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QUANTUM LOGICAL STRUCTURES FOR SIMILAR PARTICLES

FEDERICO HOLIK

Instituto de Física La Plata, CCT-CONICET, and
Departamento de Física, Facultad de Ciencias Exactas,
Universidad Nacional de La Plata, La Plata, Argentina.
Centre Leo Apostel, Vrije Universiteit Brussel, Brussels, Belgium.
olentiev2@gmail.com

IGNACIO GOMEZ

Departamento de Matemática, Facultad de Ciencias Exactas y Naturales
Universidad de Buenos Aires, Buenos Aires, Argentina.
Instituto de Física de Rosario (IFIR-CONICET), Rosario, Argentina.
stratoignacio@hotmail.com

DÉCIO KRAUSE

Departmento de Filosofia
Universidade Federal de Santa Catarina, UFSC
Florianópolis, SC, Brazil.
deciokrause@gmail.com

Abstract: In this work we discuss logical structures related to indistinguishable (or similar) particles. Most of the framework used to develop these structures was presented in previous works. We use these structures and constructions to discuss possible ontologies for identical particles. In other words, we use these structures in order to characterize the logical structure of quantum systems for the case of similar particles, and draw possible philosophical implications. We also review some proposals available in the literature which may be considered within the framework of the quantum logical tradition regarding the problem of indistinguishability. Besides these discussions and constructions, we advance novel technical results, namely, a lattice-theoretical structure for identical particles for the finite dimensional case. This approach has not been present in the scarce literature on quantum logic and similar particles.

Keywords: quantum logic, convex sets, indistinguishable particles.

ESTRUTURAS LÓGICO-QUÂNTICAS PARA PARTÍCULAS SEMELHANTES

Resumo: Neste trabalho discutimos estruturas lógicas relacionadas a partículas indistinguíveis (ou semelhantes). A maior parte do quadro teórico usado para desenvolver essas estruturas foi apresentada em trabalhos anteriores. Usamos essas estruturas e construções para discutir possíveis ontologias para partículas idênticas. Em outras palavras, usamos essas estruturas para caracterizar a estrutura lógica de sistemas quânticos para o caso de partículas semelhantes, e traçamos possíveis implicações filosóficas. Também examinamos algumas propostas disponíveis na literatura que podem ser consideradas dentro do quadro da tradição da lógica quântica concernentes ao problema da indistinguibilidade. Além dessas discussões e construções, apresentamos novos resultados técnicos relativos à estrutura da teoria de reticulado para partículas idênticas, no caso de dimensão finita. Esta abordagem não está presente na pequena literatura sobre lógica quântica e partículas semelhantes.

Palavras-chave: lógica quântica, conjuntos convexos, partículas indistinguíveis.

1. Introduction

Among the foundational problems of quantum mechanics (QM), the discussion about identical particles (so denominated for example by VAN FRAASSEN, 1991, chaps. 11 and 12) plays a central role.[1] The case of similar particles is treated separately in almost all introductory books on QM (BALLENTINE, 2000; SAKURAI, 1985) and in many works on physics (SCHRÖDINGER, 1950; MESSIAH & GREENBERG, 1964; GIRARDEAU, 1965; AMICO et al., 2008; DOWLING et al., 2006; GHIRARDI & MARINATTO, 2004; PLASTINO et al., 2009), and philosophical debates have been dedicated to this problem (SCHRÖDINGER, 1952; D'ESPAGNAT, 1976; FRENCH & KRAUSE, 2006; POST, 1963).

From the interpretational point of view, one of the most important open tasks is the right characterization of the word 'similar' (or 'identical', or 'indistinguishable') as used in this context. On the one hand, the axioms of QM, as standardly formulated, are based on classical logic and mathematics: that mathematics that can be built in a fragment of the Zermelo-Fraenkel set

[1] In this work we follow Dirac and call them *similar particles*.

theory with the axiom of choice (ZFC) (MANIN, 1977; HALMOS, 1960; KUNEN, 1980). Thus, it involves – at least at the formal level – the 'classical' theory of identity, which says that there are no indiscernible objects: indiscernible, or indistinguishable, entities must be *the very same object*, and that's all there is to it. But on the other hand, the theory does not seem to provide the means to distinguish (or label) quanta in most cases. On the contrary, it seems to point in the direction that quanta cannot be discerned at all. Thus, at the interpretational level, many authors inclined themselves towards a point of view in which *quanta cannot* be considered individuals (SCHRÖDINGER, 1950, 1952; FRENCH & KRAUSE, 2006, 2010; KRAUSE, 2003, 2010; POST, 1963). This position is usually called *the received view*.

The question of whether elementary particles are indiscernible by 'physical properties' only (but retain individuality), or whether they are absolutely indiscernible, being entities that even in God's mind cannot be discerned, either by physical or logical tools, is not a settled topic (see for example MULLER & SEEVINCK, 2009, and FRENCH & KRAUSE, 2006, for a complete discussion). We will return to this problem in this paper and use logical tools to shed light onto it.

Many authors claim that the alleged indiscernibility concerns only 'physical properties', and usually state that even if quanta cannot be discerned by physical means, they still retain their individuality as some kind of 'primitive thisness'. For advocates of this position, an ontology of individuals is possible, but the theory provides no means to discern the particles by observational methods. Thus, this alternative opens the door for some kind of hidden variables, or parameters (in this case, as a form of "hidden identity"), which would be represented at least in the logical domain of the theory – logic here encompassing mathematics – and which hide themselves when we try to manipulate them experimentally (as happens with "hidden variables" in Bohm's interpretation). No one knows the implications of this assumption, as for instance, whether there is some kind of no-go theorem concerning 'logical' properties. Confusion usually appears in crucial questions, such as testing experimentally whether quanta are individuals or not. In order to settle these questions, the problem must be properly formulated and some questions clarified. Logic may be helpful for this task, and we explore this possibility throughout the paper.

It is important to discuss here the relationship between a given formalism and its possible interpretations. The axiomatization of a physical theory depends heavily on the mathematical formalism used to describe it. And this axiomatization involves different levels of discourse: on the one hand, we have the mathematical formalism used to describe the theory; but we also have rules connecting elements of the mathematical formalism with empirical data. For example, in QM, one says that to the energy observable of the particle corresponds a self-adjoint operator in a Hilbert space.

Interpretation of the mathematical formalism plays a crucial role here and vice versa: the mathematical formalism is used to show things at the interpretational level, as for example, in the derivation of Kochen-Speckers's theorem. Theorems of a physical theory intermingle mathematics and interpretation in a complicated way.

If the received view is accepted, there is in fact a foundational problem of logical nature when the interpretation – which presupposes non-individuality – is contrasted with the standard axiomatic formulation of the theory, which as mentioned above, presupposes the classical theory of identity. In other words, if electrons are assumed to be non-individuals, it is not reasonable to deal with them as if they were collections of distinguishable objects as the ones appearing in ZFC. This situation has given rise to different kinds of criticisms, as the discussion about the "surplus structure" in REDHEAD & TELLER (1991, 1992) or the demand for a more direct formulation (avoiding the symmetrization postulate) in KRAUSE (2003) and POST (1963). In MULLER & SEEVINCK (2009) the contradiction is exploited in order to support the position that quanta can be discerned (at least in a weak form).

In this work, we remark that there is a deeper level in the axiomatization of a given theory: in the end, any mathematical formalism relies on a logical background. For example, the axioms of QM can be formulated using functional analysis, and functional analysis relies on set theory (as for example, ZFC). But set theory relies on first order classical logic. This simple fact implies that the conclusions that we extract out of a given axiomatization depend in part on this logical level.[2] What happens if the logical level is changed? As we show in section 4.2, it is possible to change the logical level of the theory in a way which is compatible with the received view. As far as we

[2] We are considering set theory, in this case ZFC, as part of the logical level.

know, in this axiomatization, particles are not individuals, and cannot be discerned.

Usually, defenders of the received view are quite aware of the surplus character of particle labels, labeled tensor products and all the elements of the theory which depend on the fact that the mathematical formalism is not prepared to deal with non-individual entities. But in MULLER & SEEVINCK (2009) the dependence of the logical dimension of mathematical proofs is forgotten, and used to regain a weak form of discernibility in the quantum formalism. One may wonder whether it is not simpler to use a different formulation of QM, like Bohmian mechanics, and just assert that particles are discernible by their trajectories. Anyway, we think that the position which asserts that elementary particles are weakly discernible is a valid one, because of metaphysical underdetermination. Our point in this work is that there is still another possibility (see section 4.2 of this work) in which particles are indiscernible non-individuals, and that particular emphasis should be put on the logical background of the theory when dealing with these matters.

Another important foundational problem in QM concerns compound systems (of which the indistinguishability problem may be considered as a particular case), in particular, the notion of entanglement (BENGTSSON & ŻYCZKOWSKI, 2006), which was considered by Schrödinger as "*the* characteristic trait of quantum mechanics" (SCHRÖDINGER, 1935, 1936; EINSTEIN, PODOLSKY & ROSEN, 1935). Entanglement has to do with quantum systems after they have interacted; inquiry into entanglement of similar particles has been growing relatively recently. Many questions remain open and, as is well known, correlations originating in entanglement are very different from those originating in statistics (exchange correlations) (GHIRARDI & MARINATTO, 2004; DOWLING et al., 2006; AMICO et al., 2008; PLASTINO et al., 2009). The singular features which appear when aggregates of systems of identical particles are studied give the subject its own particular problems. This has to be taken into account when structural properties of aggregates of quanta are studied.

In this paper, we outline a discussion on certain topics involving indiscernible particles within the scope of different mathematical structures that are motivated by quantum theory. One of them has to do with *quantum logics* (QL) and compound quantum systems of identical particles. The standard quantum logical approach to QM (BIRKHOFF & VON NEUMANN, 1936; JAUCH, 1968; PIRON, 1976;

BELTRAMETTI & CASSINELLI, 1981; DVUREČENSKIJ & PULMANNOVÁ, 2000; DALLA CHIARA et al., 2004; ENGESSER et al., 2009) uses the lattice of projections of the Hilbert space of the system as the lattice of propositions (see section 3). This approach has been useful for the study of structural properties of quantum systems by the characterization of their operational lattices, and for the clarification of differences with other theories (such as classical mechanics). Recently, an alternative proposal has been developed (DOMENECH et al., 2010; HOLIK et al., 2010; HOLIK et al., 2013) in order to solve some problems which appear in the study of compound quantum systems (see for example AERTS, 2000). After reviewing the standard formulation of the formalism of similar particles and posing its problems in section 2, we adapt the constructions presented in DOMENECH et al. (2010), HOLIK et al. (2010), and HOLIK et al. (2013) for the indistinguishable particle case in section 3. This construction is particularly suitable for an extension of the QL approach to the case of similar particles, which is difficult to accommodate in the traditional approach, and has not been explored in the literature (see however GRIGORE, 1993). It provides a new formal framework for the study of compound quantum systems in the indistinguishable case and its entanglement properties, as discussed in section 3.2.

