_{t-2r+3} & & \pi^K_{t-s}X^{t+p}_{t+p-r+2} \supseteq E_r^{s+p,t+p} \\
\downarrow & & \downarrow \\
E_{r+p}^{s-r,t-r+1} \subseteq \pi^K_{t-s+1}X^{t-r+1}_{t-2r+3-p} & \xrightarrow{\delta} & \pi^K_{t-s}X^{t+p}_{t-r+2} \supseteq E_{r+p}^{s+p,t+p}
\end{array} \qquad \square$$

While our definition and results so far are very general (and our proofs would also apply to other settings than equivariant homotopy theory), we will now formulate a result that is specific to cyclic 2-groups. For the following proposition, both the statement and the proof are variants of [34, Theorem 4.4], but also work for exotic transfers and restrictions on finite pages and circumvent a mistake in [34, Lemma 4.5].[3]

Proposition 3.10. *Let G be a cyclic 2-group, $H \subset G$ an index 2 subgroup, and $V \in RO(G)$.*

(i) Let $y \in E^{s,V}_{r+1}(G/G)$ with $a_\sigma y = 0 \in E^{s+1,V+1-\sigma}_{r+1}(G/G)$. Then y is an (exotic) transfer of filtration jump (at most) $r-1$.[4]
(ii) Let $z \in E^{s,V}_{r+1}(G/H)$ with $\mathrm{Tr}(z) = 0 \in E^{s,V}_{r+1}(G/G)$. Then z is an (exotic restriction) from $E^{s-(r-1),V-(r-1)+(1-\sigma)}_{r+1}$ of filtration jump (at most) $r-1$.

Proof. For the first part, by shifting the tower and applying suspension if necessary, we can fix the bidegree of y to be $(r-1, r-1)$. The term $E^{r-1,r-1}_{r+1}(G/G)$ injects into $\pi^G_0 X^{r-1}_0$. Smashing the long exact sequence associated with the cofiber sequence $G/H_+ \to S^0 \xrightarrow{a_\sigma} S^\sigma$ with X^{r-1}_0 and taking homotopy groups, we get the long exact sequence

$$\pi^G_1 X^{r-1}_0 \xrightarrow{a_\sigma} \pi^G_{1-\sigma} X^{r-1}_0 \xrightarrow{\mathrm{Res}} \pi^H_0 X^{r-1}_0 \xrightarrow{\mathrm{Tr}} \pi^G_0 X^{r-1}_0 \xrightarrow{a_\sigma} \pi^G_{-\sigma} X^{r-1}_0.$$

From this long exact sequence, we see that $a_\sigma y = 0$ implies $y = \mathrm{Tr}(\widetilde{w})$ with $\widetilde{w} \in \pi^H_0 X^{r-1}_0$. By definition, this defines an element $w \in E^{0,0}_{r+1}(G/H)$ such that y is an exotic transfer of w of filtration jump $r-1$.

[3] With notation as in the cited lemma, a counterexample is the following: Fix an object A. Take $A_{i,j}$ to be $\Sigma^{-1}A$, zero or A, depending on whether $i+j$ is smaller, equal or larger than 2. The $a_{i,j}$ and $b_{i,j}$ are id if possible, with the exception of $a_{2,1}$ being an arbitrary self-equivalence of A, which is not equivalent to $\pm\,\mathrm{id}$. Take further $W = A$ and $f_3 = \mathrm{id}$. Then f_1 exists (and can be taken to be id), but f_1 and f_2 cannot simultaneously exist. Strictly speaking, the cited lemma is ambiguous on whether it claims that f_1 and f_2 exist *simultaneously* if f_1 exists, but this seems to be the way that it is later used in [34, Theorem 4.4].

[4] The "at most" is actually unnecessary here, as the proof shows that y is an exotic transfer of filtration jump $r-1$. We write it for emphasis though since y might be very well also an exotic transfer of smaller filtration jump. This is related to the non-uniqueness described in Item 3 of Remark 3.5. Thus the statement is best used in conjunction with a uniqueness result like Lemma 3.7.

For the second part, we can fix the bidegree of z to be $(r-1, r-1)$ by shifting the tower and applying suspension if necessary to view z as an element in $\pi_0^H X_0^{r-1}$. Using the long exact sequence induced by $G/H_+ \to S^0 \xrightarrow{a_\sigma} S^\sigma$ again, we see that z is the restriction of some $\widetilde{v} \in \pi_{1-\sigma}^G X_0^{r-1}$. By definition, this defines an element $v \in E_{r+1}^{0,1-\sigma}(G/G)$ such that z is an exotic restriction of v of filtration jump $r-1$. □

Let us give an example of a possible workflow working with exotic transfers, which we will apply in Proposition 5.19.

Workflow 3.11. Let G be a cyclic 2-group and $H \subset G$ of index 2. Let $y \in E_r^{s+r-1,t+r-1}(G/G)$ and $r' > r$. We assume the following:

(1) $a_\sigma y$ is nonzero and is hit by a d_r-differential;
(2) y persists to a nonzero class in the $E_{r'+r-1}$-page, which we denote by the same name;
(3) every class in $E_2^{s+k,t+k}(G/H)$ for $0 < k < r-1$ is either hit by a differential of length at most $r+k-1$ or supports a differential of length at most $r-k$;
(4) $y \in E_{2r-1}^{s+r-1,t+r-1}$ is not the image of a $(2r-1)$-cycle in E_2 which is the transfer of a class in $E_2^{s+r-1,t+r-1}$.

By (1), $a_\sigma y$ vanishes on E_{r+1}. Thus, by Proposition 3.10, there exists $x \in E_{r+1}^{s,t}(G/H)$ such that $y \in E_{r+1}^{s+r-1,t+r-1}(G/G)$ is an exotic transfer of x of filtration jump $r-1$. Applying Lemma 3.7 in conjunction with (3) and (4), we see that x cannot be zero (as zero is the unique exotic transfer of zero under our assumptions); in case that there is only one non-zero element in the relevant bidegree, this already uniquely determines x. Suppose now further that:

(5) $x = d_{r'}(a)$;
(6) $d_{r'+q}(\mathrm{Tr}_H^G a) = 0$ for $0 \leq q < r-1$.

Then Proposition 3.9 implies that $d_{r'+r-1}(\mathrm{Tr}_H^G(a))$ is an exotic transfer of x in the same degree as $y \in E_{r'+r-1}^{s,t}$ and thus must be y by Lemma 3.7 again.

3.4. The behavior of norms

This section is about the behavior of norms in the (regular) slice spectral sequence and its localized variant. We will formulate a generalization of [55, Chapter I.5] and then discuss how it applies to Ullman's original setting (the regular slice spectral sequence), to the localized slice spectral sequence and the homotopy fixed point spectral sequence.

We will first work in an abstract setting: Let (X^i) be a tower of G-spectra and $E_*^{*,*}$ be the associated spectral sequence as in the preceding subsection. Set $X^\infty = \lim_i X^i$ and $X_n = X_n^\infty$.

Let $H \subset G$ be a subgroup of index h. We assume that we have maps $N_H^G X_n \to X_{hn}$ and $N_H^G X_n^n \to X_{hn}^{hn}$ that are (up to homotopy) compatible with the maps $X_n \to X_{n-1}$ and $X_n \to X_n^n$. (Here we leave the restriction maps implicit.) We call this a *norm structure*. It induces norm maps $N_H^G \colon E_2^{s,V+s} \to E_2^{hs,\operatorname{Ind}_H^G V+hs}$.

Proposition 3.12. *Let $x \in E_2(G/H)$ be an element representing zero in $E_{r+2}(G/H)$. Then $N_H^G(x)$ represents zero in $E_{rh+2}(G/G)$.*

Proof. The proof is the same as that of [55, Proposition I.5.17]. □

Example 3.13. Our first example of this setting is the regular slice tower of [55], which coincides with the slice tower of [33] for norms of $MU_{\mathbb{R}}$ and $BP_{\mathbb{R}}$ – thus there should be no danger of confusion if we use the same notation $P^i X$ for the regular slice tower.

Ullman constructs in [55, Corollaries I.5.10 and I.5.11] for every H-spectrum X natural compatible maps $N_H^G P_n X \to P_{nh} N_H^G X$ and $N_H^G P_n^n X \to P_{nh}^{nh} N_H^G X$. Moreover the square

$$\begin{array}{ccc} N_H^G P_n X & \longrightarrow & P_{hn} N_H^G X \\ \downarrow & & \downarrow \\ N_H^G P_{n-1} X & \longrightarrow & P_{hn-h} N_H^G X \end{array}$$

commutes, as $N_H^G P_n X$ is $\geq hn$ by [55, Corollary I.5.8] and both maps into $N_H^G P_n X \to P_{hn-h} N_H^G X$ are compatible with the respective maps to $N_H^G X$.

Let R be a G-spectrum with a map $N_H^G \operatorname{Res}_H^G R \to R$. The composite $N_H^G P_n \operatorname{Res}_H^G R \to P_{nh} N_H^G \operatorname{Res}_H^G R \to P_{nh} R$ and its analogue for P_n^n define a norm structure on the regular slice tower of R. This applies in particular if R is a G-commutative ring spectrum.

Example 3.14. Let R be a G-commutative ring spectrum with $G = C_{2^n}$. We will define a norm structure on the tower $X^i = a_\lambda^{-1} P^i X$ defining the localized regular slice spectral sequence. Using the observations above for the regular slice spectral sequence, it suffices to produce natural maps $N_H^G \operatorname{Res}_H^G a_\lambda^{-1} P_n R \to a_\lambda^{-1} N_H^G \operatorname{Res}_H^G P_{hn} R$ and similarly for P_n^n. As N_H^G and Res_H^G are monoidal, by Lemma 2.18 it thus suffices to provide a natural map

$$a_\lambda^{-1} \mathbb{S}_G \simeq a_\lambda^{-1} N_H^G \mathbb{S}_H \to N_H^G \operatorname{Res}_H a_{\lambda^{-1}} \mathbb{S}_G \simeq a_{\operatorname{Ind}_H^G \operatorname{Res}_H^G \lambda}^{-1} \mathbb{S}_G$$

As observed before, $\operatorname{Ind}_H^G \operatorname{Res}_H^G \lambda$ is a multiple of λ if $H \neq e$ and contains a trivial summand if $H = e$. This produces the norm structure if $H \neq e$. In contrast for $H = e$, all norms would have to be zero.

We remark that we have not used the full strength of our considerations in Section 2.5 here, but we expect that these will be necessary for deeper considerations about norms.

Example 3.15. Lastly we define a norm structure on the homotopy fixed point spectral sequence. Observe first that there is for H-spectra X a natural map

$$N_H^G X^{EH_+} \to (N_H^G X^{EH_+})^{EG_+} \xleftarrow{\simeq} (N_H^G X)^{EG_+},$$

where the latter map is an equivalence as $\mathrm{Res}_e^G N_H^G X \to \mathrm{Res}_e^G N_H^G X^{EH_+}$ is an equivalence.

Recall that the tower defining the homotopy fixed point spectral sequence for a spectrum R is defined by $X^n = (\tau_{\leq n} R)^{EG_+}$. We observe that we have natural equivalences $X_n \simeq (P_n R)^{EG_+}$ and $X_n^n \simeq (P_n^n R)^{EG_+}$ for $(P^n R)_n$ the regular slice tower. Combining these equivalences with the natural map from the last paragraph, the norm structure from Example 3.13 induces a norm structure on the homotopy fixed point spectral sequence.

We will use the following proposition without further comment.

Proposition 3.16. *Both in the regular slice spectral sequence and in the localized regular slice spectral sequence of a G-commutative ring spectrum, the norms are multiplicative: $N_H^G(xy) = N_H^G(x) N_H^G(y)$.*

Proof. This follows from the commutativity of

$$\begin{array}{ccc} N_H^G(P_m X \wedge P_n Y) & \longrightarrow & N_H^G(P_{m+n} X \wedge Y) \\ \downarrow & & \downarrow \\ P_{hm} N_H^G X \wedge P_{hn} N_H^G Y & \longrightarrow & P_{hm+hn} N_H^G(X \wedge Y) \end{array}$$

for G-spectra X and Y. This in turn follows as there is up to homotopy just one map

$$N_H^G(P_m X \wedge P_n Y) \to P_{hm+hn} N_H^G(X \wedge Y)$$

compatible with the maps to $N_H^G(X \wedge Y)$ as $N_H^G(P_m X \wedge P_n Y) \geq h(m+n)$ by [55, Corollaries I.4.2 and I.5.8]. □

Given two towers (X^n) and (Y^n) with norm structures, a morphism of towers $(X^n) \to (Y^n)$ is *compatible with the norm structures* if the diagrams

$$\begin{array}{ccc} N_H^G X_n & \longrightarrow & X_{hn} \\ \downarrow & & \downarrow \\ N_H^G Y_n & \longrightarrow & Y_{hn} \end{array}$$

commute for all n and similarly for X_n^n and Y_n^n. Such a morphism induces in particular a morphism of spectral sequence that is compatible with the norms on the E_2-terms.

Example 3.17. Given any spectrum X, there is a natural map from the regular slice tower to the tower defining the homotopy fixed point spectral sequence, namely $P^n X \to (P^n X)^{EG_+}$. In case that X is a G-commutative ring spectrum (or more generally a spectrum admitting a map $N_H^G \operatorname{Res}_H^G X \to X$), this map of towers is (essentially by construction) compatible with the norm structures introduced in Example 3.13 and Example 3.15.

3.5. Comparison of spectral sequences

When computing localizations of a norm, we can apply different spectral sequences. For instance, in the isomorphism

$$\tilde{E}\mathcal{F}[H] \wedge N_H^G X \simeq P^*_{G/H}(N_e^{G/H}\Phi^H(X))$$

of Theorem 2.9, the left hand side $\tilde{E}\mathcal{F}[H] \wedge N_H^G X$ can be computed by the localized slice spectral sequence we just built, while the right hand side can be computed by the pullback of the (G/H)-equivariant slice spectral sequence of $N_e^{G/H}\Phi^H X$. In this section, we give a comparison map between these spectral sequences, which we will use in understanding the homotopy fixed points and the Tate spectral sequence of $N_e^{G/H}\Phi^H X$.

Such comparison can only be made by regrading the slice tower. In the cases of relevance for us this takes the shape of the following doubling process: Let $P^\bullet$ be a tower, we define $\mathscr{D}P^\bullet$, the doubled tower of $P^\bullet$, as

$$\mathscr{D}P^{2n+\epsilon} := P^n$$

for $\epsilon = 0, 1$. We also use $\mathscr{D}$ as a prefix of a spectral sequence obtained from a tower as the spectral sequence of the doubled tower.

In the following theorem we will use both the slice tower $P^\bullet$ and the pullback functor P^*_{G/C_2} from Section 2.2; the double usage of P will hopefully not cause any confusion to the reader.

Theorem 3.18. *Let $G = C_{2^n}$, $X \in Sp^G$ and $Y = \Phi^{C_2} X \in Sp^{G/C_2}$. Let $P^\bullet X$ and $P^\bullet Y$ be their slice towers in the corresponding categories. Then there is a commutative diagram of towers*

$$\begin{array}{ccc} P^\bullet X & \longrightarrow & P^*_{G/C_2}\mathscr{D}P^\bullet Y \\ \downarrow & \nearrow & \\ \tilde{E}G \wedge P^\bullet X & & \end{array}$$

such that the map $\tilde{E}G \wedge P^\bullet X \to P^*_{G/C_2}\mathscr{D}P^\bullet Y$ converges to the G-equivalence $\tilde{E}G \wedge X \to P^*_{G/C_2} Y$.

In particular, the induced map on the C_2-level spectral sequences of $P^\bullet X \to P^*_{G/C_2}\mathscr{D}P^\bullet Y$ converges to the geometric fixed points map

$$\Phi^{C_2} : \pi^{C_2}_\star X \to \pi_* Y.$$

Proof. To construct the map $P^\bullet X \to P^*_{G/C_2}\mathscr{D}P^\bullet Y$, consider the composition

$$X \to a_\lambda^{-1} X \simeq P^*_{G/C_2} Y \to P^*_{G/C_2}\mathscr{D}P^{2i} Y.$$

We only need to show $P^*_{G/C_2}\mathscr{D}P^{2i}Y \leq_{Slice} 2i$ for each i (the analogous statement for $\mathscr{D}P^{2i+1}$ follows from this). This can be checked by testing against slice cells of dimension more than $2i$. By induction we can assume that our claim is true after restriction to any proper subgroup of G, so we can ignore induced slice cells. Thus it suffices to check that $[S^{k\rho_G}, P^*_{G/C_2}\mathscr{D}P^{2i}Y]_G = 0$ for $k|G| > 2i$.

The following equivalence of G-spectra is essential to our proof:

$$\tilde{E}G \wedge S^{k\rho_G} \simeq P^*_{G/C_2} S^{k\rho_{G/C_2}}.$$

It comes from the fact that both sides are equivalent to the representation sphere $S^{\infty\lambda+\rho_{G/C_2}}$. The left hand side of the equivalence is a localization of a slice cell of dimension $k|G|$ while the right hand side is a pullback of a slice cell of dimension $\frac{k|G|}{2}$. This difference is the reason of doubling the tower of Y.

Using this equivalence, we have a series of equivalences of mapping sets:

$$\begin{aligned}
[S^{k\rho_G}, P^*_{G/C_2}\mathscr{D}P^{2i}Y]_G &\cong [\tilde{E}G \wedge S^{k\rho_G}, P^*_{G/C_2}\mathscr{D}P^{2i}Y]_G \\
&\cong [P^*_{G/C_2} S^{k\rho_{G/C_2}}, P^*_{G/C_2}\mathscr{D}P^{2i}Y]_G \\
&\cong [S^{k\rho_{G/C_2}}, \mathscr{D}P^{2i}Y]_H \\
&\cong [S^{k\rho_{G/C_2}}, P^i Y]_H \\
&\cong 0.
\end{aligned}$$

The change-of-group isomorphism comes from the fact that P^*_{G/C_2} is fully faithful on homotopy categories, and the last isomorphism is because $S^{k\rho_{G/C_2}}$ is a slice cell of dimension $> i$ in G/C_2-spectra.

By construction, the map $P^\bullet X \to P^*_{G/C_2}\mathscr{D}P^\bullet Y$ converges to the map $X \to a_\lambda^{-1}X \simeq P^*_{G/C_2}Y$. Since everything in the tower $P^*_{G/C_2}\mathscr{D}P^\bullet Y$ is already a_λ-local, the tower map factors through the a_λ-localization $\tilde{E}G \wedge P^\bullet X$. □

Proposition 3.19. *Let $G = C_{2^n}$ and $X \in Sp^G$ a G-commutative ring spectrum. Then the tower $P^*_{G/C_2}\mathscr{D}P^\bullet\Phi^{C_2}X$ has a norm structure in the sense of Section 3.4 and the maps*

$$P^\bullet X \to P^*_{G/C_2}\mathscr{D}P^\bullet\Phi^{C_2}X \quad \text{and} \quad a_\lambda^{-1}P^\bullet X \to P^*_{G/C_2}\mathscr{D}P^\bullet\Phi^{C_2}X$$

from Theorem 3.18 are compatible with norms from subgroups containing C_2.

