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Integral Representations Of Sl_2(z/nz)

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INTEGRAL REPRESENTATIONS OF $\mathbf{SL}_2(\mathbb{Z}/n\mathbb{Z})$

by

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DEDICATION

To My Family and Friends

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attempted to mimic his style used in his Master's Thesis for mine.

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CONVENTIONS

We will employ the following conventions and notations:

Character and representation will be used interchangeably.

$\zeta_p = \exp(2\pi i/p)$.

ϵ and Ω_p denote the quadratic Gauss sum for an odd prime p.

$\varepsilon = (-1)^{(q-1)/2}$ where q is a prime power. $\varepsilon = \pm 1$.

δ_i is the Kronecker delta function.

δ_L is the discriminant of a lattice L.

$\mathbb{F}_q$ denotes a finite field of prime power order $q = p^n$.

$\mathbb{F}_p$ and $\mathbb{Z}/p\mathbb{Z}$ denote a finite field of order p.

C_p is the additive cyclic group of order p.

K_p is the group $C_p \times C_p$.

H_p or $H(G, p)$ is the Heisenberg group of order p^3.

$\mathfrak{s}$ is the matrix $\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$. Other definitions of this matrix will also be given.

$\mathfrak{t}$ is the matrix $\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$. Other definitions of this matrix will also be given.

A^T and $A^\intercal$ denote the transpose of a matrix or vector A.

ν_p is the cyclic group of the roots of unity: $\nu_p := \{\zeta \in \mathbb{C} : \zeta^N = 1\}$.

$R(1, +)$, $R(n, +)$, and ξ are the Principal Series Weil Characters.

$R(1, -)$, $R(n, -)$, and ζ are the Cuspidal Series Weil Characters.

St is the Steinberg representation.

V_Q is the Vandermonde matrix associated to the quadratic form Q.

INTRODUCTION

Issai Schur and Herbert Edwin Jordan computed the character tables of the special linear group of order two over a finite field, $\mathrm{SL}_2(\mathbb{F}_q)$, over a hundred years ago. The irreducible representations came a little later and while there are several techniques, we will employ the Weil representation to compute the irreducible representations of $\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$ as well as $\mathrm{SL}_2(\mathbb{F}_q)$. The Weil representation is a beautiful construct that has many other applications. It is used to study theta functions (modular forms). When generalized to a Grassmann algebra, it appears in quantum field theory. There are also many number theoretic applications as well. We will cite and extend results pertaining to the integrality of the irreducible representations by Wang[36] and Gilmer et al[13]. These results were proven in the context of topological quantum field theory (TQFT). TQFT is related to knot theory, theory of four-manifolds in algebraic topology, and theory of modulii spaces in algebraic geometry.

The Weil representation was originally motivated by theoretical physics, namely by quantization[15]. It was firstly defined on the level of Lie algebra by L. van Hove in 1951, then on the level of Lie group by I. E. Segal and D. Shale in the 1960's. On the arithmetical side, in 1964, A. Weil generalized this machinery to include all local fields. This is the main ingredient of Weil's representation-theoretic approach to theta functions. In fact, the theta functions can be interpreted as automorphic forms of a subgroup of certain metaplectic group[15]. We fully concur with H.N. Ward's statement "I have a strong fondness for the Weil representation." It is our desire that the reader will also develop a fondness for it.

This document contains six chapters. In the first chapter, we construct the Weil representation using a Heisenberg group. This treatment, while not new, is explicit enough and is suitable for advanced undergraduates and those interested in learning about the Weil representation. It also contains examples of the representations which are lacking in all the other treatments of the construction. In the second chapter, we define and discuss integral representations and provide alternate proofs regarding integrality that are more accessible.

In Chapter 3, we construct all the irreducible representations of $\mathrm{SL}_2(\mathbb{Z}/3\mathbb{Z})$ with the exception of the Steinberg (which will be covered in Chapter 4). We use the methods we learned from Nobs and Wolfart's methods[23],[24]. Nothing is left to the imagination in the construction of these representations. The calculations are long and drawn out to illustrate the nontriviality of the computations. We use the larger format of matrices/vectors for the delta functions' indicies for greater legibility and to accommodate those that may be vision-impaired. It does increase the page count but legibility was the priority. Due to formatting requirements, we use smaller matrices for some of the calculations. We do regret that compromise. In Chapter 4, we construct the six-dimensional irreducible representation of $\mathrm{SL}_2(\mathbb{Z}/5\mathbb{Z})$ and the Steinberg representation. We also construct the reducible $1 + St$ representation. We show that the Steinberg representation is integral over $\mathbb{Z}$ and give an explicit basis for it. This is a well-known fact and we learned of the integral basis from Reeder[25]. We then construct the reducible six-dimensional representation that is the direct sum of two irreducible principal series Weil representations. Samuel Wilson and Yilong Wang have created a GAP package that computes ALL of the irreducible representations of $\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$. The GAP package resides at https://snw-0.github.io/sl2-reps/. In Chapter 5, we discuss the integrality results of Wang and Zemel and ponder a conjecture by Candelori. We also attempt to prove Wang's basis/method yields the smallest ring of definition for the principal series Weil characters directly. It fails spectacularly because no method exists to ascertain the values of Legendre symbols for an arbitrary prime.

The notion of a minimal integral model is introduced in Chapter 6. Using SAGE to compute the principal series Weil representations of many primes, we noticed that the representations lie in the ring conjectured by Riese. We prove Wang's basis/method yields the minimal integral model for these representations. This is the central result of this document. The proofs require only material that is generally covered in the first year of a graduate program in mathematics. The quadratic Gauss sum and Wang's basis play pivotal roles.

CHAPTER 1 A Construction of The Weil Representation

1.1 The Groups H_p

We will use the Heisenberg group to construct the Weil representation of $\mathrm{SL}_2(\mathbb{F}_p)$. We provide the necessary background material and discuss the Heisenberg group.

1.1.1 Quadratic forms on an abelian group

We begin with several definitions.

Definition 1.1.1. *A* ***bilinear form*** *or* ***bicharacter*** *on an abelian group G with values in another abelian group E is a function $b : G \times G \to E$ satisfying:*

$$b(x, y + z) = b(x, y) + b(x, z) \quad \text{and} \quad b(x + y, z) = b(x, z) + b(y, z).$$

It is ***symmetric*** *if $b(x, y) = b(y, x)$. It is* ***nondegenerate*** *if $b(x, y) = 0$ for all $y \in G$ implies that $x = 0$ and if $b(x, y) = 0$ for all $x \in G$ implies that $y = 0$.*

Definition 1.1.2. *Let G be a finite abelian group. A* ***quadratic form*** *on a finite abelian group G is a function $q : G \to E$ such that $q(-x) = q(x)$ and the form assigned to it $b(x, y) := q(x + y) - q(x) - q(y)$ is bilinear.*

Definition 1.1.3. *Let G be an finite abelian group and $q : G \to \mathbb{Q}/\mathbb{Z}$ be a quadratic form. The* ***level*** *of q is the smallest integer N such that $Nq(x) \in \mathbb{Z}$ for all $x \in G$.*

The following examples illustrate the definitions.

Example 1.1.4. *Let $p > 2$. Consider the map $q : G = \mathbb{Z}/p\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$ that sends $x \mapsto x^2/p$. The form $b(x, y)$*

$$q(x + y) - q(x) - q(y) = \frac{(x + y)^2}{p} - \frac{x^2}{p} - \frac{y^2}{p} = \frac{2xy}{p} = b(x, y)$$

is bilinear since

$$b(x, y + a) = \frac{2x(y + a)}{p} = \frac{2xy}{p} + \frac{2xa}{p} = b(x, y) + b(x, a)$$

and

$$b(x+c,y) = \frac{2(x+c)y}{p} = \frac{2xy}{p} + \frac{2cy}{p} = b(x,y) + b(c,y).$$

It is symmetric since

$$b(x,y) = \frac{2xy}{p} = \frac{2yx}{p} = b(y,x).$$

It is nondegenerate since if $b(x,y) = 0$ *for all* y*, then* $x = 0$ *and if* $b(x,y) = 0$ *for all* x*, then* $y = 0$*.* $q(x)$ *is a quadratic form since* $b(x,y)$ *is bilinear and* $q(-x) = (-x^2)/p = x^2/p = q(x)$*.* $q(x)$ *is of level* p*.*

Example 1.1.5. *Let* $p = 2$*. Consider the map* $q : G = \mathbb{Z}/2\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$ *that sends* $x \mapsto x^2/4$*.*

$$q(x+y) - q(x) - q(y) = \frac{(x+y)^2}{4} - \frac{x^2}{4} - \frac{y^2}{4} = \frac{2xy}{4} = \frac{xy}{2} = b(x,y).$$

We are in characteristic two. If $x \neq y$*, then* $b(1,0) = b(0,1) = 0$*. If* $x = y$*, then* $b(0,0) = 0$ *and* $b(1,1) = 1 - 1/4 - 1/4 = 1/2$*. So* $b(x,y) = xy/2$ *is a is a symmetric and nondegenerate bilinear form. So* q *is a quadratic from of level 4.*

We would like to explain why Definition 1.1.3 is compatible with the standard definition of a quadratic form:[27]:

Definition 1.1.6. *Let* V *be a module over a commutative ring* A*. A function* $Q : V \to A$ *is called a quadratic form on* V *if:*

1. $Q(ax) = a^2 Q(x)$ *for* $a \in A$ *and* $x \in V$*.*
2. *the function* $(x,y) \mapsto Q(x+y) - Q(x) - Q(y)$ *is a bilinear form.*

We have to restrict a to ± 1 because otherwise $a^2 Q(x)$ is not defined if the codomain is a group.

1.1.2 The Heisenberg group $H(G,q)$

We now provide a more general definition of the Heisenberg group.

Definition 1.1.7. *Let* $N > 1$ *be an integer and* $\nu_N = \{\zeta \in \mathbb{C} : \zeta^N = 1\})$*,* G *be an* *abelian group and let q be a quadratic form on G. The Heisenberg group $H(G,q)$ is the set $\nu_N \times G \times G$ together with the group law*

$$(\zeta_1, x_1, y_1)(\zeta_2, x_2, y_2) = (\zeta_1\zeta_2 \exp(2\pi i b(x_1, y_2)), x_1 + x_2, y_1 + y_2).$$

Example 1.1.8. *Let $p > 2$, $G = \mathbb{Z}/p\mathbb{Z}$ and q be the quadratic form defined by $x \mapsto x^2/p$. Then H_p (or $H(G,q)$) has the group law*

$$\begin{aligned}(\zeta_1, x_1, y_1)(\zeta_2, x_2, y_2) &= (\zeta_1\zeta_2 \exp(2\pi i(2x_1y_2/p), x_1 + x_2, y_1 + y_2) \\ &= (\zeta_1\zeta_2\zeta_p^{2x_1y_2}, x_1 + x_2, y_1 + y_2).\end{aligned}$$

1.1.3 The Heisenberg group H_p

Let p be a prime. Let C_p be the cyclic group of order p. Let

$$K_p := C_p \times C_p.$$

This is a two dimensional $\mathbb{F}_p$-vector space, so that $\mathrm{Aut}(K_p) = \mathrm{GL}_2(\mathbb{F}_p)$. Let

$$\nu_p = \{\zeta \in \mathbb{C} : \zeta^p = 1\}$$

be set of the p-th roots of unity. Define the group H_p to be the set $\nu_p \times K_p$ with product

$$(\lambda_1, x_1, y_1)(\lambda_2, x_2, y_2) = (\lambda_1\lambda_2\zeta_p^{2x_1y_2}, x_1 + x_2, y_1 + y_2).$$

1.1.4 Important facts about H_p

We will do the following:

1) Prove H_p is a group.

2) Prove

   a) H_p is a non-trivial group extension $1 \to \nu_p \to H_p \to K_p \to 0$.

   b) $\nu_p = Z(H_p)$ (i.e., H_p is a central extension).

   c) calculate the commutator in H_p of two elements of the form $(1, x_1, y_1)$ and $(1, x_2, y_2)$.

3) Find the conjugacy classes of H_p.
4) Compute the character table of H_p by starting with the known subgroup and induce characters from those. Also, we consider quotients and inflate characters from those.
5) Determine the p-dimensional representations.

(1): H_p is a group

We now show that H_p is a group.

Well-defined: It is clear that the group operation is well-defined. Let $(\lambda_1, x_1, y_1) = (\lambda_2, x_2, y_2)$. Then for any other element of H_p, (λ, x, y), we have

$$(\lambda_1, x_1, y_1)(\lambda, x, y) = (\lambda_1\lambda\zeta_p^{2x_1y}, x_1 + x, y_1 + y)$$

and

$$(\lambda_2, x_2, y_2)(\lambda, x, y) = (\lambda_2\lambda\zeta_p^{2x_2y}, x_2 + x, y_2 + y)$$

implying

$$(\lambda_1\lambda\zeta_p^{2x_1y}, x_1 + x, y_1 + y) = (\lambda_2\lambda\zeta_p^{2x_2y}, x_2 + x, y_2 + y).$$

Even though H_p is not commutative, we still see that

$$(\lambda, x, y)(\lambda_1, x_1, y_1) = (\lambda\lambda_1\zeta_p^{2xy_1}, x + x_1, y + y_1)$$

and

$$(\lambda, x, y)(\lambda_2, x_2, y_2) = (\lambda\lambda_2\zeta_p^{2xy_2}, x + x_2, y + y_2)$$

imply

$$(\lambda_1\lambda\zeta_p^{2x_1y}, x_1 + x, y_1 + y) = (\lambda_2\lambda\zeta_p^{2x_2y}, x_2 + x, y_2 + y).$$

Closure: The group operation, product, is well-defined. We need to show that the set is closed under the operation. For $\lambda_1, \lambda_2 \in \nu_p$ and $x_1, x_2, y_1, y_2 \in C_p$, we need to show $\lambda_1\lambda_2\zeta_p^{2x_1y_2} \in \nu_p$.

$$\begin{aligned}
\lambda_1\lambda_2\zeta_p^{2x_1y_2} &= \zeta_p^a \cdot \zeta_p^b \cdot \zeta_p^{2x_1y_2} \text{ for some } a, b \in C_p \\
&= \zeta_p^{a+b+2x_1y_2} \\
&= \zeta_p^c \text{ where } c = a + b + 2x_1y_2 \\
&\in \nu_p,
\end{aligned}$$

Under modulo addition, C_p is closed, so $x_1 + x_2 \in C_p$ and $y_1 + y_2 \in C_p$. So the set under the operation, product, is closed.

Associativity: Straightforward calculations show that

$$\begin{aligned}
(\lambda_1, x_1, y_1)[(\lambda_2, x_2, y_2)(\lambda_3, x_3, y_3)] &= (\lambda_1, x_1, y_1)[(\lambda_2\lambda_3\zeta_p^{2x_2y_3}, x_2 + x_3, y_2 + y_3)] \\
&= (\lambda_1\lambda_2\lambda_3\zeta_p^{2x_2y_3}\zeta^{2x_1(y_2+y_3)}, \\
&\qquad x_1 + x_2 + x_3, y_1 + y_2 + y_3) \\
&= (\lambda_1\lambda_2\lambda_3\zeta_p^{2(x_2y_3+x_1y_2+x_1y_3)}, \\
&\qquad x_1 + x_2 + x_3, y_1 + y_2 + y_3)
\end{aligned}$$

and

$$\begin{aligned}
[(\lambda_1, x_1, y_1)(\lambda_2, x_2, y_2)](\lambda_3, x_3, y_3) &= (\lambda_1\lambda_2\zeta_p^{2x_1y_2}, x_1 + x_2, y_1 + y_2)](\lambda_3, x_3, y_3) \\
&= (\lambda_1\lambda_2\lambda_3\zeta_p^{2x_1y_2}\zeta^{2(x_1+x_2)y_3}, \\
&\qquad x_1 + x_2 + x_3, y_1 + y_2 + y_3) \\
&= (\lambda_1\lambda_2\lambda_3\zeta_p^{2(x_1y_2+x_1y_3+x_2y_3)}, \\
&\qquad x_1 + x_2 + x_3, y_1 + y_2 + y_3) \\
&= (\lambda_1\lambda_2\lambda_3\zeta_p^{2(x_2y_3+x_1y_2+x_1y_3)}, \\
&\qquad x_1 + x_2 + x_3, y_1 + y_2 + y_3)
\end{aligned}$$

are equal. Therefore associativity holds.

Identity: In order for

$$(\lambda_1, x_1, y_1)(a, b, c) = (\lambda_1 a \zeta_p^{2x_1 c}, x_1 + b, y_1 + c)$$
$$= (\lambda_1, x_1, y_1)$$

to be true, $a = 1$, $b = c = 0$. So the identity is $(1, 0, 0)$. Verification is straightforward.

$$(1, 0, 0)(\lambda_1, x_1, y_1) = (1 \cdot \lambda_1 \cdot \zeta_p^{2 \cdot 0 \cdot y_1}, 0 + x_1, 0 + y_1) = (\lambda_1, x_1, y_1),$$

$$(\lambda_1, x_1, y_1)(1, 0, 0) = (\lambda_1 \cdot 1 \cdot \zeta_p^{2 \cdot x_1 \cdot 0}, x_1 + 0, y_1 + 0) = (\lambda_1, x_1, y_1).$$

Inverses: For

$$(\lambda_1, x_1, y_1)(\lambda_2, x_2, y_2) = (\lambda_1 \lambda_2 \zeta_p^{2x_1 y_2}, x_1 + x_2, y_1 + y_2) = (1, 0, 0),$$

we require $x_1 = -x_2$, $y_1 = -y_2$, and $\lambda_1 \lambda_2 \zeta_p^{2x_1 y_2} = 1$. So, $1 = \lambda_1 \lambda_2 \zeta_p^{2x_1 y_2}$ implies

$$\lambda_2 = \lambda_1^{-1} \zeta_p^{-2x_1 y_2} = \lambda_1^{-1} \zeta_p^{2x_1 y_1}.$$

Therefore, the inverse of (λ_1, x_1, y_1) is $(\lambda_1^{-1} \zeta_p^{2x_1 y_1}, -x_1, -y_1)$. Verification is straightforward.

$$(\lambda_1, x_1, y_1)(\lambda_1^{-1} \zeta_p^{2x_1 y_1}, -x_1, -y_1) = (\lambda_1 \cdot \lambda_1^{-1} \zeta_p^{2x_1 y_1} \cdot \zeta_p^{2x_1(-y_1)}, x_1 - x_1, y_1 - y_1) = (1, 0, 0).$$

$$(\lambda_1^{-1} \zeta_p^{2x_1 y_1}, -x_1, -y_1)(\lambda_1, x_1, y_1) = (\lambda_1^{-1} \zeta^{2x_1 y_1} \lambda_1 \zeta^{2(-x_1) \cdot (y_1)}, -x_1 + x_1, -y_1 + y_1)$$
$$= (1, 0, 0).$$

So, H_p is a group under the product operation.

(2): Properties of H_p

a) A TRIVIAL group extension is an extension

$$1 \to K \to G \to H \to 1$$

that is equivalent to the extension

$$1 \to K \to K \times H \to H \to 1.$$

Since we want a non-trivial one, we could consider the semi-direct product of $\nu_p \rtimes_\phi K_p$. H_p is isomorphic to a semidirect product of ν_p and K_p if and only if there exists a short exact sequence

$$1 \to \nu_p \stackrel{\beta}{\to} H_p \stackrel{\alpha}{\to} K_p \to 0$$

and a group homomorphism $\gamma : K_p \to H_p$ such that $\alpha \circ \gamma = \mathbb{1}_{K_p}$.
$\phi : K_p \to \mathrm{Aut}(\nu_p)$ is given by $\phi(k) = \phi_k$, where $\phi_k(n) = \beta^{-1}(\gamma(k)\beta(n)\gamma(h^{-1}))$.
It's easier to directly construct the map $H_p \to K_p$. We use the "coordinates" given, compute its kernel, and then show that the exact sequence we obtain is a non-trivial central extension. We need to kill off the first "coordinate" of H_p. That's readily done by a projection. That is, for an element $(\lambda, x, y) \in H_p$,

$$(\lambda, x, y) \mapsto (x, y) \in K_p.$$

The projection is a homomorphism. $\alpha(1, 0, 0) = (0, 0)$ so it sends the identity to the identity. Second, we need to verify that $\alpha(h_1 \cdot h_2) = \alpha(h_1) + \alpha(h_2)$ for $h_1, h_2 \in H_p$. Let $h_1 = (\lambda_1, x_1, y_1), h_2 = (\lambda_2, x_2, y_2)$. Then

$$h_1 h_2 = (\lambda_1 \lambda_2 \zeta_p^{2x_1 y_2}, x_1 + x_2, y_1 + y_2).$$

So $\alpha(h_1 h_2) = (x_1 + x_2, y_1 + y_2)$. Since $\alpha(\lambda_1, x_1, y_1) = (x_1, y_1)$ and $\alpha(\lambda_2, x_2, y_2) =$

(x_2, y_2), we have

$$\alpha(\lambda_1, x_1, y_1) + \alpha(\lambda_2, x_2, y_2) = (x_1 + x_2, y_1 + y_2).$$

Hence it satisfies the homomorphic property. Since ν_p and K_p are both commutative, the trivial extension $\nu \times K_p$ would also be commutative. To see that H_p is a non-trivial extension, we need to verify that the product operation in H_p is not commutative.

Let $h_1 = (\lambda_1, x_1, y_1), h_2 = (\lambda_2, x_2, y_2)$. Then

$$h_1 h_2 = (\lambda_1 \lambda_2 \zeta_p^{2x_1 y_2}, x_1 + x_2, y_1 + y_2)$$

and

$$h_2 h_1 = (\lambda_2 \lambda_1 \zeta_p^{2x_2 y_1}, x_1 + x_2, y_1 + y_2).$$

So $h_1 h_2 \neq h_2 h_1$ if $x_1 y_2 \neq x_2 y_1$. Since K_p and ν_p are both abelian, their direct product is abelian. The given product is not abelian so the extension is not trivial.

b) An extension is called a central extension if the subgroup ν_p lies in the center of H_p. To show that ν_p is in the center of H_p, we need to find elements that commute with all elements of H_p. $\mathbb{Z}(H_p)$ is not empty as it contains the identity. Let $z = (a, b, c) \in \mathbb{Z}(H_p)$. Let $h = (\lambda, x, y) \in H_p$. Then $zh = hz$ so

$$(a\lambda\zeta_p^{2by}, b + x, c + y) = (\lambda a \zeta_p^{2xc}, x + b, y + c).$$

This implies that for fixed b, c, $\zeta_p^{2by} = \zeta_p^{2xc}$ for all $x, y \in C_p$. Hence, $b = c = 0$ and $Z(H_p) = \nu_p$. We have that H_p is a central extension.

c) Let $h_1 = (1, x_1, y_1), h_2 = (1, x_2, y_2)$. The commutator of h_1, h_2 is given by

$$\begin{aligned}
[h_1, h_2] &= h_1^{-1} h_2^{-1} h_1 h_2 \\
&= (1^{-1} \cdot \zeta_p^{2x_1 y_1}, -x_1, -y_1) \cdot (1^{-1} \cdot \zeta_p^{2x_2 y_2}, -x_2, -y_2) \cdot (1, x_1, y_1) \cdot (1, x_2, y_2)
\end{aligned}$$

$$= (\zeta^{2x_1y_1+2x_2y_2}\zeta^{2(-x_1)(-y_2)}, -x_1 - x_2, -y_1 - y_2)$$

$$\cdot (1 \cdot 1 \cdot \zeta^{2x_1y_2}, x_1 + x_2, y_1 + y_2)$$

$$= (\zeta^{2x_1y_1+2x_2y_2+2x_1y_2}, -x_1 - x_2, -y_1 - y_2) \cdot (\zeta^{2x_1y_2}, x_1 + x_2, y_1 + y_2)$$

$$= (\zeta^{2x_1y_1+2x_2y_2+2x_1y_2}\zeta^{x_1y_2}\zeta^{-2(x_1+x_2)(y_1+y_2)}, 0, 0)$$

$$= (\zeta^{2x_1y_1+2x_2y_2+2x_1y_2+2x_1y_2}\zeta^{-2(x_1y_1+x_1y_2+x_2y_1+x_2y_2)}, 0, 0)$$

$$= (\zeta^{2x_1y_2}\zeta^{-2(x_2y_1)}, 0, 0)$$

$$= (\zeta^{2(x_1y_2-x_2y_1)}, 0, 0).$$

(3): Conjugacy classes of H_p

Lets find the conjugate of an arbitrary element of H_p. For $(\lambda, x, y), (a, b, c) \in H_p$, we have

$$(\lambda, x, y)^{-1} = (\lambda^{-1}\zeta_p^{2xy}, -x, -y)$$

and

$$\begin{aligned}(\lambda, x, y)^{-1}(a, b, c)(\lambda, x, y) &= (\lambda^{-1}\zeta_p^{2xy}, -x, -y)(a, b, c)(\lambda, x, y) \\ &= (\lambda^{-1}\zeta_p^{2xy}a\zeta_p^{-2xc}, -x + b, -y + c)(\lambda, x, y) \\ &= (\lambda^{-1}\zeta_p^{2xy}a\zeta_p^{-2xc}\lambda\zeta_p^{2(-x+b)y}, b, c) \\ &= (a\zeta_p^{2(by-xc)}, b, c)\end{aligned}$$

Now lets consider some cases.

Case 1. Fix a and let $b = c = 0$. Then $(a, 0, 0)$ is conjugate to only itself as it should since it is in the center of H_p.

Case 2. Fix a, c, $c \neq 0$ and take $b = 0$. Then $(a, 0, c)$ is conjugate to $(a\zeta_p^{-2xc}, 0, c)$. The centralizer of $(a, 0, c)$ consists of all elements of the form $(\lambda, 0, y)$ since

$$(a, 0, c)(\lambda, 0, y) = (a\lambda\zeta_p^{2\cdot 0\cdot y}, 0 + 0, c + y) = (a\lambda, 0, c + y)$$

and

$$(\lambda, 0, y)(a, 0, c) = (\lambda a\zeta_p^{2\cdot 0\cdot c}, 0 + 0, y + c) = (a\lambda, 0, y + c).$$

So the centralizer of $(a, 0, c)$ contains p^2 elements. By the orbit-stabilizer theorem, there are p elements in the orbit of $(a, 0, c)$ (since $|H_p| = p^3, |C_{(a,0,c)}| = p^2$). The conjugacy class of $(a, 0, c)$ is

$$\{(a\zeta_p^{-0\cdot 2c}, 0, c), (a\zeta_p^{-2c}, 0, c), (a\zeta_p^{-4c}, 0, c), (a\zeta_p^{-6c}, 0, c), \ldots, (a\zeta_p^{-2(p-1)c}, 0, c)\}$$

Since $a = \zeta_p^k$ for $0 \leq k \leq p - 1$, we have

$$\{(\zeta_p^k, 0, c), (\zeta_p^{k-2c}, 0, c), (\zeta_p^{k-4c}, 0, c), (\zeta_p^{k-6c}, 0, c), \ldots, (\zeta_p^{k-2(p-1)c}, 0, c)\}$$

Or in simplified form

$$\{(1, 0, c), (\zeta_p, 0, c), (\zeta_p^2, 0, c), \ldots, (\zeta_p^{p-1}, 0, c).$$

Fix a, b, $b \neq 0$ and take $c = 0$. Then $(a, b, 0)$ is conjugate to $(a\zeta_p^{2by}, b, 0)$.

Case 3. Fix a, b, $b \neq 0$ and take $c = 0$. The centralizer of $(a, b, 0)$ consists of all elements of the form $(\lambda, x, 0)$ since

$$(a, b, 0)(\lambda, x, 0) = (a\lambda\zeta_p^{2b\cdot 0}, b + x, 0 + 0) = (a\lambda, b + x, 0)$$

and

$$(\lambda, x, 0)(a, b, 0) = (\lambda a\zeta_p^{2x\cdot 0}, x + b, 0 + 0) = (a\lambda, b + x, 0).$$

So the centralizer of $(a, b, 0)$ contains p^2 elements. By the orbit-stabilizer theorem, there are p elements in the orbit of $(a, b, 0)$ (since $|H_p| = p^3, |C_{(a,b,0)}| = p^2$). The conjugacy class of $(a, b, 0)$ is

$$\{(a\zeta_p^{2\cdot 0b}, b, 0), (a\zeta_p^{2\cdot 1b}, b, 0), (a\zeta_p^{2\cdot 2b}, b, 0), (a\zeta_p^{2\cdot 3b}, b, 0), \ldots, (a\zeta_p^{2(p-1)b}, 0, c)\}.$$

Since $a = \zeta_p^k$ for $0 \leq k \leq p-1$, we have

$$\{(\zeta_p^k, b, 0), (\zeta_p^{k+2b}, b, 0), (\zeta_p^{k+4b}, 0, b), (\zeta_p^{k+6b}, b, 0), \ldots, (\zeta_p^{k+2(p-1)b}, b, 0)\}.$$

Or in simplified form

$$\{(1, b, 0), (\zeta_p, b, 0), (\zeta_p^2, b, 0), \ldots, (\zeta_p^{p-1}, b, 0).$$

Case 4. Now for the last case. Fix a, b, c such that $b \neq 0$ and $c \neq 0$. Then (a, b, c) is conjugate to $(a\zeta_p^{2(by-xc)}, b, c)$. The centralizer of (a, b, c) consists of all elements of the form (λ, x, y) where only $by \equiv cx \mod p$:

$$(a, b, c)(\lambda, x, y) = (a\lambda\zeta_p^{2by}, b + x, c + y)$$

and

$$(\lambda, x, y)(a, b, c) = (\lambda a\zeta_p^{2xc}, x + b, y + c).$$

Once x is chosen, y's value is determined. This implies there are p^2 elements in the centralizer: p values for λ and p values for x. So the size of the orbit is p.

A partitioning of H_p that "agrees" with the conjugacy classes. There are $p(p-1)$ elements of the form $(a, 0, c)$ that are not in the center of H_p. There are $p(p-1)$ elements of the form $(a, b, 0)$ that are not in the center of H_p. There are p elements of the form $(a, 0, 0)$ that comprise the center of H_p. There are $p(p-1)(p-1)$ of the form (a, b, c) where $b \neq 0$ and $c \neq 0$. So,

$$\begin{aligned}
p(p-1) + p(p-1) + p + p(p-1)(p-1) &= 2p^2 - 2p + p + p(p^2 - 2p + 1) \\
&= 2p^2 + 2p - p + p^3 - 2p^2 + p \\
&= p^3 \\
&= |H_p|.
\end{aligned}$$

There are p conjugacy classes in the center of size 1. There are $p-1$ conjugacy classes of the form $(a,b,0)$, $b \neq 0$; one for each b. There are $p-1$ conjugacy classes of the form $(a,0,c)$, $c \neq 0$, one for each c. There are $(p-1)(p-1)$ conjugacy classes of the form (a,b,c), $b \neq 0$ and $c \neq 0$. We can further simplify to stating that there are p conjugacy classes in the center of size 1 and p^2-1 conjugacy classes of size p with the form (a,b,c), with $a \in \nu_p$, $b, c \in C_p$ with $b \neq 0$ AND $c \neq 0$. We will use this simplification in the construction of the character table.

(4): The char. table of H_p obtained by induction of the modulation subgroup

The subgroup of K_p that is generated by $(x,0)$ where $x \in \mathbb{F}_p$ is the called the modulation subgroup. To compute the character table for H_p, we need to determine the number of conjugacy classes. We recall that the number of irreducible representations of H_p (up to isomorphism) is equal to the number of conjugacy classes of H_p. The number of conjugacy classes for H_p is given by

$$p + (p-1) + (p-1) + (p-1)(p-1) = p + 2p - 2 + p^2 - 2p + 1 = p^2 + p - 1.$$

So our character table has $p^2 + p - 1$ rows and columns.

We computed the commutator earlier and now we use it with the fact that the degree 1 representations of H_p are in bijective correspondence with the degree 1 representations of the abelian group $H_p/H_p' = H_p/[H_p, H_p]$. That gives us $H_p/[H_p, H_p] \cong C_p \times C_p$. $C_p \times C_p$ is abelian with order p^2. This implies there are p^2 distinct one dimensional representations of H_p. Let ϕ_1 and ϕ_2 be one-dimensional characters of C_p. Then the one dimensional character of an element (λ, x, y) of H_p is given by

$$\xi(\lambda, x, y) = \phi_1(x)\phi_2(y).$$

One dimensional representations are irreducible and we have p^2 of them. That leaves $p-1$ of them to find. Using the fact that the sums of squares of the dimensions of the irreducible representations equals the order of the group and that we obtained p^2

representations of dimension 1, we have

$$|H_p| - p^2 = p^3 - p^2 = p^2(p-1).$$

So we have $p-1$ representations of degree p. We consider a subgroup of H_p of index p:

$$A = \{(\lambda, x, 0), \lambda \in \nu_p, x \in C_p\}.$$

A is abelian, has order p^2 and $A \cong \mathbb{F}_p^2$. (it is clear that $\nu_p \cong C_p \cong \mathbb{F}_p$). Given characters ϕ_1 and ϕ_2 of $\mathbb{F}_p$, we fix a one-dimensional character of A and call it ψ. ψ is defined as

$$\psi(\lambda, x, 0) = \phi_1(x)\phi_2(\lambda).$$

We induce the representation of the subgroup A to obtain a p-dimensional representation

$$\rho = \mathrm{Ind}_A^{H_p}(\psi).$$

Now we need to show that this representation is irreducible. Let B be the set of representatives of H_p/A. So

$$B = \{(1, 0, y), y \in \mathbb{F}_p\}.$$

The character of ρ is given by

$$\chi_\rho((a,b,c)) = \sum_{\substack{y \in \mathbb{F}_p \\ (1,0,y)\cdot(a,b,c)\cdot(1,0,y)^{-1} \in A}} \psi((1,0,y)\cdot(a,b,c)\cdot(1,0,y)^{-1}).$$

Since

$$\begin{aligned}
(1,0,y)(a,b,c)(1,0,y)^{-1} &= (1,0,y)(a,b,c)(1,0,-y) \\
&= (a\zeta^{2\cdot 0\cdot c}, 0+b, c+y)(1,0,-y) \\
&= (a,b,c+y)(1,0,-y) \\
&= (a\zeta^{-2by}, b, c),
\end{aligned}$$

if $c \neq 0$ then $\chi_\rho((a,b,c)) = 0$. This implies (for $c = 0$)

$$\chi_\rho((a,b,c)) = \sum_{\substack{y \in \mathbb{F}_p \\ (1,0,y)\cdot(a,b,c)\cdot(1,0,y)^{-1} \in A}} \psi((1,0,y)\cdot(a,b,c)\cdot(1,0,y)^{-1})$$

$$= \sum_{\substack{y \in \mathbb{F}_p \\ (a\zeta^{-2by},b,c) \in A}} \psi((a\zeta^{-2by},b,c))$$

$$= \sum_{y \in \mathbb{F}_p} \psi((a\zeta^{-2by},b,0)) \quad (c = 0)$$

$$= \sum_{y \in \mathbb{F}_p} \phi_1(b)\phi_2(a\zeta^{-2by})$$

$$= \phi_1(b) \sum_{y \in \mathbb{F}_p} \phi_2(a\zeta^{-2by}).$$

If $b \neq 0$ and $\phi_2 = 1$, we have $\chi_\rho(a,b,0) = p\phi_1(b)$. If $b \neq 0$, and ϕ_2 is not the trivial character, then $a\zeta^{-2by}$ runs through all of the elements of $\mathbb{F}_p$. This implies $\chi_\rho(a,b,0) = 0$. If $b = 0$, then our element is of the form $(a,0,0)$ and that lies in the center of H_p. So, $\chi_\rho(a,0,0) = \phi_1(0)\cdot p \cdot \phi_2(a) = p\phi_2(a)$. Consider the following grouping of the conjugacy classes:

(1) p conjugacy classes of size 1 in the center.