The other formal structure discussed in this work is the theory of quasi-sets (KRAUSE, 2003), which is based on a non-classical logic, namely, a non-reflexive logic, and has to do with the problem of the identity of similar particles.[3] This is done in section 4. In this section we review the main characteristics of quasi-set theory and discuss its implications. We also review how it can be used for an alternative formulation of the Fock-space formalism (DOMENECH et al., 2008; DOMENECH et al., 2009), discussing the implication of such a construction for the interpretation of QM. Next, in section 5, we pose

[3] We call 'non-reflexive' those logics which deviate from classical logic in what respects the theory of identity, in particular, by questioning the principle of identity in some formulation. In our case, we question the principle in the form $\forall x\ (x=x)$ (also called the reflexive law of identity), since we assume that there are entities to which the standard notion of identity (described by the theory governing the symbol '=') does not hold. This assumption is based on some of Schrödinger's opinions – see SCHRÖDINGER (1952) and FRENCH & KRAUSE (2006) for a detailed history and context.

the problem of identical particles in a new form, under the light of the logical structures presented in this work.

Finally, we present our conclusions in section 6, where we try to condense some ontological implications of the discussions posed in sections 3, 4 and 5.

2. The problem of identical particles

We sketch here a brief introduction to the standard formulation of the problem of identical particles. We emphasize the usual mathematical trick that is used to achieve indistinguishability, namely, permutational symmetry. The clarification of that trick opens the door – from the foundational point of view – to the idea that a different mathematical formalism would be in order. We explore the (old) idea that physics may suggest new logical schemas (see for example PUTNAM, 1968).

2.1. States and compound quantum systems

In the standard quantum mechanical formalism, for any system S, a Hilbert space $\mathcal{H}$ is assigned and observables are represented by self-adjoint ~~compact~~ operators. Let $\mathcal{B}(\mathcal{H})$ denote the set of bounded operators on a suitable Hilbert space $\mathcal{H}$, while the set of bounded self-adjoint operators is denoted by $\mathcal{A}$. $\mathcal{B}(\mathcal{H})$ is a well known example of a von Neumann algebra (RÉDEI, 1998).

States are represented (for either single or compound systems) by the set of positive trace class and self-adjoint operators of trace 1,

$$\mathcal{C} = \{\rho \in \mathcal{A} \mid \mathrm{Tr}(\rho) = 1 \text{ and } \rho \geq 0\} , \tag{2.1.1}$$

where the operators ρ are called *density operators*. They represent the more general available states, and for any observable represented by an Hermitian operator A, they assign a real number (which is interpreted as the mean value of the observable) according to the rule

$$\mathrm{Tr}(\rho A) = \langle A \rangle . \tag{2.1.2}$$

If P is a projection operator (i.e., a self-adjoint operator satisfying $P^2 = P$) intended to represent an elementary test or event (BIRKHOFF & VON NEUMANN, 1936; PIRON, 1976), then the real number $\mathrm{Tr}(\rho P)$ is interpreted as the probability of obtaining the event P given that the system is in state ρ. This is nothing but Born's rule.

There is a special subclass of states, called pure states, which are the density operators satisfying $\rho^2 = \rho$. They have a representation as normalized vectors in $|\psi\rangle \in \mathcal{H}$ in the form

$$\rho_{pure} = |\psi\rangle\langle\psi| \quad . \tag{2.1.3}$$

States which are not pure are called *mixed states*. For pure states, we have the superposition principle: any normalized linear combination of states yields a new state. In formulae, if $|\psi\rangle$ and $|\varphi\rangle$ are normalized vectors representing pure states, and if α and β are complex numbers satisfying $|\alpha|^2 + |\beta|^2 = 1$, then

$$|\phi\rangle = \alpha|\psi\rangle + \beta|\varphi\rangle \tag{2.1.4}$$

is also a state.[4]

For a compound quantum system formed by two subsystems represented by Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, we assign a Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, where "$\otimes$" denotes the *tensor product*. Observables are represented by Hermitian operators in $\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$. Let $\{|\alpha_i\rangle\}$ and $\{|\beta_j\rangle\}$ be orthonormal bases of $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. The set $\{|\alpha_i\rangle \otimes |\beta_j\rangle\}$ forms an orthonormal basis for $\mathcal{H}_1 \otimes \mathcal{H}_2$. Then, a pure state of the composite system can be written as $|\psi\rangle = \sum_{i,j} \lambda_{ij} |\alpha_i\rangle \otimes |\beta_j\rangle$. Given a state ρ of the composite system, partial states ρ_1 and ρ_2 can be defined for the subsystems. The relations between ρ, ρ_1 and ρ_2 are given by:

[4] An analogous version of this principle exists for mixed states, replacing *linear combination* by *convex combination*, and pure states by general density matrices (see for example HOLIK et al., 2010).

$$\rho_1 = \mathrm{Tr}_2(\rho) \;, \quad \rho_2 = \mathrm{Tr}_1(\rho) \;, \tag{2.1.5}$$

where Tr_i stands for the partial trace over the ith degrees of freedom. A density matrix of the composite system ρ is said to be a convex combination of product states, if there exists $\{p_i\}$ and states $\{\rho^i{}_1\}$ and $\{\rho^i{}_2\}$ such that

$$\rho \;=\; \sum_{i=1}^{N} p_i \, \rho_1^i \otimes \rho_2^i \;. \tag{2.1.6}$$

If a state of the composite system can be written as a convex combination of product states (or approximated by a sequence of them), then it is said to be *separable*. If not, it is said to be entangled (BENGTSSON & ŻYCZKOWSKI, 2006).

2.2. Similar particles

If particles are identical – in the sense of sharing all their intrinsic properties (for example, a collection in which all particles are electrons) –, we must add the condition that all pure states should be symmetrized. This is the content of the symmetrization postulate (FRENCH & KRAUSE, 2006), and this means that all states must have a definite symmetry with respect to the action of the permutation operator. For example, if $|\varphi\rangle \in \mathcal{H}_1$, $|\phi\rangle \in \mathcal{H}_2$, and $|\varphi\rangle \neq |\phi\rangle$, then the corresponding symmetrized states are

$$|\psi\rangle \;=\; \tfrac{1}{\sqrt{2}}\left(|\varphi\rangle \otimes |\phi\rangle \pm |\phi\rangle \otimes |\varphi\rangle\right) \;, \tag{2.2.1}$$

where the "+" sign stands for bosons and the "–" sign for fermions. For bosons, states of the form $|\varphi\rangle \otimes |\varphi\rangle$ (and all possible normalized linear combinations) are also possible. If $\{|\varphi_i\rangle\}_{i\in I}$ and $\{|\phi_j\rangle\}_{j\in I}$ are bases of $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively, then $\{|\varphi_i\rangle \otimes |\phi_j\rangle\}_{\langle i,j\rangle \in I\times I}$ is a basis of $\mathcal{H}$. Then, a permutation operator

$$\mathcal{P}_{12} : \mathcal{H} \rightarrow \mathcal{H}, \quad \mathcal{P}_{12}|\varphi_i\rangle \otimes |\phi_j\rangle \rightarrow |\phi_j\rangle \otimes |\varphi_i\rangle \tag{2.2.2}$$

can be defined, because it is defined on each element of this basis (and extended linearly in a trivial way). It can be shown that it is independent of the chosen basis. As $\mathcal{P}_{12}{}^2 = 1$, its eigenvalues are 1 and –1 for bosons and fermions, respectively. Then, this operator selects two special subspaces of $\mathcal{H}$ according to its eigenvalues

$$\mathcal{H}^{+} = \{ | \psi\rangle \in \mathcal{H} \mid \mathcal{P}_{12} | \psi\rangle = | \psi\rangle\} \quad , \tag{2.2.3a}$$
$$\mathcal{H}^{-} = \{ | \psi\rangle \in \mathcal{H} \mid \mathcal{P}_{12} | \psi\rangle = - | \psi\rangle\} \quad , \tag{2.2.3b}$$

which represent the possible (pure) states of the system when similar particles are involved. Quantum theory postulates that all physical states of similar particles must obey these symmetry conditions. This is an empirical statement, and up to now, no other symmetries have been found (see GIRARDEAU, 1965, and MESSIAH & GREENBERG, 1964, for a discussion of this statement). So, while the mathematical formulation of the theory opens the door for "para-statistics", it seems that none of them where found to have correspondents in nature, and we will not treat this case here.

2.3. How does the symmetrization postulate work and its open problems

It is important to make here the crucial observation of how the scheme of the symmetrization postulate works. First, particles are *labeled* by assigning them normalized vectors $| \varphi \rangle$ and $| \phi \rangle$ in their corresponding spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. If the state of the compound system were simply $| \varphi \rangle \otimes | \phi \rangle$, then, particles could be distinguished by special observables, i.e., the theory would allow for an asymmetry between both systems. But things are not so, and the symmetrization postulate must erase any observable characteristic which allows us to individuate any of the particles. Thus, the state must be symmetrized as in 2.2.1. This is how the symmetrization postulate works: by first attaching a label in order to *individuate* each particle, and then making it physically unobservable by imposing a postulate.[5] This 'mathematical' individuation might be in conflict with the standard interpretation of QM, for which physical individuation is not

[5] This form of individuation may be termed 'mathematical'.

possible. It is not difficult to realize that this trick is unavoidable in the standard formulation of QM, given that its axiomatics is formulated using standard (Zermelo-Fraenkel) mathematics, based on the classical theory of identity. Thus, the symmetrization postulate *hides* particle identities, leaving the door open to an interpretation based on non-individuals. This is one of the reasons why many authors support the received view.