Proof. Let $H \subset G$ be a subgroup of index h such that $C_2 \subset H$. Then we obtain maps

$$N_H^G P^*_{G/C_2} P_{2n} \Phi^{C_2} X \simeq P^*_{G/C_2} N_{H/C_2}^{G/C_2} P_{2n} \Phi^{C_2} X \to P^*_{G/C_2} P_{2hn} \Phi^{C_2} X$$

and

$$N_H^G P^*_{G/C_2} P^{2n}_{2n} \Phi^{C_2} X \simeq P^*_{G/C_2} N_{H/C_2}^{G/C_2} P^{2n}_{2n} \Phi^{C_2} X \to P^*_{G/C_2} P^{2hn}_{2hn} \Phi^{C_2} X,$$

which are compatible in the necessary sense. Here we use the norm structure on the regular slice tower from Example 3.13, the G/C_2-commutative ring structure on $\Phi^{C_2} X$ from Example 2.22 and the commutation of norms and pullbacks from Proposition 2.14.

To show that $P^\bullet X \to P^*_{G/C_2} \mathscr{D} P^\bullet \Phi^{C_2} X$ is compatible with norm structures, note first that the diagram

$$\begin{array}{ccc} N_H^G X & \longrightarrow & N_H^G P^*_{G/C_2} \Phi^{C_2} X \simeq P^*_{G/C_2} N_{H/C_2}^{G/C_2} \Phi^{C_2} X \\ \downarrow & & \downarrow \\ X & \longrightarrow & P^*_{G/C_2} \Phi^{C_2} X \end{array}$$

commutes since $X \to P^*_{G/C_2} \Phi^{C_2} X \simeq a_\lambda^{-1} X$ is a morphism of $\mathcal{O}$-algebras by Definition 2.16 and Example 2.22, where $\mathcal{O}$ is an N_∞-operad arising as the pullback of a G/C_2-E_∞-operad. Next consider the diagram

$$\begin{array}{ccccc} \Phi^{C_2} N_H^G P_n X & \longrightarrow & N_{H/C_2}^{G/C_2} P_{2n} \Phi^{C_2} X & \longrightarrow & N_{H/C_2}^{G/C_2} \Phi^{C_2} X \\ \downarrow & & \downarrow & & \downarrow \\ \Phi^{C_2} P_{hn} X & \longrightarrow & P_{2hn} \Phi^{C_2} X & \longrightarrow & \Phi^{C_2} X \end{array}$$

The outer rectangle is obtained from the previous diagram by applying Φ^{C_2} (and using the maps $P_n X \to X$ and $P_{hn} X \to X$) and thus commutes. Given the connectivity estimate [55, Corollary I.5.8] and the universal property of P_{2hn}, we see that $\Phi^{C_2} N_H^G P_n X \to \Phi^{C_2} X$ factors through $P_{2hn} \Phi^{C_2}$ in an essentially *unique* way, so the left square also has to commute. By the adjointness of Φ^{C_2} and P^*_{G/C_2} this implies the commutativity of

$$\begin{array}{ccc} N_H^G P_n X & \longrightarrow & N_H^G P^*_{G/C_2} P_{2n} \Phi^{C_2} X \\ \downarrow & & \downarrow \\ P_{hn} X & \longrightarrow & P^*_{G/C_2} P_{2hn} \Phi^{C_2} X \end{array}$$

The proof of the commutativity for the corresponding square for P^n_n is completely analogous. The a_λ-inverted case follows again because the target is a_λ-local. □

4. The localized slice spectral sequences of $BP^{((G))}$: summary of results

We now turn to analyze the localized slice spectral sequence of $BP^{((G))}$ for $G = C_{2^n}$. From now on, everything will be implicitly 2-localized. In this section, we list our main results and give an outline of the computation. Detailed computations of the results stated in this section are in Section 5.

As we discussed in Section 3, the Slice Theorem [33, Theorem 6.1] implies that the slice associated graded of $BP^{((C_{2^n}))}$ is

$$H\underline{\mathbb{Z}}[G \cdot \bar{t}_1, G \cdot \bar{t}_2, \ldots],$$

where $\bar{t}_i \in \pi^{C_2}_{(2^i-1)\rho_2} BP^{((C_{2^n}))}$ (see also [33, Section 2.4] for details).

For the rest of the paper, we use λ for the 2-dimensional real representation of C_{2^n} which is rotation by $\left(\frac{\pi}{2^{n-1}}\right)$, and σ for the 1-dimensional sign representation of G. We use σ_2 for the sign representation of the unique *subgroup* C_2 in G. Let $i < j \leq n$, we will use $\mathrm{Res}^{2^j}_{2^i}$, $\mathrm{Tr}^{2^j}_{2^i}$ and $N^{2^j}_{2^i}$ for restrictions, transfers and norms between C_{2^i} and C_{2^j} as subgroups of G. If their subscript and superscript are omitted, they mean the restriction, transfer and norm between C_2 and C_4.

Theorem 4.1.

(1) Let $G = C_{2^n}$ and $H = C_2$ be the subgroup of order 2 inside G. There is a $RO(G/H)$-graded spectral sequence of Mackey functors $a_\lambda^{-1}\,\mathrm{SliceSS}(BP^{((G))})$ that converges to the $RO(G/H)$-graded homotopy Mackey functor of $N_e^{G/H} H\mathbb{F}_2$. The E_2-page of this spectral sequence is

$$a_\lambda^{-1} H\underline{\mathbb{Z}}_\bigstar[G \cdot \bar{t}_1, G \cdot \bar{t}_2, \cdots].$$

(2) The integral E_2-page of $a_\lambda^{-1}\,\mathrm{SliceSS}(BP^{((G))})$ is bounded by the vanishing lines $s = (2^n-1)(t-s)$ and $s = -(t-s)$ in Adams grading. In other words, at stem $t-s$, the classes with filtrations greater than $(2^n-1)(t-s)$ or less than $-(t-s)$ are all zero.

(3) On the integral E_2-page, the a_λ-localizing map

$$\mathrm{SliceSS}(BP^{((G))}) \to a_\lambda^{-1}\,\mathrm{SliceSS}(BP^{((G))})$$

induces an isomorphism of classes in positive filtrations. The kernel of this map consists of transfer classes in $\mathrm{SliceSS}(BP^{((G))})$ from the trivial subgroup in filtration 0. These classes are all permanent cycles.

Proof. By Theorem 3.3, $a_\lambda^{-1}\,\mathrm{SliceSS}(BP^{((G))})$ computes the homotopy of $\tilde{E}G \wedge BP^{((G))}$. By Theorem 2.9 and the fact that $\Phi^{C_2}(BP_{\mathbb{R}}) \simeq H\mathbb{F}_2$,

$$\tilde{E}G \wedge BP^{((G))} \simeq P^*_{G/C_2}(N_1^{G/C_2} H\mathbb{F}_2).$$

Since the E_2-page of the slice spectral sequence of $BP^{((G))}$ has the form

$$H\underline{\mathbb{Z}}_\bigstar[G\cdot\bar{t}_1, G\cdot\bar{t}_2, \ldots],$$

the E_2-page of $a_\lambda^{-1}\,\mathrm{SliceSS}(BP^{((G))})$ is

$$a_\lambda^{-1} H\underline{\mathbb{Z}}_\bigstar[G\cdot\bar{t}_1, G\cdot\bar{t}_2, \ldots]$$

Together with Theorem 2.9 and Theorem 3.3 this proves (1).

The top vanishing line $s = (2^n-1)(t-s)$ follows from the fact that $\underline{\pi}_i(S^{k\rho_G+l\lambda}\wedge H\underline{\mathbb{Z}}) = 0$ for $k, l \geq 0$ and $i < k$ (see [33, Theorem 4.42]). For the second vanishing line $y = -x$, note that in stem $t-s$, classes in filtration less than $-(t-s)$ are contributed by slices of negative dimension, but $BP^{((G))}$ has no negative slices. This proves (2).

To prove (3), by unpacking the description of the E_2-page, we need to show that for $k, l \geq 0$, the a_λ-multiplication map

$$a_\lambda : \pi_i^G(S^{k\rho_G+l\lambda} \wedge H\underline{\mathbb{Z}}) \longmapsto \pi_i^G(S^{k\rho_G+(l+1)\lambda} \wedge H\underline{\mathbb{Z}})$$

is an isomorphism for $k \leq i < k|G| + 2l$ and is surjective with kernel consisting of transfer classes from trivial subgroup for $i = k|G| + 2l$. Using the cellular structures and their corresponding chain complexes described in [34, Section 3], we see that when $k \leq i \leq k|G| + 2l$, a_λ induces isomorphism on the cellular chain complexes, therefore it induces isomorphism on homology for $k \leq i < k|G| + 2l$ and surjection on homology for $i = k|G| + 2l$ with the kernel exactly the image of $Tr_1^{2^n}$. Since the underlying tower of the slice tower is the Postnikov tower, all the class in the trivial subgroup and their transfers are permanent cycles. □

Remark 4.2. In fact, (2) and (3) of Theorem 4.1 hold in a greater generality. For instance, they are true for any (-1)-connected G-spectrum. We will investigate properties of the localized slice spectral sequences in a future paper.

By [42] and [14], all C_{2^n} norms of $H\mathbb{F}_2$ are cofree, therefore we will not distinguish between their fixed points and homotopy fixed points.

Corollary 4.3. *The 0-th homotopy group of* $(N_1^{2^{n-1}} H\mathbb{F}_2)^{hC_{2^{n-1}}}$ *is isomorphic to* $\mathbb{Z}/2^n$.

Proof. In $a_\lambda^{-1}\,\mathrm{SliceSS}(BP^{((G))})$, the only Mackey functor contributing to the 0-stem is $\underline{\pi}_0(a_\lambda^{-1} H\underline{\mathbb{Z}})$, and we claim that

$$\pi_0^G(a_\lambda^{-1}H\underline{\mathbb{Z}})(G/G) \cong \mathbb{Z}/2^n.$$

Indeed, the maps $\pi_0^G(S^{n\lambda} \wedge H\underline{\mathbb{Z}}) \to \pi_0^G(S^{(n+1)\lambda} \wedge H\underline{\mathbb{Z}})$ are isomorphisms for $n \geq 1$ and $\pi_0^G(S^\lambda \wedge H\underline{\mathbb{Z}})$ is the cokernel of the transfer $\mathrm{Tr}_1^{2^n} : \pi_0^e H\underline{\mathbb{Z}} \to \pi_0^{C_{2^n}} H\underline{\mathbb{Z}}$, i.e. of multiplication by 2^n on $\mathbb{Z}$. □

For the rest of the paper, we focus on the case $G = C_4$.

Theorem 4.4. *The first 8 stems of $\pi_*^{C_4}(a_\lambda^{-1}BP^{((C_4))}) \cong \pi_*^{C_2}N_1^2H\mathbb{F}_2$ are shown in the following chart:*

i	0	1	2	3	4	5	6	7	8
π_i	$\mathbb{Z}/4$	$\mathbb{Z}/2$	$\mathbb{Z}/4$	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/4 \oplus \mathbb{Z}/2$	$\mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$	$\mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$

On the E_∞-page of the localized spectral sequence, the black subgroups are those generated by non-exotic transfers from $\mathcal{A}_* = \pi_*(H\mathbb{F}_2 \wedge H\mathbb{F}_2)$, and the red subgroups consist of everything else. (For interpretation of the colors in the text and figures, the reader is referred to the web version of this article.) For the Mackey functor structure, see Fig. 6.

Modulo transfers from $\mathcal{A}_*$, the homotopy groups have the following generators:

(1) π_1 is generated by $\eta = N(\bar{t}_1)a_\lambda a_\sigma$, the image of the first Hopf invariant one element under the composition $\mathbb{S} \to (BP^{((C_4))})^{C_4} \to (a_\lambda^{-1}BP^{((C_4))})^{C_4}$;
(2) π_2 is generated by $\frac{\eta^2}{2} = 2u_\lambda a_\lambda^{-1}$;
(3) π_3 is generated by $\nu = N(\bar{t}_2)a_\lambda^3 a_\sigma^3$, the image of the second Hopf invariant one element;
(4) π_6 is generated by $\frac{\nu^2}{2} = 2u_\lambda^3 a_\lambda^{-3}$;
(5) π_7 is generated by $N(\bar{t}_3)a_\lambda^7 a_\sigma^7$ and $N(\bar{t}_2)u_\lambda u_{2\sigma} a_\lambda^2 a_\sigma$, and one of them detects the third Hopf invariant one element σ.
(6) π_8 is generated by $\mathrm{Tr}_2^4(\bar{t}_2^2\bar{t}_1^2 a_{\sigma_2}^8) + \mathrm{Tr}_2^4(\bar{t}_3\bar{t}_1 a_{\sigma_2}^8) + N(\bar{t}_2)N(\bar{t}_1)u_{2\sigma}^2 a_\lambda^4$.

In [51], Rognes shows that the unit map $S^0 \to (N_1^2H\mathbb{F}_2)^{hC_2}$ induces a splitting injection on mod 2 homology as an $\mathcal{A}_*$-comodule thus a splitting injection on the E_2-page of the Adams spectral sequence. Therefore, the ring spectrum $(N_1^2H\mathbb{F}_2)^{hC_2} \simeq (a_\lambda^{-1}BP^{((C_4))})^{C_4}$ detects all Hopf invariant one elements. They all restrict to 0, since the underlying Adams spectral sequence of $H\mathbb{F}_2 \wedge H\mathbb{F}_2$ is concentrated in filtration 0. Therefore, they are detected by red subgroups in the corresponding degree.

The proof of Theorem 4.4 is by computing $a_\lambda^{-1}\,\mathrm{SliceSS}(BP^{((C_4))})$ and is given in the next section. The most relevant differentials in the spectral sequence are listed in the table below.

5. Computing the localized slice spectral sequences of $BP^{((G))}$

In this section, we compute $a_\lambda^{-1}\,\mathrm{SliceSS}(BP^{((C_4))})$ and prove Theorem 4.4. Our approach is similar to that of [34] and [31]. When going through the computations in this

Differential	Formula	Proof
d_3	$d_3(u_{2\sigma_2}) = a_{\sigma_2}^3(\bar{t}_1 + \gamma\bar{t}_1)$ $d_3(u_\lambda) = \mathrm{Tr}_2^4(a_{\sigma_2}^3\,\bar{t}_1)$	Proposition 5.8
d_5	$d_5(u_{2\sigma}) = N(\bar{t}_1)a_\lambda a_\sigma^3$	Theorem 5.7
d_5	$d_5(u_\lambda^2) = N(\bar{t}_1)u_\lambda a_\lambda^2 a_\sigma$	Proposition 5.11
d_7	$d_7(u_{2\sigma_2}^2) = a_{\sigma_2}^7(\bar{t}_2 + \bar{t}_1^3 + \gamma\bar{t}_2)$ $d_7(2u_\lambda^2) = \mathrm{Tr}_2^4(a_{\sigma_2}^7\bar{t}_1^3)$	Theorem 5.4 Proposition 5.16
d_7	$d_7(u_\lambda^4) = \mathrm{Tr}_2^4(\bar{t}_1^3 u_{2\sigma_2}^2 a_{\sigma_2}^7)$	Proposition 5.18
d_{13}	$d_{13}(u_\lambda^4 a_\sigma) = N(\bar{t}_2 + \bar{t}_1^3 + \gamma(\bar{t}_2))u_{2\sigma}^2 a_\lambda^7$	Proposition 5.22
d_{15}	$d_{15}(2u_\lambda^4) = \mathrm{Tr}_2^4(\bar{t}_3^{C_2} a_{\sigma_2}^{15})$	Proposition 5.23

section, the following guiding principles are useful to keep in mind. We hope these points would serve as a road map that will be helpful to the readers who are new to these types of computations.

(1) The E_2-page of the spectral sequence can be obtained by computing the $RO(C_4)$-graded homotopy groups of $a_\lambda^{-1}H\underline{\mathbb{Z}}$.
(2) The C_2-level spectral sequence, $a_{\sigma_2}^{-1}\,\mathrm{SliceSS}(BP_{\mathbb{R}} \wedge BP_{\mathbb{R}})$, is easy to compute, as it is completely determined by the Hill–Hopkins–Ravenel slice differentials.
(3) In the positive cone part of $a_\lambda^{-1}\,\mathrm{SliceSS}(BP^{((C_4))})$ (which includes the entire integer-graded spectral sequence), the only algebra generators that are not permanent cycles are essentially classes of forms u_V and $u_V a_\sigma$. Therefore, we only need to focus on finding differentials on these classes, and then use the Leibniz rule. This is why even though the integer-graded spectral sequence is the computation of interest, we often move to analyze certain classes in $RO(C_4)$-degrees.
(4) Many of the differentials are proven by using the C_2-level spectral sequence, and using the restrictions and transfers on the E_2-page. More precisely, if one knows that $d_r(\mathrm{Res}_{C_2}^{C_4} x) = y$, then x must support a differential of length at most r. Similarly, if $d_r(x) = y$, and $\mathrm{Tr}_{C_2}^{C_4}(y)$ is not zero on the E_2-page, then it must be killed by a differential of length at most r.
(5) The remaining differentials and extension are proven by using the Hill–Hopkins–Ravenel norm and the theory of exotic restrictions and transfers.

We would like to also remark that the differentials proven in this section determine all the differentials in the integer-graded spectral sequence in our range of interest. There are other differentials in the $RO(C_4)$-graded page (both in the positive cone and outside the positive cone) that don't influence the integer-graded page of the spectral sequence.

5.1. Computing the E_2-page

We will first give a complete algebraic description of the E_2-page of $a_\lambda^{-1}\,\mathrm{SliceSS}(BP^{((C_4))})$ in terms of generators and relations. To do so, by Theorem 4.1, we need to describe the C_2-homotopy groups $\underline{\pi}_\star(a_{\sigma_2}^{-1}H\underline{\mathbb{Z}})$ and the C_4-homotopy groups $\underline{\pi}_\star(a_\lambda^{-1}H\underline{\mathbb{Z}})$.

Proposition 5.1. *We have*

$$\pi_{\bigstar}^{C_2}(a_{\sigma_2}^{-1}H\underline{\mathbb{Z}}) = \mathbb{F}_2[u_{2\sigma_2}, a_{\sigma_2}^{\pm 1}].$$

The Mackey functor structure is determined by the contractibility of the underlying spectrum.

This proposition is proved by a standard Tate cohomology computation, see [22, Section 2.C] for details.