(2) $p-1$ conjugacy classes of size p of the form $(*,0,c)$ with $c \neq 0$.

(3) $p(p-1) = p^2 - p$ classes of size p of the form $(*,b,c)$ with $b \neq 0$.

We have two cases to check for the irreducibility of χ_ρ: $\phi_2 = 1$ and $\phi_2 \neq 1$. We compute the inner product to determine irreducibility:

Case $\phi_2 = 1$:

$$\begin{aligned}
\frac{1}{|H_p|}\sum_{t\in H_p}\bar{\chi_\rho}(t)\chi_\rho(t) &= \frac{1}{p^3}\left[p\cdot p^2 + p(p-1)\cdot(p\cdot|\phi_1(c)|)^2 + (p^2-p)\cdot p\cdot 0^3\right] \\
&= \frac{1}{p^3}\left[p^3 + (p^2-p)p^2 + 0\right] \\
&= \frac{1}{p^3}\left[p^3 + p^4 - p^3\right] \\
&= \frac{1}{p^3}\cdot p^4 \\
&= p \\
&\neq 1.
\end{aligned}$$

Case $\phi_2 \neq 1$:

$$\begin{aligned}
\frac{1}{|H_p|}\sum_{t\in H_p}\bar{\chi_\rho}(t)\chi_\rho(t) &= \frac{1}{p^3}\left[p\cdot[p\cdot|\phi_2(a)|]^2 + p(p-1)\cdot 0^2 + (p^2-p)\cdot p\cdot 0^3\right] \\
&= \frac{1}{p^3}\left[p\cdot p^2\cdot 1\right] \\
&= \frac{1}{p^3}\left[p^3\right] \\
&= 1.
\end{aligned}$$

So only for the case $\phi_2 \neq 1$, χ_ρ is irreducible.

$$\chi_\rho((a,b,c)) = \begin{cases} p\phi_2(a) & (a,b,c)\in Z(H_p) \\ & \text{otherwise.} \end{cases}$$

Recalling that the one dimensional character ξ of H_p was determined to be $\xi(a,b,c) = \phi_1(b)\phi_2(c)$ where ϕ_1 and ϕ_2 are the one-dimensional characters of $\mathbb{F}_p$, we can now summarize the character table for H_p:

(5): p-dim. representation ρ from induction of modulation subgroup

Fix a character ϕ_2. Then there exists a unique p-dimensional irreducible character representation $\rho : H_p \to \mathrm{GL}_p$ corresponding to ϕ_2 (according to the Table 1.1.1).

	$(a,0,0)$	(a,b,c), $(b,c)\neq(0,0)$
Trivial	1	1
ξ	1	$\phi_1(b)\phi_2(c)$
χ_ρ	$p\phi_2(a)$	0

Table 1.1.1: Character Table for H_p

We write an explicit model for this representation ρ. That is, given the generators $(\zeta,0,0),(1,1,0),(1,0,1)$ of H_p, we find the matrices for each generator g. We do this using the standard procedure. Recall the set of representatives of H_p/A as $B=\{(1,0,y),y\in\mathbb{F}_p\}$. Also, we note that

$$\begin{aligned}(1,0,y_1)^{-1}(a,b,c)(1,0,y_2) &= (1,0,-y_1)(a,b,c)(1,0,y_2)\\ &= (1\cdot a\cdot\zeta_p^{2\cdot 0\cdot c},0+b,c-y_1)(1,0,y_2)\\ &= (a,b,c-y_1)(1,0,y_2)\\ &= (a\zeta^{2by_2},b,c-y_1+y_2)\end{aligned}$$

Then $\psi(a\zeta^{2by_2},b,c-y_1+y_2)=0$ if $c-y_1+y_2\neq 0$. Let $y_i=(1,0,\nu_i)$ for $\nu_i\in\mathbb{F}_p$. That is, $\nu_i=i$, $i\in\mathbb{F}_p$. Then for $g=(a,b,c)\in H_p$,

$$\rho((a,b,c))=\begin{bmatrix}\psi\left(y_0^{-1}gy_0\right) & \psi\left(y_0^{-1}gy_1\right) & \cdots & \psi\left(y_0^{-1}gy_{p-1}\right)\\ \psi\left(y_1^{-1}gy_0\right) & \psi\left(y_1^{-1}gy_1\right) & \cdots & \psi\left(y_1^{-1}gy_{p-1}\right)\\ \vdots & \vdots & \vdots & \vdots\\ \psi\left(y_{p-1}^{-1}gy_0\right) & \psi\left(y_{p-1}^{-1}gy_1\right) & \cdots & \psi\left(y_{p-1}^{-1}gy_{p-1}\right)\end{bmatrix}$$

$$=\begin{bmatrix}\psi(a\zeta^{2b\cdot 0},b,c-0+0) & \cdots & \psi(a\zeta^{2b\cdot(p-1)},b,c-0+(p-1)\\ \psi(a\zeta^{2b\cdot 0},b,c-1+0) & \cdots & \psi(a\zeta^{2b\dot{(}p-1)},b,c-1+(p-1))\\ \vdots & \vdots & \vdots\\ \psi(a\zeta^{2b\cdot 0},b,c-(p-1)+0) & \cdots & \psi(a\zeta^{2b(p-1)},b,c-(p-1)+(p-1))\end{bmatrix}$$

$$= \begin{bmatrix} \psi(a,b,c) & \cdots & \psi(a\zeta^{2b(p-1)}, b, c+(p-1)) \\ \psi(a,b,c-1) & \cdots & \psi(a\zeta^{2b(p-1)}, b, c-1+(p-1)) \\ \vdots & \vdots & \vdots \\ \psi(a,b,c-(p-1)) & \cdots & \psi(a\zeta^{2b(p-1)}, b, c-(p-1)+(p+1) \end{bmatrix}$$

$$= \begin{bmatrix} \psi(a,b,c) & \cdots & \psi(a\zeta^{2b(p-1)}, b, c+(p-1)) \\ \psi(a,b,c-1) & \cdots & \psi(a\zeta^{2b(p-1)}, b, c-1+(p-1)) \\ \vdots & \vdots & \vdots \\ \psi(a,b,c-(p-1)) & \cdots & \psi(a\zeta^{2b(p-1)}, b, c) \end{bmatrix}.$$

Since $\psi(a,b,c) = 0$ for $c \neq 0$,

$$\rho((\zeta,0,0)) = \begin{bmatrix} \psi(a,b,c) & \cdots & \psi(a\zeta^{2b(p-1)}, b, c+(p-1)) \\ \psi(a,b,c-1) & \cdots & \psi(a\zeta^{2b(p-1)}, b, c-1+(p-1)) \\ \vdots & \vdots & \vdots \\ \psi(a,b,c-(p-1)) & \cdots & \psi(a\zeta^{2b(p-1)}, b, c) \end{bmatrix}$$

$$= \begin{bmatrix} \psi(\zeta,0,0) & \psi(\zeta,0,1) & \cdots & \psi(\zeta,0,(p-1)) \\ \psi(\zeta,0,0-1) & \psi(\zeta,0,0) & \cdots & \psi(\zeta,0,0-1+(p-1)) \\ \vdots & \vdots & \vdots & \vdots \\ \psi(\zeta,0,0-(p-1)) & \psi(\zeta,0,-(p-1)+1) & \cdots & \psi(\zeta,0,0) \end{bmatrix}$$

$$= \begin{bmatrix} \psi(\zeta,0,0) & 0 & \cdots & 0) \\ 0 & \psi(\zeta,0,0) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \psi(\zeta,0,0) \end{bmatrix}$$

$$= \begin{bmatrix} \phi_2(\zeta) & 0 & \cdots & 0 \\ 0 & \phi_2(\zeta) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \phi_2(\zeta) \end{bmatrix}.$$

$\mathrm{Tr}(\rho(\zeta,0,0)) = p\phi_2(\zeta)$ as required.

$$\rho((1,1,0)) = \begin{bmatrix} \psi(a,b,c) & \cdots & \psi(a\zeta^{2b(p-1)}, b, c+(p-1)) \\ \psi(a,b,c-1) & \cdots & \psi(a\zeta^{2b(p-1)}, b, c-1+(p-1)) \\ \vdots & \vdots & \vdots \\ \psi(a,b,c-(p-1)) & \cdots & \psi(a\zeta^{2b(p-1)}, b, c) \end{bmatrix}$$

$$= \begin{bmatrix} \psi(1,1,0) & \cdots & \psi(1\zeta^{2(p-1)}, 1, 0+p-1) \\ \psi(1,1,0-1) & \cdots & \psi(1\zeta^{2(p-1)}, 1, 0-1+(p-1)) \\ \vdots & \vdots & \vdots \\ \psi(1,1,0-(p-1)) & \cdots & \psi(1\zeta^{2(p-1)}, 1, 0) \end{bmatrix}$$

$$= \begin{bmatrix} \psi(1,1,0) & \cdots & \psi(\zeta^{2(p-1)},1,p-1) \\ \psi(1,1,-1) & \cdots & \psi(\zeta^{2(p-1)},1,p-1) \\ \psi(1,1,-2) & \cdots & \psi(\zeta^{2(p-1)},1,p-1) \\ \vdots & \vdots & \vdots \\ \psi(1,1,-(p-1)) & \cdots & \psi(\zeta^{2(p-1)},1,(p-1)-(p-1)) \end{bmatrix}$$

$$= \begin{bmatrix} \psi(1,1,0) & 0 & 0 & \cdots & 0 \\ 0 & \psi(\zeta^2,1,0) & 0 & \cdots & 0 \\ 0 & 0 & \psi(\zeta^4,1,0) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \psi(\zeta^{2(p-1)},1,0) \end{bmatrix}$$

$$\rho((1,1,0)) = \begin{bmatrix} \phi_2(1) & 0 & 0 & \cdots & 0 & 0 \\ 0 & \phi_2(\zeta^2) & 0 & \cdots & 0 & 0 \\ 0 & 0 & \phi_2(\zeta^4) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \phi_2(\zeta^{2(p-1)}) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \zeta^2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \zeta^4 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \zeta^{2(p-1)} \end{bmatrix}.$$

$\mathrm{Tr}(\rho(1,1,0)) = \sum_{n=0}^{p-1} \phi_2(\zeta^{2n}) = \sum_{n=0}^{p-1} \zeta^{2n} = 0$ is as required.

$$\rho((1,0,1)) = \begin{bmatrix} \psi(a,b,c) & \cdots & \psi(a\zeta^{2b(p-1)}, b, c+(p-1)) \\ \psi(a,b,c-1) & \cdots & \psi(a\zeta^{2b(p-1)}, b, c-1+(p-1)) \\ \vdots & \vdots & \vdots \\ \psi(a,b,c-(p-1)) & \cdots & \psi(a\zeta^{2b(p-1)}, b, c) \end{bmatrix}$$

$$= \begin{bmatrix} \psi(1,0,1) & \cdots & \psi(1\zeta^{2\cdot 0\cdot(p-1)}, 0, 1+(p-1)) \\ \psi(1,0,1-1) & \cdots & \psi(1\zeta^{2\cdot 0\cdot(p-1)}, 0, 0-1+(p-1)) \\ \vdots & \vdots & \vdots \\ \psi(1,0,1-(p-1)) & \cdots & \psi(1\zeta^{2\cdot 0\cdot(p-1)}, 0, 1) \end{bmatrix}$$

$$\rho((1,0,1)) = \begin{bmatrix} \psi(1,0,1) & \cdots & \psi(1,0,1+(p-1)) \\ \psi(1,0,0) & \cdots & \psi(1,0,1-1+(p-1)) \\ \vdots & \vdots & \vdots \\ \psi(1,0,1-(p-1)) & \cdots & \psi(1,0,1) \end{bmatrix}$$

$$= \begin{bmatrix} \psi(1,0,1) & \psi(1,0,2) & \cdots & \psi(1,0,0) \\ \psi(1,0,0) & \psi(1,0,1) & \cdots & \psi(1,0,(p-1)) \\ \vdots & \vdots & \vdots & \vdots \\ \psi(1,0,1-(p-1)) & \psi(1,0,2-(p-1)) & \cdots & \psi(1,0,1) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & \phi_2(1) \\ \phi_2(1) & 0 & 0 & \cdots & 0 & 0 \\ 0 & \phi_2(1) & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \phi_2(1) & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

$\mathrm{Tr}(\rho(1,0,1)) = 0$ is as required.

1.1.5 Summary

A "natural character' of ν_p is defined to be the inclusion of ν_p into $\mathbb{C}$ that corresponds to $\phi_2 = \mathbb{1}$, the identity. So we proved for each prime p, there exists a unique (up to isomorphism) irreducible representation of H_p such that ν_p acts by its natural character. This is the so-called "Stone-von-Neumann-Mackey Theorem" [12]:

Theorem 1.1.9 (Mackey-Stone-von Neumann)**.** *For fixed non-trivial (unitary) central character, up to isomorphism there is a unique irreducible (unitary) representation of the Heisenberg group with that central character. Further, any (unitary) representation with that central character is a multiple of that irreducible.*

We have proven the following theorem.

Theorem 1.1.10. *All irreducible representations of H_p can be realized over the ring $\mathbb{Z}[\zeta_p]$.*

Since H_p is solvable of exponent p, this falls under the integrality cited by Riese[26, Thm 1.], so the result is not new, but we provided explicit models for all the representations.

1.1.6 Examples for the rep. induced by the translation subgroup

To make things even more explicit, we take $p = 2$ and $p = 3$ and write down the matrices of the 2,3- dimensional representations with $\phi_2 = \mathbb{1}$.

Two Dimensional Representations with $\phi_2 = \mathbb{1}$

Take $p = 2$. $|H_2| = 8$. $\zeta_2 = -1$. The generators are $(-1, 0, 0)$, $(1, 1, 0)$, are $(1, 0, 1)$. The eight elements of H_2 and their representations are

$$(-1, 0, 0) \mapsto \begin{bmatrix} \phi_2(-1) & 0 \\ 0 & \phi_2(-1) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$(1, 0, 1) \mapsto \begin{bmatrix} 0 & \phi_2(1) \\ \phi_2(1) & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$(1, 1, 0) \mapsto \begin{bmatrix} \phi_2(1) & 0 \\ 0 & \phi_2(\zeta_2^2) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$(-1, 0, 0)(-1, 0, 0) = ((-1)(-1)\zeta_2^{2\cdot 0\cdot 0}, 0 + 0, 0 + 0) = (1, 0, 0)$$

$$(1, 0, 0) \mapsto \begin{bmatrix} \phi_2(1) & 0 \\ 0 & \phi_2(1) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$(1, 0, 1)(1, 1, 0) = ((1)(1)\zeta_2^{2\cdot 0\cdot 0}, 0 + 1, 1 + 0) = (1, 1, 1)$$

$$(1, 1, 1) \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$(-1, 0, 0)(1, 1, 0) = ((-1)(1)\zeta^{2\cdot 0\cdot 0}, 0 + 1, 0 + 0) = (-1, 1, 0)$$

$$(-1,1,0) \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$(-1,0,0)(1,0,1) = ((-1)(1)\zeta^{2\cdot 0\cdot 1}, 0+0, 0+1) = (-1,0,1)$$

$$(-1,0,1) \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and

$$(-1,1,0)(1,0,1) = ((-1)(1)\zeta^{2\cdot 1\cdot 1}, 1+0, 0+1) = (-1,1,1)$$

$$(-1,1,1) \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Table 1.1.2 summarizes the results.

Element	Representation
$(1,0,0)$ $(-1,0,0)$ $(1,1,0)$ $(-1,1,0)$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
$(-1,1,1)$ $(-1,0,1)$ $(1,1,1)$ $(1,0,1)$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Table 1.1.2: Table of representations for H_2 with $\phi_2 = \mathbb{1}$

Three Dimensional Representations with $\phi_2 = \mathbb{1}$

Take $p = 3$. $|H_3| = 27$. $\zeta_3 = \exp(2\pi i/3) = \frac{1}{2}(-1+\sqrt{3})$. The generators are $(\zeta_3, 0, 0)$, $(1,1,0)$, are $(1,0,1)$. The 27 elements of H_3 and their representations with $\phi_2 = \mathbb{1}$:

$$(\zeta_3,0,0)\mapsto\begin{bmatrix}\zeta_3&0&0\\0&\zeta_3&0\\0&0&\zeta_3\end{bmatrix},\quad(\zeta_3^2,0,0)\mapsto\begin{bmatrix}\zeta_3^2&0&0\\0&\zeta_3^2&0\\0&0&\zeta_3^2\end{bmatrix},\quad(\zeta_3^3,0,0)\mapsto\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix},$$

$$(1,0,1)\mapsto\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix},\qquad(1,1,0)\mapsto\begin{bmatrix}1&0&0\\0&\zeta_3^2&0\\0&0&\zeta_3^4\end{bmatrix}=\begin{bmatrix}1&0&0\\0&\zeta_3^2&0\\0&0&\zeta_3\end{bmatrix},$$

$$(\zeta_3,0,0)(1,0,1)=(\zeta_3(1)\zeta_3^{2\cdot0\cdot1},0,1)=(\zeta_3,0,1)$$

$$(\zeta_3,0,1)\mapsto\begin{bmatrix}\zeta_3&0&0\\0&\zeta_3&0\\0&0&\zeta_3\end{bmatrix}\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}=\begin{bmatrix}0&0&\zeta_3\\\zeta_3&0&0\\0&\zeta_3&0\end{bmatrix},$$

$$(\zeta_3,0,1)(1,0,1)=(\zeta_3(1)\zeta_3^{2\cdot0\cdot1},0+0,1+1)=(\zeta_3,0,2)$$

$$(\zeta_3,0,2)\mapsto\begin{bmatrix}0&0&\zeta_3\\\zeta_3&0&0\\0&\zeta_3&0\end{bmatrix}\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}=\begin{bmatrix}0&\zeta_3&0\\0&0&\zeta_3\\\zeta_3&0&0\end{bmatrix},$$

$$(\zeta_3^2,0,0)(1,0,1)=(\zeta_3^2(1)\zeta_3^{2\cdot0\cdot1},0,1)=(\zeta_3^2,0,1)$$

$$(\zeta_3^2,0,1)\mapsto\begin{bmatrix}\zeta_3^2&0&0\\0&\zeta_3^2&0\\0&0&\zeta_3^2\end{bmatrix}\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}=\begin{bmatrix}0&0&\zeta_3^2\\\zeta_3^2&0&0\\0&\zeta_3^2&0\end{bmatrix},$$

$$(\zeta_3,0,0)(1,1,0)=(\zeta_3(1)\zeta_3^{2\cdot0\cdot0},1,0)=(\zeta_3,1,0)$$

$$(\zeta_3, 1, 0) \mapsto \begin{bmatrix} \zeta_3 & 0 & 0 \\ 0 & \zeta_3 & 0 \\ 0 & 0 & \zeta_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \zeta_3^2 & 0 \\ 0 & 0 & \zeta_3 \end{bmatrix} = \begin{bmatrix} \zeta_3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \zeta_3^2 \end{bmatrix},$$

$$(\zeta_3^2, 0, 0)(1, 1, 0) = (\zeta_3^2(1)\zeta_3^{2\cdot 0\cdot 0}, 1, 0) = (\zeta_3^2, 1, 0)$$

$$(\zeta_3^2, 1, 0) \mapsto \begin{bmatrix} \zeta_3^2 & 0 & 0 \\ 0 & \zeta_3^2 & 0 \\ 0 & 0 & \zeta_3^2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \zeta_3^2 & 0 \\ 0 & 0 & \zeta_3 \end{bmatrix} = \begin{bmatrix} \zeta_3^2 & 0 & 0 \\ 0 & \zeta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$(1, 0, 1)(\zeta_3, 1, 0) = ((1)\zeta_3 \cdot \zeta_3^{2\cdot 0\cdot 0}, 1, 1) = (\zeta_3, 1, 1)$$

$$(\zeta_3, 1, 1) \mapsto \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \zeta_3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \zeta_3^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \zeta_3^2 \\ \zeta_3 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

$$(1, 0, 1)(1, 0, 1) = ((1)(1)\zeta_3^{2\cdot 0\cdot 1}, 0, 2) = (1, 0, 2)$$

$$(1, 0, 2) \mapsto \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix},$$

$$(1, 1, 0)(1, 1, 0) = ((1)(1)\zeta_3^{2\cdot 1\cdot 0}, 2, 0) = (1, 2, 0)$$

$$(1, 2, 0) \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & \zeta_3^2 & 0 \\ 0 & 0 & \zeta_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \zeta_3^2 & 0 \\ 0 & 0 & \zeta_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \zeta_3 & 0 \\ 0 & 0 & \zeta_3^2 \end{bmatrix},$$

$$(1, 0, 1)(1, 1, 0) = ((1)(1)\zeta_3^{2\cdot 0\cdot 0}, 1, 1) = (1, 1, 1)$$

$$(1,1,1) \mapsto \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \zeta_3^2 & 0 \\ 0 & 0 & \zeta_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \zeta_3 \\ 1 & 0 & 0 \\ 0 & \zeta_3^2 & 0 \end{bmatrix},$$

$$(\zeta_3, 0, 1)(1,1,1) = (\zeta_3(1)\zeta_3^{2\cdot 0\cdot 1}, 0+1, 1+1) = (\zeta_3, 1, 2)$$

$$(\zeta_3,1,2) \mapsto \begin{bmatrix} 0 & 0 & \zeta_3 \\ \zeta_3 & 0 & 0 \\ 0 & \zeta_3 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & \zeta_3 \\ 1 & 0 & 0 \\ 0 & \zeta_3^2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & \zeta_3^2 \\ \zeta_3 & 0 & 0 \end{bmatrix},$$

$$(1,0,2)(1,1,0) = ((1)(1)\zeta^{2\cdot 0\cdot 0}, 1, 2) = (1,1,2)$$

$$(1,1,2) \mapsto \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \zeta_3^2 & 0 \\ 0 & 0 & \zeta_3 \end{bmatrix} = \begin{bmatrix} 0 & \zeta_3^2 & 0 \\ 0 & 0 & \zeta_3 \\ 1 & 0 & 0 \end{bmatrix},$$

$$(1,0,1)(1,2,0) = ((1)(1)\zeta_3^{2\cdot 0\cdot 0}, 2, 1) = (1,2,1)$$

$$(1,2,1) \mapsto \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \zeta_3 & 0 \\ 0 & 0 & \zeta_3^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \zeta_3^2 \\ 1 & 0 & 0 \\ 0 & \zeta_3 & 0 \end{bmatrix},$$

$$(1,1,2)(1,1,0) = ((1)(1)\zeta^{2\cdot 1\cdot 0}, 2, 2) = (1,2,2)$$

$$(1,2,2) \mapsto \begin{bmatrix} 0 & \zeta_3^2 & 0 \\ 0 & 0 & \zeta_3 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \zeta_3^2 & 0 \\ 0 & 0 & \zeta_3 \end{bmatrix} = \begin{bmatrix} 0 & \zeta_3 & 0 \\ 0 & 0 & \zeta_3^2 \\ 1 & 0 & 0 \end{bmatrix},$$

$$(\zeta_3,1,0)(\zeta_3,1,0) = (\zeta_3 \cdot \zeta_3 \cdot \zeta_3^{2\cdot 1\cdot 0}, 1+1, 0) = (\zeta_3^2, 2, 0)$$

$$(\zeta_3^2,2,0)\mapsto\begin{bmatrix}\zeta_3&0&0\\0&1&0\\0&0&\zeta_3^2\end{bmatrix}\begin{bmatrix}\zeta_3&0&0\\0&1&0\\0&0&\zeta_3^2\end{bmatrix}=\begin{bmatrix}\zeta_3^2&0&0\\0&1&0\\0&0&\zeta_3\end{bmatrix},$$

$$(\zeta_3,0,1)(\zeta_3,0,1)=(\zeta_3\cdot\zeta_3\cdot{\zeta_3^{2\cdot0\cdot1}},0+0,1+1)=(\zeta_3^2,0,2)$$

$$(\zeta_3^2,0,2)\mapsto\begin{bmatrix}0&0&\zeta_3\\\zeta_3&0&0\\0&\zeta_3&0\end{bmatrix}\begin{bmatrix}0&0&\zeta_3\\\zeta_3&0&0\\0&\zeta_3&0\end{bmatrix}=\begin{bmatrix}0&\zeta_3^2&0\\0&0&\zeta_3^2\\\zeta_3^2&0&0\end{bmatrix},$$

$$(\zeta_3^2,2,0)(\zeta_3,0,1)=(\zeta_3^2\cdot\zeta_3\cdot\zeta_3^{2\cdot2\cdot1},2+0,0+1)=(\zeta_3^2,2,1)$$

$$(\zeta_3^2,2,1)\mapsto\begin{bmatrix}\zeta_3^2&0&0\\0&1&0\\0&0&\zeta_3\end{bmatrix}\begin{bmatrix}0&0&\zeta_3\\\zeta_3&0&0\\0&\zeta_3&0\end{bmatrix}=\begin{bmatrix}0&0&1\\\zeta_3&0&0\\0&\zeta_3^2&0\end{bmatrix},$$

(duplicate and it agrees with the previous result:)

$$(\zeta_3,0,1)(1,1,0)=(\zeta_3(1)\zeta_3^{0\cdot0},0+1,1+1)=(\zeta_3,1,1)$$

$$(\zeta_3,1,1)\mapsto\begin{bmatrix}0&0&\zeta_3\\\zeta_3&0&0\\0&\zeta_3&0\end{bmatrix}\begin{bmatrix}1&0&0\\0&\zeta_3^2&0\\0&0&\zeta_3\end{bmatrix}=\begin{bmatrix}0&0&\zeta_3^2\\\zeta_3&0&0\\0&1&0\end{bmatrix},$$

$$(\zeta_3^2,1,0)(\zeta_3,1,1)=(\zeta_3^2\cdot\zeta_3\cdot\zeta_3^{2\cdot1\cdot1},1+1,0+1)=(\zeta_3,2,1)$$

$$(\zeta_3,2,1)\mapsto\begin{bmatrix}\zeta_3^2&0&0\\0&\zeta_3&0\\0&0&1\end{bmatrix}\begin{bmatrix}0&0&\zeta_3^2\\\zeta_3&0&0\\0&1&0\end{bmatrix}=\begin{bmatrix}0&0&\zeta_3\\\zeta_3^2&0&0\\0&1&0\end{bmatrix},$$

$$(\zeta_3,2,1)(\zeta_3,0,2)=(\zeta_3\cdot\zeta_3\cdot\cdot\zeta_3^{2\cdot2\cdot2},2+0,1+2)=(\zeta_3,2,0)$$

$$(\zeta_3, 2, 0) \mapsto \begin{bmatrix} 0 & 0 & \zeta_3 \\ \zeta_3^2 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & \zeta_3 & 0 \\ 0 & 0 & \zeta_3 \\ \zeta_3 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \zeta_3^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \zeta_3 \end{bmatrix},$$

$$(\zeta_3, 0, 1)(\zeta_3, 1, 1) = (\zeta_3 \cdot \zeta_3 \cdot \zeta_3^{2 \cdot 0 \cdot 1}, 0+1, 1+1) = (\zeta_3^2, 1, 2)$$

$$(\zeta_3^2, 1, 2) \mapsto \begin{bmatrix} 0 & 0 & \zeta_3 \\ \zeta_3 & 0 & 0 \\ 0 & \zeta_3 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & \zeta_3^2 \\ \zeta_3 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \zeta_3 & 0 \\ 0 & 0 & 1 \\ \zeta_3^2 & 0 & 0 \end{bmatrix},$$

$$(\zeta_3, 0, 1)(\zeta_3, 2, 1) = (\zeta_3 \cdot \zeta_3 \cdot \zeta_3^{2 \cdot 0 \cdot 1}, 0+2, 1+1) = (\zeta_3^2, 2, 2)$$

$$(\zeta_3^2, 2, 2) \mapsto \begin{bmatrix} 0 & 0 & \zeta_3 \\ \zeta_3 & 0 & 0 \\ 0 & \zeta_3 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & \zeta_3 \\ \zeta_3^2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \zeta_3 & 0 \\ 0 & 0 & \zeta_3^2 \\ 1 & 0 & 0 \end{bmatrix},$$

$$(\zeta_3, 0, 1)(\zeta_3, 1, 0) = (\zeta_3 \cdot \zeta_3 \cdot \zeta_3^{2 \cdot 0 \cdot 0}, 0+1, 1+0) = (\zeta_3^2, 1, 1)$$

$$(\zeta_3^2, 1, 1) \mapsto \begin{bmatrix} 0 & 0 & \zeta_3 \\ \zeta_3 & 0 & 0 \\ 0 & \zeta_3 & 0 \end{bmatrix} \begin{bmatrix} \zeta_3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \zeta_3^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ \zeta_3^2 & 0 & 0 \\ 0 & \zeta_3 & 0 \end{bmatrix},$$

and

$$(\zeta_3, 1, 1)(\zeta_3^2, 1, 1) = (\zeta_3 \cdot \zeta_3^2 \cdot \zeta_3^{2 \cdot 1 \cdot 1}, 1+1, 1+1) = (\zeta_3, 2, 2)$$

$$(\zeta_3, 2, 2) \mapsto \begin{bmatrix} 0 & 0 & \zeta_3^2 \\ \zeta_3 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ \zeta_3^2 & 0 & 0 \\ 0 & \zeta_3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & \zeta_3 \\ \zeta_3^2 & 0 & 0 \end{bmatrix}.$$

The following table, Table 1.1.3, summarizes the representations.

$(\zeta_3, 0, 0) \mapsto \begin{bmatrix} \zeta_3 & 0 & 0 \\ 0 & \zeta_3 & 0 \\ 0 & 0 & \zeta_3 \end{bmatrix}$	$(\zeta_3, 0, 1) \mapsto \begin{bmatrix} 0 & 0 & \zeta_3 \\ \zeta_3 & 0 & 0 \\ 0 & \zeta_3 & 0 \end{bmatrix}$	$(\zeta_3, 0, 2) \mapsto \begin{bmatrix} 0 & \zeta_3 & 0 \\ 0 & 0 & \zeta_3 \\ \zeta_3 & 0 & 0 \end{bmatrix}$
$(\zeta_3, 1, 0) \mapsto \begin{bmatrix} \zeta_3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \zeta_3^2 \end{bmatrix}$	$(\zeta_3, 1, 1) \mapsto \begin{bmatrix} 0 & 0 & \zeta_3^2 \\ \zeta_3 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$	$(\zeta_3, 1, 2) \mapsto \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & \zeta_3^2 \\ \zeta_3 & 0 & 0 \end{bmatrix}$
$(\zeta_3, 2, 0) \mapsto \begin{bmatrix} \zeta_3^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \zeta_3 \end{bmatrix}$	$(\zeta_3, 2, 1) \mapsto \begin{bmatrix} 0 & 0 & \zeta_3 \\ \zeta_3^2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$	$(\zeta_3, 2, 2) \mapsto \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & \zeta_3 \\ \zeta_3^2 & 0 & 0 \end{bmatrix}$
$(\zeta_3^2, 0, 0) \mapsto \begin{bmatrix} \zeta_3^2 & 0 & 0 \\ 0 & \zeta_3^2 & 0 \\ 0 & 0 & \zeta_3^2 \end{bmatrix}$	$(\zeta_3^2, 0, 1) \mapsto \begin{bmatrix} 0 & 0 & \zeta_3^2 \\ \zeta_3^2 & 0 & 0 \\ 0 & \zeta_3^2 & 0 \end{bmatrix}$	$(\zeta_3^2, 0, 2) \mapsto \begin{bmatrix} 0 & \zeta_3^2 & 0 \\ 0 & 0 & \zeta_3^2 \\ \zeta_3^2 & 0 & 0 \end{bmatrix}$
$(\zeta_3^2, 1, 0) \mapsto \begin{bmatrix} \zeta_3^2 & 0 & 0 \\ 0 & \zeta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$(\zeta_3^2, 1, 1) \mapsto \begin{bmatrix} 0 & 0 & 1 \\ \zeta_3^2 & 0 & 0 \\ 0 & \zeta_3 & 0 \end{bmatrix}$	$(\zeta_3^2, 1, 2) \mapsto \begin{bmatrix} 0 & \zeta_3 & 0 \\ 0 & 0 & 1 \\ \zeta_3^2 & 0 & 0 \end{bmatrix}$

$(\zeta_3^2, 2, 0) \mapsto \begin{bmatrix} \zeta_3^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \zeta_3 \end{bmatrix}$	$(\zeta_3^2, 2, 1) \mapsto \begin{bmatrix} 0 & 0 & 1 \\ \zeta_3 & 0 & 0 \\ 0 & \zeta_3^2 & 0 \end{bmatrix}$	$(\zeta_3^2, 2, 2) \mapsto \begin{bmatrix} 0 & \zeta_3 & 0 \\ 0 & 0 & \zeta_3^2 \\ 1 & 0 & 0 \end{bmatrix}$
$(1, 0, 0) \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$(1, 0, 1) \mapsto \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$	$(1, 0, 2) \mapsto \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$
$(1, 1, 0) \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 1 & \zeta_3^2 & 0 \\ 0 & 0 & \zeta_3 \end{bmatrix}$	$(1, 1, 1) \mapsto \begin{bmatrix} 0 & 0 & \zeta_3 \\ 1 & 0 & 0 \\ 0 & \zeta_3^2 & 0 \end{bmatrix}$	$(1, 1, 2) \mapsto \begin{bmatrix} 0 & \zeta_3^2 & 0 \\ 0 & 0 & \zeta_3 \\ 1 & 0 & 0 \end{bmatrix}$
$(1, 2, 0) \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & \zeta_3 & 0 \\ 0 & 0 & \zeta_3^2 \end{bmatrix}$	$(1, 2, 1) \mapsto \begin{bmatrix} 0 & 0 & \zeta_3^2 \\ 1 & 0 & 0 \\ 0 & \zeta_3 & 0 \end{bmatrix}$	$(1, 2, 2) \mapsto \begin{bmatrix} 0 & \zeta_3 & 0 \\ 0 & 0 & \zeta_3^2 \\ 1 & 0 & 0 \end{bmatrix}$

Table 1.1.3: Table of representations for H_3 with $\phi_2 = \mathbb{1}$.