But the received view has been criticized in many ways. One of them has to do with a contradiction between the method of introducing the adequate symmetries and the interpretation itself. On the one hand, Schrödinger and others tell us that we must give up any intent to provide individuality for elementary particles, and this position has a considerable agreement with experience. But it is also true that particles are labeled in the *symmetrization mechanism*, and thus, they seem to posses some form of individuality. How to reconcile these views?

The problem was discussed and a possible solution was proposed in REDHEAD & TELLER (1991, 1992). The authors characterized the non-symmetrical parts of the Hilbert space, discarded by the symmetrization postulate, as *surplus structure*, i.e., a mathematical structure which plays no role in the final formulation of the theory and its predicted experience. Next, they give arguments in favor of the Fock-space formulation of QM. As is well known, it is possible to use the Fock-space formalism as an alternative approach to QM (ROBERTSON, 1973). The criticism raised against this solution reads that the Fock-space mechanism also appeals to particle labeling (FRENCH & KRAUSE, 2006), and so, it has a similar problem as that of the usual symmetrization postulate formulation.

From a different point of view, MULLER & SAUNDERS (2008) argue that fermions can be discerned in a weak form: they are weakly discernible because they have the same state (represented by the same reduced density operator), but they satisfy some symmetric and irreflexive relation. In a later work, MULLER & SEEVINCK (2009) show that elementary particles in general can be discerned. In section 4 we will see that this reduces to the fact that QM is axiomatized using ZF mathematics, and we argue that the conclusion drawn by the authors is premature, and the problem and its alternative solutions are not properly posed.

There is still a possibility for a reconciliation between the received view and a valid reformulation of QM, but one which encompasses a radical change in the logical framework of the theory. This is done in sections 4 and 6, after clarifying the problem in a clear logical and ontological form.

One question is still at stack. Is the received view really desirable? Or unavoidable? The assumption of the individuality of quanta seems to play no role in any experience or, at least, it can be removed and experiments can be successfully explained. On the contrary, if the assumption of individuality is not taken with special care and protected by suitable hypotheses, it may lead to wrong results. Then, in spite of this, why not still postulate a form of "hidden individuality", playing no role but satisfying a particular metaphysical taste? This can be done, and we discuss the consequences of this assumption when things are properly formulated, in the following sections.

3. A quantum logical formalism for identical particles

In this section, we introduce the lattice of convex subsets formalism presented in DOMENECH et al. (2010), HOLIK et al. (2010), and HOLIK et al. (2013). Next, we apply it to the case of identical particles using the preliminaries introduced above. The traditional approach to quantum logic uses the bounded projection operators (or equivalently, closed subspaces) of the Hilbert space as propositions or properties, using the direct sum as the disjunction, the intersection as the conjunction, inclusion as the order relation, and orthogonal complement as negation (BIRKHOFF & VON NEUMANN, 1936; DALLA CHIARA et al., 2004; ENGESSER et al., 2009). Contrary to that, we use a lattice formed by convex subsets of the state space. This lattice is more suitable in order to define maps that relate states of a compound system to states of the subsystems, and allows for the introduction of mixed states (something almost unavoidable when we have for example, bipartite systems of identical particles) (HOLIK et al., 2010; HOLIK et al., 2013). It is important to remark that no similar construction can be made with the usual projection lattice for the identical particles case.

3.1 The lattice of convex subsets

According to the orthodox quantum-logical approach, the propositions of classical mechanics are the *subsets* of the *set* of states (classical phase space) (DALLA CHIARA et al., 2004). In HOLIK et al. (2010) and HOLIK et al. (2013) the *convex* subsets of the *convex* set of states are considered. Convexity is a key feature of QM (see for example MIELNIK, 1968, 1969, 1974, for an axiomatization based on convex sets). Let us begin by considering the set of all convex subsets of $\mathcal{C}$ (defined in eq. 2.1.1). We restrict the discussion to finite dimensional Hilbert spaces.

Definition 3.1. $\mathcal{L}_{\mathcal{C}} := \{C \subseteq \mathcal{C} \mid C \text{ is a convex subset of } \mathcal{C}\}$.

In order to give $\mathcal{L}_{\mathcal{C}}$ a lattice structure, we introduce the following operations:

Definition 3.2. For all $C, C_1, C_2 \in \mathcal{L}_{\mathcal{C}}$:

($\wedge$) $C_1 \wedge C_2 := C_1 \cap C_2$.

($\vee$) $C_1 \vee C_2 :=$ *conv* (C_1,C_2). It is again a convex set, and it is included in $\mathcal{C}$ (using convexity).

($\neg$) $\neg\, C := C^{\perp} \cap \mathcal{C}$.

($\rightarrow$) $C_1 \rightarrow C_2 := C_1 \subseteq C_2$.

With the operations of definition 3.2, it is apparent that $(\mathcal{L}_{\mathcal{C}}; \rightarrow)$ is a poset. If we set $\varnothing = \mathbf{0}$ and $\mathcal{C} = \mathbf{1}$, then $(\mathcal{L}_{\mathcal{C}}; \rightarrow; \mathbf{0},\mathbf{1})$ will be a bounded partially ordered set (*poset*). With the operations defined in 3.5, $\mathcal{L}_{\mathcal{C}}$ will be a bounded, atomic and complete lattice.

It is possible to define maps which connect states of the compound system with states of the subsystems as follows (HOLIK et al., 2010). Given $C_1 \subseteq \mathcal{C}_1$ and $C_2 \subseteq \mathcal{C}_2$, define

$$C_1 \otimes C_2 := \{\rho_1 \otimes \rho_2 \mid \rho_1 \in C_1, \rho_2 \in C_2\} . \tag{3.1.1}$$

Using the above definition, it is possible to define the map

$$\Lambda : \mathcal{L}_{C_1} \times \mathcal{L}_{C_2} \rightarrow \mathcal{L}_C , \quad (C_1 , C_2) \mapsto conv\,(C_1 \otimes C_2) \ . \qquad (3.1.2)$$

If we use partial traces:

$$\mathrm{tr}_i : C \rightarrow C_j , \quad \rho \mapsto \mathrm{tr}_i (\rho) , \qquad (3.1.3)$$

we can also construct the maps

$$\tau_i : \mathcal{L}_C \rightarrow \mathcal{L}_{C_i} , \quad C \mapsto \mathrm{tr}_i (C) , \qquad (3.1.4)$$

which link the elements of $\mathcal{L}_C$ (compound system) to the elements of $\mathcal{L}_{C_1}$ and $\mathcal{L}_{C_2}$ (subsystems).

Next, define the product map

$$\tau : \mathcal{L}_C \rightarrow \mathcal{L}_{C_1} \times \mathcal{L}_{C_2} , \quad C \mapsto (\tau_1(C), (\tau_2(C)) \ . \qquad (3.1.5)$$

In HOLIK et al. (2010) and HOLIK et al. (2013) it is shown that using Λ and τ it is possible to link states of the compound system to the states of its subsystems (at the lattice level). This cannot be done in the standard QL formalism. In the following we will take this feature of $\mathcal{L}_C$ as an advantage for constructing a lattice for the case of identical particles; the use of mixtures then becomes unavoidable.

3.2. The identical particle lattice $\mathcal{L}_{C\pm}$ of convex subsets (bipartite case)

Let us now define a lattice for the identical particles case. For the sake of simplicity we restrict ourselves to the finite-dimensional bipartite case. We begin by building the lattice of convex subsets for symmetrized Hilbert spaces. Taking into account the principle of indistinguishability, let

Definition 3.3. $\mathcal{C}^{\pm} := \{\rho : \mathcal{H}^{\pm} \to \mathcal{H}^{\pm} \mid \mathrm{tr}(\rho)=1, \rho^{\dagger} = \rho \text{ and } \rho \geq 0\}$,
and define (in analogy with $\mathcal{L}_{\mathcal{C}}$):

Definition 3.4. $\mathcal{L}_{\mathcal{C}^{\pm}} := \{C \subseteq \mathcal{C}^{\pm} \mid C \text{ is a convex subset of } \mathcal{C}^{\pm}\}$.

$\mathcal{C}^{\pm}$ can be considered a convex subset of $\mathcal{C}$, namely $\mathcal{C}^{\pm} \in \mathcal{L}_{\mathcal{C}}$. This is because any matrix in $\mathcal{C}^{\pm}$ can be canonically extended to the whole Hilbert space. In order to provide $\mathcal{L}_{\mathcal{C}^{\pm}}$ with a lattice structure similar to that of $\mathcal{L}_{\mathcal{C}}$, we define the following operations:

Definition 3.5. For all $C, C_1, C_2 \in \mathcal{L}_{\mathcal{C}^{\pm}}$:

$(\wedge^{\pm})$ $\quad C_1 \wedge^{\pm} C_2 := C_1 \cap C_2$.

$(\vee^{\pm})$ $\quad C_1 \vee^{\pm} C_2 := \mathit{conv}\,(C_1, C_2)$.

$(\neg^{\pm})$ $\quad \neg^{\pm} C := C^{\perp} \cap \mathcal{C}^{\pm}$.

$(\rightarrow^{\pm})$ $\quad C_1 \rightarrow^{\pm} C_2 := C_1 \subseteq C_2$.

$C^{\perp}$ is the orthogonal complement of C with respect to the scalar product $\langle A,B\rangle = \mathrm{tr}(A \cdot B^{\dagger})$, namely $C^{\perp} = \{\rho \in \mathbf{C}^{N\times N} \mid \mathrm{tr}(\sigma \cdot \rho^{\dagger}) = 0, \forall \sigma \in C\}$. With these operations, it follows that $(\mathcal{L}_{\mathcal{C}^{\pm}} ; \rightarrow^{\pm})$ is a partially ordered set. And if we take $\varnothing = \mathbf{0}$ and $\mathbf{1} = \mathcal{C}^{\pm}$, then $(\mathcal{L}_{\mathcal{C}^{\pm}} ; \rightarrow^{\pm}; \mathbf{0}; \mathbf{1})$ is a bounded partially ordered set. We also notice that, because $\mathcal{C}^{\pm} \in \mathcal{L}_{\mathcal{C}^{\pm}}$, and since the operations of $\mathcal{C}^{\pm}$ are inherited from $\mathcal{C}$, then $\mathcal{L}_{\mathcal{C}^{\pm}}$ is a sublattice of $\mathcal{L}_{\mathcal{C}}$.