Let S be the subring of

$$R = \mathbb{Z}/4[a_\sigma, u_{2\sigma}^{\pm 1}, u_\lambda a_\lambda^{-1}]/(2a_\sigma, u_\lambda a_\lambda^{-1} a_\sigma^2 = 2u_{2\sigma})$$

generated by the elements $\{a_\sigma, u_{2\sigma}, u_\lambda a_\lambda^{-1}, 2u_{2\sigma}^k, u_{2\sigma}^k u_\lambda a_\lambda^{-1} \,|\, k < 0\}$, and let $M = \mathbb{Z}/2[u_{2\sigma}^{\pm 1}, u_\lambda a_\lambda^{-1}, a_\sigma^{\pm 1}]/(u_{2\sigma}^\infty, a_\sigma^\infty)$ be considered as a module over S. Here, $R[x^{\pm 1}]/(x^\infty)$ is the cokernel of the map $R[x] \to R[x^{\pm 1}]$.

Proposition 5.2. *We have*

$$\pi_{\bigstar}^{C_4}(a_\lambda^{-1}H\underline{\mathbb{Z}}) = (S \oplus \Sigma^{-1}M)[a_\lambda^{\pm 1}],$$

where $S \oplus \Sigma^{-1}M$ is the square-zero extension of M over S of degree -1.

The Green functor structure is determined by the following facts:

(1) The C_2-restriction of $a_\lambda^{-1}H\underline{\mathbb{Z}}$ is the spectrum $a_{\sigma_2}^{-1}H\underline{\mathbb{Z}}$ in Proposition 5.1.
(2) The C_2-restrictions of the classes u_λ and $u_{2\sigma}$ are $u_{2\sigma_2}$ and 1, respectively.
(3) Given $V \in RO(C_4)$, there is an exact sequence (see [34, Lemma 4.2])

$$\pi_{i_{C_2}^* V}^{C_2} X \xrightarrow{\mathrm{Tr}_2^4} \pi_V^{C_4} X \xrightarrow{a_\sigma} \pi_{V-\sigma}^{C_4} X \xrightarrow{\mathrm{Res}_2^4} \pi_{i_{C_2}^* V - 1}^{C_2} X.$$

In other words, the kernel of a_σ-multiplication is the image of the transfer from C_2 to C_4, and the image of a_σ-multiplication is the kernel of the restriction from C_4 to C_2.

The proof of Proposition 5.2 and a more explicit presentation of the Mackey functor are given in [57, Proposition 6.7]. Fortunately, in most of the paper we only need the "positive cone" of the coefficient Green functor, that is, the part $\bigstar = a + b\sigma + c\lambda$ for $b \leq 0$. The Green functor structure of this part is computed in [34, Section 3]. However, the other part also plays an important role on the computation, see for example the proofs of Proposition 5.14 and Proposition 5.21.

The relation $u_\lambda a_\lambda^{-1} a_\sigma^2 = 2u_{2\sigma}$ and its integral version $u_\lambda a_\sigma^2 = 2u_{2\sigma} a_\lambda$ are commonly called the *gold relation* (see [34, Lemma 3.6]).

Symbol	$\circ$	$\blacktriangle$	$\blacktriangledown$
Lewis Diagram	$\mathbb{Z}/4$ $1 \curvearrowright 2$ $\mathbb{Z}/2$ $\curvearrowright$ 0	$\mathbb{Z}/2$ $1 \curvearrowright 0$ $\mathbb{Z}/2$ $\curvearrowright$ 0	$\mathbb{Z}/2$ $0 \curvearrowright 1$ $\mathbb{Z}/2$ $\curvearrowright$ 0
Symbol	$\bullet$	$\hat{\bullet}$	$\bar{\bullet}$
Lewis Diagram	$\mathbb{Z}/2$ $\curvearrowright$ 0 $\curvearrowright$ 0	$\mathbb{Z}/2$ $\Delta \curvearrowright \nabla$ $\mathbb{Z}/2[C_4/C_2]$ $\curvearrowright$ 0	0 $\curvearrowright$ $\mathbb{Z}/2$ $\curvearrowright$ 0

Fig. 2. Table of C_4-Mackey functors.

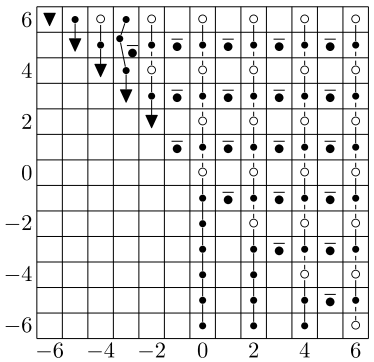


Fig. 3. $\underline{\pi}_{a+b\sigma}(a_\lambda^{-1} H\underline{\mathbb{Z}})$ for $-6 \leq a, b \leq 6$.

Fig. 2 gives the Lewis diagrams (first introduced in [38]) we use for C_4-Mackey functors, where restrictions Res^G_H map downwards and transfers Tr^G_H map upwards. These notations are consistent with [34, Section 5].

Fig. 3 shows $\underline{\pi}_{a+b\sigma}(a_\lambda^{-1} H\underline{\mathbb{Z}})$ in the range $-6 \leq a, b \leq 6$. In the figure, the horizontal coordinate is a and the vertical coordinate is b. Vertical lines are a_σ-multiplications, where solid lines are surjections and the dashed lines represent maps of the form $\mathbb{Z}/2 \hookrightarrow \mathbb{Z}/4$.

Although we mostly care the most about the C_4-equivariant homotopy groups of $a_\lambda^{-1} BP^{((C_4))}$, there are two advantages for computing $a_\lambda^{-1}\,\mathrm{SliceSS}(BP^{((C_4))})$ as a spectral sequence of Mackey functors:

(1) The Mackey functor structure can transport certain differentials on the C_2-level to differentials on the C_4-level.

(2) The Mackey functor structure and d_r-differentials can result in exotic extensions of filtration $r-1$ (see Section 3.3).

We will see (1) in the computations of d_3, d_7, and d_{15}-differentials below. (2) will be used to prove certain extensions forming the $(\mathbb{Z}/4)$s in Theorem 4.4, see Propositions 5.14 and 5.21.

Notation 5.3. Let $V \in RO(H)$ be a virtual representation that is in the image of the restriction $i_H^* : RO(G) \to RO(H)$. Then for any preimage W of V, there is a transfer map

$$\mathrm{Tr}_H^{G,W} : \pi_V^H X \to \pi_W^G X,$$

as a part of the homotopy Mackey functor structure. In our computation we will omit writing W when it is clear from the context what W is.

5.2. The C_2-spectral sequence

We start our computation with the C_2-underlying spectral sequence of $a_\lambda^{-1}\,\mathrm{SliceSS}(BP^{((C_4))})$.

Theorem 5.4.

(1) The underlying C_2-spectral sequence of $a_\lambda^{-1}\,\mathrm{SliceSS}(BP^{((C_4))})$ is $a_{\sigma_2}^{-1}\,\mathrm{SliceSS}(BP_{\mathbb{R}} \wedge BP_{\mathbb{R}})$. Its E_2-page is

$$a_{\sigma_2}^{-1} H\underline{\mathbb{Z}}_\bigstar[\overline{t}_1, \gamma\overline{t}_1, \overline{t}_2, \gamma\overline{t}_2, \cdots].$$

More precisely, the E_2-page of the underlying non-equivariant spectral sequence is trivial, and the E_2-page of the C_2-spectral sequence is

$$\mathbb{F}_2[u_{2\sigma_2}, a_{\sigma_2}^{\pm 1}][\overline{t}_1, \gamma\overline{t}_1, \overline{t}_2, \gamma\overline{t}_2, \cdots].$$

The elements $u_{2\sigma_2}, \overline{t}_i$ and $\gamma\overline{t}_i$ have filtration 0, while a_{σ_2} has filtration 1.[5]

(2) All the differentials in $a_{\sigma_2}^{-1}\,\mathrm{SliceSS}(BP_{\mathbb{R}} \wedge BP_{\mathbb{R}})$ are determined by a_{σ_2}, $\overline{t}_i$ and $\gamma\overline{t}_i$ being permanent cycles, the differentials

$$d_{2^{k+1}-1}(u_{2\sigma_2}^{2^{k-1}}) = a_{\sigma_2}^{2^{k+1}-1} \sum_{i=0}^{k} \overline{t}_{k-i}^{2^i} \gamma\overline{t}_i, \quad k \geq 1$$

[5] We recall the convention here that the filtration of an element in $\pi_V^H P_n^n X$ in the slice spectral sequence for some X is in filtration $n - \dim_{\mathbb{R}} V$. In particular the classes a_V will be always in filtration $\dim_{\mathbb{R}} V$.

and the Leibniz formula (for notational convenience, we let $\overline{t}_0 = \gamma\overline{t}_0 = 1$). The $E_{2^{k+1}}$-page has the form

$$\mathbb{F}_2[u_{2\sigma_2}^{2^k}, a_{\sigma_2}^{\pm 1}][\overline{t}_1, \gamma\overline{t}_1, \cdots]/(\overline{v}_1, \overline{v}_2, \cdots, \overline{v}_k)$$

where $\overline{v}_k = \sum_{i=0}^{k} \overline{t}_{k-i}^{2^i}\gamma\overline{t}_i$.

(3) *The E_∞-page of $a_{\sigma_2}^{-1}\,\mathrm{SliceSS}(BP_{\mathbb{R}} \wedge BP_{\mathbb{R}})$ is*

$$\mathbb{F}_2[a_{\sigma_2}^{\pm 1}][\overline{t}_1, \gamma\overline{t}_1, \cdots]/(\overline{v}_1, \overline{v}_2, \cdots)$$

In particular, in the integral grading, all the stem-n non-trivial permanent cycles are located in filtration n.

Proof. For (1), note that since $i_{C_2}^* BP^{((C_4))} = BP_{\mathbb{R}} \wedge BP_{\mathbb{R}}$, the C_2-underlying slice spectral sequence of $\mathrm{SliceSS}(BP^{((C_4))})$ is $\mathrm{SliceSS}(BP_{\mathbb{R}} \wedge BP_{\mathbb{R}})$. Moreover, $i_{C_2}^* a_\lambda = a_{\sigma_2}^2$. Therefore inverting a_λ in the C_4-spectral sequence inverts a_{σ_2} in the underlying C_2-spectral sequence.

For (2), we use the Hill–Hopkins–Ravenel slice differential theorem [33, Theorem 9.9] and the formula in [8, Theorem 1.1] that expresses the $\bar{v}_i$-generators in terms of the $\overline{t}_i$-generators (our $\overline{v}_i$ and $\overline{t}_i$ are $\overline{t}_i^{C_2}$ and $\overline{t}_i^{C_4}$ respectively in [8]). The Hill–Hopkins–Ravenel slice differential theorem states that in the slice spectral sequence of $BP_{\mathbb{R}}$, there are differentials

$$d_{2^{k+1}-1}(u_{2\sigma_2}^{2^{k-1}}) = \overline{v}_k a_{\sigma_2}^{2^{k+1}-1}, \quad k \geq 1.$$

The formula in [8, Theorem 3.1] shows that under the left unit map $BP_{\mathbb{R}} \to BP_{\mathbb{R}} \wedge BP_{\mathbb{R}}$,

$$\overline{v}_k = \sum_{i=0}^{k} \overline{t}_{k-i}^{2^i}\gamma\overline{t}_i \bmod (2, \overline{v}_1, \cdots, \overline{v}_{k-1}).$$

The left unit map induces a map

$$a_{\sigma_2}^{-1}\,\mathrm{SliceSS}(BP_{\mathbb{R}}) \longrightarrow a_{\sigma_2}^{-1}\,\mathrm{SliceSS}(BP_{\mathbb{R}} \wedge BP_{\mathbb{R}})$$

of spectral sequences. We will use naturality and induction to obtain the differentials and the description of the $E_{2^{k+1}}$-page.

To start the induction process, note that the description of the E_2-page is already given in (1). Now assume that we have obtained a description of the E_{2^k}-page. For degree reasons, the next potential differential is of length exactly $2^{k+1} - 1$. The differential formula for $a_{\sigma_2}^{-1}\,\mathrm{SliceSS}(BP_{\mathbb{R}})$ above shows that for any polynomial $P \in \mathbb{F}_2[\overline{t}_1, \gamma\overline{t}_1, \cdots]/(\overline{v}_1, \overline{v}_2, \cdots, \overline{v}_{k-1})$ and l an odd number, we have the differential

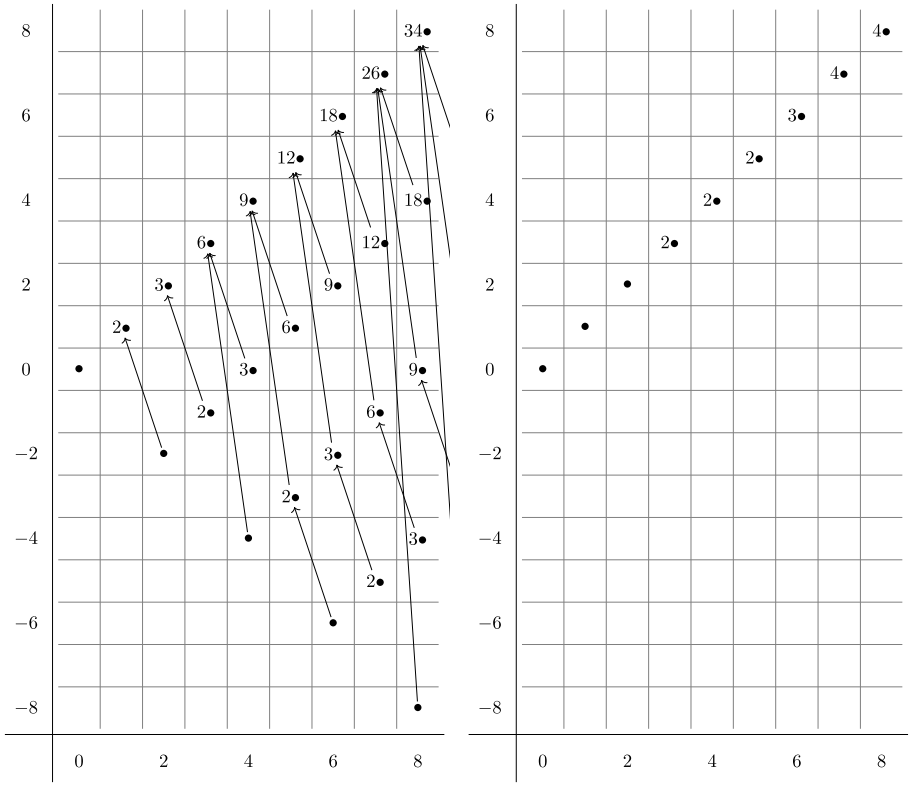


Fig. 4. The integral E_2- and E_∞-pages of $a_{\sigma_2}^{-1}\,\mathrm{SliceSS}(BP_{\mathbb{R}} \wedge BP_{\mathbb{R}})$.

$$d_{2^{k+1}-1}(Pu_{2\sigma_2}^{2^k l}) = P\overline{v}_k u_{2\sigma_2}^{2^{k-1}l} a_{\sigma_2}^{2^{k+1}-1}$$

in $a_{\sigma_2}^{-1}\,\mathrm{SliceSS}(BP_{\mathbb{R}} \wedge BP_{\mathbb{R}})$. The source and the target of this differential are always non-zero on the E_{2^k}-page because the sequence $(\overline{v}_1, \overline{v}_2, \cdots)$ is a regular sequence in the polynomial ring $\mathbb{F}_2[\overline{t}_1, \gamma\overline{t}_1, \cdots]$. Taking the quotient of the kernel and cokernel of this differential, we see that the $E_{2^{k+1}}$-page has the above description.

(3) is a direct consequence of (2) by letting $k \to \infty$. See Fig. 4 for the integral E_2 and E_∞-pages of this spectral sequence. □

Remark 5.5. In Proposition 6.2 we show that the C_2-geometric fixed points of the $\overline{t}_i$ and $\gamma\overline{t}_i$ generators are the ξ_i and ζ_i generators in the mod 2 dual Steenrod algebra $\mathcal{A}_*$. Therefore, the formula

$$\overline{v}_k = \sum_{i=0}^{k} \overline{t}_{k-i}^{2^i} \gamma\overline{t}_i \bmod (2, \overline{v}_1, \cdots, \overline{v}_{k-1})$$

reduces to Milnor's conjugation formula $0 = \sum_{i=0}^{k} \xi_{k-i}^{2^i} \zeta_i$ in $\mathcal{A}_*$.

5.3. The C_4-spectral sequence: d_3, d_5 and d_7-differentials

The rest of this section is dedicated to computing the first 8 stems of the C_4-Mackey functor homotopy groups of $a_\lambda^{-1}BP^{((C_4))}$. The result is stated in Theorem 4.4. By Section 3.4, we are free to use the norm structure from C_2 to C_4 in the localized slice spectral sequence.

As a consequence of the slice theorem [33, Theorem 6.1], the 0-th slice of $MU^{((G))}$ is $H\underline{\mathbb{Z}}$ and $\underline{\pi}_0 MU^{((G))} \cong \underline{\mathbb{Z}}$. Therefore, every Mackey functor in the (localized) slice spectral sequence and the homotopy of any $MU^{((G))}$-module is a module over $\underline{\mathbb{Z}}$. By [53, Theorem 16.5], we have the following proposition.

Proposition 5.6. *Let $K \subset H \subset G$, and x be an element in the G/H-level of a Mackey functor either in the (localized) slice spectral sequence or the homotopy of a $MU^{((G))}$-module, then*

$$\mathrm{Tr}_K^H(\mathrm{Res}_K^H(x)) = [H : K]x.$$

Before getting to the page-by-page computation, we note that all the differentials on the classes $u_{2\sigma}^{2^k}$ for $k \geq 0$ are already known by the work of Hill–Hopkins–Ravenel. Their theorem is originally formulated for the slice spectral sequence for $MU^{((C_4))}$ and the exact same statement and proof carries over to $\mathrm{SliceSS}(BP^{((C_4))})$ and $a_\lambda^{-1}\,\mathrm{SliceSS}(BP^{((C_4))})$.

Theorem 5.7 *([33, Theorem 9.9]). For $k \geq 0$ and $i < 2^{k+3} - 3$, $d_i(u_{2\sigma}^{2^k}) = 0$ and*

$$d_{2^{k+3}-3}(u_{2\sigma}^{2^k}) = N(\bar{t}_{k+1})a_\lambda^{2^{k+1}-1}a_\sigma^{2^{k+2}-1}.$$

Now we start the page-by-page computation. First, note that for degree reasons all the differential lengths will be odd.