1.2 Automorphism group of H_p

What is the automorphism group of H_p? Let $\mathrm{Aut}^{\mathrm{Sym}}_{\nu_p}(H_p)$ denote the group of automorphisms of H_p that fix the center, ν_p, of H_p. The claim is that for $\phi' \in \mathrm{Aut}(H_p)$,

$$\begin{array}{ccc} \mathrm{Aut}(H_p) & \longrightarrow\!\!\!\!\rightarrow & \mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z}) \\ & & . \\ \phi' & \longmapsto & \phi \in \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \end{array}$$

Given

$$\nu_p \subseteq H_p \xrightarrow{\ \rho\ } \mathrm{GL}_p(\mathbb{C})$$

$$\nu_p \ni \zeta_p \longmapsto \begin{bmatrix} \zeta_p & & 0 \\ & \ddots & \\ 0 & & \zeta_p \end{bmatrix}$$

where $\mu_p \to \mathbb{C}^\times$ and $\zeta_p \to \zeta_p$, we have

$$H_p \xrightarrow{\ \sigma\ } H_p \xrightarrow{\ \rho\ } \mathrm{GL}_p(\mathbb{C}) \ .$$

$\rho^\sigma := \rho \circ \sigma$ is an irreducible representation of dimension p such that ν_p acts naturally. This implies $\rho^\sigma \cong \rho$. Call this isomorphism $\mathbb{W}(\sigma) \in \mathrm{PGL}_p(\mathbb{C})$. Recall that $\mathrm{PGL}(\mathbb{C}) \cong \mathrm{GL}(\mathbb{C})/Z(\mathrm{GL}(\mathbb{C}))$. By Schur's Lemma, $\mathbb{W}(\sigma)\rho^\sigma = \rho\mathbb{W}(\sigma)$. We would like to prove the following: $\sigma \mapsto \mathbb{W}(\sigma)$ gives a representation

$$\mathrm{Aut}^{\mathrm{Sym}}_{\nu_p}(H_p) \cong \mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z}) \longrightarrow \mathrm{PGL}_p(\mathbb{C}).$$

In order to do so, we will require additional definitions and concepts.

Symplectic Group

From Wikipedia[41] we have the following definitions and facts regarding the symplectic group.

Definition 1.2.1. *A **symplectic matrix** is a $2n \times 2n$ matrix M with entries from* *a field F that satisfies the condition $M^T \Omega M = \Omega$, where Ω is a non-singular, skew-symmetric matrix. Typically, Ω is chosen to be the block matrix*

$$\Omega = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix},$$

where I_n is the $n \times n$ identity matrix. The matrix Ω has determinant $+1$ and $\Omega^{-1} = \Omega^T = -\Omega$. Every symplectic matrix has determinant $+1$.

Definition 1.2.2. *A* ***symplectic vector space*** *vector space over a field F equipped with a symplectic bilinear form. A* ***symplectic bilinear form*** *is a mapping $\omega : V \times V \to F$ that is*

- *bilinear*
- *alternating: $\omega(v, v) = 0$ holds for all $v \in V$, and*
- *nondegenerate: $\omega(u, v) = 0$ for all $v \in V$ implies that $u = 0$.*

For fields whose characteristic is not 2, alternation is equivalent to skew-symmetric. The abstract analog of a symplectic matrix is a **symplectic transformation** of a symplectic vector space. A symplectic transformation is then a linear transformation $L : V \to V$ which preserves ω: $\omega(Lu, Lv) = \omega(u, v)$.

The **symplectic group** is a classical group defined as the set of transformations of a $2n$-dimensional vector space over the field F which preserve a non-degenerate skew-symmetric bilinear form. Such a vector space is called a symplectic vector space, and the symplectic group of an abstract symplectic vector space V is denoted $\mathrm{Sp}(V)$. Upon fixing a basis for V, the symplectic group becomes the group of symplectic $2n \times 2n$ matrices with entries in F under the operation of matrix multiplication.

The symplectic group $\mathrm{Sp}(2, \mathbb{F}_p)$ is isomorphic to $\mathrm{SL}_2(\mathbb{F}_p)$. To show this we use the matrix form of the symplectic group with $n = 1$. Then for matrices M with entries from

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \nu_p & \longrightarrow & H_p & \longrightarrow & K_p & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \overline{\psi}} & & \\ 0 & \longrightarrow & \nu_p & \longrightarrow & H_p & \longrightarrow & K_p & \longrightarrow & 0 \end{array}$$

Figure 1.2.1: Commutative diagram for the construction of $\mathrm{Aut}(H_p)$

$\mathbb{F}_p$, we solve the equation $M^T \Omega M = \Omega$ with $\Omega = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$ and letting $M = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$, we have

$$\begin{aligned} M^T \Omega M &= \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} c & d \\ -a & -b \end{bmatrix} \\ &= \begin{bmatrix} ac - ac & ad - bc \\ bc - ad & bd - bd \end{bmatrix} = \begin{bmatrix} 0 & ad - bc \\ bc - ad & 0 \end{bmatrix} \\ &= \Omega \\ &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \end{aligned}$$

This implies that $ad - bc = 1$. So $\det M = 1$ and that implies $M \in \mathrm{SL}_2(\mathbb{F}_p)$ and every element of $\mathrm{SL}_2(\mathbb{F}_p)$ satisfies the equation. So $\mathrm{SL}_2(\mathbb{F}_p) \cong \mathrm{Sp}(2, \mathbb{F}_p)$.

1.2.1 Constructing $\mathrm{Aut}(H_p)$

We rewrite the material from [11] to construct the $\mathrm{Aut}(H_p)$. Some notational reference for [11]: $\mathbb{G}_{m,k}$ is ν_p, δ is p and $\mathcal{H}(\delta)$ is H_p. We denote by $\mathrm{Aut}_{\nu_p}(H_p)$ the group of automorphisms ψ of H_p inducing the identity on ν_p (that fix ν_p), that is, the group of automorphisms ψ fitting in a diagram, Figure 1.2.1, [11] of the form

The commutator pairing $e_p : K_p \times K_p \to \nu_p$ sends $(x_1, x_2) \mapsto [x_1, x_2]$. The commutator pairing is a map that does not appear to be mentioned outside of Weil Representations. The commutativity of the diagram shows that the induced automorphism $\overline{\phi}$ is symplectic with respect to the commutator pairing. For all $x_1, x_2 \in K_p$, $e_p(\overline{\psi}(x_1), \overline{\psi}(x_2)) = e_p(x_1, x_2)$. Denote by $\mathrm{Sp}(K_p)$ the group of symplectic automorphisms of K_p. In order to study the possible extensions of $\overline{\psi} \in \mathrm{Sp}(K_p)$ to an element of $\mathrm{Aut}_{\nu_p}(H_p)$ it is convenient to introduce the following definition [11, Def. 13]:

Definition 1.2.3. *[11, Def. 13] Let $\overline{\psi} \in \mathrm{Sp}(K_p)$. A $\overline{\psi}$-semi-character (or a semi-character if no confusion is possible) for the canonical pairing is a map $\chi_{\overline{\psi}} : K_p \to \nu_p$ such that for $(x_1, x_2), (x_1', x_2') \in K_p$,*

$$\chi_{\overline{\psi}}(x_1 + x_1', x_2 + x_2') = \chi_{\overline{\psi}}((x_1, x_2) \cdot \chi_{\overline{\psi}}(x_1', x_2')) \cdot [(\overline{\psi}(x_1', x_2')_2(\overline{\psi}(x_1, x_2)_1)] \cdot x_2'(x_1)^{-1}),$$

where we write $\overline{\psi}(x_1, x_2) = (\overline{\psi}(x_1, x_2)_1, \overline{\psi}(x_1, x_2)_2)$ (respectively $\overline{\psi}(x_1', x_2') = (\overline{\psi}(x_1', x_2')_1, \overline{\psi}(x_1', x_2')_2))$. in the canonical decomposition of K_p. A semi-character $\chi_{\overline{\psi}}$ is said to be symmetric if for all $(x_1, x_2) \in K_p$, $\chi_{\overline{\psi}}(-(x_1, x_2)) = \chi_{\overline{\psi}}(x_1, x_2)$.

1.2.2 Semi-characters

The definition, [11, Def. 13], needs to be modified for our application.

General Case

Let $A \in \mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})$ and let $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$. For $\psi_A \in \mathrm{Aut}^{\mathrm{Sym}}_{\nu_p}(H_p)$, we are interested in the mapping

$$\begin{aligned} \mathrm{Aut}^{\mathrm{Sym}}_{\nu_p}(H_p) &\to \mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z}) \\ \psi_A &\mapsto A. \end{aligned}$$

Let $\overline{\psi}_A$ be the image of ψ_A and let $\chi_{\overline{\psi}_A}$ denote its semi-character. Then

$$\begin{aligned} \psi_A(\lambda_1, x_1, y_1) &= \psi_A(\lambda_1, 0, 0) \cdot \psi_A(1, x_1, y_1) \\ &= (\lambda_1, 0, 0) \cdot (1 \cdot \chi_{\overline{\psi}_A}(x_1, y_1), ax_1 + by_1, cx_1 + dy_1) \end{aligned}$$

$$= (\lambda_1, 0, 0) \cdot (\chi_{\overline{\psi}_A}(x_1, y_1), ax_1 + by_1, cx_1 + dy_1)$$

$$= (\lambda \chi_{\overline{\psi}_A}(x_1, y_1), ax_1 + by_1, cx_1 + dy_1).$$

For general elements, we have

$$\psi_A(\lambda_1, x_1, y_1)\psi_A(\lambda_2, x_2, y_2) = \psi_A(\lambda_1\lambda_2 \cdot \zeta_p^{2x_1y_2}, x_1 + x_2, y_1 + y_2).$$

Since

$$\begin{aligned}\psi_A(\lambda_1\lambda_2 \cdot \zeta_p^{2x_1y_2}, x_1 + x_2, y_1 + y_2) = (&\lambda_1\lambda_2\zeta_p^{2x_1y_2}\chi_{\overline{\psi}_A}(x_1 + x_2, y_1 + y_2),\\ &a(x_1 + x_2) + b(y_1 + y_2), c(x_1 + x_2) + d(y_1 + y_2))\end{aligned}$$

and

$$\begin{aligned}\psi_A(\lambda_1, x_1, y_1)\psi_A(\lambda_2, x_2, y_2) &= (\lambda_1\chi_{\overline{\psi}_A}(x_1, y_1), ax_1 + by_1, cx_1 + dy_1)\cdot\\ &\quad (\lambda_2\chi_{\overline{\psi}_S}(x_2, y_2), ax_2 + by_2, cx_2 + dy_2)\\ &= (\lambda_1\lambda_2\chi_{\overline{\psi}_A}(x_1, y_1)\chi_{\overline{\psi}_A}(x_2, y_2) \cdot \zeta_p^{2(ax_1+by_1)(cx_2+dy_2)},\\ &\quad a(x_1 + x_2) + b(y_1 + y_2), c(x_1 + x_2) + d(y_1 + y_2)),\end{aligned}$$

it is required that

$$\begin{aligned}\lambda_1\lambda_2 \cdot \zeta_p^{2x_1y_2}\chi_{\overline{\psi}_A}(x_1 + x_2, y_1 + y_2) &= \lambda_1\lambda_2 \cdot \chi_{\overline{\psi}_A}(x_1, y_1) \cdot \chi_{\overline{\psi}_S}(x_2, y_2) \cdot \zeta_p^{2(ax_1+by_1)(cx_2+dy_2)}\\ &= \lambda_1\lambda_2 \cdot \chi_{\overline{\psi}_A}(x_1, y_1) \cdot \chi_{\overline{\psi}_S}(x_2, y_2)\\ &\quad \cdot \zeta_p^{2(acx_1x_2+bcy_1x_2+adx_1y_2+bdy_1y_2)}.\end{aligned}$$

This requires

$$\begin{aligned}\chi_{\overline{\psi}_A}(x_1 + x_2, y_1 + y_2) &= \chi_{\overline{\psi}_A}(x_1, y_1) \cdot \chi_{\overline{\psi}_S}(x_2, y_2) \cdot \zeta_p^{2(acx_1x_2+bcy_1x_2+adx_1y_2+bdy_1y_2-x_1y_2)}\\ &= \chi_{\overline{\psi}_A}(x_1, y_1) \cdot \chi_{\overline{\psi}_S}(x_2, y_2) \cdot \zeta_p^{2(acx_1x_2+bcy_1x_2+(ad-1)x_1y_2+bdy_1y_2)}\\ &= \chi_{\overline{\psi}_A}(x_1, y_1) \cdot \chi_{\overline{\psi}_S}(x_2, y_2) \cdot \zeta_p^{2(acx_1x_2+bcy_1x_2+(bc)x_1y_2+bdy_1y_2)}\\ &= \chi_{\overline{\psi}_A}(x_1, y_1) \cdot \chi_{\overline{\psi}_S}(x_2, y_2) \cdot \zeta_p^{2(acx_1x_2+bc(y_1x_2+x_1y_2)+bdy_1y_2)}.\end{aligned}$$

So our modified definition of a semi-character is

Definition 1.2.4 (Semi-Character)**.** *Let* $\overline{\psi} = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{Sp}(K_p) = \mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})$*. A* $\overline{\psi}$*-semi-character for the canonical pairing is a map* $\chi_{\overline{\psi}} : K_p \to \nu_p$ *such that for* $(x_1, x_2), (x'_1, x'_2) \in K_p$*,*

$$\chi_{\overline{\psi}}(x_1 + x'_1, x_2 + x'_2) = \chi_{\overline{\psi}}(x_1, x_2) \cdot \chi_{\overline{\psi}}(x'_1, x'_2) \cdot \zeta_p^{2(acx_1x_2 + bc(y_1x_2 + x_1y_2) + bdy_1y_2)}.$$

A semi-character $\chi_{\overline{\psi}}$ *is said to be symmetric if for all* $(x_1, x_2) \in K_p$*,* $\chi_{\overline{\psi}}(-(x_1, x_2)) = \chi_{\overline{\psi}}(x_1, x_2)$*.*

For the generator S, ζ_p^{-2xy} is a symmetric semi-character.

Let $S \in \mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})$ and let $S = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$. For $\psi_S \in \mathrm{Aut}^{\mathrm{Sym}}_{\nu_p}(H_p)$, we are interested in the mapping

$$\begin{aligned} \mathrm{Aut}^{\mathrm{Sym}}_{\nu_p}(H_p) &\mapsto \mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z}), \\ \psi_S &\to S. \end{aligned}$$

Let $\overline{\psi}_S$ be the image of ψ_S and let $\chi_{\overline{\psi}_S}$ denote its semi-character. Then

$$\begin{aligned} \psi_S(\lambda_1, x_1, y_1) &= \psi_S(\lambda_1, 0, 0) \cdot \psi_S(1, x_1, y_1) \\ &= (\lambda_1, 0, 0) \cdot (1 \cdot \chi_{\overline{\psi}_S}(x_1, y_1), -y_1, x_1) \\ &= (\lambda_1, 0, 0) \cdot (\chi_{\overline{\psi}_S}(x_1, y_1), -y_1, x_1) \\ &= (\lambda \chi_{\overline{\psi}_S}(x_1, y_1), -y_1, x_1). \end{aligned}$$

For general elements, we have

$$\psi_S(\lambda_1, x_1, y_1)\psi_S(\lambda_2, x_2, y_2) = \psi_S(\lambda_1\lambda_2 \cdot \zeta_p^{2x_1y_2}, x_1 + x_2, y_1 + y_2).$$

Since

$$\begin{aligned} \psi_S(\lambda_1\lambda_2 \cdot \zeta_p^{2x_1y_2}, x_1 + x_2, y_1 + y_2) = (\lambda_1\lambda_2 \cdot \zeta_p^{2x_1y_2} \chi_{\overline{\psi}_S}(x_1 + x_2, y_1 + y_2), \\ -(y_1 + y_2), (x_1 + x_2)) \end{aligned}$$

and

$$\begin{aligned}\psi_S(\lambda_1,x_1,y_1)\psi_S(\lambda_2,x_2,y_2) &= (\lambda_1\chi_{\overline{\psi}_S}(x_1,y_1),-y_1,x_1)(\lambda_2\chi_{\overline{\psi}_S}(x_2,y_2),-y_2,x_2)\\ &= (\lambda_1\lambda_2\cdot\chi_{\overline{\psi}_S}(x_1,y_1)\cdot\chi_{\overline{\psi}_S}(x_2,y_2)\cdot\zeta_p^{-2y_1x_2},\\ &\quad -(y_1+y_2),(x_1+x_2)),\end{aligned}$$

it is required that

$$\lambda_1\lambda_2\cdot\zeta_p^{2x_1y_2}\chi_{\overline{\psi}_S}(x_1+x_2,y_1+y_2) = \lambda_1\lambda_2\cdot\chi_{\overline{\psi}_S}(x_1,y_1)\cdot\chi_{\overline{\psi}_S}(x_2,y_2)\cdot\zeta_p^{-2y_1x_2}.$$

This requires

$$\chi_{\overline{\psi}_S}(x_1+x_2,y_1+y_2) = \chi_{\overline{\psi}_S}(x_1,y_1)\cdot\chi_{\overline{\psi}_S}(x_2,y_2)\cdot\zeta_p^{-2y_1x_2-2x_1y_2}.$$

Substituting the $a=0, b=-1, c=1, d=0$ in our definition,

$$\zeta_p^{2(acx_1x_2+bc(y_1x_2+x_1y_2)+bdy_1y_2)} = \zeta_p^{-2(y_1x_2+x_1y_2)}$$

we verify the calculation agrees with the definition. So what is $\chi_{\overline{\psi}_S}(x,y)$? Let's verify that $\chi_{\overline{\psi}_S}(x,y)=\zeta_p^{-2xy}$ is a semi-character.

$$\begin{aligned}\chi_{\overline{\psi}_S}(x_1,y_1)\chi_{\overline{\psi}_S}(x_2,y_2)\cdot\zeta_p^{-2x_1y_1-2x_2y_2} &= \zeta_p^{-2x_1y_1}\cdot\zeta_p^{-2x_2y_2}\cdot\zeta_p^{-2x_2y_1-2x_1y_2}\\ &= \zeta_p^{-2(x_1+x_2)(y_1+y_2)}\\ &= \chi_{\overline{\psi}_S}(x_1+x_2,y_1+y_2).\end{aligned}$$

$\chi_{\overline{\psi}_S}(x,y)=\zeta_p^{-2xy}$ is also symmetric since

$$\begin{aligned}\chi_{\overline{\psi}_S}(-(x,y)) &= \chi_{\overline{\psi}_S}(-x,-y)\\ &= \zeta_p^{-2(-x)(-y)}\\ &= \zeta_p^{-2xy}\\ &= \chi_{\overline{\psi}_S}(x,y).\end{aligned}$$

For the generator T, $\zeta_p^{y^2}$ is a symmetric semi-character.

Let $T \in \mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})$ and let $T = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$. For $\psi_T \in \mathrm{Aut}_{\nu_p}^{\mathrm{Sym}}(H_p)$, we are interested in the mapping

$$\begin{aligned} \mathrm{Aut}_{\nu_p}^{\mathrm{Sym}}(H_p) &\mapsto \mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z}), \\ \psi_T &\to T. \end{aligned}$$

Let $\overline{\psi}_T$ be the image of ψ_T and let $\chi_{\overline{\psi}_T}$ denote its semi-character. Then

$$\begin{aligned} \psi_T(\lambda_1, x_1, y_1) &= \psi_T(\lambda_1, 0, 0) \cdot \psi_T(1, x_1, y_1) \\ &= (\lambda_1, 0, 0) \cdot (1 \cdot \chi_{\overline{\psi}_T}(x_1, y_1), x_1 + y_1, y_1) \\ &= (\lambda_1, 0, 0) \cdot (\chi_{\overline{\psi}_T}(x_1, y_1), x_1 + y_1, y_1) \\ &= (\lambda \chi_{\overline{\psi}_T}(x_1, y_1), x_1 + y_1, y_1). \end{aligned}$$

For general elements, we have

$$\psi_T(\lambda_1, x_1, y_1)\psi_T(\lambda_2, x_2, y_2) = \psi_T(\lambda_1\lambda_2 \cdot \zeta_p^{2x_1y_2}, x_1 + x_2, y_1 + y_2).$$

Since

$$\begin{aligned} \psi_T(\lambda_1\lambda_2 \cdot \zeta_p^{x_1y_2}, x_1 + x_2, y_1 + y_2) = (&\lambda_1\lambda_2 \cdot \zeta_p^{x_1y_2}\chi_{\overline{\psi}_T}(x_1 + x_2, y_1 + y_2), \\ &(x_1 + y_1 + x_2 + y_2), (y_1 + y_2)) \end{aligned}$$

and

$$\begin{aligned} \psi_T(\lambda_1, x_1, y_1)\psi_T(\lambda_2, x_2, y_2) &= (\lambda_1\chi_{\overline{\psi}_T}(x_1, y_1), x_1 + y_1, y_1)(\lambda_2\chi_{\overline{\psi}_T}(x_2, y_2), x_2 + y_2, y_2) \\ &= (\lambda_1\lambda_2 \cdot \chi_{\overline{\psi}_T}(x_1, y_1) \cdot \chi_{\overline{\psi}_T}(x_2, y_2) \cdot \zeta_p^{2(x_1+y_1)y_2}, \\ &\qquad (x_1 + y_1 + x_2 + y_2), (y_1 + y_2)), \end{aligned}$$

it is required that

$$\lambda_1\lambda_2 \cdot \zeta_p^{2x_1y_2}\chi_{\overline{\psi}_T}(x_1 + x_2, y_1 + y_2) = \lambda_1\lambda_2 \cdot \chi_{\overline{\psi}_T}(x_1, y_1) \cdot \chi_{\overline{\psi}_T}(x_2, y_2) \cdot \zeta_p^{2(x_1+y_1)y_2}.$$

This requires

$$\chi_{\overline{\psi}_T}(x_1+x_2, y_1+y_2) = \chi_{\overline{\psi}_T}(x_1, y_1)\cdot\chi_{\overline{\psi}_T}(x_2, y_2)\cdot\zeta_p^{2y_1y_2}.$$

Letting $a=1, b=1, c=0$, and $d=1$, we see that for $\chi_{\overline{\psi}_T}$, letting

$$\zeta_p^{2(acx_1x_2+bc(y_1x_2+x_1y_2)+bdy_1y_2)} = \zeta_p^{2y_1y_2}$$

meets the definition of a semi-character. Lets determine if $\chi_{\overline{\psi}_T}(x,y) = \zeta_p^{x^2}$ is a semi-character.

$$\chi_{\overline{\psi}_T}(x_1, y_1)\chi_{\overline{\psi}_T}(x_2, y_2)\cdot\zeta_p^{2y_1y_2} = \zeta_p^{x_1^2}\cdot\zeta_p^{x_2^2}\cdot\zeta_p^{2y_1y_2} = \zeta_p^{(x_1^2+x_2^2+2y_1y_2)}$$

and

$$\chi_{\overline{\psi}_T}(x_1+x_2, y_1+y_2) = \zeta_p^{(x_1+x_2)^2} = \zeta_p^{(x_1^2+2x_1x_2+x_2^2)}$$

show that $\chi_{\overline{\psi}_T}(x,y) = \zeta_p^{x^2}$ is NOT a semi-character.

Lets determine if $\chi_{\overline{\psi}_T}(x,y) = \zeta_p^{y^2}$ is a semi-character.

$$\chi_{\overline{\psi}_T}(x_1, y_1)\chi_{\overline{\psi}_T}(x_2, y_2)\cdot\zeta_p^{2y_1y_2} = \zeta_p^{y_1^2}\cdot\zeta_p^{y_2^2}\cdot\zeta_p^{2y_1y_2} = \zeta_p^{(y_1^2+y_2^2+2y_1y_2)} = \zeta_p^{(y_1+y_2)^2}$$

and

$$\chi_{\overline{\psi}_T}(x_1+x_2, y_1+y_2) = \zeta_p^{(y_1+y_2)^2} = \zeta_p^{(y_1^2+2y_1y_2+y_2^2)}$$

show that $\chi_{\overline{\psi}_T}(x,y) = \zeta_p^{y^2}$ is a semi-character. $\chi_{\overline{\psi}_T}(x,y) = \zeta_p^{y^2}$ is also symmetric since

$$\chi_{\overline{\psi}_T}(-(x,y)) = \chi_{\overline{\psi}_T}(-x,-y) = \zeta_p^{(-y)^2} = \zeta_p^{y^2} = \chi_{\overline{\psi}_T}(x,y).$$

The ST and its semicharacter

Can we determine semicharacters of an arbitrary element of $\mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})$? We have

$$ST = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$$

and

$$\begin{aligned}\psi_{ST}(\lambda, x, y) &= (\psi_S \circ \psi_T)(\lambda, x, y)\\ &= \psi_S(\lambda\chi_{\overline{\psi}}(x, y), x + y, y))\\ &= \psi_S(\lambda\zeta_p^{y^2}, x + y, y))\\ &= (\lambda\zeta_p^{y^2}\zeta_p^{-2(x+y)(y)}, -y, (x + y))\\ &= (\lambda\zeta_p^{y^2-2xy-2y^2}, -y, (x + y))\\ &= (\lambda\zeta_p^{-y^2-2xy}, -y, (x + y)).\end{aligned}$$

Is $\zeta_p^{-y^2-2xy}$ a semicharacter for ST? We have

$$\begin{aligned}\chi_{\overline{\psi_{ST}}}(x_1 + x_2, y_1 + y_2) &= \zeta_p^{-(y_1+y_2)^2-2(x_1+y_1)(y_1+y_2)}\\ &= \zeta_p^{-y_1^2-2y_1y_2-y_2^2-2x_1y_1-2x_1y_2-2x_2y_1-2x_2y_2}\end{aligned}$$

and

$$\begin{aligned}&\chi_{\overline{\psi_{ST}}}(x_1, y_1)\chi_{\overline{\psi_{ST}}}(x_2, y_2)\zeta_p^{2(acx_1x_2+bc(y_1x_2+x_1y_2)+bdy_1y_2)} =\\ &= \chi_{\overline{\psi_{ST}}}(x_1, y_1)\chi_{\overline{\psi_{ST}}}(x_2, y_2)\zeta_p^{2(0cx_1x_2+(-1)(1)(y_1x_2+x_1y_2)+(-1)(1)y_1y_2)}\\ &= \chi_{\overline{\psi_{ST}}}(x_1, y_1)\chi_{\overline{\psi_{ST}}}(x_2, y_2)\zeta_p^{2(-y_1x_2-x_1y_2-y_1y_2)}\\ &= \zeta_p^{-y_1^2-2x_1y_1}\zeta_p^{-y_2^2-2x_2y_2}\zeta_p^{-2y_1x_2-2x_1y_2-2y_1y_2}\\ &= \zeta_p^{-y_1^2-2x_1y_1-y_2^2-2x_2y_2-2y_1x_2-2x_1y_2-2y_1y_2}\\ &= \zeta_p^{-y_1^2-2y_1y_2-y_2^2-2x_1y_1-2x_1y_2-2x_2y_1-2x_2y_2}.\end{aligned}$$

So it is a semi-character and it is also symmetric:

$$\chi_{\overline{\psi_{ST}}}(-(x, y)) = \chi_{\overline{\psi_{ST}}}(-x, -y) = \zeta_p^{(-y)^2-2(-x)(-y)} = \zeta_p^{-y^2-2xy} = \chi_{\overline{\psi_{ST}}}(x, y).$$

The above illustrates how to obtain the semi-character for a given element of $\mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})$. Definition 1.2.3 and its modification 1.2.4 were motivated by the following lemma[11, Lem. 14].

Lemma 1.2.5. *[11, Lem. 14] Let $\psi \in \mathrm{Aut}_{\nu_p} H_p$ and let $\overline{\psi}$ be the associated symplectic automorphism of K_p. Then there exists a unique semi-character $\chi_{\overline{\psi}}$ such that for all $(\alpha, (x_1, x_2)) \in H_p$,*

$$\psi : (\alpha, (x_1, x_2)) \mapsto (\alpha\chi_{\overline{\psi}}((x_1, x_2), \overline{\psi}((x_1, x_2)). \tag{1.2.1}$$

As a consequence, if $\overline{\psi} \in \mathrm{Sp}(K_p)$ there is a one on one correspondence between the set of extensions of $\overline{\psi}$ to $\mathrm{Aut}_{\nu_p} H_p$ and the set of semi-characters.

Proof. Note that (1.2.1) uniquely defines a map $\chi_{\overline{\psi}}$ given ψ, and conversely also uniquely defines a map ψ and given $\overline{\psi}$. Moreover, by writing out the definitions $\chi_{\overline{\psi}}$ is a semicharacter if and only if ψ is a homomorphism. □

1.2.3 ψ_c is an automorphism

Switching notations for convenience, let

$$K_p := \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z},$$

and define the map

$$e_p : K_p \times K_p \to \nu_p \subset \mathbb{C}^\times$$

where

$$(x_1, y_1, x_2, y_2) \mapsto \zeta_p^{2(x_1y_2 - y_1x_2)}.$$

Proposition 1.2.6. *Let $c \in K_p$ and $\psi_c((\lambda, x, y)) := (\lambda e_p(c, (x, y)), x, y)$. Then ψ_c is an automorphism that fixes the center of H_p.*

Proof. It is clear that the image of ψ_c lies in H_p. We next verify that ψ_c is a homomorphism. Fix $c = (c_1, c_2) \in K_p$.

$$\psi_c((\lambda_1, x_1, y_1))\psi_c((\lambda_2, x_2, y_2)) = (\lambda_1 e_p(c, (x_1, y_1)), x_1, y_1)(\lambda_2 e_p(c, (x_2, y_2)), x_2, y_2)$$

$$= (\lambda_1 e_p((c_1, c_2), (x_1, y_1)), x_1, y_1) \cdot$$
$$(\lambda_2 e_p((c_1, c_2), (x_2, y_2)), x_2, y_2)$$
$$= (\lambda_1 \zeta_p^{2c_1y_1 - 2c_2x_1}, x_1, y_1)(\lambda_2 \zeta_p^{2c_1y_2 - 2c_2x_2}, x_2, y_2)$$
$$= (\lambda_1 \lambda_2 \zeta_p^{2(c_1y_1 - c_2x_1)} \zeta_p^{2(c_1y_2 - c_2x_2)} \zeta^{2x_1y_2}, x_1 + x_2, y_1 + y_2)$$
$$= (\lambda_1 \lambda_2 \zeta_p^{(2c_1y_1 - 2c_2x_1 + 2c_1y_2 - 2c_2x_2 + 2x_1y_2)}, x_1 + x_2, y_1 + y_2).$$

$$\begin{aligned}
\psi_c((\lambda_1, x_1, y_1)(\lambda_2, x_2, y_2)) &= \psi_c(\lambda_1 \lambda_2 \zeta_p^{2x_1y_2}, x_1 + x_2, y_1 + y_2) \\
&= (\lambda_1 \lambda_2 \zeta_p^{2x_1y_2} e_p((c_1, c_2), (x_1 + x_2), (y_1 + y_2) \\
&= (\lambda_1 \lambda_2 \zeta_p^{2x_1y_2} \zeta_p^{2c_1(y_1+y_2) - 2c_2(x_1+x_2)} \\
&= (\lambda_1 \lambda_2 \zeta_p^{2x_1y_2} \zeta_p^{2c_1y_1 + 2c_1y_2 - 2c_2x_1 - 2c_2x_2} \\
&= (\lambda_1 \lambda_2 \zeta_p^{2x_1y_2 + 2c_1y_1 + 2c_1y_2 - 2c_2x_1 - 2c_2x_2}, x_1 + x_2, y_1 + y_2) \\
&= (\lambda_1 \lambda_2 \zeta_p^{2c_1y_1 - 2c_2x_1 + 2c_1y_2 - 2c_2x_2 + 2x_1y_2}, x_1 + x_2, y_1 + y_2) \\
&= \psi_c((\lambda_1, x_1, y_1)) \psi_c((\lambda_2, x_2, y_2)).
\end{aligned}$$

Since H_p is finite, it suffices to show that ψ_c is injective for ψ_c to be bijective. We have

$$\psi_c((\lambda_1, x_1, y_1)) = (\lambda_1 \zeta_p^{2c_1y_1 - 2c_2x_1}, x_1, y_1),$$

and

$$\psi_c((\lambda_2, x_2, y_2)) = (\lambda_2 \zeta_p^{2c_1y_2 - 2c_2x_2}, x_2, y_2).$$

If $\psi_c((\lambda_1, x_1, y_1)) = \psi_c((\lambda_2, x_2, y_2))$ then $(\lambda_1 \zeta_p^{2c_1y_1 - 2c_2x_1}, x_1, y_1) = (\lambda_2 \zeta_p^{2c_1y_2 - 2c_2x_2}, x_2, y_2)$. This implies $x_1 = x_2$ and $y_1 = y_2$. So we now have

$$(\lambda_1 \zeta_p^{2(c_1y_1 - c_2x_1)}, x_1, y_1) = (\lambda_2 \zeta_p^{2(c_1y_1 - c_2x_1)}, x_1, y_1)$$

which implies $\lambda_1 = \lambda_2$. So, ψ_c is an automorphism since it is a bijective endomorphism of H_p. It also fixes the center since for any element $(\lambda, 0, 0) \in Z(H_p)$, we have

$$\psi_c((\lambda, 0, 0)) = (\lambda \zeta_p^{2(c_1 \cdot 0 - c_2 \cdot 0)}, 0, 0) = (\lambda, 0, 0).$$

□

1.2.4 Kernel of Φ

Consider the map Φ from the group of automorphisms of H_p that fix the center to the automorphisms of K_p:

$$\Phi : \mathrm{Aut}_{\nu_p}(H_p) \to \mathrm{Aut}(K_p) \cong \mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z}).$$

We would like to determine its kernel. To do this, we will need to apply a finite dimensional version of the Riesz Representation Theorem.

Theorem 1.2.7 (Riesz Representation Theorem)**.** *Let V be a finite dimensional inner product space whose inner product $\langle \cdot, \cdot \rangle$ is non-degenerate and bilinear and sends*

$$v \mapsto (w \mapsto \langle v, w \rangle).$$

Denote this map f. Then f is an isomorphism: $f : V \overset{\cong}{\to} V^*$.