Let $\mathrm{tr}_i(\mathcal{L}_{\mathcal{C}^{\pm}}) := \{\mathrm{tr}_i(C) \mid C \in \mathcal{L}_{\mathcal{C}^{\pm}}\} \subseteq \mathcal{L}_{\mathcal{C}_j}$ $(i \neq j)$. It is possible to define the canonical projections τ_i and τ for the identical particles case as follows:

$$\tau_i^{\pm} : \mathcal{L}_{\mathcal{C}} \to \mathrm{tr}_j(\mathcal{L}_{\mathcal{C}^{\pm}}) \ , \quad C \mapsto \mathrm{tr}_j(C) \ . \tag{3.2.1}$$

and the product map

$$\tau^{\pm} : \mathcal{L}_{C\pm} \longrightarrow \mathrm{tr}_2(\mathcal{L}_{C\pm}) \times \mathrm{tr}_1(\mathcal{L}_{C\pm}) \ , \quad C \mapsto (\tau_1^{\pm}(C), (\tau_2^{\pm}(C)) \ . \tag{3.2.2}$$

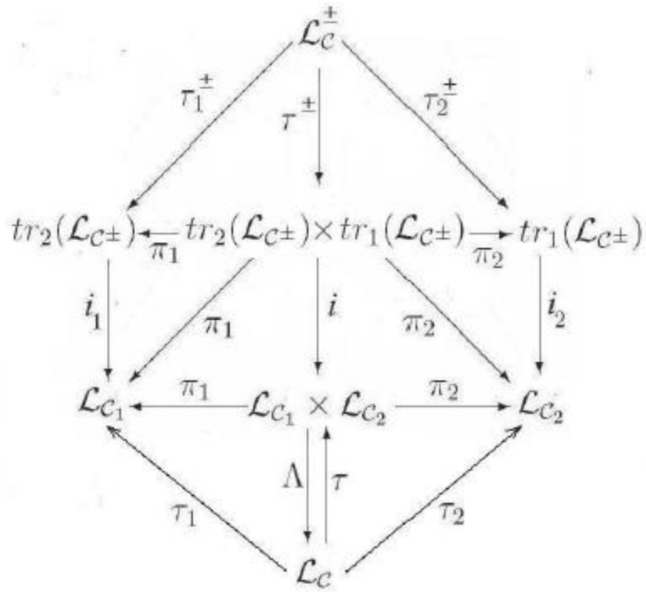

Figure 1. Canonical maps between the lattices $\mathcal{L}_C$, $\mathcal{L}_{C_1}$, $\mathcal{L}_{C_2}$, $\mathcal{L}_{C\pm}$. The arrows i and π represent inclusions and canonical projections.

In other words, the maps $\tau_1^{\pm} = \tau_i|_{\mathcal{L}_{C\pm}}$ and $\tau^{\pm} = \tau|_{\mathcal{L}_{C\pm}}$ are the restrictions of τ_i and τ respectively to $\mathcal{L}_{C\pm}$. Given that the subsystems are identical, it follows that $\tau_1^{\pm} = \tau_2^{\pm}$. The map $\tau^{\pm}$ is defined in analogy with (3.1.5), and it allows one to link states of the compound system to states of its subsystems.

It would be of great interest to find an extension $\Lambda^{\pm}$ of the canonic map Λ. However, such an extension for the identical particle case is not immediate, because the elements of the set $\mathrm{tr}_2(C_1) \otimes \mathrm{tr}_1(C_2)$ will not be symmetrized states in the general case. The connection between the lattices $\mathcal{L}_{C_1}$, $\mathcal{L}_{C_2}$ and $\mathcal{L}_C$ with the lattice of the identical particles $\mathcal{L}_{C\pm}$ is shown in Fig. 1.

4. Quasi-set theory

In this section, we basically follow the exposition of KRAUSE & ARENHART (2009); for details, see FRENCH & KRAUSE (2006, ch. 7; 2010). Quasi-set theory is a mathematical theory that enables us to deal with collections (quasi-sets) of objects that may be indiscernible without being identical (being the same object). The only way of dealing with objects of this kind in a standard theory is to allow the introduction of some *ad hoc* devices, such as the restriction of the lexicon of properties, that is, by taking a language with a finite number of them. This is Quine's famous way of defining identity, namely, by the exhaustion of the chosen (finitely many) predicates (QUINE, 1952, ch. 12). But this just defines indiscernibility relative to the language's predicates only, and not identity strictly speaking, for there may exist other predicates not in the language which could distinguish the entities.

In ZFC (the same can be said of most theories with due qualification) or in some conservative extension of it, let a be a particular set. The postulates of ZFC enable us to form the unitary set $A = \{a\}$ and define a unary "property" (a formula with just one free variable) *to be identical with a*, by posing $I_a(x)$ iff $x \in A$ (alternatively, iff $x=a$). This formula of course distinguishes (or discriminates) a from any *other* object, for only a satisfies it. Saying in other words, in the standard mathematics (in the sense mentioned in section 1), whenever we have a set with cardinal $\alpha > 1$, its elements *are* distinct. In short, if electrons are modeled as elements of ZFC, i.e., if it is assumed that a collection of electrons can be described as an element of ZFC, then, electrons are not indiscernible, except with respect to a few chosen predicates.

Quine says that objects may be *strongly discriminable* when for every object there is a formula with only one free variable that is satisfied only by the object. Two objects are *moderately discriminable* when there exists a formula with two free variables that is satisfied by the two objects in one order but not in the reverse order (QUINE, 1982, ch. 15). As he recalls, any two real numbers are strongly discriminable, although they may be 'specified', that is, definable by a formula.

But in the interpretation that supports the received view, there is no way to attribute an identity to quanta (let us call this a 'which is which criterion'), something that isolates one of them from the others in such a way that *this* chosen object remains with its identity forever. In this way, elementary particles are considered as not individuals in a deep ontological sense. This assumption of the received view implies that quanta do not obey the formal properties of identity at the ontological level. Of course, one may licitly assume the received view and still *represent*[6] quanta as elements of ZFC; but under this assumption, the *formal* properties of identity of ZFC do not represent the *ontological* properties of quanta: there is no harmony between formalism and ontology. And as is well known, the ontological assumption that quanta are utterly indiscernible has influenced the prediction of very concrete physical phenomena, like Bose-Einstein condensates and the statistical properties of quanta (Bose-Einstein, BE, and Fermi-Dirac, FD, statistics).[7] In this way, the ontological properties of quanta related to utter indiscernibility, show themselves as deeply connected with observable physical properties; the empirical predictions are influenced (or guided) by these kinds of assumptions, and the *explanation* of phenomena depends heavily on them.

Thus, how can we deal with the two electrons of a helium atom in the fundamental state, according to the perspective of the received view? We know that they have all the same properties but distinct spins in a given direction, and so they obey an irreflexive and symmetric relation: "to have different spin of". Some authors claim that they have found objects (the mentioned electrons) that are weakly discernible[8] (MULLER & SAUNDERS, 2008) and thus, not utterly indiscernible (see also MULLER & SEEVINCK, 2009, and MULLER, 2015). But this way of approaching the problem of quantum indiscernibility is not the only one (KRAUSE, 2010) and has been criticized. For example, in CAULTON (2013) it is argued that the properties used to discern (weakly) are unphysical. Similarly, DIEKS (2010, p. 29) claims that:

> All evidence points into the same direction: "identical quantum particles" behave like money units in a bank account rather than like Blackean spheres. It does not matter what external standards we introduce, they will always possess the same relations to all (hypothetically present)

[6] We could also use the word "*describe*" or "*model*" here.

[7] The fact that these phenomena can be also explained using other interpretations of QM, is irrelevant for the discussion presented here.

[8] One should not confuse 'weak discernibility' with the Quinean notion of 'moderate discernibility'.

> entities. The irreflexive relations used by Saunders and others to argue that identical quantum particles are weakly discernible individuals *lack the physical significance required to make them suitable for the job*.

The solution proposed in CAULTON (2013) is not entirely satisfactory, because particles are weakly discerned by using an observable based on their (squared) relative positions in space. And in this way, one may argue that what has been achieved is weak discernibility of relative space-time positions (because as is well known, it is problematic to attribute positions to quantum particles before measurement). With regard to weak discernibility, BIGAJ (2015, p. 118) points out that:

> One sense of discerning involves recognizing some qualitative differences (whether in the form of different properties or different relations) between the objects considered. When we discern objects in this sense, we should (at least in principle) be able to pick out one of them but not the other. Being able to discern objects in that way seems to be a prerequisite for making successful reference, or giving a unique name, to each individual object. But by discerning we can also mean recognizing objects as numerically distinct. In this sense of the word, discernment is a process by which, using some qualitative features of the objects, we make sure that there are indeed two entities and not one.

Furthermore, DIEKS (2010, p. 29) asserts that

> The analogy between quantum mechanical systems of "identical particles" and classical collections of weakly discernible objects is only superficial. There is no sign within standard quantum mechanics that "identical particles" are things at all: there is no ground for the supposition that relations between the indices in the formalism possess physical significance in the sense that they connect actual objects. Consequently, the irreflexivity of these relations is not important either. Conventional wisdom appears to have it right after all.

Thus, we see that many criticisms were raised against the weak discernibility approach in QM. Indeed, different criticisms against weak discernibility can be found in LADYMAN & BIGAJ (2010), DIEKS (2010), BIGAJ (2015), HAWLEY (2006, 2009), and VAN FRAASSEN & PESCHARD (2008). The validity of the principle of identity of indiscernibles (PII) was questioned in ARENHART (2013a), BUTTERFIELD (1993), and FRENCH & REDHEAD (1988), and more discussion about the plausibility of non-individuals can be found in ARENHART (2013b) and ARENHART & KRAUSE (2014). *In this way, we see that there exist interesting attempts to overthrow the received view. But this interpretation is still strong, because the criticisms raised against it up to now are not conclusive*. In this work, we concentrate in the study of formalisms which are in harmony with this interpretation.[9]

If electrons are formally represented using classical logic,[10] being objects (in some sense of the word) in a finite number, they can always be named, say a and b, and the above property 'to belong to $A = \{a\}$', which is true just for a, distinguishes them absolutely! In Quine's words, they can be always specified, that is, in the language (of ZFC) there is always a formula in one free variable that is uniquely satisfied by a given object (a set) (QUINE, 1982, p. 134). Of course, it is possible to argue that the property $I_a(x)$ ('being identical with a') is not a "legitimate property"of x. Or more specifically, that it is not a "legitimate *physical* property" of x. But a lot of care is to be taken here. According to the received view, quanta are not individuals and may be also indiscernible entities. In this case, $I_a(x)$ *is not* a legitimate property in an ontological sense, and this assumption has important *physical* consequences (as for example, the well known implications of the symmetrization *postulate*). The objects of the assumed ontology and their properties should not be confused with the *representation* of these objects in a mathematical formalism: $I_a(x)$ will be (or not) a legitimate property depending on the assumed ontology. And the formal properties of the mathematical language used to describe the objects may be either in agreement with their ontological properties or it is not. Many possibilities are at stake here, and the conclusions depend drastically in the assumptions involved.