Proposition 5.8.

$$d_3(u_\lambda) = \mathrm{Tr}_2^4(\bar{t}_1 a_{\sigma_2}^3)$$

Proof. By Theorem 5.4, the restriction $\mathrm{Res}_2^4(u_\lambda) = u_{2\sigma_2}$ supports the differential

$$d_3(u_{2\sigma_2}) = (\bar{t}_1 + \gamma\bar{t}_1)a_{\sigma_2}^3$$

in the C_2-spectral sequence. By naturality and degree reasons, the class u_λ must also support a d_3-differential in the C_4-spectral sequence whose target restricts to the class $(\bar{t}_1 + \gamma\bar{t}_1)a_{\sigma_2}^3$. The only class that restricts to $(\bar{t}_1 + \gamma\bar{t}_1)a_{\sigma_2}^3$ with $RO(C_4)$-degree $1 - \lambda$ is $\mathrm{Tr}_2^4(\bar{t}_1 a_{\sigma_2}^3)$. □

In Fig. 5, this proposition gives all d_3 coming out of $\circ$, namely $u_\lambda a_\lambda^{-1}$ at $(2, -2)$, $N(\bar{t}_1)^2 u_\lambda u_{2\sigma} a_\lambda$ at $(6, 2)$ and $u_\lambda^3 a_\lambda^{-3}$ at $(6, -6)$.

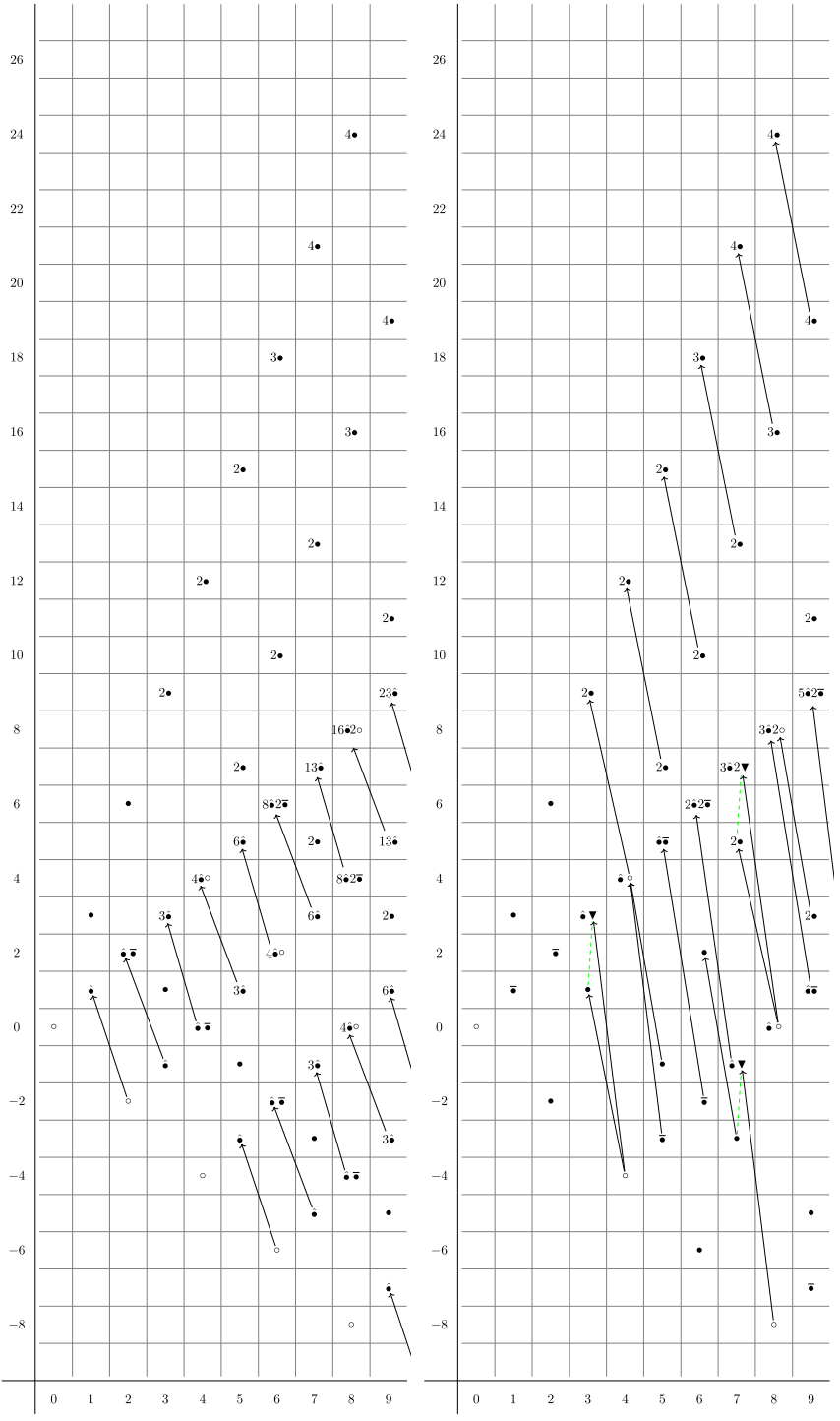


Fig. 5. Left: d_3-differentials in a_λ^{-1} SliceSS($BP^{((C_4))}$). Right: d_5- and d_7-differentials in a_λ^{-1} SliceSS($BP^{((C_4))}$).

Corollary 5.9. *Let P be a polynomial of $\overline{t}_i$, $\gamma\overline{t}_i$, a_{σ_2}, then*

$$d_3(u_\lambda^{2k+1} Tr_2^{4,V}(P)) = u_\lambda^{2k} \operatorname{Tr}_2^{4,V-\lambda+1}(P(\overline{t}_1 + \gamma\overline{t}_1)a_{\sigma_2}^3)$$

for all $k \geq 0$ and any $V \in RO(C_4)$ restricting to the $RO(C_2)$-degree of P.

Proof. This is a direct consequence of Proposition 5.8, the Frobenius relation [34, Definition 2.3] and the Leibniz rule. □

As displayed in Fig. 5, this corollary gives all other d_3-differentials. We now explain them in detail.

In terms of Mackey functors, the d_3-differentials give the following exact sequences:

$$0 \to \bullet \to \circ \xrightarrow{d_3} \hat{\bullet} \to \overline{\bullet} \to 0$$

$$0 \to \hat{\bullet} \xrightarrow{d_3} \hat{\bullet} \to 0$$

$$0 \to \overline{\bullet} \xrightarrow{d_3} \hat{\bullet} \to \blacktriangledown \to 0.$$

Here are examples of d_3-differentials corresponding to each exact sequence above:

$$d_3(u_\lambda) = \operatorname{Tr}_2^4(\overline{t}_1 a_{\sigma_2}^3)$$
$$d_3(\operatorname{Tr}_2^4(\overline{t}_1 a_{\sigma_2})u_\lambda) = \operatorname{Tr}_2^4(\overline{t}_1(\overline{t}_1 + \gamma\overline{t}_1)a_{\sigma_2}^4)$$
$$d_3(u_{2\sigma_2} a_{\sigma_2}) = (\overline{t}_1 + \gamma\overline{t}_1)a_{\sigma_2}^4.$$

Note that the last differential is a C_2-differential, but it has an effect on the C_4-level Mackey functor structure. By results in Section 3.3, the d_3-differentials also give certain exotic restrictions of filtration jump at most 2 (that is, the image of the restriction is of filtration at most 2 higher than the source). For example, consider the element $N(\overline{t}_1)u_\lambda a_\sigma$ at $(3, 1)$. This class is a d_3-cycle. By Proposition 5.8, the class $N(\overline{t}_1)u_\lambda$ supports the d_3-differential

$$d_3(N(\overline{t}_1)u_\lambda) = \operatorname{Tr}_2^4(\overline{t}_1^2\gamma\overline{t}_1 a_{\sigma_2}^3).$$

By Proposition 3.10, the class $\overline{t}_1^2\gamma\overline{t}_1 a_{\sigma_2}^3$ receives an exotic restriction of filtration jump at most 2 in integral degree, and the only possible source is $N(\overline{t}_1)u_\lambda a_\sigma$. The same argument applies to all 2-torsions classes with $(t-s, s)$-bidegrees $(3+4i+4j, 1+4i-4j)$ for $i, j \geq 0$. The exotic restrictions are represented by the vertical green dashed lines in Fig. 5.

Remark 5.10. These exotic restrictions are the first family of examples of an interesting phenomenon in the $RO(G)$-graded spectral sequence of Mackey functors. Exotic restrictions and transfers can imply nontrivial abelian group extensions. By Proposition 5.6,

the transfer of a restriction of a class must be the multiple of this class by the index of the subgroup. Therefore, As Mackey functors, these extensions are of the form

$$0 \to \bullet \to \circ \to \blacktriangledown \to 0,$$

which represents a nontrivial extension

$$0 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0$$

if one evaluates the exact sequence of Mackey functors at C_4/C_4. Notice that in the category of Mackey functors, there are essentially two nontrivial extensions between $\bullet$ and $\blacktriangle$, but only the one above fits into Proposition 5.6.

For readers who are familiar with Lubin–Tate E-theories and topological modular forms, the family of 2-extensions above is a generalization of the type of 2-extension between the class ν at $(3,1)$ and the class 2ν at $(3,3)$ in the homotopy fixed points spectral sequences of $E_2^{hC_4}$ and $TMF_0(5)$ (see [6] and [9]).

In summary, the d_3-differentials can be described as follows:

(1) On C_2-level, it is the first differential in Theorem 5.4.
(2) The Green functor structure of the spectral sequence gives d_3-differentials on the C_4-level, by Proposition 5.8 and Corollary 5.9. After these d_3-differentials, there is no room for further d_3-differentials.
(3) Every d_3-differential of the form $\overline{\bullet} \to \hat{\bullet}$ gives an extension of filtration 2 by the above remark.

Now we will prove the d_5-differentials. There are two different types of d_5-differentials. The first type is given by Theorem 5.7:

$$d_5(u_{2\sigma}) = N(\overline{t}_1)a_\lambda a_\sigma^3.$$

Since $N(\overline{t}_1)$ and a_λ are both permanent cycles, on the integral page for our range, it gives the following d_5-differential at $(4,4)$:

$$d_5(N(\overline{t}_1)^2 u_{2\sigma} a_\lambda^2) = N(\overline{t}_1)^2 a_\lambda^3 a_\sigma^3,$$

and it repeats by multiplying by $N(\overline{t}_1)a_\lambda a_\sigma$. In Fig. 5, these are the d_5-differentials with sources on or above the line of slope 1.

The second type of d_5-differentials is given by the following proposition.

Proposition 5.11.

$$d_5(u_\lambda^2) = N(\overline{t}_1)u_\lambda a_\lambda^2 a_\sigma,$$
$$d_5(u_\lambda^2 a_\sigma) = 2N(\overline{t}_1)u_{2\sigma} a_\lambda^3.$$

Proof. The restriction $\mathrm{Res}^4_2(u_\lambda^2) = u_{2\sigma_2}^2$ supports the d_7-differential

$$d_7(u_{2\sigma_2}^2) = (\overline{t}_2 + \gamma\overline{t}_2 + \overline{t}_1^3)a_{\sigma_2}^7$$

by Theorem 5.4. By naturality, u_λ^2 must support a differential of length at most 7. For degree reasons, the length of this differential can only be 5 or 7. If the length of this differential is 7, the target must restrict to the class $(\overline{t}_2 + \gamma\overline{t}_2 + \overline{t}_1^3)a_{\sigma_2}^7$. However, this class is not in the image of the restriction map Res^4_2. Therefore, u_λ^2 must support a d_5-differential. The only possible target of this d_5-differential is $N(\overline{t}_1)u_\lambda a_\lambda^2 a_\sigma$. This proves the first d_5-differential.

Multiplying with a_σ on both sides of the first d_5-differential gives

$$d_5(u_\lambda^2 a_\sigma) = N(\overline{t}_1)u_\lambda a_\lambda^2 a_\sigma^2.$$

Applying the gold relation $u_\lambda a_\sigma^2 = 2u_{2\sigma}a_\lambda$ gives the second d_5-differential. □

In Fig. 5, the d_5-differentials in Proposition 5.11 can be seen on the following classes:

(1) $u_\lambda^2 a_\lambda^{-2}$ at $(4,-4)$,
(2) $N(\overline{t}_1)u_\lambda^2 a_\lambda^{-1}a_\sigma$ at $(5,-1)$,
(3) $N(\overline{t}_1)^2 u_\lambda^2 u_{2\sigma}$ at $(8,0)$,
(4) $N(\overline{t}_1)^3 u_\lambda^2 u_{2\sigma}a_\lambda a_\sigma$ and $N(\overline{t}_2)u_\lambda^2 u_{2\sigma}a_\lambda a_\sigma$ at $(9,3)$.

Remark 5.12. Although u_λ^2 and $u_\lambda^2 a_\sigma$ support differentials of the same length, this is not true in general. For example, we will see soon that u_λ^4 supports a d_7-differential, while $u_\lambda^4 a_\sigma$ supports a d_{13}-differential.

Corollary 5.13.

$$d_5(u_\lambda^3 a_\sigma) = 2N(\overline{t}_1)u_\lambda u_{2\sigma}a_\lambda^3.$$

Proof. First, we will show that $u_\lambda a_\sigma$ is a nontrivial permanent cycle. Since the target of the d_3-differential on u_λ is a transfer class, it is killed by a_σ, and therefore $u_\lambda a_\sigma$ is a d_3-cycle. The only potential non-trivial differential that $u_\lambda a_\sigma$ can support is the d_5-differential

$$d_5(u_\lambda a_\sigma) = N(\overline{t}_1)a_\lambda^2 a_\sigma^2.$$

If this differential happens, then multiplying a_σ on both sides and using the gold relation will produce the differential

$$d_5(2u_{2\sigma}a_\lambda) = N(\overline{t}_1)a_\lambda^2 a_\sigma^3.$$

This is a contradiction to Theorem 5.7.

Applying the Leibniz rule on the first d_5-differential in Proposition 5.11 with the class $u_\lambda a_\sigma$ produces the d_5-differential

$$d_5(u_\lambda^3 a_\sigma) = u_\lambda a_\sigma d_5(u_\lambda^2) = N(\bar{t}_1)u_\lambda^2 a_\lambda^2 a_\sigma^2 = 2N(\bar{t}_1)u_\lambda u_{2\sigma} a_\lambda^3. \quad \square$$

In Fig. 5, this d_5-differential implies the d_5-differential on the class $N(\bar{t}_1)u_\lambda^3 a_\lambda^{-2} a_\sigma$ at $(7,-3)$. Notice that the class $N(\bar{t}_1)u_\lambda u_{2\sigma} a_\lambda^2$ supports a d_3-differential and the class $2N(\bar{t}_1)u_\lambda u_{2\sigma} a_\lambda^2$ is killed by a d_5-differential. In the integral grading, this happens to the $\mathbb{Z}/4$ in $(6,2)$.

There are extensions of filtration jump 4 induced by the d_5-differentials.

Proposition 5.14. *There is an exotic transfer of filtration jump* 4 *from* $(2,2)$ *to* $(2,6)$*:*

$$\mathrm{Tr}_2^4(\bar{t}_1^2 a_{\sigma_2}^2) = N(\bar{t}_1)^2 a_\lambda^2 a_\sigma^2.$$

There is an exotic restriction of filtration jump 4*, from* $(2,-2)$ *to* $(2,2)$*:*

$$\mathrm{Res}_2^4\left(2u_\lambda a_\lambda^{-1}\right) = \bar{t}_1^2 a_{\sigma_2}^2.$$

Proof. We use Proposition 3.10 to prove both extensions.

For the first claim, note that $d_5(N(\bar{t}_1)u_{2\sigma}a_\lambda) = N(\bar{t}_1)^2 a_\lambda^2 a_\sigma^3$, and $N(\bar{t}_1)^2 a_\lambda^2 a_\sigma^2$ is a nontrivial d_5-cycle. Therefore, $N(\bar{t}_1)^2 a_\lambda^2 a_\sigma^2$ is the target of an exotic transfer of filtration jump 4 in E_6, and the only possible source is $\bar{t}_1^2 a_{\sigma_2}^2$.

For the second claim, first note that by Proposition 5.2 (also see Fig. 3) and the gold relation,

$$2u_\lambda a_\lambda^{-1} = \left(\frac{u_\lambda^2}{u_{2\sigma}} a_\lambda^{-2} a_\sigma\right) a_\sigma.$$

We have the d_5-differential

$$d_5\left(\frac{u_\lambda^2}{u_{2\sigma}} a_\lambda^{-2} a_\sigma\right) = \mathrm{Tr}_2^4(\bar{t}_1^2 a_{\sigma_2}^2).$$

To prove this differential, consider the class $\frac{u_\lambda^2}{u_{2\sigma}} a_\lambda^{-2}$. This class supports a d_5-differential because after multiplying it by $u_{2\sigma}^2 a_\lambda^2$ (which is a d_5-cycle), the class $u_\lambda^2 u_{2\sigma}$ supports the d_5-differential

$$d_5(u_\lambda^2 u_{2\sigma}) = N(\bar{t}_1)u_\lambda u_{2\sigma} a_\lambda^2 a_\sigma$$

by Proposition 5.11. Therefore

$$d_5\left(\frac{u_\lambda^2}{u_{2\sigma}} a_\lambda^{-2}\right) = N(\bar{t}_1)\frac{u_\lambda}{u_{2\sigma}} a_\sigma.$$

Multiplying both sides by a_σ, we have

$$d_5\left(\frac{u_\lambda^2}{u_{2\sigma}}a_\lambda^{-2}a_\sigma\right) = N(\bar{t}_1)\frac{u_\lambda}{u_{2\sigma}}a_\sigma^2 = 2N(\bar{t}_1)a_\lambda = \mathrm{Tr}_2^4(\mathrm{Res}_2^4(N(\bar{t}_1)a_\lambda))$$
$$= \mathrm{Tr}_2^4(\bar{t}_1\gamma\bar{t}_1 a_{\sigma_2}^2) = \mathrm{Tr}_2^4(\bar{t}_1^2 a_{\sigma_2}^2)$$

The last equation holds because by Theorem 5.4, $\bar{t}_1 = \gamma\bar{t}_1$ after the d_3-differentials in the C_2-spectral sequence.

Therefore, $\bar{t}_1^2 a_{\sigma_2}^2$ must receive an exotic restriction of filtration jump 4 in the integral degree, and the only source of the restriction is $2u_\lambda a_\lambda^{-1}$. □

In Fig. 6, the exotic restrictions and transfers are the green and blue dashed lines, respectively.

Remark 5.15. Similar to Remark 5.10, the exotic restrictions and transfers also give extensions of abelian groups on the C_4-level. The situation is more subtle here because each individual exotic extension doesn't involve non-trivial extensions of abelian groups at any level. When we combine the two extensions together, however, we obtain an abelian group extension of filtration 8 from $(2,-2)$ to $(2,6)$:

$$0 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0,$$

and $2(2u_\lambda a_\lambda^{-1}) = N(\bar{t}_1)^2 a_\lambda^2 a_\sigma^2$ in homotopy. This extension is similar to the extension in the 22-stem of $E_2^{hC_4}$ and $TMF_0(5)$. (See [6, Figure 10] and [9, Section 2].)

We will now prove the d_7-differentials. While we state them in some $RO(C_4)$-graded page first, we recommend that the reader multiplies with appropriate powers of a_λ whenever possible to visualize the arguments in Fig. 5.