It is clear that $K_p = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ is 2-dim vector space over $\mathbb{F}_p$. The map

$$e_p : K_p \times K_p \to \mathbb{C}^\times$$

that sends

$$(a, b) \times (c, d) \mapsto \zeta_p^{2(ad - bc)}$$

is a symplectic bi-linear form. In other words, it is a bi-linear form (character) sometimes called a bi-character that is non-degenerate and alternating. Let $(a, b), (c, d), (m, n) \in K_p$. Then

$$e_p((a, b) + (c, d), (m, n)) = e_p((a + c, b + d), (m, n))$$

$$= \zeta_p^{2((a+c)n-(b+d)m)}$$

$$= \zeta_p^{2an+2cn-2bm-2dm}$$

$$= \zeta_p^{2(an-bm)}\zeta_p^{2(cn-dm)}$$

$$= e_p((a,b),(m,n)) \cdot e_p((c,d),(m,n)),$$

$$\begin{aligned} e_p(\lambda \cdot (a,b),(c,d)) &= e_p((\lambda a, \lambda b),(c,d)) \\ &= \zeta_p^{2\lambda ad - 2\lambda bc} \\ &= \zeta_p^{2a\lambda d - 2b\lambda c} \\ &= e_p((a,b),(\lambda c, \lambda d)) \\ &= e_p((a,b), \lambda \cdot (c,d)), \end{aligned}$$

and

$$\begin{aligned} \zeta_p^{\lambda 2(ad-bc)} &= \left(\zeta_p^{2(ad-bc)}\right)^{\lambda} \\ &= \left(e_p((a,b),(c,d))\right)^{\lambda}. \end{aligned}$$

verifies bilinearity. It is non-degenerate since if $e_p((a,b),(c,d)) = \zeta_p^{2(ad-bc)} = 1$ for all $(c,d) \in K_p$, then $(a,b) = (0,0)$ and if $e_p((a,b),(c,d)) = \zeta_p^{2(ad-bc)} = 1$ for all $(a,b) \in K_p$, then $(c,d) = (0,0)$. It is alternating since $e_p((a,b),(a,b)) = \zeta_p^{2(ab-ba)} = 1$ for all $(a,b) \in K_p$. If $p \neq 2$, then alternation is equivalent to skew-symmetric. Recall Figure 1.2.1 diagram[11]:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \nu_p & \longrightarrow & H_p & \longrightarrow & K_p & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \overline{\psi}} & & \\ 1 & \longrightarrow & \nu_p & \longrightarrow & H_p & \longrightarrow & K_p & \longrightarrow & 0 \end{array} .$$

If ψ is an automorphism of H_p, it induces an automorphism $\overline{\psi}$ of K_p. The commutativity of the diagram shows that $\overline{\psi}$ is symplectic with respect to the commutator pairing[11]. That is, for all $x, y \in K_p$, $e_p(\overline{\psi}(x), \overline{\psi}(y)) = e_p(x,y)$. It is a straightforward verification.

Let

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z}) \cong \mathrm{Sp}(K_p)$$

Let $x = (u_1, u_2)$ and $y = (v_1, v_2)$. Then

$$\overline{\psi}(x) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} au_1 + bu_2 \\ cu_1 + du_2 \end{bmatrix}$$

and

$$\overline{\psi}(y) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} av_1 + bv_2 \\ cv_1 + dv_2 \end{bmatrix}.$$

$$e_p(\overline{\psi}(x), \overline{\psi}(y)) = \zeta_p^{2(au_1+bu_2)(cv_1+dv_2)-2(cu_1+du_2)(av_1+bv_2)}$$

simplifying the exponent and noting that if $\overline{\psi}$ is symplectic,

$$\begin{aligned} &2(au_1 + bu_2)(cv_1 + dv_2) \\ &-2(cu_1 + du_2)(av_1 + bv_2) = 2(acu_1v_1 + bcu_2v_1 + adu_1v_2 + bdu_2v_2) \\ &\qquad - 2(cau_1v_1 + dau_2v_1 + cbu_1v_2 + dbu_2v_2) \\ &\qquad = 2(acu_1v_1 + bcu_2v_1 + adu_1v_2 + dbu_2v_2) \\ &\qquad - 2(acu_1v_1 + adu_2v_1 + bcu_1v_2 + dbu_2v_2) \\ &\qquad = 2(bcu_2v_1 + adu_1v_2) - 2(adu_2v_1 + bcu_1v_2) \\ &\qquad = 2u_2v_1(bc - ad) + 2u_1v_2(ad - bc) \\ &\qquad = -2u_2v_1(ad - bc) + 2u_1v_2(ad - bc)) \\ &\qquad = -2u_2v_1 + 2u_1v_2 \\ &\qquad = 2u_1v_2 - 2u_2v_1 \\ &\qquad = 2(u_1v_2 - u_2v_1). \end{aligned}$$

So,

$$\begin{aligned} e_p(\overline{\psi}(x), \overline{\psi}(y)) &= \zeta_p^{2(u_1v_2 - u_2v_1)} \\ &= e_p(x, y). \end{aligned}$$

Another way to see this is to note that the commutativity of the diagram implies the commutator map of the upper row is the pullback via $\overline{\psi}$ of the commutator map of the lower row[3][Lem 6.6.3, page 161]. But

$$\begin{aligned} &(\alpha, u_1, u_2)(\beta, v_1, v_2)(\alpha, u_1, u_2)^{-1}(\beta, v_1, v_2)^{-1} \\ &= (\alpha, u_1, u_2)(\beta, v_1, v_2)((\alpha^{-1}\zeta_p^{2u_1u_2}, -u_1, -u_2)(\beta^{-1}\zeta_p^{2v_1v_2}, -v_1, -v_2) \\ &= (\alpha\beta\zeta_p^{2u_1v_2}, u_1 + v_1, u_1 + v_2)(\alpha^{-1}\beta^{-1}\zeta_p^{2u_1u_2+2v_1v_2}\zeta_p^{2(-u_1)(-v_2)}, -u_1 - v_1, -u_2 - v_2) \\ &= (\zeta_p^{2u_1u_2+4u_1v_2+2v_1v_2} \cdot \zeta_p^{2(u_1+v_1)(-u_2-v_2)}, 0, 0) \\ &= (\zeta_p^{2u_1u_2+4u_1v_2+2v_1v_2} \cdot \zeta_p^{-2(u_1u_2+v_1u_2+u_1v_2+v_1v_2)}, 0, 0) \\ &= (\zeta_p^{2u_1u_2+4u_1v_2+2v_1v_2-2u_1u_2-2v_1u_2-2u_1v_2-2v_1v_2)}, 0, 0) \\ &= (\zeta_p^{2(u_1v_2-v_1u_2)}, 0, 0) \\ &= (\zeta_p^{2(u_1v_2-u_2v_1)}, 0, 0) \\ &= (e_p(x, y), 0, 0). \end{aligned}$$

for all $(\alpha, u_1, u_2)(\beta, v_1, v_2) \in H_p$. So the induced isomorphism (automorphism) $\overline{\psi}$ is a symplectic isormorphism with respect to the form e_p.

In order to study $\mathrm{Aut}_{\nu_p}(H_p)$, we consider the symplectic group $\mathrm{Sp}(K_p)$ consisting of all automorphims of K_p which preserve the alternating form e_p[3]. Every element of $\mathrm{Aut}_{\nu_p}(H_p)$ induces a symplectic isomorphism of K_p. This gives a homomorphism $\Phi : \mathrm{Aut}_{\nu_p}(H_p) \to \mathrm{Sp}(K_p)$. One the other hand, any $c \in K_p$ defines an automorphism $\psi_c \in \mathrm{Aut}_{\nu_p}(H_p)$, namely $\psi_c(\lambda, x_1, x_2) = (\lambda e_p(c, (x_1, x_2), x_1, x_2))$ for all $(\lambda, x_1, x_2) \in H_p$. Since e_p is nondegenerate, the assignment $c \mapsto \psi_c$ is an injective homomorphism $\Psi : K_p \to \mathrm{Aut}_{\nu_p}(H_p)$. Any $\varphi \in \ker\Phi$ is necessarily of the form $\varphi(\lambda, x_1, x_2) = (\lambda g(x_1, x_2), x_1, x_2)$. The function $g : K_p \to \mathbb{C}^\times$ is linear since φ is a homomorphism. By the Riesz representation theorem, $g(x_1, x_2) = e_p(c, (x_1, x_2))$ for some $c \in K_p$. Let ψ_c denote these automorphisms instead of φ.

So the number of such automorphisms ψ_c is p^2. We will show this set is a group with the operation being composition. It is clear that the operation is closed. The operation is associative since function composition is associative. Lets verify. For $a, b, c \in K_p$, we have

$$\begin{aligned}
((\psi_a \circ \psi_b) \circ \psi_c)(\lambda, x, y) &= ((\psi_a \circ \psi_b)(\psi_c(\lambda, x, y) \\
&= ((\psi_a \circ \psi_b)(\lambda\zeta_p^{2(c_1y-c_2x)}, x, y) \\
&= \psi_a(\psi_b((\lambda\zeta_p^{2(c_1y-c_2x)}, x, y)) \\
&= \psi_a((\lambda\zeta_p^{2(c_1y-c_2x)}\zeta_p^{2(b_1y-b_2x}), x, y)) \\
&= (\lambda\zeta_p^{2(c_1y-c_2x)}\zeta_p^{2(b_1y-b_2x)}\zeta_p^{2(a_1y-a_2x)}, x, y))
\end{aligned}$$

and

$$\begin{aligned}
((\psi_a \circ \psi_b) \circ \psi_c)(\lambda, x, y) &= (\psi_a(\psi_b(\psi_c(\lambda, x, y))) \\
&= \psi_a(\psi_b((\lambda\zeta_p^{2(c_1y-c_2x)}, x, y)) \\
&= \psi_a((\lambda\zeta_p^{2(c_1y-c_2x)}\zeta_p^{2(b_1y-b_2x)}, x, y)) \\
&= (\lambda\zeta_p^{2(c_1y-c_2x)}\zeta_p^{2(b_1y-b_2x)}\zeta_p^{2(a_1y-a_2x)}, x, y))
\end{aligned}$$

which shows associativity holds. It has the identity, taking $c = (0, 0)$ gives us the identity automorphism. $\psi_{(0,0)}(\lambda, x, y) = (\lambda \cdot \zeta_p^{2\cdot(0y-0x)}, x, y) = (\lambda, x, y)$ for all $(\lambda, x, y) \in H_p$.

ψ_c has an inverse namely ψ_{-c}.

$$\begin{aligned}
(\psi_c \circ \psi_{-c})(\lambda, x, y) &= \psi_c(\psi_{-c})(\lambda, x, y)) \\
&= \psi_c(\lambda\zeta_p^{-2c_1y_1+2c_2x_1}, x_1, y_1)
\end{aligned}$$

$$= (\lambda\zeta_p^{-2c_1y_1+2c_2x_1}\zeta_p^{2c_1y_1-2c_2x_1}, x_1, y_1)$$
$$= (\lambda, x, y)$$

shows that ψ_c has an inverse. That inverse is also unique.

So the set of ψ_c forms a group. Its order is p^2. So it is isomorphic to either $\mathbb{Z}/p^2\mathbb{Z}$ or $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$. Since $\mathbb{Z}/p^2\mathbb{Z}$ is cyclic it has a generator but our group of ψ_c does not. Fixing a $c = (c_1, c_2) \in K_p$,

$$(\psi_c)^n(\lambda, x, y) = (\lambda\zeta_p^{2n(c_1y-c_2x)}, x, y) = (\lambda\zeta_p^{2nc_1y-2nc_2x}, x, y).$$

As n ranges from 0 to $p^2 - 1$, $(2nc_1, 2nc_2)$ will not generate every element of K_p. There are $p - 1$ multiples of p in this range. So there will be $p - 1$ values of n that send $(2nc_1, 2nc_2)$ to $(0, 0)$. Therefore it will not generate the group.

Since $c \in K_p$ and K_p is not cyclic (because $\gcd(p, p) = p \neq 1$), our group of ψ_c is not cyclic. So it must be that it is isomorphic to $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} = K_p$.

So we have our desired result:

Theorem 1.2.8. *Given*

$$\Phi : \mathrm{Aut}_{\nu_p}(H_p) \to \mathrm{Sp}(K_p) \cong \mathrm{SL}_2(\mathbb{F}_p),$$

$\ker(\Phi) \cong K_p$ *and* $\psi_c \in \mathrm{Aut}_{\nu_p}(H_p)$ *is mapped to* $c \in K_p$*.*

An automorphism $\psi \in \mathrm{Aut}_{\nu_p}(H_p)$ is **symmetric** if it commutes with the action $(\alpha, x, y) \mapsto (\alpha, -x, -y)$ on H_p. We also have a theorem for symmetric automorphisms.

Theorem 1.2.9. *For symmetric automorphisms,*

$$\Phi : \mathrm{Aut}_{\nu_p}^{\mathrm{Sym}}(H_p) \to \mathrm{Sp}(K_p) \cong \mathrm{SL}_2(\mathbb{F}_p),$$

$\ker(\Phi) \cong K_p[2]$*, the subgroup of* 2*-torsion of* K_p*.*

Proof. Let $c = (c_1, c_2)$. By definition of a symmetric automorphism, we must have

$$\psi_c(\lambda, -x, -y) = \psi_c(\lambda, x, y).$$

Since

$$\psi_c(\lambda, -x, -y) = (\lambda\zeta_p^{-2c_1y+2c_2x}, -x, -y)$$

and

$$\psi_c(\lambda, x, y) = (\lambda\zeta_p^{2c_1y-2c_2x}, x, y),$$

we have

$$e_p(c, (-x, -y)) = e_p(c, (x, y)).$$

Since

$$e_p(c, (x, y))^{-1} = (\zeta_p^{2(c_1y-c_2x)})^{-1} = \zeta_p^{-2c_1y+2c_2x} = e_p(-c, (x, y)),$$

we have that

$$e_p(c, x, y) = e_p(-c, x, y)$$

which implies $c = -c$. In field whose characteristic is not two, i.e., $p \geq 3$, $c = 0$. For $p = 2$, we have $\ker(\Phi) \cong K_p[2]$. $\square$

1.3 Explicit Weil Rep. for induction by the modulation subgroup

We have H_p as our Heisenberg group. There is a unique map

$$\sigma : H_p \to \mathrm{GL}_p(\mathbb{C}) = \mathrm{GL}(V)$$

such that it is irreducible and $\nu_p \subset H_p$ acts by scalar multiplication. Explicitly,

$$V = \{ \text{ functions: } \mathbb{Z}/p\mathbb{Z} \to \mathbb{C}\}$$

where $\dim_{\mathbb{C}} V = p$ has a canonical basis of "delta functions": $\delta_x : \mathbb{Z}/p\mathbb{Z} \to \mathbb{C}$ where $0 \mapsto 0$, $1 \mapsto 0$, $\cdots$, $x \mapsto 1$, $\vdots$, $p-1 \mapsto 0$. So $\sigma : H_p \to \mathrm{GL}(V)$ has three types of maps:

(1) Scaling: $(\zeta, 0, 0) \mapsto (\delta_\nu \mapsto \zeta\delta_\nu)$ where $(\delta_\nu \mapsto \zeta\delta_\nu)$ in matrix form is

$$\begin{bmatrix} \zeta & 0 & 0 & \cdots & 0 \\ 0 & \zeta & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \zeta \end{bmatrix}.$$

(2) Translation: $(1, 0, y) \mapsto (\delta_\nu \mapsto \delta_{\nu-y})$. To see how translation works, take $p = 5$ and $y = 1$. Then

$$(1, 0, 1) \mapsto (\delta_\nu \mapsto \delta_{\nu-1})$$

What is it in matrix form? $\sigma(1, 0, 1)\delta_\nu = \delta_{\nu-1}$. In matrix form, the action is given by

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(3) Modulation: $(1, x, 0) \mapsto (\delta_\nu \mapsto \zeta^{-2\nu x}\delta_\nu)$. To see how modulation works, take $p = 5$ and $y = 3$. Then since $(1, 3, 0) = (1, 1, 0)^3$. So the action in matrix form is

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & \zeta^{-2} & 0 & 0 & 0 \\ 0 & 0 & \zeta^{-4} & 0 & 0 \\ 0 & 0 & 0 & \zeta^{-6} & 0 \\ 0 & 0 & 0 & 0 & \zeta^{-8} \end{bmatrix}^3 = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & \zeta^{-6} & 0 & 0 & 0 \\ 0 & 0 & \zeta^{-12} & 0 & 0 \\ 0 & 0 & 0 & \zeta^{-18} & 0 \\ 0 & 0 & 0 & 0 & \zeta^{-24} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & \zeta^{4} & 0 & 0 & 0 \\ 0 & 0 & \zeta^{3} & 0 & 0 \\ 0 & 0 & 0 & \zeta^{2} & 0 \\ 0 & 0 & 0 & 0 & \zeta^{1} \end{bmatrix}$$

For $p > 2$ take $\psi \in \mathrm{Aut}^{\mathrm{Sym}}_{\mathbb{C}^\times}$. Consider the commutative diagram

$$\begin{CD} H_p @>{\sigma}>> \mathrm{GL}_p(\mathbb{C}) \\ @V{\psi}VV @VV{M(\psi)}V \\ H_p @>{\sigma^\psi}>> \mathrm{GL}_p(\mathbb{C}) \end{CD}$$

Since ψ fixes the center, σ^ψ is irreducible, and ν_p acts by scalars, this implies that $\sigma \cong \sigma^\psi$. In turn this implies there exists a matrix $M(\psi) \in \mathrm{GL}_2(\mathbb{C})$ such that

$$M(\psi)\sigma = \sigma^\psi M(\psi).$$

1.3.1 Case S

Take $S = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$ and let ψ_S be the lift of S, $a := (1, x, 0)$ and $b := (1, 0, y)$ the two squares commute:

$$\begin{CD} V @>{\sigma(a)}>> V \\ @V{M(\psi_S)}VV @VV{M(\psi_S)}V \\ V @>{\sigma^{\psi_S}(a)}>> V \end{CD}\quad , \qquad \begin{CD} V @>{\sigma(b)}>> V \\ @V{M(\psi_S)}VV @VV{M(\psi_S)}V \\ V @>{\sigma^{\psi_S}(b)}>> V \end{CD}$$

We note that

$$\sigma((1, x, 0))\delta_\nu = \zeta^{-2\nu x}\delta_\nu \quad \text{and} \quad \sigma((1, 0, y))\delta_\nu = \delta_{\nu - y}.$$

Recall that $\psi_S((\lambda, x, y) = (\lambda\zeta^{-2xy}, -y, x)$. Making the following definitions

$$\sigma^{\psi_S}((1, x, 0)) := \sigma(\psi_S(1, x, 0)) = \sigma((1, 0, x)),$$

$$\sigma^{\psi_S}((1, 0, y)) := \sigma(\psi_S(1, 0, y)) = \sigma((1, -y, 0)),$$

we see that

$$M(\psi_S)\sigma((1, x, 0)) = \sigma((1, 0, x))M(\psi_S)$$

and

$$M(\psi_S)\sigma((1, 0, y)) = \sigma((1, -y, 0))M(\psi_S).$$

So, what is $M(\psi_S)$? We claim that is a discrete Fourier transform. The discrete Fourier transform, DFT, is defined to be

$$DFT(\delta_\nu) = \sum_{\mu=0}^{p-1} \zeta^{-2\nu\mu}\delta_\mu,$$

satisfies these two conditions. That is, taking $M(\psi_S)$ to be the discrete Fourier transform satisfies this property.

Explicitly,

$$M(\psi_S) = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \zeta^{-2} & \zeta^{-4} & \cdots & \zeta^{-2(p-1)} \\ 1 & \zeta^{-4} & (\zeta^{-8}) & \cdots & (\zeta^{-4(p-1)}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \zeta^{-2(p-1)} & (\zeta^{-4(p-1)} & \cdots & (\zeta^{-2(p-1)(p-1)}) \end{bmatrix}.$$

Lets verify that this is indeed the case. Take a basis vector δ_ν. Then

$$\begin{aligned} M(\psi_S)\sigma((1,x,0))\delta_\nu &= M(\psi_S)\zeta^{-2\nu x}\delta_\nu \\ &= \sum_{\mu=0}^{p-1} \zeta^{-2\nu\mu}\zeta^{-2\mu x}\delta_\mu \\ &= \sum_{\mu=0}^{p-1} \zeta^{(\nu+x)(-2\mu)}\delta_\mu \\ &= DFT(\delta_{\nu+x}), \end{aligned}$$

which is the Discrete Fourier transform of δ_ν shifted by x and so is

$$\begin{aligned} \sigma((1,0,x))M(\psi_S)\delta_\nu &= \sigma((1,0,x))\sum_{\mu=0}^{p-1} \zeta^{-2\nu\mu}\delta_\mu \\ &= DFT(\delta_{\nu+x}). \end{aligned}$$

Also, we have

$$\begin{aligned} M(\psi_S)\sigma((1,0,y))\delta_\nu &= M(\psi_S)\delta_{\nu-y} \\ &= \sum_{\mu=0}^{p-1} \zeta^{-2\nu\mu}\delta_{\mu-y} \\ &= \sum_{\mu\in\mathbb{F}_p} \zeta^{-2\nu(\mu+y)}\delta_\mu \\ &= \sum_{\mu\in\mathbb{F}_p} \zeta^{-2\nu y}\zeta^{-2\nu\mu}\delta_\mu \\ &= \zeta^{-2\nu y} DFT(\delta_\nu), \end{aligned}$$

and

$$\begin{aligned} \sigma(1,-y,0)M(\psi_S)\delta_\nu &= \sigma(1,-y,0)\sum_{\mu=0}^{p-1}\zeta^{-2\nu\mu}\delta_\mu \\ &= \zeta^{-2\nu y} DFT(\delta_\nu). \end{aligned}$$

1.3.2 Case action of modulation negative and translation positive for S

Take $S = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$ and let ψ_S be the lift of S, $a := (1,x,0)$ and $b := (1,0,y)$ the two squares commute:

$$\begin{array}{ccc} V & \xrightarrow{\sigma(a)} & V \\ {\scriptstyle M(\psi_S)}\downarrow & & \downarrow{\scriptstyle M(\psi_S)} \\ V & \xrightarrow{\sigma^{\psi_S}(a)} & V \end{array}, \qquad \begin{array}{ccc} V & \xrightarrow{\sigma(b)} & V \\ {\scriptstyle M(\psi_S)}\downarrow & & \downarrow{\scriptstyle M(\psi_S)} \\ V & \xrightarrow{\sigma^{\psi_S}(b)} & V \end{array}$$

We note that

$$\sigma((1,x,0))\delta_\nu = \zeta^{-2\nu x}\delta_\nu \quad \text{and} \quad \sigma((1,0,y))\delta_\nu = \delta_{\nu+y}.$$

Recall that $\psi_S((\lambda,x,y) = (\lambda\zeta^{-2xy}, -y, x)$. Making the following definitions

$$\sigma^{\psi_S}((1,x,0)) := \sigma(\psi_S(1,x,0)) = \sigma((1,0,x)),$$

$$\sigma^{\psi_S}((1,0,y)) := \sigma(\psi_S(1,0,y)) = \sigma((1,-y,0)),$$

we see that

$$M(\psi_S)\sigma((1,x,0)) = \sigma((1,0,x))M(\psi_S)$$

and

$$M(\psi_S)\sigma((1,0,y)) = \sigma((1,-y,0))M(\psi_S).$$

So, what is $M(\psi_S)$? We claim that is a discrete Fourier transform. The discrete Fourier transform, DFT, is defined to be

$$DFT(\delta_\nu) = \sum_{\mu=0}^{p-1} \zeta^{-2\nu\mu}\delta_\mu,$$

satisfies these two conditions. That is, taking $M(\psi_S)$ to be the discrete Fourier transform satisfies this property. Explicitly,

$$M(\psi_S) = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \zeta^{-2} & \zeta^{-4} & \cdots & \zeta^{-2(p-1)} \\ 1 & \zeta^{-4} & (\zeta^{-8}) & \cdots & (\zeta^{-4(p-1)}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \zeta^{-2(p-1)} & (\zeta^{-4(p-1)} & \cdots & (\zeta^{-2(p-1)(p-1)}) \end{bmatrix}.$$

Lets verify that this is indeed the case. Take a basis vector δ_ν. Then

$$\begin{aligned} M(\psi_S)\sigma((1,x,0))\delta_\nu &= M(\psi_S)\zeta^{-2\nu x}\delta_\nu \\ &= \sum_{\mu=0}^{p-1} \zeta^{-2\nu\mu}\zeta^{-2\mu x}\delta_\mu \\ &= \sum_{\mu=0}^{p-1} \zeta^{(\nu+x)(-2\mu)}\delta_\mu \\ &= DFT(\delta_{\nu+x}), \end{aligned}$$

which is the Discrete Fourier transform of δ_ν shifted by x but

$$\begin{aligned}\sigma((1,0,x))M(\psi_S)\delta_\nu &= \sigma((1,0,x))\sum_{\mu=0}^{p-1}\zeta^{-2\nu\mu}\delta_\mu \\ &= \sum_{\mu\in\mathbb{F}_p}\zeta^{-2\nu\mu}\delta_{\mu+x} \\ &= \sum_{\mu\in\mathbb{F}_p}\zeta^{-2\nu(\mu-x)}\delta_\mu \\ &= DFT(\delta_{\nu-x})\end{aligned}$$

is shifted by $-x$. It does not work. Also, we have

$$\begin{aligned}M(\psi_S)\sigma((1,0,y))\delta_\nu &= M(\psi_S)\delta_{\nu-y} \\ &= \sum_{\mu=0}^{p-1}\zeta^{-2\nu\mu}\delta_{\mu-y} \\ &= \sum_{\mu\in\mathbb{F}_p}\zeta^{-2\nu(\mu+y)}\delta_\mu \\ &= \sum_{\mu\in\mathbb{F}_p}\zeta^{-2\nu y}\zeta^{-2\nu\mu}\delta_\mu \\ &= \zeta^{-2\nu y}DFT(\delta_\nu),\end{aligned}$$

and it does not agree with

$$\begin{aligned}\sigma(1,-y,0)M(\psi_S)\delta_\nu &= \sigma(1,-y,0)\sum_{\mu=0}^{p-1}\zeta^{-2\nu\mu}\delta_\mu \\ &= \zeta^{2\nu y}DFT(\delta_\nu).\end{aligned}$$

1.3.3 Case T

Now for the case $T = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$. Let ψ_T be the lift of T, $a := (1, x, 0)$ and $b := (1, 0, y)$ the two squares commute:

$$\begin{array}{ccc} V & \xrightarrow{\sigma(a)} & V \\ {\scriptstyle M(\psi_T)}\big\downarrow & & \big\downarrow{\scriptstyle M(\psi_T)} \\ V & \xrightarrow{\sigma^{\psi_T}(a)} & V \end{array}, \qquad \begin{array}{ccc} V & \xrightarrow{\sigma(b)} & V \\ {\scriptstyle M(\psi_T)}\big\downarrow & & \big\downarrow{\scriptstyle M(\psi_T)} \\ V & \xrightarrow{\sigma^{\psi_T}(b)} & V \end{array}$$

Recall that $\psi_T((\lambda, x, y)) = (\lambda\zeta^{y^2}, x+y, y)$. So $\psi_T((1, x, 0)) = (1, x, 0)$ and $\psi_T((1, 0, y)) = (\zeta^{y^2}, y, y)$. Making the following definitions

$$\sigma^{\psi_T}((1, x, 0)) := \sigma(\psi_T(1, x, 0)) = \sigma((1, x, 0)),$$

$$\sigma^{\psi_T}((1, 0, y)) := \sigma(\psi_T(1, 0, y)) = \sigma((\zeta^{y^2}, y, y))$$

and noting the commutativity of the squares we see that

$$M(\psi_T)\sigma((1, x, 0)) = \sigma((1, x, 0))M(\psi_T)$$

and

$$M(\psi_T)\sigma((1, 0, y)) = \sigma((\zeta^{y^2}, y, y))M(\psi_T).$$

So, what is $M(\psi_T)$? Since $\sigma(1, x, 0)\delta_0 = \zeta^{-2\cdot x\cdot 0}\delta_0 = \delta_0$, we have

$$M(\psi_T)\sigma(1, x, 0)\delta_0 = M(\psi_T)\delta_0.$$

Since the square commutes, we have

$$\begin{aligned} \sigma^{\psi_T}((1, x, 0))M(\psi_T)\delta_0 &= \sigma((1, x, 0))M(\psi_T)\delta_0 \\ &= M(\psi_T)\sigma(1, x, 0)\delta_0 \\ &= M(\psi_T)\delta_0 \end{aligned}$$

implies that $M(\psi_T)\delta_0$ is an eigenvector of ALL the operators $\sigma((1,x,0))$ of eigenvalue 1. This implies that $M(\psi_T)\delta_0 = c \cdot \delta_0$. Take $c = 1$. So the first column of $M(\psi_T)$ is

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}^\intercal.$$

We can write $\delta_\nu = \delta_{0-(-\nu)} = \sigma((1,0,-\nu))\delta_0$. Then we have

$$\begin{aligned}
M(\psi_T)\delta_\nu &= M(\psi_T)\sigma((1,0,-\nu))\delta_0 \\
&= \sigma^{\psi_T}((1,0,-\nu))M(\psi_T)\delta_0 \\
&= \sigma((\zeta^{(-\nu)^2}, -\nu, -\nu))M(\psi_T)\delta_0 \\
&= \sigma((\zeta^{\nu^2},0,0))\sigma((1,0,-\nu))\sigma((1,-\nu,0))M(\psi_T)\delta_0 \\
&= \sigma((\zeta^{\nu^2},0,0))\sigma((1,0,-\nu))\sigma((1,-\nu,0)) \cdot c \cdot \delta_0 \\
&= \sigma((\zeta^{\nu^2},0,0))\sigma((1,0,-\nu)) \cdot c \cdot \sigma((1,-\nu,0))\delta_0 \\
&= \sigma((\zeta^{\nu^2},0,0))\sigma((1,0,-\nu)) \cdot c \cdot \delta_0 \\
&= \sigma((\zeta^{\nu^2},0,0))\sigma((1,0,-\nu)) \cdot 1 \cdot \delta_0 \\
&= \sigma((\zeta^{\nu^2},0,0))\sigma((1,0,-\nu))\delta_0 \\
&= \sigma((\zeta^{\nu^2},0,0))\delta_\nu \\
&= \zeta^{\nu^2}\delta_\nu,
\end{aligned}$$

which shows that $M(\psi_T)$ is a diagonal matrix:

$$M(\psi_T) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \zeta^1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \zeta^{2^2} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \zeta^{(p-1)^2} \end{bmatrix}.$$

1.3.4 Summary for S and T

For ψ_S, $M(\psi_S)$ a modified DFT_p and for ψ_T, $M(\psi_T)$ is diagonal. We know that $S^2 = -\mathbb{1} = (ST)^3$. Letting $M'(\psi_S) := (\Omega(p))^{-1} \cdot M(\psi_S)$, we have the equality

$$(M'(\psi_S))^2 = (M'(\psi_S)(M(\psi_T))^3.$$

CHAPTER 2 Integral Representations

Nobs and Wolfart[23][24] show us how to construct all of the irreducible representations of $\mathrm{SL}_2(\mathbb{Z}/p^\lambda\mathbb{Z})$. We will compute them for primes 3 and 5 in the following chapters. Our goal is to find the smallest rings of integers for which we can write these representations. Riese[26] characterizes those rings. We will attempt to explain those results in this chapter. For the computation of those integral representations of the Weil characters, we will use results of Wang[36] and Zemel[44]. Using Nobs and Wolfart's methods, we can construct integral representations that have "denominators", that is, $\mathbb{Z}[\zeta_p, \frac{1}{p}]$. In this section, we explain how those integral representations with denominators, e.g. over $\mathbb{Z}[\zeta_p, \frac{1}{p}]$, tell us the existence of that integral representations over $\mathbb{Z}[\zeta_p]$. We then compute the integral representation of the p-dimensional irreducible principal series representation of $\mathrm{SL}_2(\mathbb{F}_p)$ for p an odd prime. Last we construct the reducible integral representation $1 + St$.

Let G be a finite group of exponent $\exp(G) = g$. Riese[26] proves the following:

Theorem 2.0.1 (Thm 1, Riese)**.** *Let $G = \mathrm{SL}(2, q)$ for some prime power $q = p^f$. Every irreducible complex character χ of G can be written in $R = \mathbb{Z}[\mu_n]$ with $n = n(\chi)$ being a proper divisor of $\exp(G)$, except possibly when χ is a (cuspidal) character of degree $q - 1$. In the exceptional case χ can be realized over $R[\frac{1}{p}]$ with $n = \frac{1}{p}\exp(G)$.*

In the case of $q = p$, we have a tight bound for those rings.

Corollary 2.0.2 (Riese)**.** *Every character of $SL(2, p)$ can be realized over the ring of integers of the $\exp(G)$th cyclotomic field.*

2.1 Definitions and useful facts

We provide the definitions of terms[10] that we will use along with some useful facts and examples.

Definition 2.1.1 (Ring of Integers)**.** *The ring of integers[42] of an algebraic number* *field K is the ring of all integral elements contained in K. An integral element is a root of a monic polynomial with integer coefficients,*

$$x^n + c^{n-1}x_{n-1} + \cdots + c_0.$$

This ring is often denoted by O_K or $\mathcal{O}_K$.

Since any integer number belongs to K and is an integral element of K, the ring $\mathbb{Z}$ is always a subring of O_K.

Example 2.1.2 ($\mathbb{Z}$)**.** *The ring $\mathbb{Z}$ is the simplest possible ring of integers. Namely, $\mathbb{Z} = O_{\mathbb{Q}}$ where $\mathbb{Q}$ is the field of rational numbers. In algebraic number theory the elements of $\mathbb{Z}$ are often called the "rational integers" because of this.*

Definition 2.1.3 (Lattice)**.** *For any finite dimensional K-space V, a full R-lattice in V is a finitely generated R-submodule M in V such that $K \cdot M = V$, where*

$$K \cdot M = \left\{ \sum \alpha_i m_i (\textit{finite sum}) \ : \ \alpha_i \in K, m_i \in M \right\}$$

In ring theory, a lattice is a module over a ring which is embedded in a vector space over a field, giving an algebraic generalization of the way a lattice group is embedded in a real vector space.

Definition 2.1.4 (Order)**.** *An R-order in the K-algebra A is a subring Λ of A, having the same unity element as A, and such that Λ is a full R-lattice in A. Λ is both left and right noetherian, since Λ is finitely generated over the noetherian domain R.*

An **order** in the sense of ring theory is a subring $\mathcal{O}$ of a ring A, such that

(1) A is a finite-dimensional algebra over the rational number field $\mathbb{Q}$

(2) $\mathcal{O}$ spans A over $\mathbb{Q}$, and

(3) $\mathcal{O}$ is a $\mathbb{Z}$-lattice in A.