Quasi-set theory offers a different alternative to deal with these questions: with this theory, the ontology of the received view can be put in

[9] Notice that this does not implies that we deny or reject other possible interpretations.

[10] The mentioned authors assume that they are using ZFC and their conclusions and definitions depend heavily on this assumption. These conclusions could be very different if quanta were represented using a different mathematical formalism or different ontological properties were assumed.

harmony with the formalism used to describe it. Although electrons do present a difference due to Pauli's exclusion principle (for instance, differences in their spins), any permutation of them does not change the relevant probabilities, as is well known; in other words, there is no 'which is which criterion' for electrons in certain situations, say for the electrons of a helium atom in its fundamental state. So, it seems that it is perfectly possible to say that the electrons may be indiscernible without collapsing them in being the very same object. A paradigmatic example may be that of atoms (or molecules) in a Bose-Einstein condensate. Thus, we need to avoid that indiscernibility implies identity (in the philosophical sense of being the same entity). To cope with this idea, we "separate" the concepts: indiscernibility, or indistinguishability, is a relation that holds (being true or not) for all objects of our domain, but identity does not. Identity simply does not apply for certain objects. Thus, certain objects may be indiscernible without turning out to be the same object, as implied by the standard theory of identity, and they may form collections with cardinals greater than one (this is achieved by the postulates of the theory), but in a way that they cannot be identified, named, labeled, counted in the standard way.[11] There is no space here to give the details of the theory, so we suggest FRENCH & KRAUSE (2006, 2010) for detailed references and for the axioms. Anyhow, in the next subsection we outline, without the details, the main ideas of the theory.

4.1. Basic ideas of the theory $\mathcal{Q}$

Intuitively speaking, a quasi-set is a collection of objects such that some of them may be indistinguishable without being identical.

As mentioned above, quasi-set theory $\mathcal{Q}$ derives its main motivations from some insights advanced by SCHRÖDINGER (1952, p. 17-18), for whom the concept of identity makes no sense when applied to elementary particles. Another motivation is (in our opinion) the need, stemming from philosophical worries, of dealing with collections of absolutely indiscernible but not identical

[11] We mean: a series of, say, five objects can be counted by proving that the set having them as elements (and no other element) is equinumerous to a finite ordinal, in the case, the ordinal $5 = \{0, \ldots, 4\}$. We remark that in order to define the bijection, we need to distinguish among the elements being counted – they need to be *individuals*.

items. Spatio-temporal differences could be used in this case, you may say, and this serves do distinguish them. Without discussing the role of spatio-temporal properties here (but see FRENCH & KRAUSE, 2006), we can argue that these objects are invariant by permutations; in other words, the world does not present differences after substituting one of them for the other. Thus, it would be difficult to say that they have some form of identity, for they (in principle) lack any identifying characteristic. Objects of this kind act like those that obey Bose-Einstein statistics, that is, bosons (we should remember that the indiscernibility hypothesis was essential in the derivation of Planck's formula – see FRENCH & KRAUSE, 2006).

Thus, the first point is to guarantee that identity and indistinguishability (or indiscernibility) will not collapse into one another when the theory is formally developed. We assume that identity (symbolized by '=') is not a primitive relation, but we use a weaker primitive concept of indistinguishability (symbolized by '≡') instead. This is just an equivalence relation and holds among all objects of the considered domain. If the objects of the theory are divided up into groups, namely, the *m*-objects (standing for 'micro-objects') and *M*-objects (for 'macro-objects') – these are ur-elements – and quasi-sets of them (probably having other quasi-sets as elements as well), then identity (having all the properties of standard identity of ZF) can be defined for *M*-objects and quasi-sets having no *m*-objects in their transitive closure (this concept is like the standard one). Thus, if we take just the part of the theory obtained by ruling out the *m*-objects and collections (quasi-sets) having them in their transitive closure, we get a copy of ZFU (ZF with *Urelemente*); if we further eliminate the *M*-objects, we get a copy of the 'pure' ZFC. It is important to remark that all observations made above concerning ZFC can be applied to ZFU as well.

Technically, expressions such as $x=y$ are not always well formed, for they are not formulas when either x or y denote *m*-objects (entities satisfying the unary primitive predicate *m*). We express that by informally saying that the concept of identity *does not always make sense* for all objects (it should be emphasized that this is just a way of speaking). The objects (the *m*-objects) to which the defined concept of identity does not apply are termed *non-individuals* for historical reasons (see FRENCH & KRAUSE, 2006). As a result (from the axioms of the theory), we can form collections of *m*-objects which have no identity; these collections may have a cardinal (termed its 'quasi-cardinal') but not an associated ordinal. Thus, the concept of ordinal and of cardinal are taken as independent, as in some formulations of ZFC proper. So, informally speaking, a quasi-set of *m*-objects may be such that its elements cannot be identified by names, counted, ordered, although there is a sense in saying that these collections have a cardinal (that cannot be defined by means of ordinals, as usual, which presupposes identity).

When $\mathcal{Q}$ is used in connection with quantum physics, the *m*-objects are thought of as representing quanta (henceforth, *q-objects*), but they are not necessarily 'particles' in the standard sense (associated with classical physics or even with orthodox QM), but may be waves, field excitations (the 'particles' in quantum field theory – QFT), perhaps even strings or whatever entities supposed indiscernible that can be taken as possible interpretations of the *m*-objects. Generally speaking, any 'objects' sharing the property of being indistinguishable can also be values of the variables of $\mathcal{Q}$ (see FALKENBURG, 2007, ch. 6, for a survey on the various different meanings that the word 'particle' has acquired in connection to quantum physics).

Another important feature of $\mathcal{Q}$ is that standard mathematics can be developed using its resources, for the theory is conceived in such a way that ZFU (and hence also ZFC) is a subtheory of $\mathcal{Q}$. In other words, the theory is constructed so that it extends standard Zermelo-Fraenkel with *Urelemente* (ZFU); thus standard sets (of ZFU) can be viewed as particular qsets (that is, there are qsets that have all the properties of the sets of ZFU, while the objects in $\mathcal{Q}$ corresponding to the *Urelemente* of ZFU are termed *M*-atoms; these satisfy the primitive unary predicate *M*). The 'sets' in $\mathcal{Q}$ will be called $\mathcal{Q}$-sets, or just *sets* for short. An object is a qset when it is neither an *m*-object nor an *M*-object. In order to make the distinction, the language of $\mathcal{Q}$ encompasses a third unary predicate *Z*, such that *Z(x)* says that *x* is a qset which is a copy of a *set* of ZFU (these objects obeying *Z* correspond to the objects of ZFU). It is also possible to show that there is a translation from the language of ZFU into the language of $\mathcal{Q}$, so that the translations of the postulates of ZFU turn to be theorems of $\mathcal{Q}$; thus, there is a 'copy' of ZFU in $\mathcal{Q}$, and we refer to it as the 'classical' part of $\mathcal{Q}$. In this copy, all the usual mathematical concepts can be stated, as for instance, the concept of ordinal (for sets).

$\mathcal{Q}$-sets are qsets whose transitive closure (defined as usual) does not contain *m*-atoms (in other words, they are 'constructed' in the classical part of the theory – see Fig. 2).

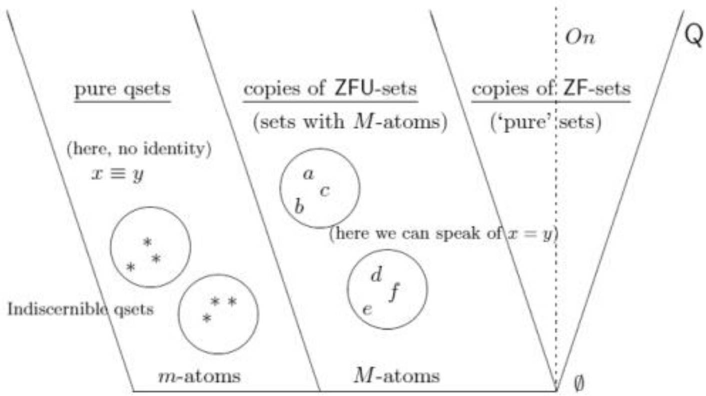

Figure 2. The quasi-set universe Q. *On* is the class of ordinals, defined in the classical part of the theory. See KRAUSE & ARENHART (2009).

In order to distinguish between $\mathcal{Q}$-sets and qsets that may have *m*-atoms in their transitive closure, we write (in the metalanguage) $\{x : \varphi(x)\}$ for $\mathcal{Q}$-sets and $[x : \varphi(x)]$ for qsets. In $\mathcal{Q}$, we term 'pure' those qsets that have only *m*-objects as elements (although these elements may not always be indistinguishable from one another, that is, the theory is consistent with the assumption of the existence of different kinds of *m*-atoms), and for them it is assumed that the usual notion of identity cannot be applied (that is, let us recall, $x=y$, as well as its negation, $x \neq y$, are not well formed formulas if either x or y stand for *m*-objects). Notwithstanding, the primitive relation $\equiv$ applies to them, and it has the properties of an equivalence relation.

But, in order to restore the notion of identity for the 'classical' objects, the theory has a defined concept of *extensional identity* and it has all the properties of standard identity of ZFU. More precisely, we write $x =_E y$ (read 'x and y are extensionally identical') iff they are both qsets having the same elements (that is, $\forall z\, (z \in x \leftrightarrow z \in y)$) or they are both M-atoms and belong to the same qsets (that is, $\forall z\, (x \in z \leftrightarrow y \in z)$). From now on, we shall not bother to always write $=_E$, using simply the symbol "=" for the extensional equality.