Proposition 5.16. *We have the following d_7-differentials*

$$d_7(2u_\lambda^2) = \mathrm{Tr}_2^{4,3-2\lambda}(a_{\sigma_2}^7\bar{t}_1^3),$$
$$d_7(2u_\lambda^2 u_{2\sigma}) = \mathrm{Tr}_2^{4,5-2\lambda-2\sigma}(a_{\sigma_2}^7\bar{t}_1^3)$$

(see Notation 5.3 for the transfer notations).

Proof. We will prove the first differential. The second differential is proven by the exact same method. On the C_2-level, we have the d_7-differential

$$d_7(u_{2\sigma_2}^2) = (\bar{t}_2 + \bar{t}_1^3 + \gamma\bar{t}_2)a_{\sigma_2}^7$$

by Theorem 5.4. Taking transfer on the target and using naturality, the class

$$\mathrm{Tr}_2^{4,3-2\lambda}(a_{\sigma_2}^7(\overline{t}_2+\overline{t}_1^3+\gamma\overline{t}_2))=\mathrm{Tr}_2^{4,3-2\lambda}(a_{\sigma_2}^7\overline{t}_1^3)$$

must be killed by a differential of length at most 7. For degree reasons, it must be the d_7-differential with source $2u_\lambda^2$. □

Remark 5.17. These differentials can also be proved by combining Proposition 3.9 and Remark 5.10. One sets the exotic w-operation to be multiplication by 2, and Remark 5.10 shows that the exotic restriction gives such an exotic multiplication.

In Fig. 5, The d_7-differentials in Proposition 5.16 and the underlying C_2-level d_7-differentials in Theorem 5.4 are supported by the classes at $(4+i,-4+i)$ for $i\geq 0$.

Proposition 5.18.

$$d_7(u_\lambda^4)=u_\lambda^2\,\mathrm{Tr}_2^4(\overline{t}_1^3a_{\sigma_2}^7).$$

Proof. We will prove in Proposition 5.22 that there is a nontrivial d_{13}-differential on the class $u_\lambda^4a_\sigma$ (we can already prove it at this point, but for organization reasons we prove it later). This implies that the class u_λ^4 must support a differential of length at most 13. For degree reasons, the claimed d_7-differential is the only possibility. □

In Fig. 5, the d_7-differential in Proposition 5.18 gives the d_7-differential supported by the class $u_\lambda^4a_\lambda^{-4}$ at $(8,-8)$.

5.4. The C_4-spectral sequence: higher differentials and extensions

We will now prove the higher differentials in our range (see Fig. 6). The next possible differential is a d_{13}-differential from Theorem 5.7:

$$d_{13}(u_{2\sigma}^2)=N(\overline{t}_2)a_\lambda^3a_\sigma^7.$$

However, we won't see this differential in Fig. 6. This is because its first appearance in the integer graded spectral sequence is on the class $(10,14)$, which is outside of our range. Note also that even though some classes at $(8,8)$ contain $u_{2\sigma}^2$, they don't support d_{13}-differentials. We will give a detailed discussion of the classes at $(8,8)$ in Section 5.5.

Proposition 5.19.

$$d_{13}(u_\lambda^4u_{2\sigma})=N(\overline{t}_2)u_\lambda u_{2\sigma}^2a_\lambda^6a_\sigma$$

Proof. On the C_2-level, the restriction $\mathrm{Res}_2^4(u_\lambda^4u_{2\sigma})=u_{2\sigma_2}^4$ supports a d_{15}-differential hitting the class $\overline{v}_3a_{\sigma_2}^{15}=(\overline{t}_3+\overline{t}_2^2\overline{t}_1+\overline{t}_1^4\gamma\overline{t}_2+\gamma\overline{t}_3)a_{\sigma_2}^{15}$. Since this class is not in the image of the restriction after the d_3-differentials, by naturality the class $u_\lambda^4u_{2\sigma}$ must support a

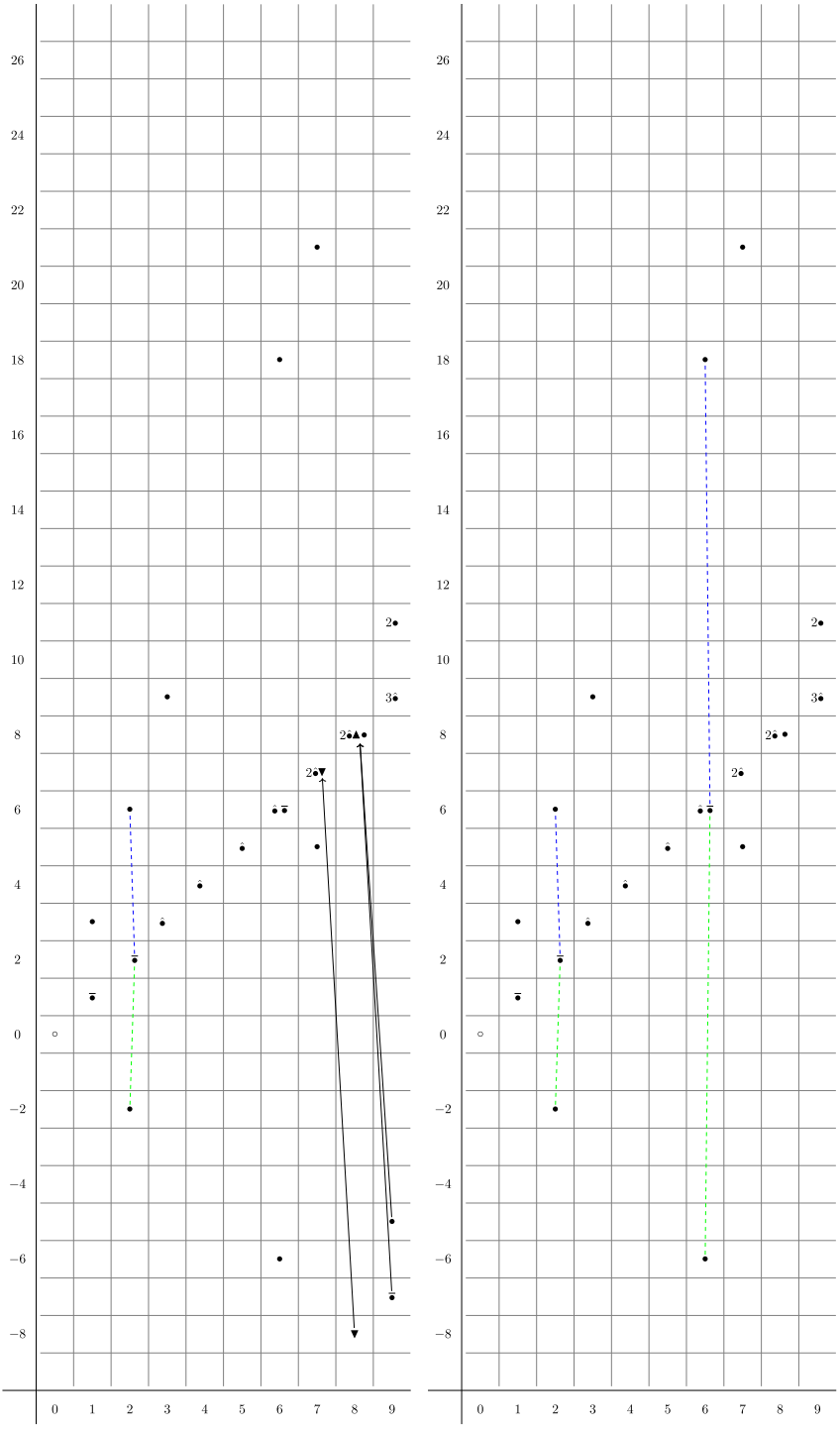


Fig. 6. Left: d_{13}- and d_{15}-differentials in $a_\lambda^{-1}\,\mathrm{SliceSS}(BP^{((C_4))})$. Right: E_∞-page of $a_\lambda^{-1}\,\mathrm{SliceSS}(BP^{((C_4))})$ with all extensions.

differential of length shorter than 15. After computing the first few pages, we see that for degree reasons the potential targets are the following classes:

(1) $\mathrm{Tr}_2^4((\bar{t}_2+\bar{t}_1^3+\gamma\bar{t}_2)u_{2\sigma_2}^2a_{\sigma_2}^7)$ in filtration 7;
(2) $N(\bar{t}_1)^3u_\lambda u_{2\sigma}^2a_\lambda^6a_\sigma$ in filtration 13;
(3) $N(\bar{t}_2)u_\lambda u_{2\sigma}^2a_\lambda^6a_\sigma$ in filtration 13.

We will first prove that the class $\mathrm{Tr}_2^4((\bar{t}_2+\bar{t}_1^3+\gamma\bar{t}_2)u_{2\sigma_2}^2a_{\sigma_2}^7)$ supports the d_{11}-differential

$$d_{11}(\mathrm{Tr}_2^4((\bar{t}_2+\bar{t}_1^3+\gamma\bar{t}_2)u_{2\sigma_2}^2a_{\sigma_2}^7))=N(\bar{t}_1)^4u_{2\sigma}^2a_\lambda^8a_\sigma^2.$$

To prove this, first note that

$$\mathrm{Tr}_2^4((\bar{t}_2+\bar{t}_1^3+\gamma\bar{t}_2)u_{2\sigma_2}^2a_{\sigma_2}^7)=\mathrm{Tr}_2^4(\bar{t}_1^3u_{2\sigma_2}^2a_{\sigma_2}^7)$$

since the class $(\bar{t}_2+\gamma\bar{t}_2)a_{\sigma_2}$ transfers to 0 in the homotopy. On the C_2-level, we have the d_7-differential

$$d_7(\bar{t}_1^3u_{2\sigma_2}^2a_{\sigma_2}^7)=\bar{t}_1^3(\bar{t}_2+\bar{t}_1^3+\gamma\bar{t}_2)a_{\sigma_2}^{14}.$$

The transfer of the target, $\mathrm{Tr}_2^4(\bar{t}_1^3(\bar{t}_2+\bar{t}_1^3+\gamma\bar{t}_2)a_{\sigma_2}^{14})=\mathrm{Tr}_2^4(\bar{t}_1^6a_{\sigma_2}^{14})$, is zero. This is because after the C_2-level d_3-differentials, the class $\bar{t}_1^6a_{\sigma_2}^{14}$ is identified with the class $\bar{t}_1^3\gamma\bar{t}_1^3a_{\sigma_2}^{14}$, which transfers to 0. We will show that the class $\bar{t}_1^6a_{\sigma_2}^{14}$ actually supports an exotic transfer of filtration jump 4. Let $x=N(\bar{t}_1)^3a_\lambda^7u_{2\sigma}^3$. We have the d_5-differential from Theorem 5.7

$$d_5(x)=N(\bar{t}_1)^4u_{2\sigma}^2a_\lambda^8a_\sigma^3.$$

By Proposition 3.10, $N(\bar{t}_1)^4u_{2\sigma}^2a_\lambda^8a_\sigma^2$ receives an exotic transfer of jump 4, and the only possible source is $\bar{t}_1^6a_\sigma^{14}$. Applying Proposition 3.9 to this exotic transfer and the C_2-level d_7, we prove the claimed d_{11}.

The class $N(\bar{t}_1)^3u_\lambda u_{2\sigma}^2a_\lambda^6a_\sigma$ in filtration 13 is killed by a d_5-differential from Proposition 5.11:

$$N(\bar{t}_1)^3u_\lambda u_{2\sigma}^2a_\lambda^6a_\sigma=d_5(N(\bar{t}_1)^2u_\lambda^2u_{2\sigma}^2a_\lambda^4).$$

It follows that the class $N(\bar{t}_2)u_\lambda u_{2\sigma}^2a_\lambda^6a_\sigma$ is the only possible target. □

Remark 5.20. The class $u_\lambda^4u_{2\sigma}$ is a permanent cycle in the homotopy fixed points spectral sequence of $E_2^{hC_4}$ (see [6, Proposition 5.23]) because $N(\bar{t}_2)$ is zero there.

Although this d_{13} doesn't imply any differentials in our range, it is used in proving extensions.

Proposition 5.21.

(1) There is an exotic transfer in stem 6 *of filtration* 12,

$$\mathrm{Tr}_2^4(\overline{t}_2\gamma\overline{t}_2a_{\sigma_2}^6) = N(\overline{t}_2)^2a_\lambda^6a_\sigma^6.$$

(2) There is an exotic restriction in stem 6 *of filtration* 12,

$$\mathrm{Res}_2^4(2u_\lambda^3a_\lambda^{-3}) = \overline{t}_2\gamma\overline{t}_2a_{\sigma_2}^6.$$

Proof. The proof is similar to that of Proposition 5.14. The exotic transfer comes from applying Proposition 3.10 to the d_{13}-differential

$$d_{13}\left(N(\overline{t}_2)u_{2\sigma}^2a_\lambda^3\right) = N(\overline{t}_2)^2a_\lambda^6a_\sigma^7$$

in Theorem 5.7.

For the exotic restriction, first note that $2u_\lambda^3a_\lambda^{-3} = \left(\frac{u_\lambda^4}{u_{2\sigma}}a_\lambda^{-4}a_\sigma\right)a_\sigma$ by the gold relation. We will prove that the class $\frac{u_\lambda^4}{u_{2\sigma}}a_\lambda^{-4}a_\sigma$ supports a d_{13}-differential. To do so, we multiply this class by $u_{2\sigma}^2a_\lambda^4$. After multiplying the differential in Proposition 5.19 by a_σ, we have

$$d_{13}(u_\lambda^4u_{2\sigma}a_\sigma) = 2N(\overline{t}_2)u_{2\sigma}^3a_\lambda^7.$$

As by the gold relation u_λ^2 kills $d_{13}(u_{2\sigma}^2a_\lambda^4)$, we can use the Leibniz rule to obtain the d_{13}-differential

$$d_{13}\left(\frac{u_\lambda^4}{u_{2\sigma}}a_\lambda^{-4}a_\sigma\right) = 2N(\overline{t}_2)u_{2\sigma}a_\lambda^3.$$

On the E_2-page, $2N(\overline{t}_2)u_{2\sigma}a_\lambda^3 = Tr_2^4(\overline{t}_2\gamma\overline{t}_2a_{\sigma_2}^6)$. By Proposition 3.10, $\overline{t}_2\gamma\overline{t}_2a_{\sigma_2}^6$ must receive an exotic restriction of filtration jump 12, and the only possible source is $2u_\lambda^3a_\lambda^{-3}$ (see Fig. 6). □

In Fig. 6, they are the exotic restriction from the class $(6,-6)$ to $(6,6)$ and the exotic transfer from $(6,6)$ to $(6,18)$. Since these extensions involve elements containing $\overline{t}_2$, we expect similar extensions in the homotopy fixed points spectral sequence of $E_4^{hC_4}$ by [8, Theorem 1.1].

Proposition 5.22.

$$d_{13}(u_\lambda^4a_\sigma) = N(\overline{t}_2 + \overline{t}_1^2\gamma\overline{t}_1 + \gamma\overline{t}_2)u_{2\sigma}^2a_\lambda^7.$$

Proof. Consider the C_2-differential

$$d_7(u_{2\sigma_2}^2) = \overline{t}_2^{C_2} a_{\sigma_2}^7.$$

Applying Proposition 3.12 to its target, we see that its norm $N(\overline{t}_2+\overline{t}_1^2\gamma\overline{t}_1+\gamma(\overline{t}_2))a_\lambda^7$ must be killed by a differential of length 13 or shorter. Since the restriction of this element is killed by d_7, it must be killed by a differential of length between 7 and 13. Since $u_{2\sigma}^2$ supports a d_{13}, if $d_r(x) = N(\overline{t}_2+\overline{t}_1^2\gamma\overline{t}_1+\gamma(\overline{t}_2))a_\lambda^7$ happens for $r<13$, one can multiply both sides by $u_{2\sigma}^2$. However, for degree reasons $N(\overline{t}_2+\overline{t}_1^2\gamma\overline{t}_1+\gamma\overline{t}_2)u_{2\sigma}^2 a_\lambda^7$ cannot be hit by a differential shorter than a d_{13}. Thus this element and hence also $N(\overline{t}_2+\overline{t}_1^2\gamma\overline{t}_1+\gamma(\overline{t}_2))a_\lambda^7$ must be hit by a d_{13} and the only possible source is $u_\lambda^4 a_\sigma$. □

On the integer graded page, this contributes to the d_{13}-differential supported by the class $N(\overline{t}_1)u_\lambda^4 a_\lambda^{-3}a_\sigma$ at $(9,-5)$.

The last differential in our range is a d_{15}-differential.

Proposition 5.23. *We have the d_{15}-differential*

$$d_{15}(2u_\lambda^4) = \mathrm{Tr}_2^4(\overline{t}_3^{C_2} a_{\sigma_2}^{15}).$$

Proof. In the C_2-spectral sequence, we have the d_{15}-differential

$$d_{15}(u_{2\sigma_2}^4) = \overline{t}_3^{C_2} a_{\sigma_2}^{15}.$$

Applying the transfer shows that the class $\mathrm{Tr}_2^4(\overline{t}_3^{C_2} a_{\sigma_2}^{15})$ must be killed by a differential of length at most 15. By naturality and degree reasons, the only possible source is the class $2u_\lambda^4 = \mathrm{Tr}_2^4(u_{2\sigma_2}^4)$. □

In Fig. 6, this contributes to the d_{15}-differential supported by the class $2u_\lambda^4 a_\lambda^{-4}$ at $(8,-8)$ (the d_{15}-differential supported by the class at $(9,-7)$ is a C_2-level differential).

These are all the differentials and extensions in the first 8 stems. Now we will discuss in detail the generators and relations in degree $(8,8)$ after each differential in order to illustrate the technical aspect of tracking differentials in the localized slice spectral sequences.

5.5. The classes at $(8,8)$

Since our discussion here focuses on a single degree, we will omit the powers of a_V and u_V classes on each monomial, except in formulas of differentials. That is, we omit $u_{2\sigma}^2 a_\lambda^4$ on C_4-classes and $a_{\sigma_2}^8$ on C_2-classes.