Example 2.1.5 (Maximal $\mathbb{Z}$-order)**.** *Let τ be the golden ratio $(1+\sqrt{5})/2$. For any positive integer n, $\Lambda_n = \mathbb{Z} + \mathbb{Z} \cdot n\sqrt{5}$ is a $\mathbb{Z}$-order in $\mathbb{Q}(\sqrt{5})$, and $\mathbb{Z} + \mathbb{Z} \cdot \tau$ is the unique* *maximal $\mathbb{Z}$-order.*

The ring of integers of an algebraic number field is the unique maximal order in the field.

Property 2.1.6 ($\mathcal{O}_K$)**.** *The ring of integers $\mathcal{O}_K$ is a finitely generated $\mathbb{Z}$-module. It is a free $\mathbb{Z}$-module and therefore has an integral basis: $b_1, \ldots, b_n \in O_K$ of the $\mathbb{Q}$-vector space K such that each element x in O_K can be uniquely represented as*

$$x = \sum_{i-1}^{n} a_i b_i$$

with $a_i \in \mathbb{Z}$. The rank n of O_K as a free $\mathbb{Z}$-module is equal to the degree of K over $\mathbb{Q}$. The rings of integers in number fields are Dedekind domains.

Example 2.1.7 (Integral Basis)**.** *If p is a prime ζ a pth root of unity and $K = \mathbb{Q}(\zeta)$ is the corresponding cyclotomic field, then an integral basis of $\mathcal{O}_K = \mathbb{Z}[\zeta]$ is given by $(1, \zeta, \zeta^2, \ldots, \zeta^{p-2})$.*

Let $K = \mathbb{Q}(\sqrt{p})$, where p is an odd prime. The *minimal polynomial* of $\sqrt{d}$ over $\mathbb{Q}$ is $X^2 - p$ which has roots $\pm\sqrt{p}$. The extension $K/\mathbb{Q}$ is *Galois* and the Galois group has order two: it consists of the identity automorphism and automorphism σ that maps $a + b\sqrt{p}$ to $a - b\sqrt{p}$.

Lemma 2.1.8. *[2, Lem 7.2.1] If a and b are rational numbers (in $\mathbb{Q}$), then $a + b\sqrt{p}$ is an algebraic integer if and only if $2a$ and $a^2 - pb^2$ belong to $\mathbb{Z}$. In this case $2b$ is also in $\mathbb{Z}$.*

Proof. Let $x = a + b\sqrt{p}$ and then $\sigma(x) = a - b\sqrt{p}$. This implies $x + \sigma(x) = 2a \in \mathbb{Q}$ and $x\sigma(x) = a^2 - pb^2 \in \mathbb{Q}$. If x is an algebraic integer, then x is a root of a monic polynomial $f \in \mathbb{Z}[X]$. Since σ is an automorphism, $f(\sigma(x)) = \sigma(f(x))$, $\sigma(x)$ is also a root of f and hence an algebraic integer. $2a$ and $a^2 - pb^2$ are also algebraic integers as well as rational numbers. Since $\mathbb{Z}$ is integrally closed, $2a$ and $a^2 - pb^2$ belong to $\mathbb{Z}$. The converse holds because $a + b\sqrt{p}$ is a root of $(X - a)^2 = pb^2$ or equivalently, $X^2 - 2aX + a^2 - pb^2 = 0$.

If $2a$ and $a^2 - pb^2$ are rational integers, then $(2a)^2 - p(2b)^2 = 4(a^2 - pb^2) \in \mathbb{Z}$. If $2b \notin \mathbb{Z}$, then its denominator would include a prime factor p_1, which would appear as p_1^2 in the denominator of $(2b)^2$. Multiplication by $(2b)^2$ by p cannot cancel the p_1^2 because p is a prime, and the result follows. $\square$

We modify an argument from [2, Ch. 7] to obtain the following proposition which we will require later.

Proposition 2.1.9. *Let $p \equiv 1 \mod 4$ be a prime. Then 1 and $\frac{1}{2}(1+\sqrt{p})]$ form an integral basis of the algebraic integers of $\mathbb{Q}(\sqrt{p})$.*

Proof. We claim that if $p \equiv 1 \mod 4$ the set of algebraic integers of $\mathbb{Q}(\sqrt{p})$ consists of all $\frac{u}{2} + \frac{v}{2}\sqrt{p}$, $u, v \in \mathbb{Z}$, where u and v have the same parity (both even or both odd). Lemma 2.1.8 tells the algebraic integers of $\mathbb{Q}(\sqrt{p})$ are of the form $\frac{u}{2} + \frac{v}{2}\sqrt{p}$ where $u, v \in \mathbb{Z}$ and $\frac{u^2}{4} - \frac{pv^2}{4} \in \mathbb{Z}$. So we have $u^2 - pv^2 \equiv 0 \mod 4$ and that implies u and v have the same parity because the square of an even number is congruent to $0 \mod 4$ and the square of an odd number is congruent to $1 \mod 4$. The case where both u and v are odd can only occur when $p \equiv \mod 4$. The case where both are even is equivalent to $\frac{u}{2}$ and $\frac{v}{2}$ are in $\mathbb{Z}$.

To see that 1 and $\frac{1}{2}(1+\sqrt{p})$ span the set of algebraic integers of $\mathbb{Q}(\sqrt{p})$, consider $\frac{1}{2}(u + v\sqrt{p})$ where u and v have the same parity. Then

$$\frac{1}{2}(u+\sqrt{p}) = \left(\frac{u-v}{2}\right)(1) + v \cdot \left(\frac{1}{2}(1+\sqrt{p})\right)$$

with $(u-v)/2$ and $v \in \mathbb{Z}$. Next to show linear independence, assume that $a, b \in \mathbb{Z}$ and

$$a + b\left(\frac{1}{2}(1+\sqrt{p})\right) = 0.$$

Then $2a + b + b\sqrt{p} = 0$ which forces $a = b = 0$. $\square$

Remark 2.1.10. *Proposition 2.1.9 gives us the ring of integers $\mathcal{O}$ of $\mathbb{Q}(\sqrt{p})$ to be $\mathbb{Z}[\frac{1}{2}(1+$*

$\sqrt{p}$) for $p \equiv 1 \mod 4$. If $p \equiv 3 \mod 4$ then $-p \equiv 1 \mod 4$. So for $p \equiv 3 \mod 4$, the ring of integers of $\mathbb{Q}(\sqrt{-p})$ is $\frac{1}{2}(1+\sqrt{-p})$. We will need this observation later.

Proposition 2.1.9 can be modified to for any square-free integer d as illustrated by the next example.

Example 2.1.11 (Integral basis for a ring of quadratic integers)**.** *If d is a square free integer and $K = \mathbb{Q}(\sqrt{d})$ is the corresponding quadratic field, then O_K is a ring of quadratic integers and its integral basis is given by $(1, (1+\sqrt{d})/2)$ if $d \equiv 1$ (mod) 4 and by $(1, \sqrt{d})$ if $d \equiv 2, 3$ (mod) 4.*

Definition 2.1.12 (Realizable Character)**.** *$\chi \in Irr(G)$ is "realizable" over a ring R if χ is the character of a representation $\rho_\chi : G \to \mathrm{GL}(R)$.*

Definition 2.1.13 (R-lattice)**.** *K is field, Fix W a vector space over K. An R-lattice for W is an R-module U such that $U \otimes_R K \cong W$.*

Let G be a group of finite order g, and let K be a commutative ring. We denote by $K[G]$, the group **algebra over** K[28]; this algebra has a basis indexed by the elements of G, and most of the time we identify this basis with G. Each element f of $K[G]$ can then be uniquely written in the form

$$f = \sum_{s \in G} a_s s, \quad \text{with} \quad a_s \in K,$$

and multiplication in $K[G]$ extends that in G.

Let V be a K-module and let $\rho : G \to \mathrm{GL}(V)$ be a linear representation of G in V. For $s \in G$ and $x \in V$, set $sx = \rho_s x$; by linearity this defines fx for $f \in K[G]$ and $x \in V$. Thus V is endowed with the structure of a **left** $K[G]$**-module**; conversely, such a structure defines a linear representation of G in V. In what follows we will indiscriminately use the terminology "linear representation" or "module".

Definition 2.1.14 (Algebra over a field)**.** *[43] Let F be a field. An algebra A over F is a ring which has a structure of a F-vector space which is compatible with the ring* *multiplication in the following sense:* $(\lambda a)b = \lambda(ab) = a(\lambda b)$ *for all* $\lambda \in F$ *and* $a, b \in A$. *An algebra is finite dimensional (one also says of finite rank) if its dimension as* F*-vector spaces is finite. A homomorphism of algebras is naturally a ring homomorphism which is also a linear transformation.*

As an algebra over a field is a ring, we can look at modules over it, which will be automatically endowed with the structure of an F-vector space.

Definition 2.1.15 (Ideal Class Group (or Class Group))**.** *In number theory, the* ***ideal class group (or class group)*** *of an algebraic number field* K *is the quotient group* J_K/P_K *where* J_K *is the group of fractional ideals of the ring of integers of* K *and* P_K *is its subgroup of principal ideals. The class group is a measure of the extent to which unique factorization fails in the ring of integers of* K*. The order of the group, which is finite, is called the* ***class number*** *of* K*.*

If R is a Dedekind domain and M is a finitely generated R-module, then M has no torsion iff M is projective (= a direct summand of a free R-module), iff M is a direct sum of a free R-module and a fractional ideal J of R. The class of J in the class group $\mathrm{Cl}(R)$ is an invariant called the Steinitz class of M.

The following material is from Conrad[7]. The ring of integers of a number field is free as a $\mathbb{Z}$-module. It is a module not just over $\mathbb{Z}$, but also over any intermediate ring of integers. That is, if $E \supset F \supset \mathbb{Q}$ we can consider $\mathcal{O}_E$ as an $\mathcal{O}_F$-module. Since $\mathcal{O}_E$ is finitely generated over $\mathbb{Z}$, it is also finitely generated over $\mathcal{O}_F$ (just a larger ring of scalars), but $\mathcal{O}_E$ may or may not have a basis over $\mathcal{O}_F$.

When we treat $\mathcal{O}_E$ as a module over Ox_F, rather than over $\mathbb{Z}$, we speak about a **relative** extension of integers. If $\mathcal{O}_F$ is a PID, then $\mathcal{O}_E$ will be a free $\mathcal{O}_F$-module, so $\mathcal{O}_E$ will have a basis over $\mathcal{O}_F$. Such a basis is called a **relative integral basis** for E over F.

What we are after is a classification of finitely generated torsion-free modules over a

Dedekind domain, which will then be applied in a number field setting to describe $\mathcal{O}_E$ as an $\mathcal{O}_F$ module. The extent to which $\mathcal{O}_E$ could fail to have an $\mathcal{O}_F$-basis will be related to ideal classes in F. A technical concept we need to describe modules over a Dedekind domain is projective modules.

Definition 2.1.16 (Projective Module)**.** *Let A be any commutative ring. An A-module P is called **projective** if every surjective linear map $f : M \twoheadrightarrow P$ from any A-module M onto P looks like a projection out of a direct sum: there is an isomorphism $h : M \cong P \oplus N$ for some A-module N such that $h(m) = (f(m), *)$ for all $m \in M$. The condition $h(m) = (f(m), *)$ means $f(m) = 0$ if and only if $h(m)$ is in $\{0\} \oplus N$, which means h restricts to an isomorphism between* $\ker f$ *and $\{0\} \oplus N \cong N$.*

When A is a domain, any submodule of A^n is torsion-free, so a finitely generated projective module over a domain is torsion-free. Therefore, a finitely generated module over a domain that has torsion is not projective. **The important thing for us is that fractional ideals in a Dedekind domain are projective modules**.

Lemma 2.1.17. *[7, Lem. 5] For a domain A, any invertible fractional A-ideal is a projective A-module. In particular, when A is a Dedekind domain all fractional A-ideals are projective A-modules.*

Theorem 2.1.18. *[7, Thm. 6] Every finitely generated torsion-free module over a Dedekind domain A is isomorphic to a direct sum of ideals in A.*

Remark 2.1.19. *[7, Rem. 7] Using equations rather than isomorphisms, Theorem 6 says $M = M_1 \oplus \cdots \oplus M_d$ where each M_i is isomorphic to an ideal in A. Those ideals need not be principal, so M_i need not have the form Am_i. If M is inside a vector space over the fraction field of A, then $M = \bigoplus_{i=1}^{d} \mathfrak{a}_{ie_i}$ for some linearly independent e_i's,**but be careful:** If $\mathfrak{a}_i$ is a proper ideal in A then e_i is is not in M since $1 \notin \mathfrak{a}_i$. The e_i's are **not** a spanning set for M as a module since their coefficients are not running through A.*

Lemma 2.1.20. *[7, Lem. 8] Let A be a Dedekind domain. For fractional A-ideals $\mathfrak{a}$ and $\mathfrak{b}$, there is an A-module isomorphism $\mathfrak{a} \oplus \mathfrak{b} \cong A \oplus \mathfrak{a}\mathfrak{b}$.*

Example 2.1.21. *[7, Ex. 5] For $A = \mathbb{Z}[\sqrt{-5}]$, let $\mathfrak{p}_2 = (2, 1 + \sqrt{-5})$, so $\mathfrak{p}_2$ is not principal but $\mathfrak{p}_2^2 = 2A$ is principal. Then there is an A-module isomorphism $\mathfrak{p}_2 \oplus \mathfrak{p}_2 \cong A \oplus \mathfrak{p}_2^2 \cong A \oplus A$. This is intriguing: $\mathfrak{p}_2$ does not have an A-basis but $\mathfrak{p}_2 \oplus \mathfrak{p}_2$ does!. Working through the proof of Lemma 2.1.20 will show one how to write down a basis of $\mathfrak{p}_2 \oplus \mathfrak{p}_2$ explicitly.*

Recall that the definition of an A-module homomorphism: Let A be a ring and let M and N be A-modules. A map $f : M \to N$ is an A-module homomorphism if it respects the A-module structures of M and N:

(a) $f(x + y) = f(x) + f(y)$ for all $x, y \in M$ and

(b) $f(ax) = af(x)$ for all $a \in A,\ x \in M$.

An A-module homomorphism is an **isomorphism** of A-modules if it is both injective and surjective.

We claim that $f : A \oplus A \to A \oplus 2A$ defined by $(a, b) \mapsto (a, 2b)$ $a, b \in A$, is an A-module isomorphism. Let $x = (a_1, b_1)$ and $y = (a_2, b_2)$. Then $f(x + y) = f(a_1 + a_2, b_1 + b_2) = (a_1 + a_2, 2(b_1 + b_2) = (a_1, 2b_1) + (a_2, 2b_2) = f(x) + f(y)$. Let $c \in A$, Then $f(cx) = f(ca_1, cb_1) = ca_1 + 2cb_2 = c(a_1 + 2b_2) = cf(x)$. So f is an A-module homomorphism. Next, f is clearly surjective. Since it sends only $(0, 0)$ to $(0, 0)$, that is, since $\ker f = 0$, it is injective. So f is a bijection A-module homomorphism. We are done.

Theorem 2.1.22. *[7, Thm. 10] Let A be a Dedekind domain. For fractional A-ideals $\mathfrak{a}_1, \ldots, \mathfrak{a}_d$, there is an A-module isomorphism $\mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_d \cong A^{d-1} \oplus \mathfrak{a}_1 \cdots \mathfrak{a}_d$.*

Corollary 2.1.23. *[7, Cor. 11] Let E/F be a finite extension of number fields with $[E : F] = n$. As an $\mathcal{O}_F$-module, $\mathcal{O}_E \cong \mathcal{O}_F^{n-1} \oplus \mathfrak{a}$ for some non-zero ideal $\mathfrak{a}$ in $\mathcal{O}_F$.*

Proof. Since $\mathcal{O}_E$ is a finitely generated $\mathbb{Z}$-module it is a finitely generated $\mathcal{O}_F$-module and obviously has no torsion, so Theorems 6 and 10 imply $\mathcal{O}_E \cong Ox_F^{d-1} \oplus \mathfrak{a}$ for some $d \geq 1$ and nonzero ideal $\mathfrak{a}$ in $\mathcal{O}_F$. Letting $m = [F : \mathbb{Q}]$, both $\mathcal{O}_F$ and $\mathfrak{a}$ are free of rank m over $\mathbb{Z}$ while $\mathcal{O}_E$ is free of rank mn over $\mathbb{Z}$. Computing the rank of $\mathcal{O}_E$ and $\mathcal{O}_F^{d-1} \oplus \mathfrak{a}$ over $\mathbb{Z}$, $mn = m(d-1) + m = md$, so $d = n$. □

Thus $\mathcal{O}_E$ is almost a free $\mathcal{O}_F$-module. If $\mathfrak{a}$ is principal then $\mathcal{O}_E$ is free. As an $\mathcal{O}_F$-module up to isomorphism, $\mathcal{O}_F^{n-1} \oplus \mathfrak{a}$ only depends on $\mathfrak{a}$ through its ideal class, since $\mathfrak{a}$ and any $x\mathfrak{a}$ ($x \in F^\times$) are isomorphic $\mathcal{O}_F$-modules. Does $\mathcal{O}_F^{n-1} \oplus \mathfrak{a}$ as an $\mathcal{O}_F$-module, depend on $\mathfrak{a}$ exactly through its ideal class? That is, if $\mathcal{O}_F^{n-1} \oplus \mathfrak{a} \cong \mathcal{O}_F^{n-1} \oplus \mathfrak{b}$ as $\mathcal{O}_F$-modules, does $[\mathfrak{a}] = [\mathfrak{b}]$ in $\mathrm{Cl}(F)$? The next two theorems say the answer is YES.

Theorem 2.1.24. *[7, Thm. 12] Let A be a domain with fraction field F. For fractional A-ideals $\mathfrak{a}$ and $\mathfrak{b}$ in F, $\mathfrak{a} \cong \mathfrak{b}$ as A-modules if and only if $\mathfrak{a} = x\mathfrak{b}$ for some $x \in F^\times$.*

Theorem 2.1.25. *[7, Thm. 13] For nonzero ideals $\mathfrak{a}_1, \ldots, \mathfrak{a}_m$ and $\mathfrak{b}_1, \ldots, \mathfrak{b}_n$ in a Dedekind domain A, we have $\mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_m \cong \mathfrak{b}_1 \oplus \cdots \oplus \mathfrak{b}_n$ if and only if $m = n$ and $[\mathfrak{a}_1 \ldots \mathfrak{a}_m] = [\mathfrak{b}_1 \ldots \mathfrak{b}_n]$ in $\mathrm{Cl}(A)$.*

Example 2.1.26. *[7, Ex. 14] $F = \mathbb{Q}(\sqrt{-6})$, $E = F(\sqrt{-3})$, and $\mathcal{O}_E \cong \mathcal{O}_F \oplus \mathfrak{p}$ where $\mathfrak{p} = (3, \sqrt{-6})$. We can show $\mathcal{O}_E$ is not a free $\mathcal{O}_F$-module: if it were free then $\mathcal{O}_E \cong \mathcal{O}_F^2$, so $\mathcal{O}_F \oplus \mathfrak{p} \cong \mathcal{O}_F \oplus \mathcal{O}_F$ as $\mathcal{O}_F$-modules. Then Theorem 13 implies $\mathfrak{p} \cong \mathcal{O}_F$ as $\mathcal{O}_F$-modules, so $\mathfrak{p}$ is principal, but $\mathfrak{p}$ is nonprincipal. This is a contradiction.*

We can now associate to any finite extension of number fields E/F a canonical class in $\mathrm{Cl}(F)$, namely $[\mathfrak{a}]$ where $\mathcal{O}_E \cong \mathcal{O}_F^{n-1} \oplus \mathfrak{a}$ as $\mathcal{O}_F$-modules. Theorem 2.1.25 assures us $[\mathfrak{a}]$ is well-defined. Since the construction of $[\mathfrak{a}]$ is due to Steinitz (1912), $[\mathfrak{a}]$ is called the ***Steinitz class*** of E/F.

2.2 An Existence Theorem for Integral Representations

We will use the following result, including the proof as it is instructive, from Keith Conrad's notes[9].

Lemma 2.2.1. *[9, Lem. 1] Let R be a domain with fraction field K. For a finitely* *generated R-module U, $K \otimes_R U$ is finite dimensional as a K-vector space and* $\dim_K(K \otimes_R U)$ *is the maximal number of R-linearly independent elements in U and is a lower bound on the size of a spanning set for U. In particular, the size of each linearly independent subset of U is less than or equal to the size of each spanning set of U.*

Proof. If $x_1, \ldots, x_n$ is a spanning set for U as an R-module then $1 \otimes x_1, \ldots, 1 \otimes x_n$ span $K \otimes_R U$ as a K-vector space, so $\dim_K(K \otimes_R U) \leq n$.

Let $y_1, \ldots, y_d$ be R-linearly independent in U. Suppose $\sum_{i-1}^{d} c_i(1 \otimes y_i) = 0$ with $c_i \in K$. Write $c_i = a_i/b$ using a common denominator b in R. Then $0 = 1/b \otimes \sum_{i=1}^{d} a_i y_i$ in $K \otimes_R U$. This implies that $\sum_{i=1}^{d} a_i y_i$ belongs to a torsion submodule of U, so $\sum_{i=1}^{d} r a_i y_i = 0$ in U for some nonzero $r \in R$. By linear independence of the y_i's over R every ra_i is 0, so every a_i is 0 (since R is a domain). Thus every $c_i = a_i/b$ is 0. So $\{1 \otimes y_i\}$ is K-linearly independent in $K \otimes_R U$ and therefore $d \leq \dim_K(K \otimes_R U)$.

Now to show U has a linearly independent subset of size $\dim_K(K \otimes_R U)$. Let $\{e_1, \ldots, e_d\}$ be a linearly independent subset of U, where d is maximal. (Since $d \leq \dim_K(K \otimes_R U)$, there is a maximal d.) For every $u \in U$, $\{e_1, \ldots, e_d, u\}$ has to be linearly dependent, so there is a nontrivial R-linear relation $a_1 e_1 + \cdots + a_d e_d + au = 0$ with $a \neq 0$. If $a = 0$, then all the a_i's are 0 by linear independence of the e_i's. In $K \otimes_R U$,

$$\sum_{i=1}^{d} a_i(1 \otimes e_i) + a(1 \otimes m) = 0$$

and from the K-vector space structure on $K \otimes_R U$ we can solve for $1 \otimes u$ as a K-linear combination of the $1 \otimes e_i$'s. Therefore $\{1 \otimes e_i\}$ spans $K \otimes_R U$ as a K-vector space. This set is also linearly independent over K by the previous paragraph, so it is a basis and therefore $d = \dim_K(K \otimes_R U)$. □

We will use a definition and a theorem from Curtis and Reiner's[10] treatment with some modifications for notational convenience. Let R be a Dedekind domain with quotient field K and let G be a finite group.

Definition 2.2.2 (R-order)**.** *Let A be a finite dimensional algebra over K with a unity element e. An R-order in A is a subset X of A with satisfies*

(a) X is a subring of A,

(b) $e \in X$,

(c) X contains a K-basis of A,

(d) X is a finitely generated R-module.

If the representations of a finite group H is given by matrices with entries in R, then RH is an R-order in KH. As another example, the ring of all algebraic integers in an algebraic number field L is a $\mathbb{Z}$-order in L.

We embed K in KG by the mapping $\alpha \mapsto \alpha e, \alpha \in K$; this also embeds R in KG. Given an KG-module U^*, we shall always make U^* into a K-module by setting $\alpha m = (\alpha e)m,\ \alpha \in K,\ m \in U^*$. Since $em = m$, $m \in U^*$, this makes U^* into a vector space over K.

A RG-module U can be embedded in the KG-module U^* where $U^* = K \otimes_R U$. The embedding of U in U^* is given by $u \mapsto 1 \otimes u$. Lemma 1 tells us that $K \otimes_R U$ is a finite dimensional vector space. That is, U^* is a finite dimensional K-space; we have defined the R-rank of U [denoted by $(U : R)$] to be $(U^* : K)$ which is the dimension of the vector space U^*. Then $(U : R)$ is just the maximal number of R-free elements of U. The action of KG on U^* is given by $(\alpha x)(\beta \otimes u) = \alpha\beta \otimes xu$. $\alpha, \beta \in K$, $x \in RG$, $u \in U$. We write $U^* = KU$ to indicate that U^* consists of all K-linear combinations of U (in this case U^* is a full lattice). Now for the theorem from Curtis and Reiner[10].

Theorem 2.2.3. *[10, Thm. 75.2] Let U^* be an KG-module that is of dimension n as a K-vector space. Then U^* contains a RG-module U of R-rank n such that $U^* = KU$.*

Proof. Let $U^* = Ku_1^* \oplus \cdots \oplus Ku_n^*$ where the u_i^* are the basis elements of U^* and U_j^* is the j-th summand of U^*. Define

$$U = \sum_{j=1}^{n} (RG)U_j^* = \left\{ \sum_j x_j u_j^* \ : \ x_j \in RG \right\}.$$

Then

$$(RG)U \subset \sum (RG)^2 u_j^* \subset \sum (RG) u_j^* = U$$

since RG is a ring. Next U is a finitely generated R-module because RG is. Further $e \in RG$ and that each $u_j^* \in U$, so that $U^* = KU$. □

We can now prove the following theorem for integral representations.

Theorem 2.2.4. *Let G be a finite group. Let $K = \mathbb{Q}(\zeta_N)$ be the cyclotomic field with its ring of integers $R = \mathbb{Z}[\zeta_N] \subset K$ ($R = \{a_0 + a_1\zeta_N + a_2\zeta_N^2 + \cdots + a_{N-1}\zeta_N^{N-1}\}$). V is a finitely generated KG-module that is free as a K-module (since it is a finite dimensional vector space over K). Then there exists an RG-module U that is projective as an R-module, such that $V \cong U \otimes_R K$.*

Proof. Since R is the ring of integers of K, R is a Dedekind domain and it is finitely generated. Using the definition of U from Theorem 75.2, we see that U is a finitely generated R-module. Noting that vector spaces are torsion free, U is a subset of a finite dimensional vector space U^*, and R is a subset of K we have that U is a torsion-free R-module. So U is a projective R-module since finitely generated modules over Dedekind domains are projective if and only if they are torsion-free. We then apply Lemma 1 to show that $U \otimes_R K$ is indeed a vector space. It is clear that $U \otimes_R K \cong K \otimes_R U$. Next Theorem 75.2 constructs U explicitly which proves its existence. □

2.3 Riese's Lemma 3 and an alternate proof

Lemma 2.3.1. *[26, Lem. 3] Let $\widetilde{R} = R[\frac{1}{p}]$ for some rational prime p. If U is an RG-lattice then $\widetilde{U} = \widetilde{R}U \cong \widetilde{R} \otimes_R U$ is an $\widetilde{R}G$-lattice, and to every $\widetilde{R}G$-lattice $\widetilde{U}$ there exists* *an RG-lattice U with this property. $\widetilde{U}$ is $\widetilde{R}$-free if and only if the Steinitz class of U can be represented by an ideal of R lying above p.*

Proof. Recall that in a Dedekind domain, R, every non-zero fractional ideal is invertible. So the set of fractional ideals forms a group under multiplication. Also, every ideal in R has a unique factorization as a product of prime ideals. So the prime ideals of R can be viewed as generators of the group $\mathrm{Frac}(R)$. To be more precise, the ideal group of R is the free abelian group generated by its nonzero prime ideals $\mathfrak{p}$.

Let $\widetilde{R} = R[\frac{1}{p}]$. Next define the map $\phi : \mathrm{Frac}(R) \to \mathrm{Frac}(\widetilde{R})$ by where it sends its prime ideals: $\mathfrak{p} \mapsto \widetilde{R}\mathfrak{p}$. Then $\ker\phi$ is generated by those prime ideals $\mathfrak{p}$ of R which contain p ($\mathfrak{p} \cap S \neq \emptyset$). ϕ is a split epimorphism. Defining $\psi : \mathrm{Frac}(\widetilde{R}) \to \mathrm{Frac}(R)$ by $\widetilde{\mathfrak{p}} \mapsto \widetilde{\mathfrak{p}} \cap R$, we have the following split short exact sequence

$$0 \longrightarrow \ker(\phi) \hookrightarrow \mathrm{Frac}(R) \underset{\underset{\psi}{\dashleftarrow}}{\overset{\phi}{\longrightarrow}} \mathrm{Frac}(\widetilde{R}) \xrightarrow{\mathrm{Im}\,\phi} 0$$

since $\phi \circ \psi = \mathbb{1}$. That is, $\widetilde{\mathfrak{p}} \mapsto (\widetilde{\mathfrak{p}} \cap R) \mapsto \widetilde{R}(\widetilde{\mathfrak{p}} \cap R) \cong \widetilde{\mathfrak{p}}$.

We have the canonical homomorphisms $\Phi_1 : \mathrm{Frac}(R) \to \mathrm{Cl}(R)$ where $\mathfrak{p} \mapsto [\mathfrak{p}]$ and $\Phi_2 : \mathrm{Frac}(\widetilde{R}) \to \mathrm{Cl}(\widetilde{R})$ where $\widetilde{\mathfrak{p}} \mapsto [\widetilde{\mathfrak{p}}]$. This gives us the following commutative square:

$$\begin{array}{ccc} \mathrm{Frac}(R) & \xrightarrow{\phi} & \mathrm{Frac}(\widetilde{R}) \\ \downarrow{\scriptstyle \Phi_1} & & \downarrow{\scriptstyle \Phi_2} \\ \mathrm{Cl}(R) & \xrightarrow{\varphi} & \mathrm{Cl}(\widetilde{R}) \end{array}$$

If $\widetilde{\mathfrak{p}} = (\widetilde{\pi})$ is a principal prime ideal of $\widetilde{R}$ and $\mathfrak{p} = \widetilde{\mathfrak{p}} \cap R$, then

$$\widetilde{R}\mathfrak{p} = \widetilde{R}(\widetilde{\mathfrak{p}} \cap R) = \widetilde{R}\widetilde{\mathfrak{p}} \cap \widetilde{R}R = \widetilde{\mathfrak{p}} \cap \widetilde{R} = \widetilde{\mathfrak{p}}$$

and $\widetilde{\pi} = \dfrac{\pi}{p^k}$ for some $\pi \in \mathfrak{p}$ and some $k \geq 0$. It follows that

$$\widetilde{R}\pi = \widetilde{R}p^k\widetilde{\pi} = \widetilde{R}\widetilde{\mathfrak{p}} = \widetilde{\mathfrak{p}}$$

and that $R\pi = \widetilde{\mathfrak{p}}J$ for some ideal J of R containing p. Thus we have an epimorphism

$\mathrm{Cl}(R) \twoheadrightarrow \mathrm{Cl}(\widetilde{R})$ whose kernel is generated by the classes of the (prime) ideals of R lying above p. Diagrammatically we have the following:

$$\begin{array}{ccccccccc} & & & & \mathrm{Frac}(R) & \xrightarrow{\phi} & \mathrm{Frac}(\widetilde{R}) & & \\ & & & & \downarrow{\scriptstyle \Phi_1} & & \downarrow{\scriptstyle \Phi_2} & & \\ 0 & \longrightarrow & \ker(\varphi) & \hookrightarrow & \mathrm{Cl}(R) & \xrightarrow{\varphi} & \mathrm{Cl}(\widetilde{R}) & \longrightarrow & 0. \end{array}$$

2.4 An Isomorphism Theorem of Class Groups

Given the short exact sequence

$$0 \longrightarrow \ker(\varphi) \hookrightarrow \mathrm{Cl}(R) \xrightarrow{\varphi} \mathrm{Cl}(\widetilde{R}) \longrightarrow 0$$

with $R = \mathbb{Z}[\zeta_p]$ and $\widetilde{R} = \mathbb{Z}[1/p, \zeta_p]$, we want to show that φ is an isomorphism. From the previous discussion, φ is surjective so we need to show that it is injective, that is, $\ker\varphi = [0]$. We will use two lemmas from Washington[37] to show that (p) is not a prime ideal in $\mathbb{Z}[\zeta_p]$. □

2.4.1 Lemmas from Washington

Lemma 2.4.1. *[37, Lem.1.3] Suppose r and s are integers with $(p, rs) = 1$. Then $(\zeta_p^r - 1)/(\zeta_p^s - 1)$ is a unit of $\mathbb{Z}[\zeta_p]$.*

Proof. (Noting that GCD is multiplicative, $1 = (p, rs) = (p, r)(p, s)$ implies $(p, r) = 1$ and $(p, s) = 1$. So ζ_p^r and ζ_p^s are primitive roots of unity). Let $r \equiv st \mod p$ for some t.

$$\frac{\zeta_p^r - 1}{\zeta_p^s - 1} = \frac{\zeta_p^{st} - 1}{\zeta_p^s - 1} = 1 + \zeta_p^s + \cdots + \zeta_p^{s(t-1)} \in \mathbb{Z}[\zeta_p].$$

Similarly, writing $s \equiv rv \mod p$ for some v,

$$\frac{\zeta_p^s - 1}{\zeta_p^r - 1} \in \mathbb{Z}[\zeta_p].$$

□

Lemma 2.4.2. *[37, Lem.1.4] The ideal $(1 - \zeta_p)$ is a prime ideal of $\mathbb{Z}[\zeta_p]$ and $(1 - \zeta_p)^{p-1} =$*

(p). Therefore p is totally ramified in $\mathbb{Q}(\zeta_p)$ (prime ideal factoring in an extension so as to give some repeated prime ideal factors).