Since *m*-atoms cannot be identified in the formalism, it is not possible in general to attribute an ordinal to qsets of such elements. Thus, for certain qsets, it is not possible to define a notion of cardinal number by means of ordinals. The theory uses a primitive concept of *quasi-cardinal* instead, which intuitively stands for the 'quantity' of objects in a collection.[12] The theory has still an 'axiom of weak extensionality', which states (informally speaking) that those quasi-sets that have the same quantity of elements of the same sort (in the sense that they belong to the same equivalence class of indistinguishable objects) are indistinguishable by their own. One of the interesting consequences of this axiom is related to the unobservability of permutations in quantum physics, which is one of the most basic facts regarding indistinguishable quanta (for a discussion on this point, see FRENCH & RICKLES, 2003). In standard set theories, if $w \in x$, then of course $(x \setminus \{w\}) \cup \{z\} = x$ iff $z = w$. That is, we can 'exchange' (without modifying the original arrangement) two elements iff they are *the same* elements, by force of the axiom of extensionality. In Q we can prove the following theorem, where $[[z]]$ (and similarly $[[w]]$) stand for a quasi-set with quasi-cardinal 1 whose only element is indistinguishable from z (respectively, from w – the reader shouldn't think that this element *is identical to either* z or w):

Theorem 1 (Unobservability of Permutations). Let x be a finite quasi-set such that x does not contain all indistinguishable from z, where z is an *m*-atom such that $z \in x$. If $w \equiv z$ and $w \notin x$, then there exists $[[w]]$ such that $(x - [[z]]) \cup [[w]] \equiv x$.

Informally speaking, supposing that x has n elements, then if we 'exchange' their elements z by corresponding similar elements w (set theoretically, this means performing the operation $(x - [[z]]) \cup [[w]]$, then the resulting quasi-set remains *indistinguishable* from the one we started with). In a certain sense, it does not matter whether we are dealing with x or with $(x - [[z]]) \cup [[w]]$. So, within Q, we can express that 'permutations are not observable', without necessarily introducing symmetry postulates, and in particular derive 'in a natural way' quantum statistics (see FRENCH & KRAUSE, 2006, ch. 7).

[12] But a notion of finite quasi-cardinal can be defined for finite qsets (see DOMENECH & HOLIK, 2007).

4.2 The $\mathcal{Q}$-space

As we have seen in section 2, if particles are similar, the state of their composite system can only be a symmetrized one. But particles are labeled in this construction, and this procedure was criticized (section 2.3). It has been claimed that the Fock-space formalism poses a solution to the questions raised by this criticism (REDHEAD & TELLER, 1991, 1992). But the Fock-space formalism also makes use of particle labeling in order to obtain the correct states (FRENCH & KRAUSE, 2006).

How can we avoid this problem of the Fock-space formulation of QM? If we could avoid the individuation of the particles at every step of the construction of a Fock-like formulation of QM, we would give a positive answer to the problem posed in REDHEAD & TELLER (1991, 1992) (recalled in the Introduction) that is not affected by the criticisms linked to it. Quasi-set theory can be used for this purpose, and in fact, this construction has been done (DOMENECH et al., 2008; DOMENECH et al., 2009). There, an alternative proposal is presented that resembles the Fock-space formalism but is based on $\mathcal{Q}$. This avoids artificial labeling. We give a sketch of the construction here, mainly following DOMENECH et al. (2008) and DOMENECH et al. (2009). In the following section, we also see that this kind of construction not only allows us to solve the problem posed above, they also give rise to interesting foundational issues.

Let us consider a set $\varepsilon = \{\varepsilon_i\}_{i \in I}$ (that is, a "set" in $\mathcal{Q}$), where I is an arbitrary collection of indices (this makes sense in the 'classical part' of $\mathcal{Q}$). Suppose that the elements ε_i represent the eigenvalues of a physical observable. Next, quasi-functions f are constructed, such that $f : \varepsilon \to \mathcal{F}_p$, where $\mathcal{F}_p$ is the quasi-set formed of finite and pure quasi-sets. f is a quasi-set formed by ordered pairs $\langle \varepsilon_i ; x \rangle$ with $\varepsilon_i \in \varepsilon$ and $x \in \mathcal{F}_p$.

These quasi-functions are chosen in such a way that whenever $\langle \varepsilon_{i(k)} ; x \rangle$ and $\langle \varepsilon_{i(k')} ; y \rangle$ belong to f and $k \neq k'$, then $x \cap y = \emptyset$. It is further assumed that the sum of the quasi-cardinals of the quasi-sets which appear in the image of each of these quasi-functions is finite. This means that $qc(x) = 0$ for every x in the image of f, except for a finite number of elements of ε. These quasi-functions form a quasi-set called $\mathcal{F}$.

A pair $\langle \varepsilon_i ; x \rangle$ is interpreted as the statement "the energy level ε_i has occupation number $qc(x)$". Quasi-functions of this kind are represented by expressions such as $f_{\varepsilon_{i(1)}\varepsilon_{i(2)}\ldots\varepsilon_{i(m)}}$. If the symbol $\varepsilon_{i(k)}$ appears j-times, the level $\varepsilon_{i(k)}$ has occupation number j. The levels that do not appear have occupation number zero.

At this point of the construction, the indexes appearing in $f_{\varepsilon_{i(1)}\varepsilon_{i(2)}\ldots\varepsilon_{i(m)}}$ have no preferred order at all. But an order can be defined as follows. Given a quasi-function $f \in \mathcal{F}$, let $\{\varepsilon_{i(1)}\varepsilon_{i(2)}\ldots\varepsilon_{i(m)}\}$ be the quasi-set formed by the elements of ε such that $\langle \varepsilon_{i(k)} ; x \rangle \in f$ and $qc(x) \neq 0$ ($k = 1\ldots m$). This quasi-set is denoted $supp(f)$. Consider now the pair $\langle o; f \rangle$, where o is a bijective quasi-function $\mathrm{o} : \{\varepsilon_{i(1)}\varepsilon_{i(2)}\ldots\varepsilon_{i(m)}\} \rightarrow \{1,2,\ldots,m\}$. Each one of the quasi-functions o defines an order on $supp(f)$. Let $O\mathcal{F}$ denote the quasi-set formed by all the pairs $\langle o,f \rangle$. $O\mathcal{F}$ is the quasi-set formed by all the quasi-functions of $\mathcal{F}$ with ordered support. Using a similar notation as before (and also repeating indexes according to the occupation number), $f_{\varepsilon_{i(1)}\varepsilon_{i(2)}\ldots\varepsilon_{i(m)}} \in O\mathcal{F}$ refers to a quasi-function $f \in \mathcal{F}$ and a special ordering of $\{\varepsilon_{i(1)}\varepsilon_{i(2)}\ldots\varepsilon_{i(m)}\}$. But now, the order of the indicies must not be understood as a labeling of particles, because it can be shown that the permutation of particles does not give place to a new element of $O\mathcal{F}$ (DOMENECH et al., 2008).

Consider next the collection of quasi-functions C which assign to every $f \in \mathcal{F}$ (or $f \in O\mathcal{F}$) a complex number. A quasi-function $c \in C$ is a collection of ordered pairs $\langle f;\lambda \rangle$, where $f \in \mathcal{F}$ (or $f \in O\mathcal{F}$) and λ a complex number. Let C_0 be the subset of C such that, if $c \in C_0$, then $c(f) = 0$ for almost every $f \in O\mathcal{F}$ (i.e., $c(f) = 0$ for every $f \in O\mathcal{F}$ except for a finite number of quasi-functions). A sum and a product can be defined in C_0 as follows.

Definition 4.1. Given $\alpha, \beta, \gamma \in C$, and $c, c_1, c_2 \in C_0$, then

$$\gamma(c(f)) := (\gamma * c)(f) \text{ and } (c_1 + c_2)(f) := c_1(f) + c_2(f).$$

Using the above definitions, $(C_0,+,\ *)$ is endowed with a complex vector space structure. Given a quasi-function $c \in C_0$ such that $c(f_i) = \lambda_i$ ($i = 1,\ldots,n$) for some finite set of quasi-functions $\{f_i\}$ belonging to $\mathcal{F}$ or $O\mathcal{F}$ the following association is done:

$$(\lambda_1 f_1 + \lambda_2 f_2 + ... + \lambda_n f_n) \ := \ c\,. \qquad (4.2.1)$$

Thus, a quasi-function $c \in C_0$ is interpreted as a linear combination of the quasi-functions f_i (representing a quantum superposition).

Scalar products must be introduced in order to reproduce the quantum mechanical machinery of computation of probabilities. It is possible to define two of them, one for bosons ("∘") and one for fermions ("•"). In this way (and using norm completion), two Hilbert spaces $(\mathcal{V}_Q, \circ)$ and $(\mathcal{V}_Q, \bullet)$ are obtained. The scalar product for bosons is defined as follows.

Definition 4.2. Let $\delta(n,m)$ be the Kronecker symbol and $f_{\varepsilon_{i(1)}\varepsilon_{i(2)}...\varepsilon_{i(n)}}$ and $f_{\varepsilon_{i'(1)}\varepsilon_{i'(2)}...\varepsilon_{i'(m)}}$ two basis vectors, then:

$$f_{\varepsilon_{i(1)}\varepsilon_{i(2)}...\varepsilon_{i(n)}} \circ f_{\varepsilon_{i'(1)}\varepsilon_{i'(2)}...\varepsilon_{i'(m)}} \ := \ \delta(n,m) \sum_p \delta_{i(1)pi'(1)}\, \delta_{i(2)pi'(2)} \ldots \delta_{i(n)pi'(n)}\ .$$

The sum is extended over all the permutations of the set $i' = (i'(1), i'(2)..., i'(n))$ and for each permutation p, $pi' = (pi'(1), pi'(2),..., pi'(n))$.

For fermions:

Definition 4.3. Let $\delta(n,m)$ be the Kronecker symbol, $f_{\varepsilon_{i(1)}\varepsilon_{i(2)}...\varepsilon_{i(n)}}$ and $f_{\varepsilon_{i'(1)}\varepsilon_{i'(2)}...\varepsilon_{i'(m)}}$ two basis vectors, then:

$$f_{\varepsilon_{i(1)}\varepsilon_{i(2)}...\varepsilon_{i(n)}} \bullet f_{\varepsilon_{i'(1)}\varepsilon_{i'(2)}...\varepsilon_{i'(m)}} \ := \ \delta(n,m) \sum_p s^p \delta_{i(1)pi'(1)}\, \delta_{i(2)pi'(2)} \ldots \delta_{i(n)pi'(n)}\ ,$$

where: $s^p = +1$ if p is even, and $s^p = -1$ if p is odd.