On the E_3-page, there are 2 $\circ$ and 16 $\hat{\bullet}$. The 2 $\circ$ are $N(\overline{t}_1)^4$ and $N(\overline{t}_2)N(\overline{t}_1)$. The 16 $\hat{\bullet}$ are

(1) $\mathrm{Tr}_2^4(\overline{t}_1^8)$, $\mathrm{Tr}_2^4(\overline{t}_1^7\gamma\overline{t}_1)$, $\mathrm{Tr}_2^4(\overline{t}_1^6\gamma\overline{t}_1^2)$, $\mathrm{Tr}_2^4(\overline{t}_1^5\gamma\overline{t}_1^3)$;
(2) $\mathrm{Tr}_2^4(\overline{t}_2\overline{t}_1^5)$, $\mathrm{Tr}_2^4(\overline{t}_2\overline{t}_1^4\gamma\overline{t}_1)$, $\mathrm{Tr}_2^4(\overline{t}_2\overline{t}_1^3\gamma\overline{t}_1^2)$, $\mathrm{Tr}_2^4(\overline{t}_2\overline{t}_1^2\gamma\overline{t}_1^3)$, $\mathrm{Tr}_2^4(\overline{t}_2\overline{t}_1\gamma\overline{t}_1^4)$, $\mathrm{Tr}_2^4(\overline{t}_2\gamma\overline{t}_1^5)$;
(3) $\mathrm{Tr}_2^4(\overline{t}_2^2\overline{t}_1^2)$, $\mathrm{Tr}_2^4(\overline{t}_2^2\overline{t}_1\gamma\overline{t}_1)$, $\mathrm{Tr}_2^4(\overline{t}_2^2\gamma\overline{t}_1^2)$;
(4) $\mathrm{Tr}_2^4(\overline{t}_2\gamma\overline{t}_2\overline{t}_1^2)$;
(5) $\mathrm{Tr}_2^4(\overline{t}_3\overline{t}_1)$, $\mathrm{Tr}_2^4(\overline{t}_3\gamma\overline{t}_1)$.

At the C_2-level, the d_3-differentials identifies $\overline{t}_1$ with $\gamma\overline{t}_1$. At the C_4-level, the effects of the d_3-differentials are as follows:

(1) All the classes in (1) are identified with $2N(\overline{t}_1)^4$;
(2) all the classes in (2) are identified to be the same;
(3) all the classes in (3) are identified to be the same;
(4) the class $\mathrm{Tr}_2^4(\overline{t}_2\gamma\overline{t}_2\overline{t}_1^2)$ is identified with $2N(\overline{t}_2)N(\overline{t}_1)$;
(5) all the classes in (5) are identified to be the same.

Therefore after the d_3-differentials, there are 2 $\circ$, generated by $N(\overline{t}_1)^4$ and $N(\overline{t}_2)N(\overline{t}_1)$, and 3 $\hat{\bullet}$, generated by $\mathrm{Tr}_2^4(\overline{t}_2\overline{t}_1^5)$, $\mathrm{Tr}_2^4(\overline{t}_2^2\overline{t}_1^2)$, and $\mathrm{Tr}_2^4(\overline{t}_3\overline{t}_1)$.

On the E_5-page, by Proposition 5.11, we have the following two d_5-differentials:

$$d_5(N(\overline{t}_1)^3u_\lambda^2u_{2\sigma}a_\lambda a_\sigma) = 2N(\overline{t}_1)^4u_{2\sigma}^2a_\lambda^4,$$
$$d_5(N(\overline{t}_2)u_\lambda^2u_{2\sigma}a_\lambda a_\sigma) = 2N(\overline{t}_2)N(\overline{t}_1)u_{2\sigma}^2a_\lambda^4.$$

It follows that after the d_5-differentials, the 2 $\circ$ become 2 $\blacktriangle$, with the same generators. In total, there are 2 $\blacktriangle$ and 3 $\hat{\bullet}$ at $(8,8)$ after the d_5-differentials (with the same generator as before).

Now we will discuss the d_7-differentials. At $(9,1)$, there are two classes on the E_7-page: a $\hat{\bullet}$ generated by $\mathrm{Tr}_2^4(\overline{t}_2\overline{t}_1^2)$ and a $\overline{\bullet}$ generated by $\overline{t}_1^5$ (it only exists on the C_2-level). Since $\overline{v}_2 = \overline{t}_2 + \overline{t}_1^3 + \gamma\overline{t}_2$, the d_7-differential on the class $\mathrm{Tr}_2^4(\overline{t}_2\overline{t}_1^2)$ hits the class

$$\mathrm{Tr}_2^4(\overline{t}_2\overline{t}_1^2(\overline{t}_2+\overline{t}_1^3+\gamma\overline{t}_2)) = \mathrm{Tr}_2^4(\overline{t}_2^2\overline{t}_1^2) + \mathrm{Tr}_2^4(\overline{t}_2\overline{t}_1^5) + \mathrm{Tr}_2^4(\overline{t}_2\gamma\overline{t}_2\overline{t}_1^2) = \mathrm{Tr}_2^4(\overline{t}_2^2\overline{t}_1^2) + \mathrm{Tr}_2^4(\overline{t}_2\overline{t}_1^5).$$

In other words, it identifies the classes $\mathrm{Tr}_2^4(\overline{t}_2^2\overline{t}_1^2)$ and $\mathrm{Tr}_2^4(\overline{t}_2\overline{t}_1^5)$.

The d_7-differential on the class $\overline{t}_1^5$ hits the class

$$\begin{aligned}\overline{t}_1^5(\overline{t}_2+\gamma\overline{t}_2+\overline{t}_1^3) &= \overline{t}_2\overline{t}_1^5 + \gamma\overline{t}_2\overline{t}_1^5 + \overline{t}_1^8\\ &= \mathrm{Res}_2^4(\mathrm{Tr}_2^4(\overline{t}_2\overline{t}_1^5)) + \mathrm{Res}_2^4(N(\overline{t}_1)^4)\\ &= \mathrm{Res}_2^4(\mathrm{Tr}_2^4(\overline{t}_2^2\overline{t}_1^2)) + \mathrm{Res}_2^4(N(\overline{t}_1)^4).\end{aligned}$$

As Mackey functors, we have

$$\hat{\bullet}\overline{\bullet} \xrightarrow{d_7} 2\blacktriangle 3\hat{\bullet} \twoheadrightarrow \bullet\blacktriangle 2\hat{\bullet}.$$

In the quotient we need to choose our generators carefully: The $\bullet$ is generated by $N(\bar{t}_1)^4 + \mathrm{Tr}_2^4(\bar{t}_2^2\bar{t}_1^2)$, because the image of $\bar{\bullet}$ identifies the restriction of $N(\bar{t}_1)^4$ with the restriction of $\mathrm{Tr}_2^4(\bar{t}_2^2\bar{t}_1^2)$. Therefore their sum is the unique element in C_4-level that has trivial restriction. The $\blacktriangle$ is generated by $N(\bar{t}_2)N(\bar{t}_1)$, as it still has nontrivial restriction. The two $\hat{\bullet}$ are generated by $\mathrm{Tr}_2^4(\bar{t}_2^2\bar{t}_1^2)$ and $\mathrm{Tr}_2^4(\bar{t}_3\bar{t}_1)$.

The next differential is a d_{13}-differential supported by the class $N(\bar{t}_1)u_\lambda^4 a_\lambda^{-3}a_\sigma$ at $(9,-5)$. By Proposition 5.22, the target of this differential is the class $N(\bar{t}_1)N(\bar{t}_2+\bar{t}_1^3+\gamma\bar{t}_2)u_{2\sigma}^2 a_\lambda^4$. The restriction of this class is

$$\bar{t}_1\gamma\bar{t}_1(\bar{t}_2+\bar{t}_1^3+\gamma\bar{t}_2)(\gamma\bar{t}_2+\gamma\bar{t}_1^3-\bar{t}_2),$$

which, after the d_3-differentials, is

$$\bar{t}_2^2\bar{t}_1^2+\gamma\bar{t}_2^2\bar{t}_1^2+\bar{t}_1^8=\mathrm{Res}_2^4(\mathrm{Tr}_2^4(\bar{t}_2^2\bar{t}_1^2))+\mathrm{Res}_2^4(N(\bar{t}_1)^4).$$

As we have discussed above, this class is killed by the d_7-differentials supported by the class $\bar{t}_1^5$. It follows that the target of the d_{13}-differential is the generator of $\bullet$, the unique nontrivial element that restricts to 0.

There is another possible d_{13}-differential supported by some classes at $(8,8)$ that is induced by the differential

$$d_{13}(u_{2\sigma}^2)=N(\bar{t}_2)a_\lambda^3 a_\sigma^7.$$

However, in $(8,8)$ every monomial containing $u_{2\sigma}^2$ also contains $N(\bar{t}_1)$. By [33, Corollary 9.13],

$$d_{13}(N(\bar{t}_1)u_{2\sigma}^2)=N(\bar{t}_1)N(\bar{t}_2)a_\lambda^3a_\sigma^7=d_5(N(\bar{t}_2)u_{2\sigma}a_\lambda^2a_\sigma^4).$$

This makes all elements containing $u_{2\sigma}^2$ in $(8,8)$ d_{13}-cycles.

In summary, after the d_{13}-differentials, we have two $\hat{\bullet}$, generated by $\mathrm{Tr}_2^4(\bar{t}_2^2\bar{t}_1^2)$ and $\mathrm{Tr}_2^4(\bar{t}_3\bar{t}_1)$, and $\blacktriangle$, generated by $N(\bar{t}_2)N(\bar{t}_1)$.

Our final differential is a d_{15}-differential on the C_2-level supported by the class at $(9,-7)$:

$$\begin{aligned}
d_{15}(\bar{t}_1u_{2\sigma_2}^4a_{\sigma_2}^{-7}) &= \bar{t}_1(\bar{t}_3+\bar{t}_2^2\bar{t}_1+\gamma\bar{t}_2\bar{t}_1^4+\gamma\bar{t}_3)a_{\sigma_2}^8\\
&= (\bar{t}_3\bar{t}_1+\gamma\bar{t}_3\bar{t}_1)a_{\sigma_2}^8+(\bar{t}_2^2\bar{t}_1^2+\gamma\bar{t}_2\bar{t}_1^5)a_{\sigma_2}^8\\
&= \mathrm{Tr}_2^4(\bar{t}_3\bar{t}_1)+(\bar{t}_2^2\bar{t}_1^2\gamma\bar{t}_2^2\bar{t}_1^2+\bar{t}_2\gamma\bar{t}_2\bar{t}_1^2)a_{\sigma_2}^8\\
&= \mathrm{Tr}_2^4(\bar{t}_3\bar{t}_1)+\mathrm{Tr}_2^4(\bar{t}_2^2\bar{t}_1^2)+\mathrm{Tr}_2^4\,\mathrm{Res}_2^4(N(\bar{t}_2)N(\bar{t}_1)).
\end{aligned}$$

The map in Mackey functors is

$$\bar{\bullet}\xrightarrow{d_{15}}\blacktriangle 2\hat{\bullet}\twoheadrightarrow\bullet 2\hat{\bullet}.$$

On the E_∞-page, $(8,8)$ is given by $\bullet 2\hat{\bullet}$. The generators for the two $\hat{\bullet}$ are $\mathrm{Tr}_2^4(\bar{t}_2^2\bar{t}_1^2)$ and $\mathrm{Tr}_2^4(\bar{t}_3\bar{t}_1)$. The generator for $\bullet$ is $\mathrm{Tr}_2^4(\bar{t}_2^2\bar{t}_1^2) + \mathrm{Tr}_2^4(\bar{t}_3\bar{t}_1) + N(\bar{t}_2)N(\bar{t}_1)$.

5.6. A family of permanent cycles

We will now present families of nontrivial permanent cycles in $a_\lambda^{-1}\,\mathrm{SliceSS}(BP^{((C_4))})$. These families will be used in the proof of Theorem 6.6.

Lemma 5.24. *In $\pi_\star^{C_4} a_\sigma^{-1}\mathbb{S}$, the element a_λ is invertible.*

Proof. We have the following commutative diagram of pointed C_4-spaces

$$\begin{array}{ccc} S^0 & \xrightarrow{a_\lambda} & S^\lambda \\ & \searrow^{a_\sigma^2} & \downarrow \theta \\ & & S^{2\sigma} \end{array}$$

where θ is the C_4-equivariant 2-folded branched cover. Since $\theta a_\lambda = a_\sigma^2$ is invertible, a_λ is invertible. □

Proposition 5.25. *In $\pi_\star^{C_4} a_\lambda^{-1} BP^{((C_4))}$, the classes $N(\bar{t}_k)a_\sigma^i$ for $k > 0$ and $0 \le i < 2^{k+1}-1$ are non-zero.*

Proof. By Lemma 5.24 we have a map of spectral sequences

$$a_\lambda^{-1}\,\mathrm{SliceSS}(BP^{((C_4))}) \longrightarrow a_\sigma^{-1}\,\mathrm{SliceSS}(BP^{((C_4))}).$$

Notice that in $a_\sigma^{-1}\,\mathrm{SliceSS}(BP^{((C_4))})$, the differentials in Theorem 5.7 completely determine the spectral sequence (see [33, Remark 9.11]). In particular, we have the following differentials in $a_\sigma^{-1}\,\mathrm{SliceSS}(BP^{((C_4))})$:

$$d_{2^{k+2}-3}(u_{2\sigma}^{2^{k-1}} a_\lambda^{-(2^k-1)} a_\sigma^{-(2^{k+1}-1)+i}) = N(\bar{t}_k)a_\sigma^i.$$

On $E_{2^{k+2}-3}$-page, this is the only differential happens in this degree.

By Proposition 5.2 and the gold relation, the class $u_{2\sigma}^{2^{k-1}} a_\sigma^{-(2^{k+1}-1)+i}$ is in the image of

$$\pi_\star^{C_4} a_\lambda^{-1} H\underline{\mathbb{Z}} \to \pi_\star^{C_4} a_\sigma^{-1} H\underline{\mathbb{Z}}$$

only when a_σ has a non-negative power, i.e. $i \ge 2^{k+1}-1$. Therefore by naturality, if the class $N(\bar{t}_k)a_\sigma^i$, $0 \le i < 2^{k+1}-1$ is killed in $a_\lambda^{-1}\,\mathrm{SliceSS}(BP^{((C_4))})$, the differential killing it must be of length longer than $2^{k+2}-3$. However, by Proposition 5.2 and Theorem 4.1,

the potential source of such a differential must be trivial in the E_2-page. Therefore the classes $N(\bar{t}_k)a_\sigma^i$ for $k > 0$ and $0 \leq i \leq 2^{k+1} - 1$ are nontrivial permanent cycles. □

Remark 5.26. After inverting a_λ, the element $N(\bar{t}_k)a_\sigma^{2^{k+1}-1}$ is zero by Theorem 5.7.

6. The Tate spectral sequence of $N_1^2 H\mathbb{F}_2$

The goal of this section is to advance our knowledge of the Tate spectral sequence of $N_1^2 H\mathbb{F}_2$. We compute it in a range and also give all differentials originating from the first diagonal of slope -1. Our main method is comparison with the localized slice spectral sequence, a method we describe first.

There is a canonical map $\mathrm{SliceSS}(X) \to \mathrm{HFPSS}(X)$ which is an isomorphism on the underlying level [55]. When $X = N_1^2 H\mathbb{F}_2 \simeq (BP^{((C_4))})^{\Phi C_2}$, combining with Theorem 3.18, we obtain the following comparison map of spectral sequences

$$a_\lambda^{-1}\,\mathrm{SliceSS}(BP^{((C_4))}) \to P_{4/2}^*\mathscr{D}\mathrm{SliceSS}(N_1^2 H\mathbb{F}_2) \to P_{4/2}^*\mathscr{D}\mathrm{HFPSS}(N_1^2 H\mathbb{F}_2),$$

where we use $P_{4/2}^*$ as a short-hand for the pullback functor P_{C_4/C_2}^* from Section 2.2. Both maps of spectral sequences are compatible with the norm $N_2^4 = N_{C_2}^{C_4}$ by Example 3.17 and Proposition 3.19. On the C_2-level the composition sends permanent cycles from $\mathrm{SliceSS}(BP^{((C_4))})$ to their C_2-geometric fixed points.

We localize the map

$$a_\lambda^{-1}\,\mathrm{SliceSS}(BP^{((C_4))}) \to P_{4/2}^*\mathscr{D}\mathrm{HFPSS}(N_1^2 H\mathbb{F}_2)$$

further at a_σ. Since σ is the pullback of the sign representation on C_4/C_2,

$$a_\sigma^{-1} P_{4/2}^*\mathscr{D}\mathrm{HFPSS}(N_1^2 H\mathbb{F}_2) \simeq P_{4/2}^* a_\sigma^{-1}\mathscr{D}\mathrm{HFPSS}(N_1^2 H\mathbb{F}_2).$$

Notice that localizing at a_σ in C_4/C_2-spectra is exactly smashing with $\tilde{E}\mathcal{F}[C_4/C_2]$, which turns the homotopy fixed points into the Tate fixed points. Therefore

$$P_{4/2}^* a_\sigma^{-1}\mathscr{D}\mathrm{HFPSS}(N_1^2 H\mathbb{F}_2) \simeq P_{4/2}^*\mathscr{D}\mathrm{TateSS}(N_1^2 H\mathbb{F}_2).$$

The above argument, along with Lemma 5.24, gives the following comparison square, which is central to our computation in this section.

$$\begin{array}{ccc} a_\lambda^{-1}\,\mathrm{SliceSS}(BP^{((C_4))}) & \longrightarrow & P_{4/2}^*\mathscr{D}\mathrm{HFPSS}(N_1^2 H\mathbb{F}_2) \\ \downarrow{\scriptstyle a_\sigma^{-1}(-)} & & \downarrow{\scriptstyle a_\sigma^{-1}(-)} \\ a_\sigma^{-1}\,\mathrm{SliceSS}(BP^{((C_4))}) & \longrightarrow & P_{4/2}^*\mathscr{D}\mathrm{TateSS}(N_1^2 H\mathbb{F}_2) \end{array} \tag{3}$$

Proposition 6.1. *In the comparison square, both horizontal maps converge to isomorphisms in homotopy groups.*

Proof. The top horizontal map is the composition

$$a_\lambda^{-1}\,\mathrm{SliceSS}(BP^{((C_4))}) \to P^*_{4/2}\mathscr{D}\mathrm{SliceSS}(N_1^2H\mathbb{F}_2) \to P^*_{4/2}\mathscr{D}\mathrm{HFPSS}(N_1^2H\mathbb{F}_2).$$

The first map converges to an isomorphism by Theorem 3.18. By the Segal conjecture, $N_1^2H\mathbb{F}_2$ is cofree, that is, the map $N_1^2H\mathbb{F}_2 \to F(EC_{2+}, N_1^2H\mathbb{F}_2)$ is an equivalence. By construction, the second map of the spectral sequences converges to this map on homotopy.

The bottom map is the a_σ-localization of the top map, thus also converges to an isomorphism. □

The bottom map in the comparison square is particularly interesting: We completely understand $a_\sigma^{-1}\,\mathrm{SliceSS}(BP^{((C_4))})$, which is determined by the fact that it computes $\pi_*H\mathbb{F}_2$. All differentials are derived from the slice differential theorem [33, Theorem 9.9, Remark 9.11]. On the other hand, the Tate spectral sequence of $N_1^2H\mathbb{F}_2$ is very mysterious: its E_2-page is determined by the Tate cohomology $\hat{H}^*(C_2;\mathcal{A}_*)$, for which we do not know a closed formula yet. Nevertheless, the Segal conjecture shows that the Tate spectral sequence converges to $\pi_*H\mathbb{F}_2$, meaning almost everything kills each other by differentials. Using Theorem 3.18, we can apply our understanding of the slice spectral sequence to understand partially how differentials work in the Tate spectral sequence.