Proof. Since

$$X^{p-1} + X^{p-2} + \cdots + X + 1 = \prod_{i=1}^{p-1}(X - \zeta_p^i),$$

we let $X = 1$ to obtain

$$p = \prod_{i=1}^{p-1}(1 - \zeta_p^i).$$

From Lemma 2.4.1, we see that $(1 - \zeta_p)$ and $(1 - \zeta_p^i)$ are associate, so we have equality of ideals $(1 - \zeta_p) = (1 - \zeta_p^i)$. Therefore $(p) = (1 - \zeta_p)^{p-1}$. Since (p) can have at most $(p - 1) = \deg(\mathbb{Q}(\zeta_p)/\mathbb{Q})$ prime factors in $\mathbb{Q}(\zeta)$, it follows that $(1 - \zeta)$ must be a prime ideal of $\mathbb{Z}[\zeta_p]$. Alternatively, if $(1 - \zeta_p) = A \cdot B$, then $p = N(1 - \zeta_p) = NA \cdot NB$ so either $NA = 1$ or $NB = 1$ since p is prime in $\mathbb{Z}$. Therefore the ideal $(1 - \zeta)$ does not factor in $\mathbb{Z}[\zeta_p]$. □

Remark 2.4.3. *Robert Ash[1] also proves these two lemmas and are worth reading.*

Theorem 2.4.4. *With $R = \mathbb{Z}[\zeta_p]$ and $\widetilde{R} = \mathbb{Z}\left[\frac{1}{p}, \zeta_p\right]$, $\mathrm{Cl}(R) \cong \mathrm{Cl}(\widetilde{R})$.*

Proof. So we have $(p) = (1 - \zeta_p)^{p-1}$ and Lemma 1.4 showed that $(1 - \zeta_p)$ is a prime ideal in $\mathbb{Z}[\zeta_p]$. Since $\mathbb{Z}[\zeta_p]$ is a Dedekind domain, it is also a maximal ideal. It is clear that (p) is not a prime ideal in $\mathbb{Z}[\zeta_p]$ nor is it prime in $\mathbb{Z}[\frac{1}{p}, \zeta_p]$ either since p is a unit in $\mathbb{Z}[\frac{1}{p}, \zeta_p]$. Now $(p) = (1 - \zeta_p)^{p-1} \subset (1 - \zeta_p)$ since $(1 - \zeta_p)$ divides $(1 - \zeta_p)^{p-1}$. $(1 - \zeta_p)$ is a principal ideal and since the class group is a quotient of the fractional ideals by the principal ideals, $(1 - \zeta_p) \in [0]$. Since $(1 - \zeta_p)$ is maximal, nothing else, namely no non-principal ideal divides (contains) $(1 - \zeta_p)$ so $\ker(\varphi) = [0]$. Thus φ is an isomorphism of class groups. □

What this isomorphism tells us is if we have an integral basis for the representations for $\widetilde{R}$, then we have an integral basis over R. We will calculate the representations of $\mathrm{SL}_2(\mathbb{Z}/3\mathbb{Z})$ and $\mathrm{SL}_2(\mathbb{Z}/5\mathbb{Z})$ using Nobs and Wolfart's methods[24], using bases developed by Wang[36], and using bases developed by Zemel[44]. We will also compare the results with Riese's[26] results. Riese refers to $R(1,+)$ (as well as $R(n,+)$) as the Weil character ξ of degree $\frac{1}{2}(p+1)$ and $R(1,-)$ (as well as $R(n,-)$) as the Weil character ζ of degree $\frac{1}{2}(p-1)$. He proves the following for these two characters (where μ_p is the p-th root of unity).

Proposition 2.4.5. *[26, Prop. 2] The Weil characters can be realized over $R = \mathbb{Z}[\mu_p]$ by representations which are stable under all field automorphisms of $G = \mathrm{SL}(2,q)$.*

Proposition 2.4.6. *[26, Prop. 3] Assume that $q \equiv 3 \mod 4$. Then the Weil character ξ can be realized over $R = \mathbb{Z}\left[\frac{1+\sqrt{-p}}{2}\right]$.*

Proposition 2.4.7. *[26, Prop. 4] Suppose that $q = p^n$ is a rational square or that $q \equiv 5 \mod 8$. Then the Weil character ξ can be realized over $R = \mathbb{Z}\left[\frac{1+\sqrt{p}}{2}\right]$.*

The non-trivial irreducible representations for $\mathrm{SL}_2(\mathbb{Z}/p^\lambda\mathbb{Z})$ are the Steinberg St, principal series ($R(1,\pm), R(n,\pm)$, and $N_1(\chi)$), and cuspidal representations.

2.5 The $p+1$-dimensional irreducible principal series integral representation

We can also construct the p-dimensional principal series representation of $\mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})$ by inducing a non-trivial one-dimensional representation of the Borel subgroup of $\mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})$. Its entries will be over $\mathbb{Z}[\zeta_{p-1}]$. This agrees with Riese[26, Prop. 1]. We can do so by inducing the one dimensional character of its Borel subgroup and the Bruhat decomposition. We will use use this method in Chapter 4 Section 2 to compute the representation that is integral over $\mathbb{Z}[\zeta_4] = \mathbb{Z}[i]$.

2.5.1 Non-trivial one-dimensional character of the Borel subgroup of $SL(2,p)$

Kirby[16] gives a complete calculation of the character tables of T, the subgroup of diagonal matrices, and B, the Borel subgroup. With p an odd prime, let $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$. Let

$G = \mathrm{SL}_2(\mathbb{F}_p)$. Its Borel subgroup is given by

$$B = \left\{ \begin{bmatrix} u & x \\ 0 & u^{-1} \end{bmatrix} \ : \ u \in \mathbb{F}_p^\times, \ x \in \mathbb{F}_p \right\}.$$

With $\xi := \exp(2\pi i/(p-1))$, and $I_2 = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$, the trivial and non-trivial characters of B are given in Table 2.5.1.

B	εI_2	$\begin{bmatrix} a^i & 0 \\ 0 & a^{-i} \end{bmatrix}$	$\begin{bmatrix} \varepsilon & 1 \\ 0 & \varepsilon \end{bmatrix}$	$\begin{bmatrix} \varepsilon & a \\ 0 & \varepsilon \end{bmatrix}$
	$\varepsilon \in \{\pm I_2\}$	$a^i \in \mathbb{F}_p^\times \setminus \{\pm I_2\}$		$a \in f_p^\times$, a **is a non-square**
|class|	1	p	$(p-1)/2$	$(p-1)/2$
#classes	2	$p-3$	2	2
χ_0	1	1	1	1
χ_j	ε^j	ξ^{ji}	ε^j	ε^j

Table 2.5.1: Character Table of the Borel Subgroup

2.5.2 Inducing the Borel subgroup

We review the notions of restricting a representation and inducing a representation.

Definition 2.5.1. *Let H be a subgroup of G and let $\pi : G \to \mathrm{GL}(V)$ be a representation of G. The* ***restriction of*** *π* ***to*** *H, denoted $Res_H^G(\pi)$, is defined by*

$$[Res_H^G(\pi)](h) = \phi(h)$$

for all $h \in H$. If χ is the character of π, write $res_H^G(\chi)$ for the character of $Res_H^G(\chi)$.

Definition 2.5.2. *Let G be a finite group, H subgroup of G and let $\sigma : H \to \mathbb{C}^\times$ be a one dimensional character (representation) of H. The* ***induced representation****, $Ind_H^G(\sigma)$, has vector space*

$$V = Ind_H^G(\sigma) = \{f : G \to \mathbb{C} : f(hg) = \sigma(h)f(g), \ \forall g \in G, \ h \in H\}$$

and representation $\rho : G \to \mathrm{GL}(V)$, given by the G-action $[\rho(g)f](x) = f(xg)$ for all

$x, g \in G$. The dimension of the induced representation is given by the index of G and H, i.e.,

$$\dim(Ind_H^G(\sigma)) = [G : H].$$

The character of $\mathrm{Ind}_H^G(\sigma))$ is given by

$$[\mathrm{ind}_H^G(\sigma)](g) = \sum_{\substack{x \in H\backslash G \\ g \in x^{-1}Hx}} \sigma(xgx^{-1}).$$

In particular, $[\mathrm{ind}_H^G(\sigma)](g) = 0$ unless g is conjugate to an element in H and if $g = 1$, then $[\mathrm{ind}_H^G(\sigma)](1) = \dim(V)$.

2.5.3 The Bruhat Decomposition

From Lang[20], we learned of the Bruhat decomposition. We corrected/clarified the conclusion of lemma. Let F be a field. A **Borel subgroup** of GL_2 or SL_2 is a subgroup which is conjugate to the **standard subgroup** consisting of all matrices

$$\begin{bmatrix} a & b \\ 0 & d \end{bmatrix},$$

(with $d = a^{-1}$ in the case of SL_2). We let U be the group of matrices

$$u(b) = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}, \quad b \in F.$$

We let A be the group of diagonal matrices, $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$. We let

$$s(a) = \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}, \quad a \in F^{\times}, \quad w = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

For the rest of this section we let $G = \mathrm{SL}_2(F)$, or $\mathrm{GL}_2(F)$.

Lemma 2.5.3. *The matrices*

$$X(b) = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}, \quad \textit{and} \quad Y(c) = \begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}$$

generate $\mathrm{SL}_2(F)$.

Proof. Multiplying an arbitrary element of $\mathrm{SL}_2(F)$ by matrices of the above type on the right and on the left corresponds to elementary row and column operations (e.g., adding a scalar multiple of a row to the other, etc.). Thus a given matrix can always be brought into a form

$$\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}$$

by such multiplications. Let $W(a) = X(a)Y(-a^{-1})$ (correction to the lemma begins here). Then,

$$\begin{aligned} W(a)W(-1) &= X(a)Y(-a^{-1})X(-1)Y(-(-1)^{-1}) \\ &= X(a)Y(-a^{-1})X(-1)Y(1) \\ &= \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -a^{-1} & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & a \\ -a^{-1} & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} a & a \\ 1 & 1 + a^{-1} \end{bmatrix} \end{aligned}$$

and since

$$
\begin{aligned}
W(a)W(-1)X(-1)Y(-a) &= \begin{bmatrix} a & a \\ 1 & 1+a^{-1} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -a & 1 \end{bmatrix} \\
&= \begin{bmatrix} a & 0 \\ 1 & a^{-1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -a & 1 \end{bmatrix} \\
&= \begin{bmatrix} a & 0 \\ 1 & a^{-1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -a & 1 \end{bmatrix} \\
&= \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix},
\end{aligned}
$$

we are done. □

If we let $\overline{U}$ be the group of lower matrices,

$$
\begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}
$$

then we see that

$$
wUw^{-1} = \overline{U}.
$$

Let V be an element of U. Then

$$
\begin{aligned}
wVw^{-1} &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 \\ -1 & -c \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ -c & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}
\end{aligned}
$$

$$= \begin{bmatrix} 1 & 0 \\ -c & 1 \end{bmatrix} \in U.$$

Also note the commutation relation

$$\begin{aligned} w \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} w^{-1} &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & d \\ -a & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} d & 0 \\ 0 & a \end{bmatrix} \end{aligned} \tag{2.5.1}$$

shows that w normalizes A. Similarly,

$$wBw^{-1} = \overline{B}.$$

Letting $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in B$, we have

$$\begin{aligned} w \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} w^{-1} &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & d \\ -a & -b \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} d & 0 \\ -b & a \end{bmatrix} \in \overline{B} \end{aligned}$$

is the group of lower triangular matrices. We note that $B = AU = UA$ and also that A normalizes U. There is a decomposition of G into disjoint subsets, $G = B \sqcup BwB$. Indeed, view G as operating on the left of column vectors. The isotropy (little or stabilizer) group of

$$e^1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

is obviously U. The orbit Be^1 consists of all column vectors whose second component is zero. On the other hand,

$$we^1 = \begin{bmatrix} 0 \\ -1 \end{bmatrix},$$

and therefore the orbit Bwe^1 consists of all vectors whose second component is $\neq 0$, and whose first component is arbitrary. Since these two orbits of B and BwB cover the orbit Ge^1, it follows that the union of B and BwB is equal to G (because the isotropy group U is contained in B), and they are obviously disjoint. This decomposition is called the **Bruhat decomposition**.

Remark 2.5.4. *Sury[33] has the most general proof and cleanest example for us. It is completely trivial to deduce a corresponding Bruhat decomposition*

$$\mathrm{SL}_n(K) = \sqcup_{w \in S_n} B_0 w B_0.$$

Here, $B_0 = B \cap \mathrm{SL}_n(K)$*. For* $n = 2$*, this is explicitly given as follows. If* $g = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{SL}_2(K)$*, then* $g \in B_0$ *if* $c = 0$*. If* $c \neq 0$*, then*

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -c^{-1} & -a \\ 0 & -c \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & dc^{-1} \\ 0 & 1 \end{bmatrix}.$$

2.5.4 Cosets of $\mathrm{SL}(2, p)/B$

We need to compute the cosets of $\mathrm{SL}_2(\mathbb{F}_p)/B$ in order to induce the one-dimensional representation of B. The Bruhat decomposition gives a glimpse of what they may be. Let $G = \mathrm{SL}_2(\mathbb{F}_p)$ and B its Borel subgroup. $|G| = p(p-1)(p+1)$ and $|B| = p(p-1)$. So, $[G : B] = p + 1$ tells us that there are $p + 1$ cosets.

The first one is the easiest one, IB. So let I be its representative.

The second coset has a representative of the form

$$\begin{bmatrix} 0 & -x \\ x^{-1} & y \end{bmatrix}$$

where $x \in \mathbb{F}_p^\times$ and $y \in \mathbb{F}_p$. Call this representative M_2. For this case all matrices of the this form are in the same coset.

$$M_2B_1 = \begin{bmatrix} 0 & -x \\ x^{-1} & y \end{bmatrix} \begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} = \begin{bmatrix} 0 & -xa^{-1} \\ ax^{-1} & bx^{-1} + ya^{-1} \end{bmatrix}.$$

There are $p(p-1)$ elements mapped to $p(p-1)$ elements. If we let $B_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, then

$$M_2B_1 = \begin{bmatrix} 0 & -x \\ x^{-1} & y \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -x \\ x^{-1} & y + x^{-1} \end{bmatrix}.$$

So we have 2 of the $p+1$ cosets. The claim is that remaining coset representatives take the form (fix a and range c over $\mathbb{F}_p^\times$):

$$\begin{bmatrix} a^{-1} & 0 \\ c & a \end{bmatrix}.$$

Let $P_c = \begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}$ and $P_d = \begin{bmatrix} 1 & 0 \\ d & 1 \end{bmatrix}$ be two of the $p-1$ representatives. Then

$$(P_d)^{-1}P_c = \begin{bmatrix} 1 & 0 \\ -d & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ d & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ d-c & 1 \end{bmatrix}.$$

So $(P_d)^{-1}P_c \notin B$ if $d \neq c$.

$$\begin{bmatrix} x^{-1} & 0 \\ y & x \end{bmatrix} B = \begin{bmatrix} x^{-1} & 0 \\ y & x \end{bmatrix} \begin{bmatrix} a^{-1} & b \\ 0 & a \end{bmatrix} = \begin{bmatrix} ax^{-1} & bx^{-1} \\ a\,y & b\,y + xa^{-1} \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 0 \\ y & 1 \end{bmatrix} B = \begin{bmatrix} 1 & 0 \\ y & 1 \end{bmatrix} \begin{bmatrix} a^{-1} & b \\ 0 & a \end{bmatrix} = \begin{bmatrix} a & b \\ a\,y & b\,y + a^{-1} \end{bmatrix}.$$

Let $X_0 \cdots X_p$ be the coset representatives for $\mathrm{SL}(2,p)/B$. Let χ denote χ_j with $j = 1$, the non-trivial character of the Borel subgroup 2.5.1. Then the p-dimensional integral and irreducible representations,$D_1(\chi)(S)$ and $D_1(\chi)(T)$, for the generators of $\mathrm{SL}(\mathbb{Z}/p\mathbb{Z})$ are given by

$$D_1(\chi)(S) = \begin{bmatrix} \chi(X_0^{-1}SX_0) & \chi(X_0^{-1}SX_1) & \cdots & \chi(X_0^{-1}SX_p) \\ \chi(X_1^{-1}SX_0) & \chi(X_1^{-1}SX_1) & \cdots & \chi(X_1^{-1}SX_p) \\ \vdots & \vdots & \vdots & \vdots \\ \chi(X_p^{-1}SX_0) & \chi(X_p^{-1}SX_1) & \cdots & \chi(X_p^{-1}SX_p) \end{bmatrix} \tag{2.5.2}$$

and

$$D_1(\chi)(T) = \begin{bmatrix} \chi(X_0^{-1}TX_0) & \chi(X_0^{-1}TX_1) & \cdots & \chi(X_0^{-1}TX_p) \\ \chi(X_1^{-1}TX_0) & \chi(X_1^{-1}TX_1) & \cdots & \chi(X_1^{-1}TX_p) \\ \vdots & \vdots & \vdots & \vdots \\ \chi(X_p^{-1}TX_0) & \chi(X_p^{-1}TX_1) & \cdots & \chi(X_p^{-1}TX_p) \end{bmatrix} \tag{2.5.3}$$

We will explicitly compute the p-dimensional principal series representation for $p = 3$ and $p = 5$.

2.6 Method for constructing the reducible integral representation of $1 + St$

We can construct a reducible $p+1$-dimensional representation of $\mathrm{SL}(2,p)$ by inducing the trivial character of the Borel subgroup[16]. Again, we $X_0 \cdots X_p$ be the coset representatives for $\mathrm{SL}(2,p)/B$. Let χ denote the trivial character of the Borel subgroup 2.5.1.

Then the p-dimensional integral and irreducible representations,$D_1(\chi)(S)$ and $D_1(\chi)(T)$, for the generators of $\mathrm{SL}(\mathbb{Z}/p\mathbb{Z})$ are given by

$$D_1(\chi)(S) = \begin{bmatrix} \chi(X_0^{-1}SX_0) & \chi(X_0^{-1}SX_1) & \cdots & \chi(X_0^{-1}SX_p) \\ \chi(X_1^{-1}SX_0) & \chi(X_1^{-1}SX_1) & \cdots & \chi(X_1^{-1}SX_p) \\ \vdots & \vdots & \vdots & \vdots \\ \chi(X_p^{-1}SX_0) & \chi(X_p^{-1}SX_1) & \cdots & \chi(X_p^{-1}SX_p) \end{bmatrix} \tag{2.6.1}$$

and

$$D_1(\chi)(T) = \begin{bmatrix} \chi(X_0^{-1}TX_0) & \chi(X_0^{-1}TX_1) & \cdots & \chi(X_0^{-1}TX_p) \\ \chi(X_1^{-1}TX_0) & \chi(X_1^{-1}TX_1) & \cdots & \chi(X_1^{-1}TX_p) \\ \vdots & \vdots & \vdots & \vdots \\ \chi(X_p^{-1}TX_0) & \chi(X_p^{-1}TX_1) & \cdots & \chi(X_p^{-1}TX_p) \end{bmatrix} \tag{2.6.2}$$

where for $0 \leq i \leq p$ and $0 \leq i \leq p$,

$$\chi(X_iSX_j) = \begin{cases} 0 & \text{if } X_iSX_j \notin B \\ 1 & \text{if } X_iSX_j \in B \end{cases} \tag{2.6.3}$$

and

$$\chi(X_iTX_j) = \begin{cases} 0 & \text{if } X_iTX_j \notin B \\ 1 & \text{if } X_iTX_j \in B \end{cases}. \tag{2.6.4}$$

We will explicitly compute this $p+1$-dimensional principal series representation for $p = 3$ and $p = 5$. We will then construct the integral (over $\mathbb{Z}$) Steinberg representations for $p = 3$ and $p = 5$. We will do both in the following chapters.

CHAPTER 3 The Irreducible Representations Of SL(2,3)

3.1 The Weil Representation

We will use Nobs and Wolfart II[24], Kloosterman[18] and what we learned from Reeder[25]. $|\,\mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})| = p(p^2-1)$. So $|\,\mathrm{SL}_2(\mathbb{Z}/3\mathbb{Z})| = 3(9-1) = 24$. Using the fact that the order of the group is equal to sum of the squares of the dimensions of the irreducible representations, we have

$$24 = 1^2 + 1^2 + 1^2 + 2^2 + 2^2 + 2^2 + 3^2 = 1 + 1 + 1 + 4 + 4 + 4 + 9,$$

giving us three irreducible representations of degree 1, three of degree 2, and one of degree 3. It agrees with Nobs and Wolfart[24] (Nobs and Wolfart do not list the trivial representation).

Representations of Level 1, $p = 3$		Degree	Number	Remarks
$D_1(\chi)$	$\chi \in \mathfrak{B}$	$p+1=4$	$\frac{1}{2}(p-3)=0$	NONE. Theorem 1[24]
$N_1(\chi)$	$\chi \in \mathfrak{B}$	$p-1=2$	$\frac{1}{2}(p-1)=1$	Theorem 2[24]
$R_1(1,\pm), R_1(n,\pm)$	$\left(\dfrac{n}{p}\right) = -1$	$\dfrac{p\pm 1}{2} = 2,\ 1$	4	Theorem 4
$N_1(\chi_1)$		$p=3$	1	"Steinberg Representation"

Table 3.1.1: The type and number of irreducible representations of SL(2, 3)

So how are they constructed from the Weil representation? Using Theorem 2 and Eq(7) from Nobs[23]: we have $Q(x) = x^2/p$, $B(x,y) = 2xy/p$, $A_\lambda = M = \mathbb{Z}/p^\lambda\mathbb{Z}$. With $\lambda = 1$, $p = 3$, we have $A_1 = \mathbb{Z}/p\mathbb{Z} = \mathbb{Z}/3\mathbb{Z}$, $|M| = 3$, $r = 1$.

With $S_Q(a) = |M|^{-1/2} \sum_{x \in M} \mathbf{e}(-aQ(x))$, we have

$$\begin{aligned}
S_Q(-1) &= 3^{-1/2} \sum_{x \in M} \mathbf{e}(-(-1)Q(x)) \\
&= 3^{-1/2} \sum_{x \in M} \mathbf{e}(Q(x)) \\
&= 3^{-1/2} \left(\exp(2\pi i(0^2/p)) + \exp(2\pi i(1^2/p)) + \exp(2\pi i(2^2/p))\right) \\
&= 3^{-1/2} \left(\exp(0) + \exp(2\pi i/3) + \exp(8\pi i/3)\right) \\
&= 3^{-1/2} \left(1 - \frac{1}{2} + i\frac{\sqrt{3}}{2} - \frac{1}{2} + i\frac{\sqrt{3}}{2}\right) \\
&= 3^{-1/2} \left(1 - 1 + i\sqrt{3}\right) \\
&= i
\end{aligned}$$

which agrees with Nobs' I, Lemma 1: with $r = 1, p = 3$, $\lambda = 1$,

$$\Lambda(a) = \left(\frac{a}{p}\right)^{\lambda}, \quad S_Q(-1) = \begin{cases} 1 & \text{if } \lambda \text{ even} \\ \left(\frac{r}{p}\right) \varepsilon(p) & \text{if } \lambda \text{ odd} \end{cases},$$

distinguishes $\varepsilon(d) = 1$ or i, depending on whether $d \equiv 1$ or $3 \mod 4$. So, $S_Q(-1) = \left(\frac{r}{p}\right) \varepsilon(p) = \left(\frac{1}{p}\right) \varepsilon(p) = 1 \cdot i = i$ where $\varepsilon(p) = i$ since $p \equiv 3 \mod 4$. With M an abelian module and Q a quadratic form on M, Nobs[23] calls the pair (M, Q) a **quadratic module**. According to the comment after Nobs[23] Definition 3, the representations of $\mathrm{SL}_2(A_\lambda)$ in the case $p \neq 2$ correspond to the quadratic modules with $M = A_\lambda, Q(x) = p^{-\lambda} r x^2$ $r \neq 0 \mod p$, $\lambda \geq 1$ are called $R_\lambda(r)$.

Recall the definition of a quadratic form on an abelian group. We now define a Gauss sum[32].

Definition 3.1.1. *Let $q(x) = x^2/p$ be a quadratic form. Let $\zeta_p = \exp(2\pi i/p)$. Then*

$$\Omega(p) := \sum_x \zeta_p^{x^2} = \begin{cases} i \cdot \sqrt{p} & \textit{if } p \equiv 3 \mod 4 \\ \sqrt{p} & \textit{if } p \equiv 1 \mod 4 \end{cases}$$

is a Gauss sum.

Remark 3.1.2. *We have the following relationship between the Gauss sum and $S_Q(-1)$:*

$$\Omega(p) = \left(\frac{S_Q(-1)}{\sqrt{M}}\right)^{-1} = \frac{\sqrt{M}}{S_Q(-1)}.$$

We will use this fact when we perform computations using SAGE.

3.1.1 Representation of the action by T

Theorem 2 from Nobs[23]tells us that

$$\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right] f(x) = \mathbf{e}(b(Q(x)) \cdot f(x) \text{ for all } b \in A_\lambda.$$

Taking $f(x) = \delta_x$ and $b = 1$, we have the action of the T matrix

$$\begin{aligned} T\delta_x &= \left[\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right] \delta_x \\ &= \mathbf{e}((Q(x)) \cdot \delta_x \\ &= \exp(2\pi i x^2/p)\delta_x \\ &= \zeta_p^{x^2} \delta_x. \end{aligned}$$

It is a diagonal matrix as before.

3.1.2 Every rep. of a finite group is equivalent to a unitary rep.

We will make use of the following lemma[31] for our calculation for S. That is, relating Nobs' "S", $\left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$ to the standard "S", $\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$.

Lemma 3.1.3. *Every representation of a finite group is equivalent to a unitary representation.*

Proof. Let G be a finite group. Let $\varphi : G \to \mathrm{GL}(V)$ be a representation with $\dim V = n$. Since very vector space has a basis, choose a basis B for V. Let $T : V \to \mathbb{C}^n$ be the isomorphism taking coordinates with respect to B. Defining $\rho_g = T\varphi_g T^{-1}$ for $g \in G$ yields a representation $\rho : G \to \mathrm{GL}_n(\mathbb{C})$ equivalent to φ. Let $\langle \cdot, \cdot \rangle$ be the standard inner product (Euclidean dot product) on $\mathbb{C}^n$. Next define

$$(v, w) = \sum_{g \in G} \langle \rho_g v, \rho_g w \rangle.$$

It meets the definition of an inner product since it has the three properties:

1. Linearity

$$\begin{aligned}
(c_1 v_1 + c_2 v_2, w) &= \sum_{g \in G} \langle \rho_g (c_1 v_1 + c_2 v_2), \rho_g w \rangle \\
&= \sum_{g \in G} [c_1 \langle \rho_g v_1, \rho_g w \rangle + (c_2 \langle \rho_g v_2, \rho_g w \rangle] \\
&= c_1 \sum_{g \in G} \langle \rho_g v_1, \rho_g w \rangle + c_2 \sum_{g \in G} \langle \rho_g v_2, \rho_g w \rangle \\
&= c_1 (v_1, w) + c_2 (v_2, w).
\end{aligned}$$

2. Conjugate Symmetry

$$\begin{aligned}
(w, v) &= \sum_{g \in G} \langle \rho_g w, \rho_g v \rangle \\
&= \sum_{g \in G} \overline{\langle \rho_g w, \rho_g v \rangle} \\
&= \overline{(v, w)}.
\end{aligned}$$

3. Positive Definiteness. Since each term $\langle \rho_g v, \rho_g v \rangle \geq 0$,

$$(v, v) = \sum_{g \in G} \langle \rho_g v, \rho_g v \rangle \geq 0.$$

If $(v, v) = 0$, then $\sum_{g \in G} \langle \rho_g v, \rho_g v \rangle = 0$ which implies $\langle \rho_g v, \rho_g v \rangle = 0$ for all $g \in G$. So $0 = \langle \rho_g v, \rho_g v \rangle = \langle \rho_1 v, \rho_1 v \rangle = \langle v, v \rangle$. So $v = 0$.

Next, we verify the representation is unitary with respect to this inner product.

$$(\rho_h v, \rho_h w) = \sum_{g \in G} \langle \rho_g \rho_h v, \rho_g \rho_h w \rangle = \langle \rho_{gh} v, \rho_{gh} \rho_{gh} w \rangle.$$

Substituting $x = gh$, x ranges over all the elements of G since G ranges over all G. To see this, let $k \in G$, then if $g = kh^{-1}$, $x = k$. So, we have our desired result:

$$(\rho_h v, \rho_h w) = \sum_{x \in G} \langle \rho_x v, \rho_x w \rangle = (v, w).$$

□

3.1.3 The representation of the action by S

Taking $f(x) = \delta_x$, applying the fact that the inverse of a unitary matrix is its conjugate transpose, recognizing that the right hand-side of the equation is a symmetric matrix,

$$\left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right] f(x) = S_Q(-1)|M|^{-1/2} \sum_{y \in M} \mathbf{e}(B(x,y)) \cdot \delta_y,$$

allows us to apply Lemma 3.1.3 to give

$$\begin{aligned} S\delta_x &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \delta_x \\ &= S_Q(1)|M|^{-1/2} \sum_{y \in M} \mathbf{e}(-B(x,y)) \cdot \delta_y \\ &= S_Q(1)|M|^{-1/2} \sum_{y \in M} \exp(-2\pi i/p \cdot (2xy)) \cdot \delta_y \\ &= \frac{-i}{\sqrt{M}} \sum_{y \in M} \zeta_p^{-2xy} \cdot \delta_y \end{aligned}$$

which agrees with our previous result.

3.2 Explicit Computation of the action of S from Theorem 2 for $p = 3$

We will brute force compute the action of S from Theorem 2 for $p = 3$ from Nobs[23]. Take $M = \mathbb{Z}/3\mathbb{Z}$. $|M| = 3$. $p = 3$. $\zeta = \zeta_3$ and $f(x) = \delta_x$. Since

$$\begin{aligned} \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right] f(x) &= S_Q(-1)|M|^{-1/2} \sum_{y \in M} \mathbf{e}(B(x,y)) \cdot f(y) \\ &= \frac{i}{\sqrt{3}} \left(\zeta^{2x \cdot 0} f(0) + \zeta^{2x \cdot 1} f(1) + \zeta^{2x \cdot 2} f(2)\right) \end{aligned}$$

$$= \frac{i}{\sqrt{3}} \left(1 \cdot f(0) + \zeta^{2x \cdot 1} f(1) + \zeta^{x} f(2) \right),$$

we have

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^2 f(x) = \frac{i}{\sqrt{3}} \left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} f(0) + \zeta^{2x} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} f(1) + \zeta^{x} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} f(2) \right).$$

Computing each summand gives

$$\begin{aligned} \frac{i}{\sqrt{3}} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} f(0) &= \frac{i}{\sqrt{3}} \frac{i}{\sqrt{3}} \left(\zeta^{2 \cdot 0 \cdot 0} f(0) + \zeta^{2 \cdot 0 \cdot 1} f(1) + \zeta^{2 \cdot 0 \cdot 2} f(2) \right), \\ &= \frac{-1}{3} \left(f(0) + f(1) + f(2) \right), \\ \frac{i}{\sqrt{3}} \cdot \zeta^{2x} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} f(1) &= \frac{-1}{3} \cdot \zeta^{2x} \cdot \left(\zeta^{2 \cdot 1 \cdot 0} f(0) + \zeta^{2 \cdot 1 \cdot 1} f(1) + \zeta^{2 \cdot 1 \cdot 2} f(2) \right), \\ &= \frac{-1}{3} \cdot \zeta^{2x} \cdot \left(f(0) + \zeta^{2} f(1) + \zeta^{1} f(2) \right), \\ \frac{i}{\sqrt{3}} \cdot \zeta^{x} \cdot \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} f(2) &= \frac{-1}{3} \cdot \zeta^{x} \cdot \left(\zeta^{2 \cdot 2 \cdot 0} f(0) + \zeta^{2 \cdot 2 \cdot 1} f(1) + \zeta^{2 \cdot 2 \cdot 2} f(2) \right). \\ &= \frac{-1}{3} \cdot \zeta^{x} \cdot \left(f(0) + \zeta^{1} f(1) + \zeta^{2} f(2) \right). \end{aligned}$$

So,

$$\begin{aligned} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^2 f(x) = \frac{-1}{3} [& (f(0) + f(1) + f(2)) \\ & + \zeta^{2x} \cdot \left(f(0) + \zeta^{2} f(1) + \zeta^{1} f(2) \right) \\ & + \zeta^{x} \cdot \left(f(0) + \zeta^{1} f(1) + \zeta^{2} f(2) \right)]. \end{aligned}$$

Next,

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^3 f(x) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \frac{-1}{3} [(f(0) + f(1) + f(2))$$

$$+\,\zeta^{2x}\cdot\big(f(0)+\zeta^2 f(1)+\zeta^1 f(2)\big)$$

$$+\,\zeta^{x}\cdot\big(f(0)+\zeta^1 f(1)+\zeta^2 f(2)\big)\ ]$$

$$=\frac{-1}{3}\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}[\ (1+\zeta^{2x}+\zeta^x)f(0)+(1+\zeta^{2x}\zeta^2+\zeta^x\zeta^1)f(1)$$

$$+\,(1+\zeta^{2x}\zeta^1+\zeta^x\zeta^2)f(2)\ ]$$

$$=\frac{-i}{\sqrt[3]{3}}[\ (1+\zeta^{2x}+\zeta^x)(f(0)+f(1)+f(2))$$

$$+\,(1+\zeta^{2x}\zeta^2+\zeta^x\zeta^1)(\big(f(0)+\zeta^2 f(1)+\zeta^1 f(2)\big))$$

$$+\,(1+\zeta^{2x}\zeta^1+\zeta^x\zeta^2)(\big(f(0)+\zeta^1 f(1)+\zeta^2 f(2)\big))\ ]$$

$$=\frac{-i}{\sqrt[3]{3}}[\ (1+\zeta^{2x}+\zeta^x)f(0)+(1+\zeta^{2x}+\zeta^x)f(1)+(1+\zeta^{2x}+\zeta^x)f(2)$$

$$+\,(1+\zeta^{2x}\zeta^2+\zeta^x\zeta^1)f(0)+(1+\zeta^{2x}\zeta^2+\zeta^x\zeta^1)\zeta^2 f(1)$$

$$+\,(1+\zeta^{2x}\zeta^2+\zeta^x\zeta^1)\zeta^1 f(2)+(1+\zeta^{2x}\zeta^1+\zeta^x\zeta^2)f(0)+$$

$$+\,(1+\zeta^{2x}\zeta^1+\zeta^x\zeta^2)\zeta^1 f(1)+(1+\zeta^{2x}\zeta^1+\zeta^x\zeta^2)\zeta^2 f(2)\ ]$$

$$=\frac{-i}{\sqrt[3]{3}}[\ (1+\zeta^{2x}+\zeta^x+1+\zeta^{2x}\zeta^2+\zeta^x\zeta^1+1+\zeta^{2x}\zeta^1+\zeta^x\zeta^2)f(0)$$

$$+\,((1+\zeta^{2x}+\zeta^x)+(1+\zeta^{2x}\zeta^2+\zeta^x\zeta^1)\zeta^2$$

$$+\,(1+\zeta^{2x}\zeta^1+\zeta^x\zeta^2)\zeta^1)f(1)$$

$$+\,((1+\zeta^{2x}+\zeta^x)+(1+\zeta^{2x}\zeta^2+\zeta^x\zeta^1)\zeta^1+$$

$$+\,(1+\zeta^{2x}\zeta^1+\zeta^x\zeta^2)\zeta^2)f(2)\ ]$$

$$=\frac{-i}{\sqrt[3]{3}}[\ (3+\zeta^{2x}+\zeta^x+\zeta^{2x+2}+\zeta^{x+1}+\zeta^{2x+1}+\zeta^{x+2})f(0)$$

$$+\,(1+\zeta^1+\zeta^2+\zeta^{2x}+\zeta^x+\zeta^{2x+4}+\zeta^{x+3}+\zeta^{2x+2}+\zeta^{x+3})f(1)$$

$$+\,(1+\zeta^1+\zeta^2+\zeta^{2x}+\zeta^x+\zeta^{2x+3}+\zeta^{x+2}+\zeta^{2x+3}+\zeta^{x+4})f(2)\ ]$$

$$=\frac{-i}{\sqrt[3]{3}}[\ (3+\zeta^{2x}+\zeta^x+\zeta^{2x+2}+\zeta^{x+1}+\zeta^{2x+1}+\zeta^{x+2})f(0)$$

$$+\,(\zeta^{2x}+\zeta^x+\zeta^{2x+1}+\zeta^x+\zeta^{2x+2}+\zeta^x)f(1)$$

$$+ (\zeta^{2x} + \zeta^{x} + \zeta^{2x} + \zeta^{x+2} + \zeta^{2x} + \zeta^{x+1}) f(2)\ ]$$

$$= \frac{-i}{\sqrt[3]{3}} [\ (3 + \zeta^{x} + \zeta^{x+1} + \zeta^{x+2} + \zeta^{2x} + \zeta^{2x+1} + \zeta^{2x+2}) f(0)$$

$$+ (3\zeta^{x} + \zeta^{2x} + \zeta^{2x+1} + \zeta^{2x+2}) f(1)$$

$$+ (\zeta^{x} + \zeta^{x+1} + \zeta^{x+2} + 3\zeta^{2x}) f(2)\ ]$$

$$= \frac{-i}{\sqrt[3]{3}} [\ (3 + (\zeta^{0} + \zeta^{1} + \zeta^{2})\zeta^{x} + (1 + \zeta^{1} + \zeta^{2})\zeta^{2x}) f(0)$$

$$+ (3\zeta^{x} + (\zeta^{0} + \zeta^{1} + \zeta^{2})\zeta^{2x}) f(1)$$

$$+ ((\zeta^{0} + \zeta^{1} + \zeta^{2})\zeta^{x} + 3\zeta^{2x}) f(2)\ ]$$

$$= \frac{-i}{\sqrt[3]{3}} [\ (3f(0) + (3\zeta^{x}) f(1) + (3\zeta^{2x}) f(2)\ ]$$

$$= \frac{-i}{\sqrt{3}} \left[(f(0) + \zeta^{x} f(1) + \zeta^{2x} f(2)\right]$$

$$= \frac{-i}{\sqrt{3}} \left[f(0) + \zeta^{-2x} f(1) + \zeta^{-4} f(2)\right]$$

$$= \frac{-i}{\sqrt{3}} \sum_{y \in M} \zeta^{-2xy} f(y).$$

So,

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^{3} f(x) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} f(x) = \frac{-i}{\sqrt{3}} \sum_{y \in M} \zeta^{-2xy} f(y)$$

agrees with our previous calculation for S.