These products can be easily extended to all linear combinations. The second product • is an antisymmetric sum of the indexes which appear in the quasi-functions and the quasi-functions must belong to $\mathcal{OF}$. If the occupation number of a product is greater or equal than two, then, it can be shown that the vector has null norm, thus reproducing Pauli's exclusion principle for fermions (DOMENECH et al., 2008).

With these constructions within Q, the formalism of QM can be rewritten giving a positive answer to the problem of giving a formulation of QM in which intrinsic indistinguishability is taken into account from the beginning, without introducing artificial labels (DOMENECH et al., 2008; DOMENECH et al., 2009).

5. Stating the problem in an adequate form

Once the formal setting is determined, we are now ready to come back to the questions posed in section 2.3 in a more formal way. In this section we state the problem of identical particles from a new perspective.

5.1 Metaphysical underdetermination

As is well known, there are several interpretations of QM, the received view being only one among others (perhaps, the most popular one). The simple fact that there exists an interpretation such as Bohm's – in which particles possess definite trajectories – represents a problem for someone who wants to extract metaphysical constructions out of physical theories: how to reconcile incompatible but plausible interpretations of a given formalism by means of which one calculates the measurement results obtained in the laboratory, such as the VON NEUMANN ([1932] 1996) formulation of QM? While in the Bohmian interpretation particles have definite trajectories, there are no trajectories at all in the standard interpretation. Particles are individuals for Bohm and non-individuals for the received view. This is the problem of metaphysical underdetermination, discussed in detail in FRENCH & KRAUSE (2006). We review this problem here.

While Schrödinger used the BE and FD statistics as an argument to support the received view, other authors, interpreted these "strange" statistics as a new form of non-local correlation between particles (considered as individuals). MULLER & SAUNDERS (2008) use the labeled tensor product Hilbert space formalism to show that quanta are weakly discernible (see also MULLER & SEEVINCK, 2009). Then, there seem to be reasonable arguments for different positions: while incompatible, these positions seem to be valid, in the sense that they do not contradict (up to now) empirical data. The received view has historical problems, such as stating clearly what a non-individual means. But there are concrete solutions to this problem, see for example FRENCH & KRAUSE (2006).

In addition to this *metaphysical* underdetermination, in section 4.2 we showed that an alternative formulation of QM may be given, but with a different underlying logic, based on quasi-set theory. This implies that there also exists a kind of *logical* underdetermination, i.e., there is no preferred logic, if the aim is to formulate the theory in an axiomatic way.

Regarding *logical underdetermination* it is important to specify what we mean by the word "logical". As is well known, language has different layers. The axioms of a physical theory, stated in a mathematical form, have an underlying logic, which is usually first order predicate logic enriched with standard set theory, but it may be formulated in a different framework, such as category theory. But here we speak of ZFC set theory only. Then, the word "logic" means the axioms used in the mathematical formulation of the theory. ZFC set theory has a deeper logical level, which are the axioms of first order classical logic.

But physics not only concerns mathematical formulation. Interpretation may be regarded as part of the theory or not; it is certain that it is unavoidable to have, at least, a minimal interpretational framework in order to connect theory with experience (and, eventually, to explain it). The language used for speaking (and thinking) about the concepts related to the word "experience", as well as the theoretical terms representing the entities involved in a given interpretation, is not just mathematical, and it is also not an artificial language based on first order classical logic. This language has its own "underlying logic", which may not necessarily be a formalized one. What Schrödinger meant by a non-individual entity, may be not clear or formal, but it is clear that the logic underlying such an interpretation seems to be not the classical one. Quasi-set theory would provide a formal framework in order to give us a formal logical basis for that notion. The logic at the level of the axioms of the theory and the underlying logic of the interpretation may coincide or not. The second one is the case of the received view, susceptible to all the criticisms mentioned above. QM formulated in the Q-space formalism seems to be in harmony with the underlying logic of the received view.

We are thus faced with incompatible alternatives to take, with different possibilities. How to make a choice? Which attitude is to be taken in view of this fact?

5.2 Ockham's razor revisited

In order to answer the question posed in the previous section, let us review a traditional example of quantum theory. Bohm's interpretation presupposes individual trajectories for quantum particles, guided by pilot waves. The latter are in *configuration space* (not in space-time, where the trajectories are). Quantum statistics would thus be ruled out by hidden variables: trajectories exist, but one is never able to predict them. The same interpretation comes endowed with a "concealment mechanism", which forbids experimental control of the postulated hidden variables. Thus, these hidden variables, while compatible with predicted experience, play no role in any experiment, i.e., they are completely *dispensable* (for the received view). They play only an explanatory role in Bohm's approach: fairies or elves may be responsible for the values that these hidden variables take. *But the impossibility of settling this question experimentally is an a priori requirement for many advocates of the Bohmian interpretation of QM.*[13]

In view of the discussion of the previous section, the word "explanatory" should be understood as follows: to make experience compatible with an ontological (or metaphysical) "preference" (or "prejudice"). Any interpretation seems to have theoretical terms which may be suppressed in order to endorse a minimal interpretation. But this is not the point that we want to stress: *dispensability is not our problem*. What we want to remark is that hidden variables in Bohm's theory cannot be measured, nor controlled in any laboratory experience. This is precisely an unavoidable requirement of the Bohmian interpretation, in order to be empirically equivalent to standard QM. Otherwise, if these variables could be controlled, or some crucial experiment based on them could be designed, standard QM would be wrong (and the

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[13] While many advocates of the Bohmian interpretation put emphasis on the equivalence of Bohmian mechanics and standard QM, Bohm had expectations in finding different empirical predictions. For more discussion about this, see for example GHOSE (2003).

supposed empirical equivalence between Bohm's theory and the orthodox formulation of QM would no longer hold). Like a quantum state or the electron charge, or even a field – which are all theoretical terms –, hidden variables cannot be prepared nor measured, simply because this is what "hidden" is intended to mean.

There are no limits, in principle, for adding hidden entities or properties. Given a metaphysical preference, we can always add as many "hidden" theoretical terms as we want, always taking care that they should not make predictions incompatible with experience (which is regulated by formalism plus a minimal interpretation). But it seems reasonable to assume that science should not be concerned with notions that are – as a matter of principle – impossible to control in any experiment, adding neither new predictions nor postulating states of affairs which are *by definition* impossible to regulate experimentally. It could be the case that a notion used to explain phenomena, such as Boltzmann's particles, could not be observed or clearly studied in any experimental set up at a certain stage of a theory, *but this impossibility cannot be part of the definition of this notion*.

As is well known, the existence of Bohm's interpretation and the fact that its hidden variables are non-local, led John S. Bell to question himself whether there exists an interpretation based on local hidden variables. These questions led to the well known story about Bell's inequalities and Bell's theorem. Any interpretation based on hidden variables compatible with quantum predictions is attached to non-locality, i.e., *hidden variables must be non-local.* And this was tested experimentally (in favor of QM). Thus, Bell's theorem shows that the acceptance of hidden variables leads also to "hidden non-locality", which of course, cannot be used to send information instantaneously.

One of the conclusions that we extract from the story of hidden variables and Bell's theorem, is as follows. It is always possible to make different interpretations of a given theory, and they may be incompatible. Where does this metaphysical underdetermination come from? We will not discuss this in detail here, but we stress one point. It may be possible that metaphysical underdetermination of physical theories is a general characteristic that language itself manifests, even at the level of simple examples of logic: think about models built within set theory. There may be several models of the same axioms, and nothing determines a preferred one. Another example is one of the Gödel's results, which asserts that any axiomatic system – with a certain degree of complexity and formulated in a certain way – has true but undecidable propositions: there is always something beyond the scope of the axioms. And if this happens already at the logical level, nothing prevents this to happen in more complex languages, such as the one employed in the formulation and interpretation of a physical theory. The complete "language" of a theory involves a complex mixture of laboratory assertions, theoretical concepts (many of them not necessarily completely or rigourously defined), and if we wish, the formal language of the axiomatic level too.

Thus, it may well be that, as it happens with Gödel's theorem, there will always exist assertions which cannot be decided using the axioms of the theory plus the available experimental data at a given historical moment, and no one knows how to state the problem in order to design the adequate experiments to decide them. But as it happened with hidden variables and Bell's theorem – by adding the adequate experimental evidence – further non-trivial information about hidden variables was extracted by stating the problem in an adequate form. Adequate formulation of the problem may involve introducing new axioms and definitions (as well as theoretical constructions), in order to distinguish between several alternatives. This is one of the great merits of Bell.

A similar program may be delineated for similar particles. Even if we do not know how to make an experimental test in order to decide whether quanta are individuals or not, working on formal structures and considering ontological specification of the involved entities may serve to design new experiments. And even if these experiments don't rule out a given possibility, they may impose restrictions on the validity of the assumption of quantum individuality. With Bohmian mechanics, hidden variables were shown to be non-local, and as this fact is in a certain sense incompatible with special theory of relativity, it gives us more elements to make a choice (although we are not required to consider it). A similar analysis can be made for the Kochen-Specker theorem.

We now have at hand examples of how these ideas work for identical particles. In order to explain BE and FD statistics, one may assume non-individuality, as usual. But other authors explain this phenomenon by postulating non-independent correlations (which are different for fermions and bosons) (VAN FRAASSEN, 1998). In spite of these small steps, we think that an analogue to Kochen-Specker or Bell's theorems for non-individuality is in order. This should be added to the program for the investigation of identical particles in another work.

It is worth analyzing what happens with individuality in classical mechanics (CM). Trajectories can be measured in CM, and thus one may postulate that to any trajectory there corresponds one particle or one body. By means of this association, individuality plays a direct role in observation. Of course, nothing prohibits us to break the link between particles and trajectories and postulate that a strange (unobservable) particle permutation may happen between the trajectories, and then, individuality could be questioned. We could go further and presuppose that classical particles of the same mass and form, charge, etc., are completely indistinguishable, and that they do not have individuality at all, but their trajectories do. In this case, we force the interpretation in order to satisfy a certain metaphysical preference, but none of these extra assumptions can be manipulated experimentally. This example in CM is the reverse of that one in the quantum case: as in Bohmian hidden variables theory, individuality of quanta is not only dispensable, but it is also impossible to manipulate experimentally.