Fig. 7 consists of the integral E_2-pages of the four spectral sequences in the comparison square. Red elements in the homotopy fixed points and the Tate spectral sequences are those in the image of the horizontal maps. We prove these claims in Corollary 6.5.

Using the comparison square, we establish an infinite family of differential in the Tate spectral sequence. We also compute all differentials in the Tate spectral sequence in the same range we computed $a_\lambda^{-1}\,\mathrm{SliceSS}(BP^{((C_4))})$ in Section 5. Specifically, we show all differentials hitting elements from stem 0 to 8 which map non-trivially into the homotopy orbit spectral sequence. In the (doubled) Tate spectral sequence of Fig. 7, they are elements below slope 1 from stem 0 to 8.

Because of the comparison square, we make our statements and arguments entirely in the spectral sequences $P^*_{4/2}\mathscr{D}\mathrm{HFPSS}$ and $P^*_{4/2}\mathscr{D}\mathrm{TateSS}$. The translation back to the C_2-homotopy fixed points and the Tate spectral sequence is straightforward. As a reference, $\mathrm{TateSS}(N_1^2H\mathbb{F}_2)$ with known differentials is shown as Fig. 9.

To start the computation, we want to understand how the maps of the comparison square behave on the E_2-page. By Theorem 3.18, they are determined by the C_2-geometric fixed points of elements in $\pi_\bigstar^{C_2}BP^{((C_4))}$.

Proposition 6.2. *Under the equivalence $\Phi^{C_2}BP^{((C_4))} \simeq H\mathbb{F}_2 \wedge H\mathbb{F}_2$, the C_2-geometric fixed points of $\overline{t}_i$ and $\gamma\overline{t}_i$ in $\pi^{C_2}_{(2^i-1)\rho_2}BP^{((C_4))}$ are ξ_i and ζ_i, the Milnor generators and their conjugates in the dual Steenrod algebra, respectively.*

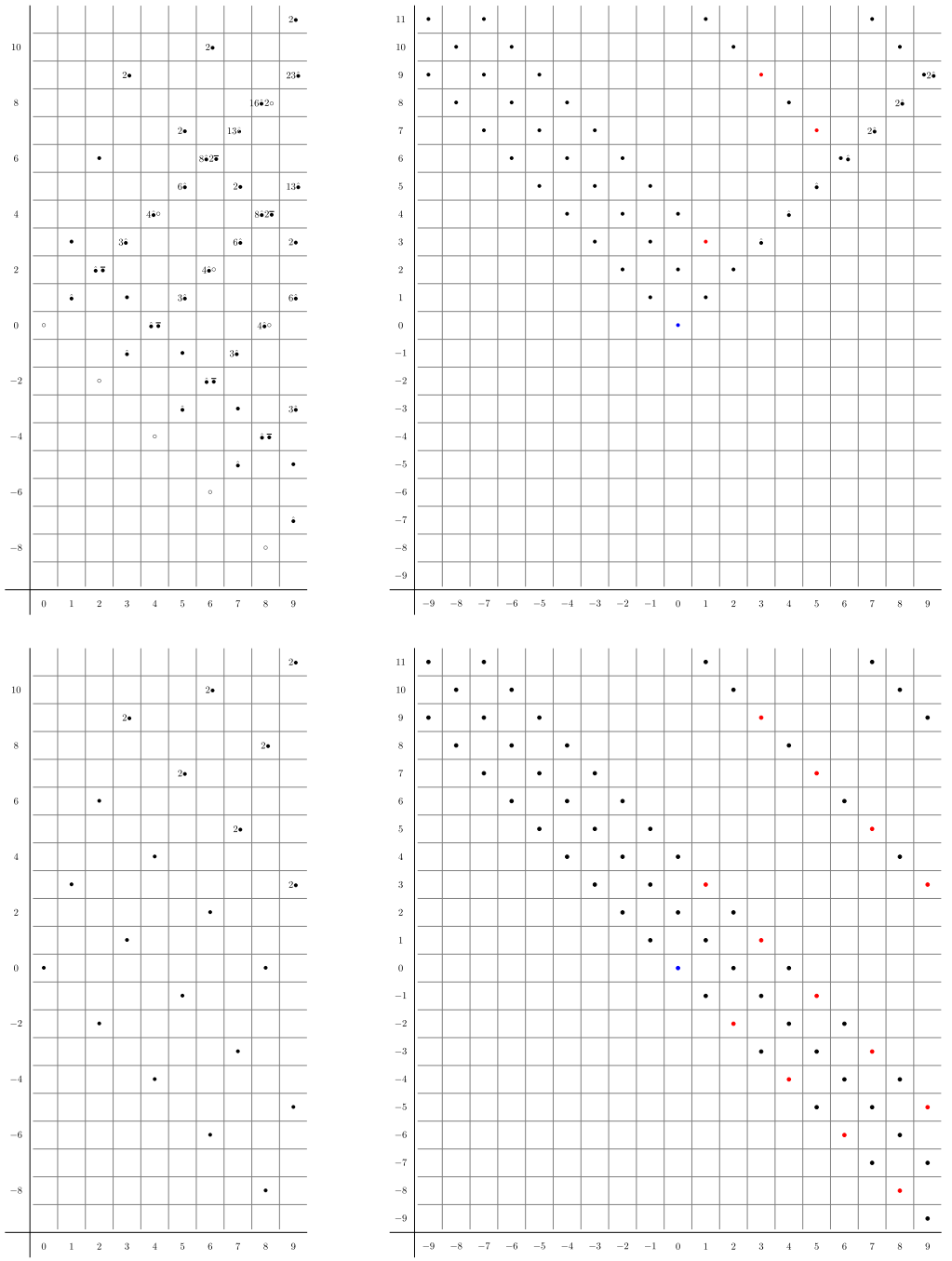


Fig. 7. The comparison square (3).

Proof. We will show $\Phi^{C_2}(\overline{t}_i) = \xi_i$; the formula $\Phi^{C_2}(\gamma\overline{t}_i) = \zeta_i$ follows from the fact that the residue C_4/C_2-action on $\Phi^{C_2}BP^{((C_4))}$ becomes the conjugate action on the dual Steenrod algebra.

Let $e : S^0 \to BP_{\mathbb{R}}$ be the unit map and F_1 and F_2 be the formal groups laws on $\pi^{C_2}_{*\rho_2}BP_{\mathbb{R}} \wedge BP_{\mathbb{R}}$ induced by the map

$$BP_{\mathbb{R}} \wedge S^0 \xrightarrow{id\wedge e} BP_{\mathbb{R}} \wedge BP_{\mathbb{R}}$$

and

$$S^0 \wedge BP_{\mathbb{R}} \xrightarrow{e\wedge id} BP_{\mathbb{R}} \wedge BP_{\mathbb{R}}$$

respectively, and let $\tilde{x}_1$, $\tilde{x}_2$ be the corresponding power series generators. As in Section 3, the elements $\overline{t}_i$ are defined as

$$\tilde{x}_2 = \sum_{i=0}^{\infty}{}^{F_2}\ \overline{t}_i \tilde{x}_1^{2^i}.$$

Taking Φ^{C_2} maps $\tilde{x}_1$ and $\tilde{x}_2$ to the two MO-orientation on $H\mathbb{F}_2 \wedge H\mathbb{F}_2$. The following lemma completes the proof. □

Lemma 6.3. *Let $x_1, x_2 \in (H\mathbb{F}_2 \wedge H\mathbb{F}_2)^1(\mathbb{R}P^\infty)$ be the MO-orientations corresponding to the maps*

$$H\mathbb{F}_2 \wedge S^0 \xrightarrow{id\wedge\mu} H\mathbb{F}_2 \wedge H\mathbb{F}_2$$

and

$$S^0 \wedge H\mathbb{F}_2 \xrightarrow{\mu\wedge id} H\mathbb{F}_2 \wedge H\mathbb{F}_2$$

respectively. Then we have

$$x_2 = \sum_{i=0}^{\infty} \xi_i x_1^{2^i}.$$

Proof. Identify $(H\mathbb{F}_2 \wedge H\mathbb{F}_2)^*(\mathbb{R}P^\infty)$ with $\mathcal{A}_*[\![x_1]\!]$, and write $x_2 = \sum\limits_{j=0}^{\infty} a_j x_1^{j+1}$. We will show that $a_{2^i-1} = \xi_i$ and all other a_j's are 0. First, since the power series $\sum\limits_{j=0}^{\infty} a_j x_1^{j+1}$ is an automorphism of the additive formal group law in an $\mathbb{F}_2$-algebra, we must have $a_0 = 1$, and $a_j = 0$ for $j \neq 2^i - 1$.

Let I be an admissible sequence and define

$$\theta_I : (H\mathbb{F}_2 \wedge H\mathbb{F}_2)^*(\mathbb{R}P^\infty) \to (H\mathbb{F}_2 \wedge H\mathbb{F}_2)^{*+|I|}(\mathbb{R}P^\infty)$$

to be the composition

$$\mathbb{R}P^\infty \to H\mathbb{F}_2 \wedge H\mathbb{F}_2 \xrightarrow{id\wedge Sq^I} \Sigma^{|I|} H\mathbb{F}_2 \wedge H\mathbb{F}_2.$$

One can verify directly that θ_I has the following properties:

- $\theta_I(x_2) = x_2^{2^n}$ if $I = (2^{n-1}, 2^{n-2}, ..., 2, 1)$, and $\theta_I(x_2) = 0$ otherwise.
- $\theta_I(x_1) = x_1$ if $I = (0)$ and $\theta_I(x_1) = 0$ otherwise.
- On homotopy, the map $H\mathbb{F}_2 \wedge H\mathbb{F}_2 \xrightarrow{id\wedge Sq^I} \Sigma^{|I|} H\mathbb{F}_2 \wedge H\mathbb{F}_2$ induces the map

  $$\theta_I : \mathcal{A}_* \longrightarrow \mathcal{A}_{*-|I|}.$$

  For any $\xi \in \mathcal{A}_*$, $\theta_I(\xi)$ is the cap product

  $$\mathcal{A}_* \xrightarrow{\Delta} \mathcal{A}_* \otimes \mathcal{A}_* \xrightarrow{id\otimes\langle -, Sq^I\rangle} \mathcal{A}_*$$

  between ξ and Sq^I. In the case when $|\xi| = |I|$, $\theta_I(\xi) = \langle \xi, Sq^I\rangle$, the pairing between the Steenrod algebra and its dual.
- When $I = (i)$, θ_I satisfies the Cartan formula:

  $$\theta_i(ab) = \sum_{j=0}^{i} \theta_{i-j}(a)\theta_j(b).$$

Now, let $I = (2^{n-1}, 2^{n-2}, ..., 2, 1)$, and apply θ_I to $x_2 = \sum_{i=0}^{\infty} a_{2^i-1} x_1^{2^i}$. The left hand side becomes

$$x_2^{2^n} = \left(\sum_{i=0}^{\infty} a_{2^i-1} x_1^{2^i}\right)^{2^n} = \sum_{i=0}^{\infty} a_{2^i-1}^{2^n} x_1^{2^{n+i}},$$

and the right hand side becomes

$$\theta_I\left(\sum_{i=0}^{\infty} a_{2^i-1} x_1^{2^i}\right) = \sum_{j=0}^{\infty} \theta_I(a_{2^i-1}) x_1^{2^i}.$$

Comparing the coefficient of $x_1^{2^n}$ in both expressions, we see that $1 = \theta_I(a_{2^n-1}) = \langle a_{2^n-1}, Sq^I\rangle$. Now, if I is any other admissible sequence with $|I| = 2^n - 1$, then $\theta_I(x_2) = 0$ and thus $0 = \theta_I(a_{2^n-1}) = \langle a_{2^n-1}, Sq^I\rangle$. This is exactly the definition of ξ_n, see [48, Chapter 6, Proposition 1]. □

We pause here to clarify notations in $P^*_{4/2}\mathscr{D}\mathrm{HFPSS}(N_1^2 H\mathbb{F}_2)$ and $P^*_{4/2}\mathscr{D}\mathrm{TateSS}(N_1^2 H\mathbb{F}_2)$. The C_2-level of this spectral sequence is the spectral sequence of

the doubled Postnikov tower of $H\mathbb{F}_2 \wedge H\mathbb{F}_2$, treated as a C_2-equivariant spectral sequence whose underlying level is trivial (and thus a_{σ_2} acts invertibly). Therefore, given an element $x \in \mathcal{A}_*$, there are elements in different $RO(G)$-grading differing by powers of a_{σ_2} that deserve the name x. We name the corresponding element in the integral grading by x, and name all others by $a^i_{\sigma_2}x$ for some $i \in \mathbb{Z}$. Notice that in this way, ξ_n has stem and filtration $2^n - 1$ since we are working in the doubled spectral sequence. Under this notation, the map

$$a_\lambda^{-1}\,\mathrm{SliceSS}(BP^{((C_4))}) \to P^*_{4/2}\mathscr{D}\mathrm{HFPSS}(N_1^2 H\mathbb{F}_2)$$

sends $\overline{t}_i$ to $a_{\sigma_2}^{-(2^i-1)}\xi_i$ (on the C_2-level), as follows from Theorem 3.18 and Proposition 6.2: since the target spectral sequence collapses on C_2-level, the image of $\overline{t}_i$ is determined by its $RO(C_2)$-degree and its image under Φ^{C_2}.

In the C_4-level, we need to be extra careful. By taking $N = N_2^4$ on $\overline{t}_i \mapsto a_\sigma^{-(2^i-1)}\xi_i$, we see that

$$N(\overline{t}_i) \mapsto a_\lambda^{-(2^i-1)}N(\xi_i),$$

where $N(\xi_i)$ is in $RO(C_4)$-degree $(2^i-1)(1+\sigma)$ and filtration $2(2^i-1)$. The complication comes from the fact that there are other generators of Tate cohomology than $N(\xi_i)$. For example, the element ξ_1 is invariant under the conjugate action and thus gives a generator of $\hat{H}^0(C_2;(\mathcal{A}_*)_1)$. For such generators in degree i of $\mathcal{A}_*$, we will use the notation b_i, and define that they are in the integral grading. For example, the generator of $\hat{H}^0(C_2;(\mathcal{A}_*)_1)$ is named b_1, and has bidegree $(1,1)$ in the double of the homotopy fixed points and the Tate spectral sequence. Since the square of b_1 restricts to $\xi_1^2 = \xi_1\zeta_1$, we have (for degree reasons) a multiplicative relation

$$b_1^2 = N(\xi_1)u_\sigma,$$

where u_σ is a generator of the Tate cohomology of trivial module

$$\hat{H}^\bigstar(C_2;\mathbb{F}_2) \cong \mathbb{F}_2[a_\sigma^\pm, u_\sigma^\pm].$$

The generator u_σ has degree $1-\sigma$ and a_σ has degree $-\sigma$. The classical integral graded Tate cohomology

$$\hat{H}^*(C_2;\mathbb{F}_2) \cong \mathbb{F}_2[x^\pm]$$

with degree 1 generator x is related to the $RO(C_2)$-graded cohomology via $x = u_\sigma a_\sigma^{-1}$. Since the sign representation on C_4/C_2 pulls back to the sign representation on C_4, we use the same notations u_σ and a_σ in the pullback of the homotopy fixed points and the Tate spectral sequence.

In summary, the C_2-level of $P^*_{4/2}\mathscr{D}\mathrm{HFPSS}(N_1^2 H\mathbb{F}_2)$ has the form

$$\mathbb{F}_2[a^{\pm}_{\sigma_2}][\xi_1, \xi_2, \cdots]$$

where ξ_i has both degree and filtration $2^i - 1$. The C_4-level of $P^*_{4/2}\mathscr{D}\mathrm{HFPSS}(N_1^2 H\mathbb{F}_2)$ has the form

$$H^0(C_2; \mathcal{A}_*)[a^{\pm}_{\lambda}, a_\sigma, u^{\pm}_{\sigma}]/(Tr(x)a_\sigma, \forall x \in \mathcal{A}_*),$$

where an element in $H^0(C_2; (\mathcal{A}_*)_i)$ has both degree and filtration i if it doesn't restrict to elements of the form $\xi_k\zeta_k$; if this happens, the integral graded element is named by $N(\xi_k)u_\sigma^{2^i-1}$. The class u_σ is of stem $1-\sigma$ and of filtration 0, while a_σ is of stem $-\sigma$ and filtration 1.

The C_4-level of $P^*_{4/2}\mathscr{D}\mathrm{TateSS}(N_1^2 H\mathbb{F}_2)$ has the form

$$\hat{H}^0(C_2; \mathcal{A}_*)[a^{\pm}_{\lambda}, a^{\pm}_{\sigma}, u^{\pm}_{\sigma}],$$

with names of elements in $\hat{H}^0$ from the image of the surjective map $H^0(C_2; \mathcal{A}_*) \to \hat{H}^0(C_2; \mathcal{A}_*)$.

Proposition 6.4. *In the Tate cohomology $\hat{H}^0(C_2; \mathcal{A}_*)$ the following elements are nontrivial:*

$$\xi_1\zeta_1, \xi_2\zeta_2, (\xi_2\zeta_2)^2, \xi_3\zeta_3, (\xi_3\zeta_3)^2, (\xi_3\zeta_3)^3, (\xi_i\zeta_i)^k$$

for $i \geq 4$ and $k \leq 4$.

The proof is purely combinatorical and is irrelevant to other parts of the paper. It uses computations and ideas from [18].

Proof. We argue by monomial degrees in $\mathcal{A}_* = \mathbb{F}_2[\xi_1, \xi_2, \cdots]$ and the Milnor conjugate formula

$$\zeta_i = \sum_{j=0}^{i-1} \xi_{i-j}^{2^j} \zeta_j.$$

The conjugate formula tells us that the transfer of a monomial (i.e. the sum of the monomial and its conjugate) in the ξ_i can only increase its monomial degree. It also tells us that the monomial with minimal monomial degree in $(\xi_i\zeta_i)^k$ is ξ_i^{2k}. Therefore, $(\xi_i\zeta_i)^k$ being in the image of transfer can only happen when ξ_i^{2k} appears in the transfer of a monomial, which has smaller monomial degree and the same topological degree.

To streamline the computation, we define that a monomial P has bidegree (a, b) if P has monomial degree b and topological degree $a - b$. In this way, ξ_i has bidegree $(2^i, 1)$.

To find monomials which have smaller monomial degree and the same topological degree, we can look at the binary expansion of a. Let t be a positive integer, and $\mathrm{Bin}(t)$ be the number of 1s in the binary expansion of t. If a monomial has bidegree (a, b), then both a must be even and $Bin(a) \leq b$. Since ξ_i has bidegree $(2^i, 1)$, any monomial whose transfer contains ξ_i^{2k} must be in bidegree (a, b) where $a - b = 2k(2^i - 1)$.

We will only check the highest power of $\xi_i\zeta_i$ listed in the statement of the proposition, since if $(\xi_i\zeta_i)^k$ is nontrivial in Tate cohomology, then $(\xi_i\zeta_i)^j$ for $j \leq k$ are all nontrivial.