This illustrates that the action of $\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$ on basis elements δ_x results in the the conjugation of the action by $\left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$. It was already symmetric so the transpose had no effect. Applying Lemma 3.1.3 is much easier. But this was an instance in which brute force worked reasonably well.

3.3 $R_1(1, \pm)$

Lemma 1 from Nobs[23] tells us that the quadratic module (M, Q) generates the Weil representation $W(M, Q)$ of $\mathrm{SL}_2(A_\lambda)$. According to the comment after Nobs[23] Definition 3, the representations of $\mathrm{SL}_2(A_\lambda)$ in the case $p \neq 2$ correspond to the quadratic modules with $M = A_\lambda, Q(x) = p^{-\lambda} r x^2 \ r \neq 0 \mod p$, $\lambda \geq 1$ are called $R_\lambda(r)$.

Recall,

$$S\delta_x = \frac{-i}{\sqrt{M}} \sum_{y \in M} \exp(-2\pi i/p \cdot (2xy)) \cdot \delta_y = \frac{-i}{\sqrt{M}} \sum_{y \in M} \zeta^{-2xy} \cdot \delta_y$$

and

$$T\delta_x = \zeta_p^{x^2} \delta_x.$$

Nobs[23] provides two methods for the decomposition of Weil's representations.

3.3.1 $\mathfrak{U} = \{-1, 1\}$

$\mathrm{Aut}(M, Q)$ is the group of automorphisms of M invariant under Q, i.e., for every $\varphi \in \mathrm{Aut}(M, Q)$, $Q(\varphi(x)) = Q(x)$ for all $x \in M$. So since $M = \mathbb{Z}/3\mathbb{Z}$ and $Q(x) = x^2/p$, $\mathrm{Aut}(M, Q)$ will consist of the identity and the inverse maps. $\mathrm{Aut}(M, Q) = \{1, -1\} \cong \mathbb{Z}/2\mathbb{Z} = C_2$. $\mathrm{Aut}(M, Q)$ has the trivial subgroup and itself as the only two subgroups since it is of prime order. So let $\mathfrak{U} = \{1, -1\}$ and let χ be a character of $\mathfrak{U}$, then

$$V(\chi) := \{f \in \mathbb{C}^M \mid f(\varepsilon x) = \chi(\varepsilon) f(x) \ \forall \varepsilon \in \mathfrak{U}, \ \forall x \in M\}$$

is a subspace of $V = \mathbb{C}^M$ that is invariant under $\mathrm{SL}_2(A_\lambda)$. If you write $W(M, Q, \chi)$ for the sub-representation of $W(M, Q)$ in the space $V(\chi)$, then

$$W(M, Q) = \oplus_\chi W(M, Q, \chi),$$

where χ runs through all the characters from $\mathfrak{U}$.

3.3.2 Basis Choice 1

For our case, there are two characters, χ_1 the trivial character and χ_2 the non-trivial character of the abelian group $\mathfrak{U}$.

Let's start with χ_1.

$$V(\chi_1) = \{f \in \mathbb{C}^M \mid f(\varepsilon x) = \chi(\varepsilon) f(x) \ \forall \varepsilon \in \mathfrak{U}, \ \forall x \in M\}$$

$$= \{f \in \mathbb{C}^M \mid f(\varepsilon x) = f(x)\ \forall \varepsilon \in \mathfrak{U},\ \forall x \in M\}$$

$$= \{f \in \mathbb{C}^M \mid f(1 \cdot x) = f(x) \text{ and } f(-1 \cdot x) = f(x)\ \forall x \in M\}$$

$$= \{f \in \mathbb{C}^M \mid f(-x) = f(x)\ \forall x \in M\}$$

So even functions meet the criterion. Let $f_0(x) = \delta_0(x)$. Then $\delta_0(-0) = \delta_0(0) = 1$, $\delta_0(-1) = \delta_0(2) = \delta_0(1) = 0$, and $\delta_0(-2) = \delta_0(1) = \delta_0(2) = 0$. It holds.

Let $f_1(x) = \delta_0(x) - \delta_1(x) - \delta_2(x)$. Then

$f_1(-0) = \delta_0(-0) - \delta_1(-0) - \delta_2(0) = 1 - 0 - 0 = f_1(0) = \delta_0(0) - \delta_1(0) - \delta_2(0)$.

$f_1(-1) = f_1(2) = \delta_0(2) - \delta_1(2) - \delta_2(2) = 0 - 0 - 1 = f_1(1) = \delta_0(1) - \delta_1(1) - \delta_2(1) = -1$.

$f_1(-2) = f_1(1) = \delta_0(1) - \delta_1(1) - \delta_2(1) = 0 - 1 - 0 = f_1(2) = \delta_0(2) - \delta_1(2) - \delta_2(2) = -1$.

It too holds.

$V(\chi_1) = \{f_0, f_1\}$. So it's a two dimensional subspace.

Next we have $\chi_2(1) = 1$ and $\chi_2(-1) = -1$. Our subspace is given by

$$V(\chi_2) = \{f \in \mathbb{C}^M \mid f(\varepsilon x) = \chi_2(\varepsilon) f(x)\ \forall \varepsilon \in \mathfrak{U},\ \forall x \in M\}.$$

Applying the relation gives $f(1 \cdot 0) = \chi_2(1) \cdot f(0) = 1 \cdot f(0) = f(0)$

$f(1 \cdot 1) = \chi_2(1) \cdot f(1) = 1 \cdot f(1) = f(1)$

$f(1 \cdot 2) = \chi_2(1) \cdot f(2) = 1 \cdot f(2) = f(2)$

$f(-1 \cdot 0) = f(0) = \chi_2(-1) \cdot f(0) = -1 \cdot f(0) = -f(0)$

$f(-1 \cdot 1) = f(-1) = f(2) = \chi_2(-1) \cdot f(1) = -1 \cdot f(1) = -f(1)$

$f(-1 \cdot 2) = f(-2) = f(1) = \chi_2(-1) \cdot f(2) = -1 \cdot f(2) = -f(2)$.

So for $\varepsilon = 1$, we have $f(x) = f(x)$. For $\varepsilon = -1$, $f(0) = -f(0)$ which implies $f(0) = 0$, $f(1) = -f(2) = -f(-1)$. So odd functions meet the criterion.

Let $f_2(x) = \delta_1(x) - \delta_2(x)$. Then

$f_2(-1 \cdot 0) = f_2(0) = \delta_1(0) - \delta_2(0) = 0 = \chi_2(-1) f_2(0) = -(\delta_1(0) - \delta_2(0)) = -0 = -f_1(0)$,

$$f_2(-1\cdot 1) = f_2(2) = \delta_1(2) - \delta_2(2) = -1 = \chi_2(-1)f_2(1) = -(\delta_1(1) - \delta_2(1)) = -1,$$

and

$$f_2(-1\cdot 2) = f_2(1) = \delta_1(1) - \delta_2(1) = 1 = \chi_2(-1)f_2(2) = -(\delta_1(2) - \delta_2(2)) = 1.$$

$V(\chi_2) = \{f_2\}$. So it's a one dimensional subspace. We need to perform a change of basis on matrix T_b

$$T_b = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \zeta_3^b & 0 \\ 0 & 0 & \zeta_3^b \end{bmatrix}$$

that results by the action of $\left[\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right]$. Let

$$M = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & -1 \end{bmatrix}.$$

Then M contains as column the vectors of the new basis $B = \{f_0, f_1, f_2\}$ with respect to the canonical basis $\{\delta_0, \delta_1, \delta_2\}$. M represents the matrix of change of basis from B to the canonical.

To determine the new coordinates with respect to the new basis,

$$\begin{aligned} T_f &= M^{-1}T_bM \\ &= \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \zeta_3^b & 0 \\ 0 & 0 & \zeta_3^b \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & -\zeta_3^b & \zeta_3^b \\ 0 & -\zeta_3^b & -\zeta_3^b \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} 1 & 1-\zeta_3^b & 0 \\ 0 & \zeta_3^b & 0 \\ 0 & 0 & \zeta_3^b \end{bmatrix}.$$

So we have our desired block diagonal matrix for the matrix representation of the action by $T = \left[\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right]$ with respect to the basis vectors $\{f_0, f_1, f_2\}$:

$$T_f = \left[\begin{array}{cc|c} 1 & 1-\zeta_3^b & 0 \\ 0 & \zeta_3^b & 0 \\ \hline 0 & 0 & \zeta_3^b \end{array}\right]$$

$R_1(1,+)$ corresponds to the 2×2 block and $R_1(1,-)$ corresponds to the 1×1 block.

Letting $T = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$, we see that

$$\begin{aligned} R_1(1,+)(T) &= \begin{bmatrix} 1 & 1-\zeta_3 \\ 0 & \zeta_3 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1-\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right) \\ 0 & -\frac{1}{2}+i\frac{\sqrt{3}}{2} \end{bmatrix} \\ &= \begin{bmatrix} 1 & \frac{3}{2}-i\frac{\sqrt{3}}{2} \\ 0 & -\frac{1}{2}+i\frac{\sqrt{3}}{2} \end{bmatrix} \end{aligned}$$

and

$$R_1(1,-)(T) = -\frac{1}{2} + i\frac{\sqrt{3}}{2}.$$

Now for the action by $\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$. Call this matrix S_b and it is

$$S_b = \frac{-i}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \zeta_3 & \zeta_3^2 \\ 1 & \zeta_3^2 & \zeta_3 \end{bmatrix}.$$

Applying the similarity transformation gives us the matrix with respect to the new basis

$$\begin{aligned}
S_f &= M^{-1}S_bM \\
&= \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \frac{-i}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \zeta_3 & \zeta_3^2 \\ 1 & \zeta_3^2 & \zeta_3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & -1 \end{bmatrix} \\
&= \frac{-i}{\sqrt{3}} \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 1 & -\zeta_3^2 - \zeta_3 + 1 & \zeta_3 - \zeta_3^2 \\ 1 & -\zeta_3^2 - \zeta_3 + 1 & \zeta_3^2 - \zeta_3 \end{bmatrix} \\
&= \frac{-i}{\sqrt{3}} \begin{bmatrix} 2 & -\zeta_3^2 - \zeta_3 & 0 \\ -1 & \zeta_3^2 + \zeta_3 - 1 & 0 \\ 0 & 0 & \zeta_3 - \zeta_3^2 \end{bmatrix} \\
&= \frac{-i}{\sqrt{3}} \begin{bmatrix} 2 & 1 & 0 \\ -1 & -2 & 0 \\ 0 & 0 & \zeta_3 - \zeta_3^2 \end{bmatrix}.
\end{aligned}$$

In block diagonal form,

$$S_f = \frac{-i}{\sqrt{3}} \left[\begin{array}{cc|c} 2 & 1 & 0 \\ -1 & -2 & 0 \\ \hline 0 & 0 & \zeta_3 - \zeta_3^2 \end{array}\right],$$

$R_1(1,+)$ corresponds to the 2×2 block and $R_1(1,-)$ corresponds to the 1×1 block. Letting $S = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$, we have

$$R_1(1,+)(S) = \begin{bmatrix} \frac{-2i\sqrt{3}}{3} & \frac{-i\sqrt{3}}{3} \\ \frac{i\sqrt{3}}{3} & \frac{2i\sqrt{3}}{3} \end{bmatrix}$$

and

$$R_1(1,-)(S) = \frac{-i\sqrt{3}}{3}(\zeta_3 - \zeta_3^2)$$

$$
\begin{aligned}
&= \frac{-i\sqrt{3}}{3}\left[\frac{-1+i\sqrt{3}}{2} - \frac{-1-i\sqrt{3}}{2}\right] \\
&= \frac{-i\sqrt{3}}{3} \cdot i\sqrt{3} \\
&= 1.
\end{aligned}
$$

3.3.3 Basis Choice 2

For our case, there are two characters, χ_1 the trivial character and χ_2 the non-trivial character of the abelian group $\mathfrak{U}$. Let's start with χ_1.

$$
\begin{aligned}
V(\chi_1) &= \{f \in \mathbb{C}^M \mid f(\varepsilon x) = \chi(\varepsilon) f(x) \ \forall \varepsilon \in \mathfrak{U},\ \forall x \in M\} \\
&= \{f \in \mathbb{C}^M \mid f(\varepsilon x) = f(x) \ \forall \varepsilon \in \mathfrak{U},\ \forall x \in M\} \\
&= \{f \in \mathbb{C}^M \mid f(1 \cdot x) = f(x) \text{ and } f(-1 \cdot x) = f(x) \ \forall x \in M\} \\
&= \{f \in \mathbb{C}^M \mid f(-x) = f(x) \ \forall x \in M\}
\end{aligned}
$$

So even functions meet the criterion. Let $f_0(x) = \delta_0(x)$. Then $\delta_0(-0) = \delta_0(0) = 1$, $\delta_0(-1) = \delta_0(2) = \delta_0(1) = 0$, $\delta_0(-2) = \delta_0(1) = \delta_0(2) = 0$. It holds.

Let $f_1(x) = \delta_1(x) + \delta_2(x)$. Then

$$f_1(-0) = \delta_1(-0) + \delta_2(-0) = 0 + 0 = 0 = \delta_1(0) + \delta_2(0)$$

$$f_1(-1) = f_1(2) = \delta_1(2) + \delta_2(2) = 0 + 1 = f_1(1) = \delta_1(1) + \delta_2(1),$$

and

$$f_1(-2) = f_1(1) = \delta_1(1) + \delta_2(1) = 1 = f_1(2) = \delta_1(2) + \delta_2(2) = 1.$$

It too holds and $V(\chi_1) = \{f_0, f_1\}$. So it's a two dimensional subspace.

Next we have $\chi_2(1) = 1$ and $\chi_2(-1) = -1$.

$$V(\chi_2) = \{f \in \mathbb{C}^M \mid f(\varepsilon x) = \chi_2(\varepsilon)f(x)\ \forall \varepsilon \in \mathfrak{U},\ \forall x \in M\}.$$

Computations show $f(1 \cdot 0) = \chi_2(1) \cdot f(0) = 1 \cdot f(0) = f(0)$,

$f(1 \cdot 1) = \chi_2(1) \cdot f(1) = 1 \cdot f(1) = f(1)$,

$f(1 \cdot 2) = \chi_2(1) \cdot f(2) = 1 \cdot f(2) = f(2)$,

$f(-1 \cdot 0) = f(0) = \chi_2(-1) \cdot f(0) = -1 \cdot f(0) = -f(0)$,

$f(-1 \cdot 1) = f(-1) = f(2) = \chi_2(-1) \cdot f(1) = -1 \cdot f(1) = -f(1)$,

$f(-1 \cdot 2) = f(-2) = f(1) = \chi_2(-1) \cdot f(2) = -1 \cdot f(2) = -f(2)$,

that for $\varepsilon = 1$, we have $f(x) = f(x)$. For $\varepsilon = -1$, $f(0) = -f(0)$ which implies $f(0) = 0$, $f(1) = -f(2) = -f(-1)$. So odd functions meet the criterion.

Let $f_2(x) = \delta_1(x) - \delta_2(x)$. Then

$$f_2(-1 \cdot 0) = f_2(0) = \delta_1(0) - \delta_2(0) = 0 = \chi_2(-1)f_2(0) = -(\delta_1(0) - \delta_2(0)) = -0 = -f_1(0),$$

$$f_2(-1 \cdot 1) = f_2(2) = \delta_1(2) - \delta_2(2) = -1 = \chi_2(-1)f_2(1) = -(\delta_1(1) - \delta_2(1)) = -1,$$

and

$$f_2(-1 \cdot 2) = f_2(1) = \delta_1(1) - \delta_2(1) = 1 = \chi_2(-1)f_2(2) = -(\delta_1(2) - \delta_2(2)) = 1$$

show that $V(\chi_2) = \{f_2\}$. So it's a one dimensional subspace. We need to perform a change of basis on matrix T_b

$$T_b = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \zeta_3^b & 0 \\ 0 & 0 & \zeta_3^{-b} \end{bmatrix}$$

that results by the action of $= \left[\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right]$. Let

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}.$$

Then M contains as column the vectors of the new basis $B = \{f_0, f_1, f_2\}$ with respect to the canonical basis $\{\delta_0, \delta_1, \delta_2\}$. M represents the matrix of change of basis from B to the canonical.

To determine the new coordinates with respect to the new basis,

$$\begin{aligned} T_f &= M^{-1} T_b M \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \zeta_3^b & 0 \\ 0 & 0 & \zeta_3^b \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \zeta_3^b & \zeta_3^b \\ 0 & \zeta_3^b & -\zeta_3^b \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \zeta_3^b & 0 \\ 0 & 0 & \zeta_3^b \end{bmatrix}. \end{aligned}$$

So we have our desired block diagonal matrix for the matrix representation of the action by $\left[\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right]$ with respect to the basis vectors $\{f_0, f_1, f_2\}$:

$$T_f = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & \zeta_3^b & 0 \\ \hline 0 & 0 & \zeta_3^b \end{array}\right]$$

Letting $T = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$, we see that

$$R_1(1,+)(T) = \begin{bmatrix} 1 & 0 \\ 0 & \zeta_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} + i\frac{\sqrt{3}}{2} \end{bmatrix}$$

and

$$R_1(1,-)(T) = -\frac{1}{2} + i\frac{\sqrt{3}}{2}.$$

Now for the action by $\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$. Call this matrix S_b and it is

$$S_b = \frac{-i}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \zeta_3 & \zeta_3^2 \\ 1 & \zeta_3^2 & \zeta_3 \end{bmatrix}.$$

Applying the similarity transformation gives us the matrix with respect to the new basis

$$\begin{aligned} S_f &= M^{-1} S_b M \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \frac{-i}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \zeta_3 & \zeta_3^2 \\ 1 & \zeta_3^2 & \zeta_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \\ &= \frac{-i}{\sqrt{3}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 1 & \zeta_3^2 + \zeta_3 & -\zeta_3^2 + \zeta_3 \\ 1 & \zeta_3^2 + \zeta_3 & \zeta_3^2 - \zeta_3 \end{bmatrix} \\ &= \frac{-i}{\sqrt{3}} \begin{bmatrix} 1 & 2 & 0 \\ 1 & \zeta_3^2 + \zeta_3 & 0 \\ 0 & 0 & -\zeta_3^2 + \zeta_3 \end{bmatrix} \end{aligned}$$

$$= \frac{-i}{\sqrt{3}} \begin{bmatrix} 1 & 2 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -\zeta_3^2 + \zeta_3 \end{bmatrix}$$

In block diagonal form,

$$S_f = \frac{-i}{\sqrt{3}} \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 1 & -1 & 0 \\ \hline 0 & 0 & \zeta_3 - \zeta_3^2 \end{array}\right],$$

$R_1(1,+)$ corresponds to the 2×2 block and $R_1(1,-)$ corresponds to the 1×1 block. Comparing the results from Basis Choice 1, the characters agree as they should. Letting $S = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$, we have

$$R_1(1,+)(S) = \begin{bmatrix} \frac{-i\sqrt{3}}{3} & \frac{-2i\sqrt{3}}{3} \\ \frac{-i\sqrt{3}}{3} & \frac{i\sqrt{3}}{3} \end{bmatrix}$$

and

$$\begin{aligned} R_1(1,-)(S) &= \frac{-i\sqrt{3}}{3}(\zeta_3 - \zeta_3^2) \\ &= \frac{-i\sqrt{3}}{3}\left[\frac{-1+i\sqrt{3}}{2} - \frac{-1-i\sqrt{3}}{2}\right] \\ &= \frac{-i\sqrt{3}}{3} \cdot i\sqrt{3} \\ &= 1. \end{aligned}$$

So these representations are realized over $\mathbb{Z}\left[\frac{1}{2}(1+\sqrt{-3}), \frac{1}{3}\right] = \mathbb{Z}\left[\zeta_3, \frac{1}{3}\right]$. By Theorem 2.4.4, integral representations over $\mathbb{Z}[\zeta_3]$ exist. We construct them in the next section.

3.4 Wang's Basis for $R(1,+)$

Wang[36] provides an integral basis for the $R(1,+)$ and $R(n,+)$ representations from the Weil representation of $T = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$. Let $v = \sum_{i=0}^{p-1} \delta_p$ and $\rho(T)$ be the Weil representation of T. Then an integral basis over $\mathbb{Z}[\zeta_p]$ is given by

$$\left\{v, \rho(T)\cdot v, (\rho(T))^2\cdot v, \cdots, (\rho(T))^{(p-3)/2}\cdot v, (\rho(T))^{(p-1)/2}\cdot v\right\}.$$

Using SAGE, we verify that the representations are integral over $\mathbb{Z}[\zeta_3]$. Since $\zeta_3 = \frac{1}{2}(1+\sqrt{-3})$, they are also integral over $\mathbb{Z}[\frac{1}{2}(1+\sqrt{-3})]$:

$$R(1,+)(S) = \begin{bmatrix} -\zeta_3 - 1 & -\zeta_3 \\ -\zeta_3 & \zeta_3 + 1 \end{bmatrix} \text{ and } R(1,+)(T) = \begin{bmatrix} 0 & -\zeta_3 \\ 1 & \zeta_3 + 1 \end{bmatrix},$$

and

$$R(1,-)(S) = 1 \text{ and } R(1,-)(T) = \zeta_3.$$

3.5 Candelori's Bases for $R(1,+)$ and $R(1,-)$

Let $\rho(S)$ and $\rho(T)$ be the Weil representations for $S = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$ and $T = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$ respectively. Define $v_+ := \delta_1 + \delta_{(p-1)}$ and $v_- := \delta_1 - \delta_{(p-1)}$. Define $\rho(U) := \rho(S)\cdot\rho(T)\cdot\rho(S)$. Then a basis for $R(n,+)$ is conjectured to be

$$\left\{v_+,\ \rho(U)\cdot v_+,\ (\rho(U))^2\cdot v_+, \cdots, (\rho(U))^{(p-3)/2}\cdot v_+,\ (\rho(T))^{(p-1)/2}\cdot v_+\right\}$$

and a basis for $R(n,-)$ is conjectured to be

$$\left\{v_-,\ \rho(U)\cdot v_-,\ (\rho(U))^2\cdot v_-, \cdots, (\rho(U))^{(p-3)/2}\cdot v_-,\ (\rho(U))^{(p-1)/2}\cdot v_-\right\}.$$

Using SAGE, we verify that the representations are integral over $\mathbb{Z}[\zeta_3] = \mathbb{Z}[\frac{1}{2}(1+\sqrt{-3})]$:

$$R(1,+)(S) = \begin{bmatrix} \zeta_3 & 1 \\ \zeta_3 & -\zeta_3 \end{bmatrix} \text{ and } R(1,+)(T) = \begin{bmatrix} \zeta_3 & 1 \\ 0 & 1 \end{bmatrix},$$

and

$$R(1,-)(S) = 1 \text{ and } R(1,-)(T) = \zeta_3.$$

3.6 $R_1(2, \pm)$

Since $0^2 \equiv 0 \mod 3$, $1^2 \equiv 1 \mod 2$, and $2^2 = 4 \equiv 1 \mod 3$, we have that 2 is a quadratic non-residue modulo 3. So, $\left(\dfrac{2}{3}\right) = -1$. We take $r = 2$. Our quadratic form is $Q(x) = 3^{-1}2 \cdot x^2 = \dfrac{2}{3}x^2$ and the associated bilinear form is

$$B(x,y) = Q(x+y) - Q(x) - Q(y) = \frac{2}{3}(x^2 + 2xy + y^2) - \frac{2}{3}x^2 - \frac{2}{3}y^2 = \frac{4}{3}xy.$$

Using Theorem 2 and Eq(7) from Nobs[23] [24]: We have $Q(x) = 2x^2/p$, $B(x,y) = 4xy/3$, $\lambda = 1$, $p = 3$, $A_\lambda = M = \mathbb{Z}/p\mathbb{Z} = A_1 = \mathbb{Z}/3\mathbb{Z}$, $|M| = 3$, $r = 2$.

$$\begin{aligned}
S_Q(a) &= |M|^{-1/2} \sum_{x \in M} \mathbf{e}(-aQ(x)) \\
S_Q(-1) &= 3^{-1/2} \sum_{x \in M} \mathbf{e}(-(-1)Q(x)) \\
&= 3^{-1/2} \sum_{x \in M} \mathbf{e}(Q(x)) \\
&= 3^{-1/2} \big(\exp(2\pi i(2 \cdot 0^2/3)) + \exp(2\pi i(2 \cdot 1^2/3)) + \exp(2\pi i(2 \cdot 2^2/3))\big) \\
&= 3^{-1/2} \left(\exp(0) + \exp(4\pi i/3) + \exp(16\pi i/3)\right) \\
&= 3^{-1/2} \left(1 - \frac{1}{2} - i\frac{\sqrt{3}}{2} - \frac{1}{2} - i\frac{\sqrt{3}}{2}\right) \\
&= 3^{-1/2} \left(1 - 1 - i\sqrt{3}\right) \\
&= -i
\end{aligned}$$

which agrees with Nobs' I, Lemma 1: with $r = 2, p = 3$, $\lambda = 1$,

$$\Lambda(a) = \left(\frac{a}{p}\right)^\lambda, \quad S_Q(-1) = \begin{cases} 1 & \text{if } \lambda \text{ even} \\ \left(\dfrac{r}{p}\right)\varepsilon(p) & \text{if } \lambda \text{ odd} \end{cases},$$

distinguishes $\varepsilon(d) = 1$ or i, depending on whether $d \equiv 1$ or $3 \mod 4$. We have $S_Q(-1) = \left(\dfrac{r}{p}\right)\varepsilon(p) = \left(\dfrac{2}{p}\right)\varepsilon(p) = -1 \cdot i = -i$ where $\varepsilon(p) = i$ since $p \equiv 3 \mod 4$. So our actions by $S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $T = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$ are

$$\begin{aligned}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \delta_x &= S\delta_x \\ &= \overline{S_Q(-1)}|M|^{-1/2} \sum_{y \in M} \mathbf{e}(-B(x,y)) \cdot \delta_y \\ &= \frac{i}{\sqrt{|M|}} \sum_{y \in M} \zeta_p^{-4xy} \cdot \delta_y.\end{aligned}$$

For our given M and $p = 3$,

$$\begin{aligned}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \delta_0 &= \frac{i}{\sqrt{3}} \sum_{y \in M} \zeta_3^{-4 \cdot 0 \cdot y} \cdot \delta_y \\ &= \frac{i}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \\ \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \delta_1 &= \frac{i}{\sqrt{3}} \sum_{y \in M} \zeta_3^{-4 \cdot 1 \cdot y} \cdot \delta_y \\ &= \frac{i}{\sqrt{3}} \begin{bmatrix} 1 \\ \zeta_3^{-4 \cdot 1 \cdot 1} \\ \zeta_3^{-4 \cdot 1 \cdot 2} \end{bmatrix} \\ &= \frac{i}{\sqrt{3}} \begin{bmatrix} 1 \\ \zeta_3^2 \\ \zeta_3 \end{bmatrix}, \\ \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \delta_2 &= \frac{i}{\sqrt{3}} \sum_{y \in M} \zeta_3^{-4 \cdot 2 \cdot y} \cdot \delta_y\end{aligned}$$

$$= \frac{i}{\sqrt{3}} \begin{bmatrix} 1 \\ \zeta_3^{-4\cdot 2\cdot 1} \\ \zeta_3^{-4\cdot 2\cdot 2} \end{bmatrix}$$

$$= \frac{i}{\sqrt{3}} \begin{bmatrix} 1 \\ \zeta_3 \\ \zeta_3^2 \end{bmatrix},$$

implying

$$S_b = \frac{i}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \zeta_3^2 & \zeta_3 \\ 1 & \zeta_3 & \zeta_3^2 \end{bmatrix}.$$

And

$$\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \delta_x = T\delta_x$$

$$= \mathbf{e}(bQ(x)) \cdot \delta_x$$

$$= \zeta_p^{2bx^2} \cdot \delta_x$$

gives

$$T_b = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \zeta_3^{2b} & 0 \\ 0 & 0 & \zeta_3^{2b} \end{bmatrix}.$$

3.6.1 $\mathfrak{U} = \{-1, 1\}$

Aut(M, Q) is the group of automorphisms of M invariant under Q, i.e., for every $\varphi \in$ Aut(M, Q), $Q(\varphi(x)) = Q(x)$ for all $x \in M$. So since $M = \mathbb{Z}/3\mathbb{Z}$ and $Q(x) = 2x^2/p$, Aut(M, Q) will consist of the identity and the inverse maps. Aut$(M, Q) = \{1, -1\} \cong \mathbb{Z}/2\mathbb{Z} = C_2$. Aut$(M, Q)$ has the trivial subgroup and itself as the only two subgroups since it is of prime order. So let $\mathfrak{U} = \{1, -1\}$ and let χ be a character of $\mathfrak{U}$, then

$$V(\chi) := \{f \in \mathbb{C}^M \mid f(\varepsilon x) = \chi(\varepsilon) f(x)\ \forall \varepsilon \in \mathfrak{U},\ \forall x \in M\}$$

is a subspace of $V = \mathbb{C}^M$ that is invariant under $\mathrm{SL}_2(A_\lambda)$. If you write $W(M, Q, \chi)$ for the sub-representation of $W(M, Q)$ in the space $V(\chi)$, then

$$W(M, Q) = \oplus_\chi W(M, Q, \chi),$$

where χ runs through all the characters from $\mathfrak{U}$.

3.6.2 Basis Choice 1

For our case, there are two characters, χ_1 the trivial character and χ_2 the non-trivial character of the abelian group $\mathfrak{U}$. Let's start with χ_1.

$$\begin{aligned} V(\chi_1) &= \{f \in \mathbb{C}^M \mid f(\varepsilon x) = \chi(\varepsilon) f(x)\ \forall \varepsilon \in \mathfrak{U},\ \forall x \in M\} \\ &= \{f \in \mathbb{C}^M \mid f(\varepsilon x) = f(x)\ \forall \varepsilon \in \mathfrak{U},\ \forall x \in M\} \\ &= \{f \in \mathbb{C}^M \mid f(1 \cdot x) = f(x) \text{ and } f(-1 \cdot x) = f(x)\ \forall x \in M\} \\ &= \{f \in \mathbb{C}^M \mid f(-x) = f(x)\ \forall x \in M\}. \end{aligned}$$

So even functions meet the criterion. Let $f_0(x) = \delta_0(x)$. Since $\delta_0(-0) = \delta_0(0) = 1$, $\delta_0(-1) = \delta_0(2) = \delta_0(1) = 0$, and $\delta_0(-2) = \delta_0(1) = \delta_0(2) = 0$, it holds.

Let $f_1(x) = \delta_0(x) - \delta_1(x) - \delta_2(x)$. Then

$$f_1(-0) = \delta_0(-0) - \delta_1(-0) - \delta_2(0) = 1 - 0 - 0 = f_1(0) = \delta_0(0) - \delta_1(0) - \delta_2(0),$$

$$f_1(-1) = f_1(2) = \delta_0(2) - \delta_1(2) - \delta_2(2) = 0 - 0 - 1 = f_1(1) = \delta_0(1) - \delta_1(1) - \delta_2(1) = -1,$$

and

$$f_1(-2) = f_1(1) = \delta_0(1) - \delta_1(1) - \delta_2(1) = 0 - 1 - 0 = f_1(2) = \delta_0(2) - \delta_1(2) - \delta_2(2) = -1$$

show it holds and that $V(\chi_1) = \{f_0, f_1\}$. So it's a two dimensional subspace.

Next we have $\chi_2(1) = 1$ and $\chi_2(-1) = -1$. With

$$V(\chi_2) = \{f \in \mathbb{C}^M \mid f(\varepsilon x) = \chi_2(\varepsilon)f(x)\ \forall \varepsilon \in \mathfrak{U},\ \forall x \in M\}$$

$f(1 \cdot 0) = \chi_2(1) \cdot f(0) = 1 \cdot f(0) = f(0)$,

$f(1 \cdot 1) = \chi_2(1) \cdot f(1) = 1 \cdot f(1) = f(1)$,

$f(1 \cdot 2) = \chi_2(1) \cdot f(2) = 1 \cdot f(2) = f(2)$,

$f(-1 \cdot 0) = f(0) = \chi_2(-1) \cdot f(0) = -1 \cdot f(0) = -f(0)$,

$f(-1 \cdot 1) = f(-1) = f(2) = \chi_2(-1) \cdot f(1) = -1 \cdot f(1) = -f(1)$,

and

$f(-1 \cdot 2) = f(-2) = f(1) = \chi_2(-1) \cdot f(2) = -1 \cdot f(2) = -f(2)$,

show that for $\varepsilon = 1$, we have $f(x) = f(x)$. For $\varepsilon = -1$, $f(0) = -f(0)$ which implies $f(0) = 0$, $f(1) = -f(2) = -f(-1)$. So odd functions meet the criterion.