A possible attitude towards this "metaphysical freedom" could be: use Ockham's razor and discard individuality of quanta. But there exists an alternative and (we think) more fruitful possibility: given extra assumptions (such as the individuality of quanta), we must provide more precise definitions, add extra postulates and design adequate crucial experiments in order to discriminate and discard between possible metaphysical alternatives.

Before entering into the general conclusions of this work, let us make an interesting remark. Bohm's interpretation postulates trajectories, hidden variables, and pilot waves. At the end of the story, when these assumptions are fully analyzed, we find that hidden variables are related in a non-local way, and that the pilot waves suffer of similar problems as those of the traditional Schrödinger wave functions. It is as if the "problematic" aspects of the standard interpretation of quantum mechanics reemerge in the Bohmian interpretation in a new fashion, or as if there was a "principle of conservation of problematic aspects". Again, this kind of analysis of the "failure" of the Bohmian program (i.e., the failure to recover a completely classical picture) does not suffice to discard the whole interpretation. But it sheds light on the consequences of our metaphysical preferences, and by studying these consequences, we gain a lot of information of how things really work. Perhaps the interesting question is not "which is the true ontology or interpretation?", but questions such as "which are the consequences of each of them?", and "which of them are in conflict with the observed phenomena and which of them are compatible?". This is perhaps the most interesting attitude towards the "metaphysical freedom" originated in metaphysical underdetermination.

6. Final discussion

As we have seen, non-classical logics and algebraic structures may arise also from insights taken from science. Non-reflexive logics, in the sense posed above, constitute a typical example. But, what can we say about ontology? Some philosophers may guess that this question is not well posed, for ontology is the study of the basic stuff of the world and it would be indifferent to the theory we are using in such an investigation. We may say that, in this old sense, a philosopher would say that ontology is the study of the basic furniture of the world, and in this sense all we need is to put some light on this world's stuff. But, at least since QUINE (1953), we have become familiarized with the talk of an ontology associated to a theory, and in order to speak about ontology, we need to look at our best theories and consider what they say about the world. This *naturalized ontology* is today well accepted by most philosophers of science, and we believe that this way of speaking is not contrary to the usual assumptions made by the physicists. Thus, taking into account quantum theory (either relativistic or non-relativistic versions – quantum field theories), we can have some insight for instance about the very nature of the basic constituents of the world, the 'elementary particles'. By 'elementary particles' we mean whatever entity postulated or assumed by the theory in its foundations. They can even be named: electrons, protons, neutrinos, quarks, and so on, and refer to them indiscriminately as *quanta*.

Just as logical constructions may be inspired on physical theories, if we adopt the above point of view about a *naturalized ontology*, we may use these logical constructions to draw conclusions about the possible ontological commitments of physical theories. Let us discuss next the implications of the existence of the formal structures presented in this paper.

6.1. *Q* and non-individuality

Regarding the discussion we have made in this paper, we can make the following assertions:

(1) Standard formulations and both non-relativistic and relativistic QM use both classical logic and standard mathematics (say, one built in ZFC). Hence, no theory founded on such a basis can contradict its theorems, unless particular assumptions are made. For example, for the case of similar particles, non-symmetric states must be disregarded as 'surplus structure'.

(2) The extracted conclusions depend also on the interpretation of the given mathematical formalism and on the specific features of this formalism itself.

(3) If the formalism of the theory is changed using quasi-set theory, quanta of the same species may be (in certain situations) absolutely indiscernible by all means provided by the theory.

(4) In this way, there is a specific sense in saying that, in certain situations, quanta cannot be said to have an identity, a permanent label that distinguishes each one of them from the others, even of the similar kind.

From these four claims, we can draw some conclusions. From (1), the standard formalism of QM always distinguishes between two quanta, even if they are of the same kind and regarded as indistinguishable. If the distinction cannot be achieved in the physical theory proper (the specific postulates of QM we are using), they can be discerned by the underlying mathematics. Thus, even the atoms in a Bose-Einstein condensate, when represented in the mathematical model (QM) constructed this way, can be distinguished. From the logical point of view, we cannot agree *in totum* with the Nobel Prize winner Wolfgang Ketterle, when he says that:

> If we have a gas of ideal gas particles at high temperature, we may imagine those particles to be billiard balls [...]. They race around in the container and occasionally collide. This is a classical picture. However, if we use the hypothesis of de Broglie that particles are matter waves, then we have to think of particles as wave packets. The size of a wave packet is approximately given by the de Broglie wavelength λ_{dB}, which is related

> to the thermal velocity v of the particles as $\lambda_{dB} = h/mv$. Here m is the mass of the particles and h is Planck's constant. Now, as long as the temperature is high, the wavepacket is very small and the concept of indistinguishability is irrelevant, because we can still follow the trajectory of each wavepacket and use classical concepts. However, a real crisis comes when the gas is cooled down: the colder the gas, the lower the velocity, and the longer the de Broglie wavelength. When individual wave packets overlap, then we have an identity crisis, because we can no longer follow trajectories and say which particle is which. At that point, quantum indistinguishability becomes important and we need quantum statistics. (KETTERLE, 2007, p. 159).

In fact, within standard logic and mathematics, there is no identity crisis! This leads us to (3) and (4), which may be taken as emphasizing the same question, namely, the way to go around standard logic (and mathematics) in order to acknowledge that there is in fact an identity crisis involving quanta in certain situations. As we have said above, there are two options: to confine ourselves to a certain protected region within standard ZFC (say a mathematical structure) and speak of *some* properties and relations only. Thus the structure may be nonrigid[14] and we can define indiscernible objects as those which are led to one another by one of the non-trivial automorphisms of the structure. This is the classical solution: symmetric and anti-symmetric vectors do the job, and assuming identity as defined *a la* Quine completes the crime (QUINE, 1986).[15] But, let us recall, any structure built in ZFC can be extended to a rigid structure. In this structure, the very nature of our alleged indiscernible quanta would be revealed, and in the background, we can see them as individuals, as entities having identity.

Thus, we conclude that if we aim at to speak of an ontology of indiscernible quanta, we need to go out from the classical frameworks and adopt an alternative logic, and quasi-set theory is one of the options. And in fact, as we have shown in section 4.2, the construction presented can be used to

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[14] That is, a structure possessing automorphisms different than that of the identity function.

[15] Quine defines identity by the exhaustion of all predicates of the language, taken always in a finite number. In our opinion, this strategy defines only indiscernibility regarding the chosen predicates.

reformulate QM in a logical background different from that of ZFC (meaning standard mathematics and classical logic). An important consequence of the existence of such a reformulation is that the conclusions about the identity of quanta posed in (1) above are not a characteristic of *any* formulation of QM, but only of those formulations based on ZFC or similar "classical" theories. The construction sketched above shows us that the answer to the problem posed in REDHEAD & TELLER (1991, 1992) and recalled in section 1 is in the affirmative, and that our construction is more in accordance with the interpretations of QM which claim that particles are not individuals. The formulation of QM discussed in section 4.2 *supports* this "received view" (see FRENCH & KRAUSE, 2006).

It is not our aim in this article to deny hidden variable theories (or haecceities), neither weak discernibility. We only stress the point that there are different interpretations of QM and according to this fact, different mathematical formulations, each of which is more or less compatible with a given interpretation.

6.2. Quantum logic for compound systems of identical particles

The quantum logical approach to physics does not restrict itself to QM. It is a general operational framework which allows one to include a huge family of physical theories. Using the method developed by PIRON (1976), it is possible to define questions on an arbitrary system, in such a way that these questions form propositions. It is possible to show that these propositions form a lattice. By imposing suitable axioms on this lattice, one may recover QM, CM, or, in principle, any arbitrary theory. This was called the Operational Quantum Logic approach to physics (OQL). This kind of approach allows us to study the structure of the propositional lattices of any given system; in particular, it allows us to study the structure formed by elementary tests in QM. As is well known since the work of BIRKHOFF & VON NEUMANN (1936), these tests are isomorphic to the projection lattice $\mathcal{P}(\mathcal{H})$. At the same time, this structural characterization allows for a comparison between theories: although classical mechanics and QM are very different theories, the OQL approach allows one to compare them in a same formal framework (that is, the lattice theoretical one) in order to look for analogies and differences. The most striking difference is perhaps that the propositional lattice of QM is not distributive, but modular (finite dimensional $\mathcal{H}$) or ortho-modular (infinite dimensional $\mathcal{H}$).

The difficulties appear when we realize that the OQL approach has problems when applied to compound systems. It is true that, from a foundational point of view, it gives useful information about the structure of compound systems, but it works in a negative way: this is the content of the results of Aerts and others (AERTS, 1984, 2000; AERTS & DAUBECHIES, 1978). The fact that no product of lattices exist, tell us a lot about the structure of compound quantum systems, but this is because of the incapability of the approach to describe them. This incapability comes from the fact that when we have an entangled state of the compound system, the reduced state of each subsystem will not be pure, and thus it will not be possible to link the state of the compound system to the states of the subsystems at the level of lattices. This happens simply because the (possibly) mixed states of the subsystems cannot be represented as elements of the corresponding lattices (DOMENECH et al., 2010; HOLIK et al., 2010; HOLIK et al., 2013). This is also the case for similar particles: if we depart from a pure symmetrized state of the bipartite system of fermions, we will always obtain mixed states for the subsystems. This is essentially the reason why the OQL approach presents some disadvantages for the study of entanglement and, even more, for the case of identical particles.

The constructions presented in DOMENECH et al. (2010), HOLIK et al. (2010), and HOLIK et al. (2013) overcome this difficulty, by incorporating mixed states as atoms of a new lattice. This allows one to link states of the compound system to states of the subsystems, and shows us the structure of the propositions thus formed. Convex subsets of the convex set of states may be interpreted as probability spaces (HOLIK et al., 2013): our constructions allow us to look at the structure of these probability spaces. This was not possible using the traditional OQL approach.

In this work (section 3.2), we have shown that a quantum logical structure for compound quantum systems of identical particles can be realized, something which was not present in the literature except for scarce examples (GRIGORE, 1993; AERTS, 1981). Our structure captures the maps which link the states of a compound system formed by two identical particles to the states of its subsystems, thus providing a formal framework in which we can study how compound quantum systems of identical particles behave. In particular, they allow us to see the divergences with classical structures. We hope that these structures allow us to study the formal structure of compound quantum systems of identical particles in future work.

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