The class of $\xi_1\zeta_1$ is obviously nontrivial in Tate cohomology, so we start our argument with $(\xi_2\zeta_2)^2$. Writing it as a polynomial in the ξ_i, the leading term is ξ_2^4, which has bidegree $(16, 4)$. We only need to check if there is a nontrivial monomial in bidegree $(14, 2)$. Since $\mathrm{Bin}(14) = 3 > 2$, there is no monomial in this degree. Therefore $(\xi_2\zeta_2)^2$ is nontrivial in the Tate cohomology.

Next we consider $(\xi_3\zeta_3)^3$. The leading term is ξ_3^6, which has bidegree $(48, 6)$. In $(46, 4)$ there is only one monomial $\xi_5\xi_3\xi_2\xi_1$. It is direct to check that

$$\xi_5\xi_3\xi_2\xi_1 + \zeta_5\zeta_3\zeta_2\zeta_1 \neq (\xi_3\zeta_3)^3.$$

In $(42, 2)$, there is no monomial since $Bin(42) = 3 > 2$.

For $(\xi_i\zeta_i)^4$ where $i > 3$, a similar argument applies. When $i = 4$ there is a monomial in $(126, 6)$, namely $\xi_6\xi_5\xi_4\xi_3\xi_2\xi_1$, but it cannot transfer to $(\xi_4\zeta_4)^4$. And there is no monomial with smaller monomial degree with the same topological degree. When $i > 4$ there is simply no suitable monomial below $(2^{i+3}, 8)$ since $\mathrm{Bin}(2^{i+3} - t)$ for $t = 2, 4, 6$ are all greater than $8 - t$. □

The proof can certainly be generalized. For example, $(\xi_i\zeta_i)^8$ are nontrivial in the Tate cohomology for $i > 11$. However, what we proved is sufficient for our computation.

Recall that the element $u_{2\sigma}$ in $\pi_{2-2\sigma}^{C_4} H\underline{\mathbb{Z}}$ maps to u_σ^2 in Tate cohomology.

Corollary 6.5. *Under the map*

$$a_\sigma^{-1}\,\mathrm{SliceSS}(BP^{((C_4))}) \to P_{4/2}^*\mathscr{D}\mathrm{TateSS}(N_1^2 H\mathbb{F}_2),$$

the classes

$$N(\bar{t}_i)^j a_\lambda^k a_\sigma^l u_{2\sigma}^m$$

map to

$$N(\xi_i)^j a_\lambda^{k-(2^i-1)} a_\sigma^l u_\sigma^{2m},$$

for $j \geq 0$ and $k, l, m \in \mathbb{Z}$. The image is nontrivial if and only if $(\xi_i\zeta_i)^j$ represents a nontrivial element in $\hat{H}^0(C_2; \mathcal{A}_*)$.

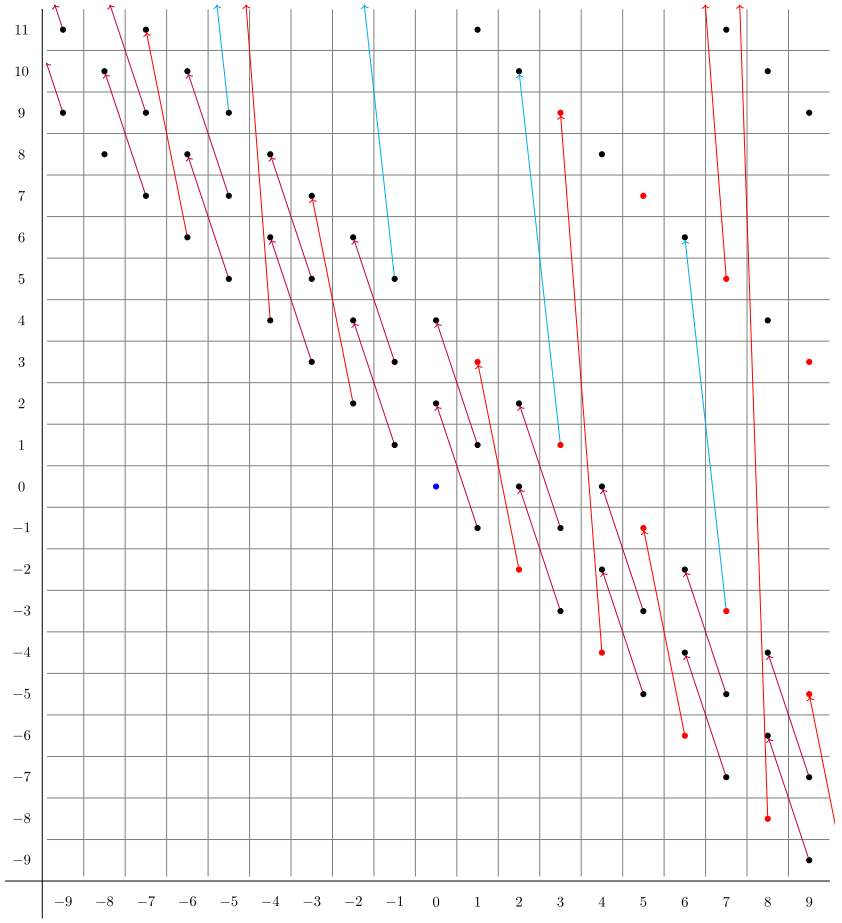


Fig. 8. $P^*_{4/2}\mathscr{D}\mathrm{TateSS}(N_1^2 H\mathbb{F}_2)$ with differentials.

The slice differentials in [33] completely describe $a_\sigma^{-1}\,\mathrm{SliceSS}(BP^{((C_4))})$. By understanding its image in $P^*_{4/2}\mathscr{D}\mathrm{TateSS}(N_1^2 H\mathbb{F}_2)$, we can deduce many differentials in the Tate spectral sequence. We prove all differentials in their most natural $RO(C_4)$-degree. They can be translated into the integral degree by invertible a_λ and a_σ multiplications. Fig. 8 presents $P^*_{4/2}\mathscr{D}\mathrm{TateSS}(N_1^2 H\mathbb{F}_2)$ with differentials proved below. For reference, Fig. 9 presents the original Tate spectral sequence $\mathrm{TateSS}(N_1^2 H\mathbb{F}_2)$ with the same differentials.

Theorem 6.6. *In $P^*_{4/2}\mathscr{D}\mathrm{TateSS}(N_1^2 H\mathbb{F}_2)$, we have differentials*

$$d_{2^{k+2}-3}(u_\sigma^{2^k}) = N(\xi_k) a_\sigma^{2^{k+1}-1}.$$

for all $k \geq 1$.

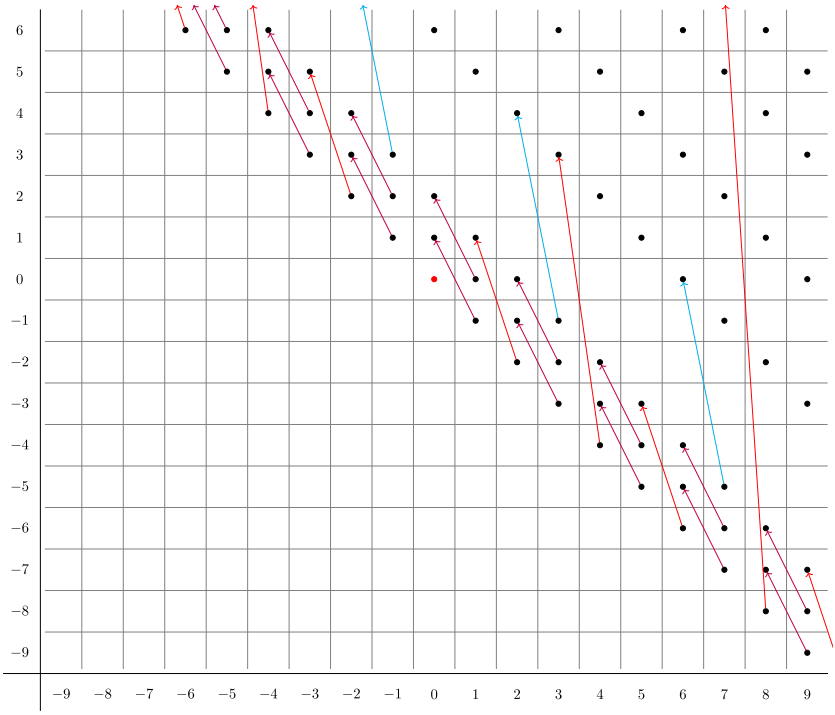


Fig. 9. $\mathrm{TateSS}(N_1^2 H\mathbb{F}_2)$ with differentials.

Proof. By Proposition 6.4, both the source and the target are in the image of

$$a_\sigma^{-1}\,\mathrm{SliceSS}(BP^{((C_4))}) \to P^*_{4/2}\mathscr{D}\mathrm{TateSS}(N_1^2 H\mathbb{F}_2).$$

There is no element below $u_\sigma^{2^k}$ in the E_2-page, so $N(\xi_k)a_\sigma^{2^{k+1}-1}$, which is in stem $2^k - 1 - 2^k\sigma$ and filtration $2^{k+2}-3$, can only be killed by a differential of length at most $2^{k+2}-3$. Therefore we only need to show that $N(\xi_k)a_\sigma^{2^{k+1}-1}$ is not killed by a differential of length less than $2^{k+2}-3$. To prove this, we show that if such a shorter differential happens, it implies $N(\xi_k)a_\sigma^{2^{k+1}-2} = 0$ in $\pi_\star^{C_2} N_1^2 H\mathbb{F}_2$. Combining with Proposition 6.2, it contradicts Proposition 5.25.

Assume that there is a differential $d_l(x) = N(\xi_k)a_\sigma^{2^{k+1}-1}$ in $P^*_{4/2}\mathscr{D}\mathrm{TateSS}(N_1^2 H\mathbb{F}_2)$ for $l < 2^{k+2}-3$. To show $N(\xi_k)a_\sigma^{2^{k+1}-2} = 0$ in $\pi_\star^{C_2} N_1^2 H\mathbb{F}_2$, we work with the map

$$P^*_{4/2}\mathscr{D}\mathrm{HFPSS}(N_1^2 H\mathbb{F}_2) \to P^*_{4/2}\mathscr{D}\mathrm{TateSS}(N_1^2 H\mathbb{F}_2)$$

in $RO(C_4)$-grading. By writing an arbitrary element y of stem $m + n\sigma$ in $P^*_{4/2}\mathscr{D}\mathrm{TateSS}(N_1^2 H\mathbb{F}_2)$ in the form $zu_\sigma^f a_\sigma^g$ with $z \in H^0(C_2, \pi_{|z|}N_1^2 H\mathbb{F}_2)$, we see that y is in the image from the spectral sequence $P^*_{4/2}\mathscr{D}\mathrm{HFPSS}(N_1^2 H\mathbb{F}_2)$ if and only if $g \geq 0$. This happens if and only if the filtration $|z|+g$ is at least $m+n = (|z|+f)+(-f-g) = |z|-g$.

The class x is of stem $2^k - 2^k\sigma$ and filtration $2^{k+2} - 3 - l$; the filtration is at least 2 since l must be odd because of the doubling operator $\mathscr{D}$. Thus xa_σ^{-1} is still in the image from $P^*_{4/2}\mathscr{D}\mathrm{HFPSS}(N_1^2 H\mathbb{F}_2)$ and likewise is $N(\xi_k)a_\sigma^{2^{k+1}-1}a_\sigma^{-1} = N(\xi_k)a_\sigma^{2^{k+1}-2}$. Thus, we have $d_l(xa_\sigma^{-1}) = N(\xi_k)a_\sigma^{2^{k+1}-2}$ in $P^*_{4/2}\mathscr{D}\mathrm{HFPSS}(N_1^2 H\mathbb{F}_2)$, unless the target is killed by a shorter differential. In any case, $N(\xi_k)a_\sigma^{2^{k+1}-2} = 0$ in $\pi^{C_2}_\star N_1^2 H\mathbb{F}_2$. □

The exact same argument gives the following differentials.

Corollary 6.7. *In $P^*_{4/2}\mathscr{D}\mathrm{TateSS}(N_1^2 H\mathbb{F}_2)$, we have differentials:*

$$d_{2^{k+2}-3}(N(\xi_k)^j u_\sigma^{2^k}) = N(\xi_k)^{j+1} a_\sigma^{2^{k+1}-1}.$$

These differentials and their propagation are the red differentials in Fig. 9. Notice that $N(\xi_k)^{j+1}$ can be zero in the E_2-page of the Tate spectral sequence. For example, $N(\xi_1)^2 = 0$ in the Tate cohomology, since

$$\xi_1^2\zeta_1^2 = \xi_1^4 = \xi_1(\xi_2 + \zeta_2) = Tr(\xi_1\xi_2)$$

As a result, the target of $d_5(N(\xi_1)u_\sigma^2)$ predicted by the corollary is zero. Instead, $N(\xi_1)u_\sigma^2$ supports a nontrivial d_9, see Proposition 6.9

As an interesting consequence, we can bound the length of differentials on elements of the first diagonal in the Tate spectral sequence of $N_1^2 R$ for a large family of ring spectra.

Corollary 6.8. *Let R be a non-equivariant (-1)-connected homotopy ring spectrum with $\pi_0(X) \cong \mathbb{Z}_S$ being a localization of $\mathbb{Z}$ such that $\frac{1}{2} \notin \mathbb{Z}_S$. Let $v \in \hat{H}^2(C_2; \pi_0 N_1^2 X) \cong \hat{H}^2(C_2; \mathbb{Z}_S)$ be the generator of the Tate cohomology. Then v^{2^k} supports a non-trivial differential of length l_k with*

$$\rho(2^{k+1}) \leq l_k \leq 2^{k+2} - 1.$$

Here $\rho(n)$ is the Radon–Hurwitz number (with $\rho(n) - 1$ the maximal number of independent vector fields on S^{n-1}): for $n = k2^{4b+c}$ with k odd and $0 \leq c < 4$, it is defined as $\rho(n) = 8b + 2^c$.[6]

Proof. Consider the sequence of non-equivariant ring maps

$$S^0 \to R \to H\mathbb{F}_2,$$

where the last map is the composition of the 0-Postnikov section and the mod 2 map.

[6] Beware that the lower bound is based on a classical result for which we don't know of a published reference.

Take the norm N_1^2 and take the Tate spectral sequence, we have

$$\mathrm{TateSS}(S^0) \to \mathrm{TateSS}(N_1^2 R) \to \mathrm{TateSS}(N_1^2 H\mathbb{F}_2).$$

Note that $\pi_0 N_1^2 X \cong \mathbb{Z}_S \otimes \mathbb{Z}_S \cong \mathbb{Z}_S$. Since v maps to $u_\sigma^2 a_\sigma^{-2}$ in $\mathrm{TateSS}(N_1^2 H\mathbb{F}_2)$, Theorem 6.6 gives the upper bound. The lower bound is given by the corresponding differential in $\mathrm{TateSS}(S^0)$. Positive powers of v lie in the homotopy orbit spectral sequence part of $\mathrm{TateSS}(S^0)$, and the homotopy orbit spectral sequence can be identified with the Atiyah–Hirzebruch spectral sequence of $\mathbb{R}P^\infty$ with homology theory π_*. This spectral sequence is the stabilization of the EHP spectral sequence. The element corresponding to v^{2^k} in the EHP spectral sequence supports differentials related to the vector fields of sphere problems: it supports a differential of length $\rho(2^{k+1})$, with target in the image of J. (See [28, Lectures 20 and 21].) □

Now we discuss the rest of differentials in the range we are concerned with, which are differentials hitting elements in Fig. 9 below slope 1 and in stem 0 to 8. In this range, the ring $\hat{H}^0(C_2; \mathcal{A}_*)$ is presented by the following generators:

$$\begin{aligned} b_1 &\text{ restricts to } \xi_1 \\ N(\xi_1)u_\sigma &\text{ restricts to } \xi_1\zeta_1 \\ N(\xi_2)u_\sigma^3 &\text{ restricts to } \xi_2\zeta_2 \end{aligned}$$

with relations:

$$\begin{aligned} b_1^2 + N(\xi_1)u_\sigma &= 0 \\ b_1^3 &= 0 \\ b_1 N(\xi_2)u_\sigma^3 &= 0 \end{aligned}$$

The following differentials present the only remaining differentials in this range.

Proposition 6.9. *In $P^*_{4/2}\mathscr{D}\mathrm{TateSS}(N_1^2 H\mathbb{F}_2)$, we have differentials:*

$$\begin{aligned} d_3(u_\sigma) &= b_1 u_\sigma^{-1} a_\sigma^2, \\ d_3(b_1) &= N(\xi_1) u_\sigma^{-1} a_\sigma^2, \\ d_9(N(\xi_1)u_\sigma^2) &= N(\xi_2) u_\sigma^{-1} a_\sigma^5. \end{aligned}$$

Proof. For the first differential, since u_σ^2 supports a d_5 by Theorem 6.6, u_σ must support a shorter differential, and the d_3 is the only possibility.

For the second differential, consider the class $N(\xi_1)u_\sigma$, which is the class in $(2, 2)$ in Fig. 9. Its preimage in the homotopy fixed points spectral sequence is the only class in stem 2 that can support a nontrivial restriction. By Theorem 4.4, π_2 of the homotopy

fixed points is $\mathbb{Z}/4$, therefore its generator must support a nontrivial restriction (Proposition 5.6). Thus $N(\xi_1)u_\sigma$ doesn't support a differential in the homotopy fixed points spectral sequence, and it must be killed by a differential in the Tate spectral sequence. The only possible source is $b_1 u_\sigma^2 a_\sigma^{-2}$ and the differential is a d_3. Since u_σ^2 supports a d_5, we obtain $d_3(b_1) = N(\xi_1)u_\sigma^{-1}a_\sigma^2$ by multiplication by $u_\sigma^{-2}a_\sigma^2$.

For the last differential, we only need to show that the class at $(6,6)$ in Fig. 9, which has the name $N(\xi_2)u_\sigma^3$, is a cycle. By the same argument as above, its preimage is the only class in stem 6 that can support a restriction, and this indeed happens by Theorem 4.4. In the Tate spectral sequence, the only possible differential killing it has the form

$$d_9(N(\xi_1)u_\sigma^6 a_\sigma^{-5}) = N(\xi_2)u_\sigma^3.$$

Multiplying both sides by $u_\sigma^{-4}a_\sigma^5$, we obtain the last differential. □

In Fig. 9, the first two differentials and their propagation are colored purple. The last differential is colored blue.

The computation of the Tate spectral sequence is largely limited by the complexity of the Tate cohomology $\hat{H}^*(C_2; \mathcal{A}_*)$. A better understanding of the Tate cohomology shall allow us to compute most differentials in the Tate spectral sequence via comparison to the localized slice spectral sequence, but can also feed back to the computation of the slice spectral sequence of $BP^{((C_4))}$.

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