Let $f_2(x) = \delta_1(x) - \delta_2(x)$. Then

$$f_2(-1 \cdot 0) = f_2(0) = \delta_1(0) - \delta_2(0) = 0 = \chi_2(-1)f_2(0) = -(\delta_1(0) - \delta_2(0)) = -0 = -f_1(0),$$

$$f_2(-1 \cdot 1) = f_2(2) = \delta_1(2) - \delta_2(2) = -1 = \chi_2(-1)f_2(1) = -(\delta_1(1) - \delta_2(1)) = -1,$$

and

$$f_2(-1 \cdot 2) = f_2(1) = \delta_1(1) - \delta_2(1) = 1 = \chi_2(-1)f_2(2) = -(\delta_1(2) - \delta_2(2)) = 1$$

shows that $V(\chi_2) = \{f_2\}$. So it's a one dimensional subspace.

We need to perform a change of basis on matrix T_b

$$T_b = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \zeta_3^{2b} & 0 \\ 0 & 0 & \zeta_3^{2b} \end{bmatrix}$$

that results by the action of $\left[\begin{smallmatrix} 1 & b \\ 1 & 0 \end{smallmatrix}\right]$. Let

$$M = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & -1 \end{bmatrix}.$$

Then M contains as column the vectors of the new basis $B = \{f_0, f_1, f_2\}$ with respect to the canonical basis $\{\delta_0, \delta_1, \delta_2\}$. M represents the matrix of change of basis from B to the canonical.

To determine the new coordinates with respect to the new basis,

$$\begin{aligned} T_f &= M^{-1} T_b M \\ &= \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \zeta_3^{2b} & 0 \\ 0 & 0 & \zeta_3^{2b} \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & -\zeta_3^{2b} & \zeta_3^{2b} \\ 0 & -\zeta_3^{2b} & -\zeta_3^{2b} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1-\zeta_3^{2b} & 0 \\ 0 & \zeta_3^{2b} & 0 \\ 0 & 0 & \zeta_3^{2b} \end{bmatrix} \end{aligned}$$

So we have our desired block diagonal matrix for the matrix representation of the action by $\left[\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right]$ with respect to the basis vectors $\{f_0, f_1, f_2\}$:

$$T_f = \left[\begin{array}{cc|c} 1 & 1-\zeta_3^{2b} & 0 \\ 0 & \zeta_3^{2b} & 0 \\ \hline 0 & 0 & \zeta_3^{2b} \end{array}\right].$$

Letting $T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, we see that

$$\begin{aligned} R_1(2,+)(T) &= \begin{bmatrix} 1 & 1-\zeta_3^2 \\ 0 & \zeta_3^2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1-\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right) \\ 0 & -\frac{1}{2}-i\frac{\sqrt{3}}{2} \end{bmatrix} \\ &= \begin{bmatrix} 1 & \frac{3}{2}+i\frac{\sqrt{3}}{2} \\ 0 & -\frac{1}{2}-i\frac{\sqrt{3}}{2} \end{bmatrix} \end{aligned}$$

and

$$R_1(2,-)(T) = -\frac{1}{2} - i\frac{\sqrt{3}}{2}.$$

Now for the action by $\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$. Call this matrix S_b and it is

$$S_b = \frac{i}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \zeta_3^2 & \zeta_3 \\ 1 & \zeta_3 & \zeta_3^2 \end{bmatrix}.$$

Applying the similarity transformation gives us the matrix with respect to the new basis

$$\begin{aligned} S_f &= M^{-1} S_b M \\ &= \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \frac{i}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \zeta_3^2 & \zeta_3 \\ 1 & \zeta_3 & \zeta_3^2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & -1 \end{bmatrix} \\ &= \frac{i}{\sqrt{3}} \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 1 & -\zeta_3^2-\zeta_3+1 & \zeta_3^2-\zeta_3 \\ 1 & -\zeta_3^2-\zeta_3+1 & \zeta_3-\zeta_3^2 \end{bmatrix} \end{aligned}$$

$$= \frac{i}{\sqrt{3}} \begin{bmatrix} 2 & -\zeta_3^2 - \zeta_3 & 0 \\ -1 & \zeta_3^2 + \zeta_3 - 1 & 0 \\ 0 & 0 & \zeta_3^2 - \zeta_3 \end{bmatrix}$$

$$= \frac{i}{\sqrt{3}} \begin{bmatrix} 2 & 1 & 0 \\ -1 & -2 & 0 \\ 0 & 0 & \zeta_3^2 - \zeta_3 \end{bmatrix}.$$

In block diagonal form,

$$S_f = \frac{i}{\sqrt{3}} \left[\begin{array}{cc|c} 2 & 1 & 0 \\ -1 & -2 & 0 \\ \hline 0 & 0 & \zeta_3^2 - \zeta_3 \end{array}\right],$$

$R_1(2,+)$ corresponds to the 2×2 block and $R_1(2,-)$ corresponds to the 1×1 block.

Letting $S = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$, we have

$$R_1(2,+)(S) = \begin{bmatrix} \frac{2i\sqrt{3}}{3} & \frac{i\sqrt{3}}{3} \\ \frac{-i\sqrt{3}}{3} & \frac{-2i\sqrt{3}}{3} \end{bmatrix}$$

and

$$\begin{aligned} R_1(2,-)(S) &= \frac{i\sqrt{3}}{3}(\zeta_3^2 - \zeta_3) \\ &= \frac{i\sqrt{3}}{3}\left[\frac{-1 - i\sqrt{3}}{2} - \frac{-1 + i\sqrt{3}}{2}\right] \\ &= \frac{i\sqrt{3}}{3} \cdot (-i\sqrt{3}) \\ &= 1. \end{aligned}$$

3.6.3 Basis Choice 2

For our case, there are two characters, χ_1 the trivial character and χ_2 the non-trivial character of the abelian group $\mathfrak{U}$.

Let's start with χ_1.

$$\begin{aligned} V(\chi_1) &= \{f \in \mathbb{C}^M \mid f(\varepsilon x) = \chi(\varepsilon) f(x)\ \forall \varepsilon \in \mathfrak{U},\ \forall x \in M\} \\ &= \{f \in \mathbb{C}^M \mid f(\varepsilon x) = f(x)\ \forall \varepsilon \in \mathfrak{U},\ \forall x \in M\} \\ &= \{f \in \mathbb{C}^M \mid f(1 \cdot x) = f(x) \text{ and } f(-1 \cdot x) = f(x)\ \forall x \in M\} \\ &= \{f \in \mathbb{C}^M \mid f(-x) = f(x)\ \forall x \in M\}. \end{aligned}$$

So even functions meet the criterion. Let $f_0(x) = \delta_0(x)$. Then $\delta_0(-0) = \delta_0(0) = 1$, $\delta_0(-1) = \delta_0(2) = \delta_0(1) = 0$, and $\delta_0(-2) = \delta_0(1) = \delta_0(2) = 0$ show it holds.

Let $f_1(x) = \delta_1(x) + \delta_2(x)$. Then

$f_1(-0) = \delta_1(-0) + \delta_2(-0) = 0 + 0 = 0 = \delta_1(0) + \delta_2(0)$,

$f_1(-1) = f_1(2) = \delta_1(2) + \delta_2(2) = 0 + 1 = f_1(1) = \delta_1(1) + \delta_2(1)$, and

$f_1(-2) = f_1(1) = \delta_1(1) + \delta_2(1) = 1 = f_1(2) = \delta_1(2) + \delta_2(2) = 1$

shows it holds and $V(\chi_1) = \{f_0, f_1\}$. So it's a two dimensional subspace.

Next we have $\chi_2(1) = 1$ and $\chi_2(-1) = -1$.

$$V(\chi_2) = \{f \in \mathbb{C}^M \mid f(\varepsilon x) = \chi_2(\varepsilon) f(x)\ \forall \varepsilon \in \mathfrak{U},\ \forall x \in M\}.$$

Then $f(1 \cdot 0) = \chi_2(1) \cdot f(0) = 1 \cdot f(0) = f(0)$,

$f(1 \cdot 1) = \chi_2(1) \cdot f(1) = 1 \cdot f(1) = f(1)$,

$f(1 \cdot 2) = \chi_2(1) \cdot f(2) = 1 \cdot f(2) = f(2)$,

$f(-1 \cdot 0) = f(0) = \chi_2(-1) \cdot f(0) = -1 \cdot f(0) = -f(0)$,

$f(-1 \cdot 1) = f(-1) = f(2) = \chi_2(-1) \cdot f(1) = -1 \cdot f(1) = -f(1)$,

and $f(-1 \cdot 2) = f(-2) = f(1) = \chi_2(-1) \cdot f(2) = -1 \cdot f(2) = -f(2)$ show that for $\varepsilon = 1$, we have $f(x) = f(x)$. For $\varepsilon = -1$, $f(0) = -f(0)$ which implies $f(0) = 0$, $f(1) = -f(2) = -f(-1)$. So odd functions meet the criterion.

Let $f_2(x) = \delta_1(x) - \delta_2(x)$. Then

$f_2(-1\cdot 0) = f_2(0) = \delta_1(0) - \delta_2(0) = 0 = \chi_2(-1)f_2(0) = -(\delta_1(0) - \delta_2(0)) = -0 = -f_1(0)$.

$f_2(-1\cdot 1) = f_2(2) = \delta_1(2) - \delta_2(2) = -1 = \chi_2(-1)f_2(1) = -(\delta_1(1) - \delta_2(1)) = -1$.

$f_2(-1\cdot 2) = f_2(1) = \delta_1(1) - \delta_2(1) = 1 = \chi_2(-1)f_2(2) = -(\delta_1(2) - \delta_2(2)) = 1$.

$V(\chi_2) = \{f_2\}$. So it's a one dimensional subspace.

We need to perform a change of basis on matrix T_b

$$T_b = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \zeta_3^{2b} & 0 \\ 0 & 0 & \zeta_3^{2b} \end{bmatrix}$$

that results by the action of $\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$. Let

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}.$$

Then M contains as column the vectors of the new basis $B = \{f_0, f_1, f_2\}$ with respect to the canonical basis $\{\delta_0, \delta_1, \delta_2\}$. M represents the matrix of change of basis from B to the canonical.

To determine the new coordinates with respect to the new basis,

$$T_f = M^{-1}T_bM$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \zeta_3^{2b} & 0 \\ 0 & 0 & \zeta_3^{2b} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \zeta_3^{2b} & \zeta_3^{2b} \\ 0 & \zeta_3^{2b} & -\zeta_3^{2b} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \zeta_3^{2b} & 0 \\ 0 & 0 & \zeta_3^{2b} \end{bmatrix}.$$

So we have our desired block diagonal matrix for the matrix representation of the action by $\left[\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right]$ with respect to the basis vectors $\{f_0, f_1, f_2\}$:

$$T_f = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & \zeta_3^{2b} & 0 \\ \hline 0 & 0 & \zeta_3^{2b} \end{array}\right]$$

Letting $T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, we see that

$$R_1(2,+)(T) = \begin{bmatrix} 1 & 0 \\ 0 & \zeta_3^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} - i\frac{\sqrt{3}}{2} \end{bmatrix}$$

and

$$R_1(2,-)(T) = -\frac{1}{2} - i\frac{\sqrt{3}}{2}.$$

Now for the action by $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Call this matrix S_b and it is

$$S_b = \frac{i}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \zeta_3^2 & \zeta_3 \\ 1 & \zeta_3 & \zeta_3^2 \end{bmatrix}.$$

Applying the similarity transformation gives us the matrix with respect to the new basis

$$\begin{aligned}
S_f &= M^{-1} S_b M \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \frac{i}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \zeta_3^2 & \zeta_3 \\ 1 & \zeta_3 & \zeta_3^2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \\
&= \frac{i}{\sqrt{3}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 1 & \zeta_3^2 + \zeta_3 & \zeta_3^2 - \zeta_3 \\ 1 & \zeta_3^2 + \zeta_3 & -\zeta_3^2 + \zeta_3 \end{bmatrix} \\
&= \frac{i}{\sqrt{3}} \begin{bmatrix} 1 & 2 & 0 \\ 1 & \zeta_3^2 + \zeta_3 & 0 \\ 0 & 0 & \zeta_3^2 - \zeta_3 \end{bmatrix} \\
&= \frac{i}{\sqrt{3}} \begin{bmatrix} 1 & 2 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & \zeta_3^2 - \zeta_3 \end{bmatrix}.
\end{aligned}$$

In block diagonal form,

$$S_f = \frac{i}{\sqrt{3}} \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 1 & -1 & 0 \\ \hline 0 & 0 & \zeta_3^2 - \zeta_3 \end{array}\right],$$

$R_1(2,+)$ corresponds to the 2×2 block and $R_1(2,-)$ corresponds to the 1×1 block.

Comparing the results from Basis Choice 1, the characters agree as they should. Letting $S = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$, we have

$$R_1(2,+)(S) = \begin{bmatrix} \frac{i\sqrt{3}}{3} & \frac{2i\sqrt{3}}{3} \\ \frac{i\sqrt{3}}{3} & \frac{-i\sqrt{3}}{3} \end{bmatrix}$$

and

$$\begin{aligned} R_1(2,-)(S) &= \frac{i\sqrt{3}}{3}(\zeta_3^2 - \zeta_3) \\ &= \frac{i\sqrt{3}}{3}\left[\frac{-1-i\sqrt{3}}{2} - \frac{-1+i\sqrt{3}}{2}\right] \\ &= \frac{i\sqrt{3}}{3} \cdot (-i\sqrt{3}) \\ &= 1. \end{aligned}$$

3.7 Wang's Basis for $R(2,+)$

Wang[36] provides an integral basis for the $R(n,+)$ and $R(n,+)$ representations from the Weil representation of $T = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$. Let $v = \sum_{i=0}^{p-1} \delta_p$ and $\rho(T)$ be the Weil representation of T. Then an integral basis over $\mathbb{Z}[\zeta_p]$ is given by

$$\left\{v, \rho(T) \cdot v, (\rho(T))^2 \cdot v, \cdots, (\rho(T))^{(p-3)/2} \cdot v, (\rho(T))^{(p-1)/2} \cdot v\right\}.$$

Using SAGE, we verify that the representations are integral over $\mathbb{Z}[\zeta_3]$. Since $\zeta_3 = \frac{1}{2}(1+\sqrt{-3})$, they are also integral over $\mathbb{Z}[\frac{1}{2}(1+\sqrt{-3})]$:

$$R(2,+)(S) = \begin{bmatrix} \zeta_3 & \zeta_3+1 \\ \zeta_3+1 & -\zeta_3 \end{bmatrix} \text{ and } R(2,+)(T) = \begin{bmatrix} 0 & \zeta_3+1 \\ 1 & -\zeta_3 \end{bmatrix},$$

and

$$R(1,-)(S) = 1 \text{ and } R(1,-)(T) = \zeta_3.$$

3.8 Candelori's Bases for $R(2,+)$ and $R(2,-)$

Let $\rho(S)$ and $\rho(T)$ be the Weil representations for $S = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$ and $T = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$ respectively. Define $v_+ := \delta_1 + \delta_{(p-1)}$ and $v_- := \delta_1 - \delta(p-1)$. Define $\rho(U) := \rho(S) \cdot \rho(T) \cdot \rho(S)$. Then a basis for $R(n,+)$ is conjectured to be

$$\left\{v_+,\ \rho(U) \cdot v_+,\ (\rho(U))^2 \cdot v_+, \cdots, (\rho(U))^{(p-3)/2} \cdot v_+,\ (\rho(T))^{(p-1)/2} \cdot v_+\right\}$$

and a basis for $R(n,-)$ is conjectured to be

$$\left\{v_-,\ \rho(U) \cdot v_-,\ (\rho(U))^2 \cdot v_-, \cdots, (\rho(U))^{(p-3)/2} \cdot v_-,\ (\rho(U))^{(p-1)/2} \cdot v_-\right\}.$$

Using SAGE, we verify that the representations are integral over $\mathbb{Z}[\zeta_3] = \mathbb{Z}[\frac{1}{2}(1+\sqrt{-3})]$:

$$R(2,+)(S) = \begin{bmatrix} -\zeta_3 - 1 & 1 \\ -\zeta_3 - 1 & \zeta_3 + 1 \end{bmatrix} \text{ and } R(1,+)(T) = \begin{bmatrix} -\zeta_3 - 1 & 1 \\ 0 & 1 \end{bmatrix},$$

and

$$R(2,-)(S) = 1 \text{ and } R(1,-)(T) = -\zeta_3 - 1 = \zeta_3^2.$$

3.9 $N_1(\chi)$

Our table in Section I, (from Nobs and Wolfart[24]) tells us that

$N_1(\chi) \cong R_1(1,-) \oplus R_1(n,-)$ for $\chi \not\equiv 1$ and $\chi^2 \equiv 1$.

N_λ is also computed using $M = \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$, $Q(x) = p^{-\lambda} x_1 x_2$ with $\lambda \geq 1$.

N_λ is called the unbranched Weil representation[24].

For our case, $\lambda = 1$, $A_\lambda = \mathbb{Z}/3\mathbb{Z}$, $M = \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$, $Q(x) = 3^{-1} x_1 x_2$.

$$\begin{aligned} B(x,y) &= Q(x+y) - Q(x) - Q(y) \\ &= 3^{-1}((x_1+y_1)(x_2+y_2) - (x_1x_2) - (y_1y_2)) \\ &= 3^{-1}(x_1x_2 + y_1x_2 + x_1y_2 + y_1y_2 - (x_1x_2) - (y_1y_2)) \end{aligned}$$

$$= 3^{-1}(y_1x_2 + x_1y_2)$$

Aut(M, Q) is the group of automorphisms of M invariant under Q, i.e., for every $\varphi \in$ Aut(M, Q), $Q(\varphi(x)) = Q(x)$ for all $x \in M$. Aut(M, Q) will consist of the identity, the inverse maps, and $\kappa : (x, y) \mapsto (y, x)$. The effect of $a \in A_\lambda^\times$ on M will be defined by $a : (x, y) \mapsto (a^{-1}x, ay)$. For $a = 1$, this is the identity map. For $a = 2$, $a^{-1} = 2$ since a is a unit of $\mathbb{Z}/3\mathbb{Z}$, the map is multiplication by 2. Multiplication by 2 is the same as the inverse map:

$2 \cdot (0, 0) = (0, 0) + (0, 0) = (0, 0) = -(0, 0)$, $2 \cdot (0, 1) = (0, 1) + (0, 1) = (0, 2) = -(0, 1)$
$2 \cdot (0, 2) = (0, 2) + (0, 2) = (0, 1) = -(0, 2)$, $2 \cdot (1, 0) = (1, 0) + (1, 0) = (2, 0) = -(1, 0)$
$2 \cdot (1, 1) = (1, 1) + (1, 1) = (2, 2) = -(1, 1)$, $2 \cdot (1, 2) = (1, 2) + (1, 2) = (2, 1) = -(1, 2)$
$2 \cdot (2, 0) = (2, 0) + (2, 0) = (1, 0) = -(2, 0)$, $2 \cdot (2, 1) = (2, 1) + (2, 1) = (1, 2) = -(2, 1)$
$2 \cdot (2, 2) = (2, 2) + (2, 2) = (1, 1) = -(2, 2)$.

We summarize the calculations in the following tables.

$a \in M$	$Q(a)$	$\kappa(a)$	$(\kappa(a))^{-1}$	$Q\left((\kappa(a))^{-1}\right)$
$(0, 0)$	0	$(0, 0)$	$(0, 0)$	0
$(0, 1)$	0	$(1, 0)$	$(2, 0)$	0
$(0, 2)$	0	$(2, 0)$	$(1, 0)$	0
$(1, 0)$	0	$(0, 1)$	$(0, 2)$	0
$(1, 1)$	$\frac{1}{3}$	$(1, 1)$	$(2, 2)$	$\frac{1}{3}$
$(1, 2)$	$\frac{2}{3}$	$(2, 1)$	$(1, 2)$	$\frac{1}{3}$
$(2, 0)$	0	$(0, 2)$	$(0, 1)$	0
$(2, 1)$	$\frac{2}{3}$	$(1, 2)$	$(2, 1)$	$\frac{2}{3}$
$(2, 2)$	$\frac{1}{3}$	$(2, 2)$	$(1, 1)$	$\frac{1}{3}$

Table 3.9.1: Part I of calculations for Aut(M, Q)

$a \in M$	$Q(a)$	a^{-1}	$(\kappa(a^{-1})$	$Q\left((\kappa(a^{-1})\right)$
$(0,0)$	0	$(0,0)$	$(0,0)$	0
$(0,1)$	0	$(0,2)$	$(2,0)$	0
$(0,2)$	0	$(0,1)$	$(1,0)$	0
$(1,0)$	0	$(2,0)$	$(0,2)$	0
$(1,1)$	$\frac{1}{3}$	$(2,2)$	$(2,2)$	$\frac{1}{3}$
$(1,2)$	$\frac{2}{3}$	$(2,1)$	$(1,2)$	$\frac{1}{3}$
$(2,0)$	0	$(1,0)$	$(2,0)$	0
$(2,1)$	$\frac{2}{3}$	$(1,2)$	$(2,1)$	$\frac{2}{3}$
$(2,2)$	$\frac{1}{3}$	$(1,1)$	$(1,1)$	$\frac{1}{3}$

Table 3.9.2: Part II of calculations for $\text{Aut}(M,Q)$

We see that the κ automorphism commutes with the inverse. Their composition is of order two as well. With the exception of the identity, every automorphism is of order two. So, $\text{Aut}(M,Q) \cong V_4$ the Klein 4-group. So $\mathfrak{U}$ can be the trivial subgroup, a subgroup of order two, or the entire group. So let $\mathfrak{U} = \{1,-1\}$ and let χ be a character of $\mathfrak{U}$, then

$$V(\chi) := \{f \in \mathbb{C}^M \mid f(\varepsilon x) = \chi(\varepsilon) f(x) \ \forall \varepsilon \in \mathfrak{U},\ \forall x \in M\}$$

is a subspace of $V = \mathbb{C}^M$ that is invariant under $\text{SL}_2(A_\lambda)$. If you write $W(M,Q,\chi)$ for the sub-representation of $W(M,Q)$ in the space $V(\chi)$, then

$$W(M,Q) = \oplus_\chi W(M,Q,\chi),$$

where χ runs through all the characters from $\mathfrak{U}$.

For our case, there are two characters, χ_1 the trivial character and χ_2 the non-trivial character of the abelian group $\mathfrak{U}$.

Let's start with χ_1.

$$\begin{aligned} V(\chi_1) &= \{f \in \mathbb{C}^M \mid f(\varepsilon x) = \chi(\varepsilon) f(x)\ \forall \varepsilon \in \mathfrak{U},\ \forall x \in M\} \\ &= \{f \in \mathbb{C}^M \mid f(\varepsilon x) = f(x)\ \forall \varepsilon \in \mathfrak{U},\ \forall x \in M\} \\ &= \{f \in \mathbb{C}^M \mid f(1 \cdot x) = f(x) \text{ and } f(-1 \cdot x) = f(x)\ \forall x \in M\} \\ &= \{f \in \mathbb{C}^M \mid f(-x) = f(x)\ \forall x \in M\}. \end{aligned}$$

Again, even functions meet the criterion.

$f(-1 \cdot (0,0)) = f((0,0)) = f((0,0))$,

$f(-1 \cdot (0,1)) = f((0,2)) = f((0,1))$, and

$f(-1 \cdot (0,2)) = f((0,1)) = f((0,2))$.

$f(-1 \cdot (1,0)) = f((2,0)) = f((1,0))$,

$f(-1 \cdot (1,1)) = f((2,2)) = f((1,1))$, and

$f(-1 \cdot (1,2)) = f((2,1)) = f((1,2))$.

$f(-1 \cdot (2,0)) = f((1,0)) = f((2,0))$,

$f(-1 \cdot (2,1)) = f((1,2)) = f((2,1))$, and

$f(-1 \cdot (2,2)) = f((1,1)) = f((2,2))$.

Using the definition of δ_ξ from Nobs and Wolfart in the proof of Theorem 2[24], let

$$\delta_\xi(\eta) = \begin{cases} 1 & \text{for } \xi = \eta \in M \\ 0 & \text{otherwise} \end{cases}$$

Then our basis functions are: $f_0 = \delta_{(0,0)}$, $f_1 = \delta_{(0,1)} + \delta_{(0,2)}$, $f_2 = \delta_{(1,0)} + \delta_{(2,0)}$, $f_3 = \delta_{(1,1)} + \delta_{(2,2)}$, and $f_4 = \delta_{(1,2)} + \delta_{(2,1)}$.

But this does not meet the dimension requirements of the irreducible representation given in the table. So we will use another method. We will use material from

Strömberg[32].

Let $M_1 = \mathbb{Z}/3\mathbb{Z}$. Then $N_1 = \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \cong M_1 \oplus \widehat{M_1} = M_1 \oplus M_2$. We have established earlier that $Q(x) = xy/3$ for N_1. For M_1, let $Q_1 = x^2/3$, M_2, let $Q_2 = 2x^2/3$. Then Lemma 2.10 and Proposition 2.11[32] tells us that

$$\begin{aligned} W(N_1, Q) &\cong W(M_1, Q_1) \otimes W(M_2, Q_2) \\ &\cong (R_1(1,+) \oplus R_1(1,-)) \otimes (R_1(2,+) \oplus R_1(2,-)) \\ &\cong (R_1(1,+) \otimes R_1(2,+)) \oplus (R_1(1,+) \otimes R_1(2,-)) \\ &\quad \oplus (R_1(1,-) \otimes R_1(2,+)) \oplus (R_1(1,-) \otimes R_1(2,-)) \end{aligned}$$

Summarizing what we have so far:

Representation	Basis 1 Generator S	Basis 1 Generator T	Basis 2 Generator S	Basis 2 Generator T
$R_1(1,+)$	$\begin{bmatrix} \frac{-2i\sqrt{3}}{3} & \frac{-i\sqrt{3}}{3} \\ \frac{i\sqrt{3}}{3} & \frac{2i\sqrt{3}}{3} \end{bmatrix}$	$\begin{bmatrix} 1 & \frac{3}{2} - i\frac{\sqrt{3}}{2} \\ 0 & \frac{-1}{2} + i\frac{\sqrt{3}}{2} \end{bmatrix}$	$\begin{bmatrix} \frac{-i\sqrt{3}}{3} & \frac{-2i\sqrt{3}}{3} \\ \frac{-i\sqrt{3}}{3} & \frac{i\sqrt{3}}{3} \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & \frac{-1}{2} + i\frac{\sqrt{3}}{2} \end{bmatrix}$
$R_1(1,-)$	1	$\frac{-1}{2} + i\frac{\sqrt{3}}{2}$	1	$\frac{-1}{2} + i\frac{\sqrt{3}}{2}$
$R_1(2,+)$	$\begin{bmatrix} \frac{2i\sqrt{3}}{3} & \frac{i\sqrt{3}}{3} \\ \frac{-i\sqrt{3}}{3} & \frac{-2i\sqrt{3}}{3} \end{bmatrix}$	$\begin{bmatrix} 1 & \frac{3}{2} + i\frac{\sqrt{3}}{2} \\ 0 & \frac{-1}{2} - i\frac{\sqrt{3}}{2} \end{bmatrix}$	$\begin{bmatrix} \frac{i\sqrt{3}}{3} & \frac{2i\sqrt{3}}{3} \\ \frac{i\sqrt{3}}{3} & \frac{-i\sqrt{3}}{3} \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & \frac{-1}{2} - i\frac{\sqrt{3}}{2} \end{bmatrix}$
$R_1(2,-)$	1	$\frac{-1}{2} - i\frac{\sqrt{3}}{2}$	1	$\frac{-1}{2} - i\frac{\sqrt{3}}{2}$

Table 3.9.3: $R_1(1,\pm)$ and $R_1(2,\pm)$ for $\mathrm{SL}_2(\mathbb{F}_3)$

The conjugacy class representatives of $\mathrm{SL}_2(\mathbb{F}_3)$ and their sizes are given in Table 3.9.4.

Conjugacy Class Representative	Generated By	Conjugacy Class Size
$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	S^4	1
$-I = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$	S^2	1
$u = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$	T	4
$u' = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$	T^2	4
$-u' = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$	T^2S^2	4
$-u = \begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix}$	TS^2	4
$s = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$	S	6

Table 3.9.4: Conjugacy class representatives of $\mathrm{SL}_2(\mathbb{F}_3)$ and their sizes

3.9.1 Basis Choice 1

For Basis 1, the four dimensional representation of the generator $S = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$ is

$$
\begin{aligned}
R_1(1,+) \otimes R_1(2,+))(S) &= \begin{bmatrix} \frac{-2i\sqrt{3}}{3} & \frac{-i\sqrt{3}}{3} \\ \frac{i\sqrt{3}}{3} & \frac{2i\sqrt{3}}{3} \end{bmatrix} \otimes \begin{bmatrix} \frac{2i\sqrt{3}}{3} & \frac{i\sqrt{3}}{3} \\ \frac{-i\sqrt{3}}{3} & \frac{-2i\sqrt{3}}{3} \end{bmatrix} \\
&= \begin{bmatrix} \frac{-2i\sqrt{3}}{3} \begin{bmatrix} \frac{2i\sqrt{3}}{3} & \frac{i\sqrt{3}}{3} \\ \frac{-i\sqrt{3}}{3} & \frac{-2i\sqrt{3}}{3} \end{bmatrix} & \frac{-i\sqrt{3}}{3} \begin{bmatrix} \frac{2i\sqrt{3}}{3} & \frac{i\sqrt{3}}{3} \\ \frac{-i\sqrt{3}}{3} & \frac{-2i\sqrt{3}}{3} \end{bmatrix} \\ \frac{i\sqrt{3}}{3} \begin{bmatrix} \frac{2i\sqrt{3}}{3} & \frac{i\sqrt{3}}{3} \\ \frac{-i\sqrt{3}}{3} & \frac{-2i\sqrt{3}}{3} \end{bmatrix} & \frac{2i\sqrt{3}}{3} \begin{bmatrix} \frac{2i\sqrt{3}}{3} & \frac{i\sqrt{3}}{3} \\ \frac{-i\sqrt{3}}{3} & \frac{-2i\sqrt{3}}{3} \end{bmatrix} \end{bmatrix}
\end{aligned}
$$

$$= \begin{bmatrix} \frac{4}{3} & \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & -\frac{4}{3} & -\frac{1}{3} & -\frac{2}{3} \\ -\frac{2}{3} & -\frac{1}{3} & -\frac{4}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} & \frac{4}{3} \end{bmatrix}.$$

The trace of this matrix raised to the fourth power is 4. The trace of the matrix squared is also 4, corresponding to the trace of $S^4 = \mathbb{1}$ and $S^2 = -\mathbb{1}$. The trace of this matrix is zero. This agrees with the trace of representation $1 + St$ (S corresponds to s in Reeder's character table).

Now for $T = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$ (which corresponds to u in Reeder's character table).

$$R_1(1,+) \otimes R_1(2,+))(T) = \begin{bmatrix} 1 & \frac{3}{2} - i\frac{\sqrt{3}}{2} \\ 0 & \frac{-1}{2} + i\frac{\sqrt{3}}{2} \end{bmatrix} \otimes \begin{bmatrix} 1 & \frac{3}{2} + i\frac{\sqrt{3}}{2} \\ 0 & \frac{-1}{2} - i\frac{\sqrt{3}}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 \begin{bmatrix} 1 & \frac{3}{2} + i\frac{\sqrt{3}}{2} \\ 0 & \frac{-1}{2} - i\frac{\sqrt{3}}{2} \end{bmatrix} & (\frac{3}{2} - i\frac{\sqrt{3}}{2}) \begin{bmatrix} 1 & \frac{3}{2} + i\frac{\sqrt{3}}{2} \\ 0 & \frac{-1}{2} - i\frac{\sqrt{3}}{2} \end{bmatrix} \\ 0 \begin{bmatrix} 1 & \frac{3}{2} + i\frac{\sqrt{3}}{2} \\ 0 & \frac{-1}{2} - i\frac{\sqrt{3}}{2} \end{bmatrix} & (\frac{-1}{2} + i\frac{\sqrt{3}}{2}) \begin{bmatrix} 1 & \frac{3}{2} + i\frac{\sqrt{3}}{2} \\ 0 & \frac{-1}{2} - i\frac{\sqrt{3}}{2} \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \frac{3}{2} + i\frac{\sqrt{3}}{2} & \frac{3}{2} - i\frac{\sqrt{3}}{2} & 3 \\ 0 & -\frac{1}{2} - i\frac{\sqrt{3}}{2} & 0 & -\frac{3}{4} + i\frac{\sqrt{3}}{4} - 3i\frac{\sqrt{3}}{4} - \frac{3}{4} \\ 0 & 0 & -\frac{1}{2} + i\frac{\sqrt{3}}{2} & -\frac{3}{4} - i\frac{\sqrt{3}}{4} + 3i\frac{\sqrt{3}}{4} - \frac{3}{4} \\ 0 & 0 & 0 & \frac{1}{4} + \frac{3}{4} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \frac{3}{2} + i\frac{\sqrt{3}}{2} & \frac{3}{2} - i\frac{\sqrt{3}}{2} & 3 \\ 0 & -\frac{1}{2} - i\frac{\sqrt{3}}{2} & 0 & -\frac{3}{2} + i\frac{\sqrt{3}}{2} \\ 0 & 0 & -\frac{1}{2} + i\frac{\sqrt{3}}{2} & \frac{3}{2} - i\frac{\sqrt{3}}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Neither are in block diagonal form so we will need to perform another basis change. But we can verify that this indeed $\mathbb{1} \oplus St$ since it agrees with the characters on the conjugacy classes and since it decomposes into two irreducible representations. It is readily verified using a computer. Let's try the second choice for the basis.

3.9.2 Basis Choice 2

For Basis 2, the four dimensional representation of the generator $S = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$ is

$$(R_1(1,+) \otimes R_1(2,+))(S) = \begin{bmatrix} \frac{-i\sqrt{3}}{3} & \frac{-2i\sqrt{3}}{3} \\ \frac{-i\sqrt{3}}{3} & \frac{i\sqrt{3}}{3} \end{bmatrix} \otimes \begin{bmatrix} \frac{i\sqrt{3}}{3} & \frac{2i\sqrt{3}}{3} \\ \frac{i\sqrt{3}}{3} & \frac{-i\sqrt{3}}{3} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-i\sqrt{3}}{3} \begin{bmatrix} \frac{i\sqrt{3}}{3} & \frac{2i\sqrt{3}}{3} \\ \frac{i\sqrt{3}}{3} & \frac{-i\sqrt{3}}{3} \end{bmatrix} & \frac{-2i\sqrt{3}}{3} \begin{bmatrix} \frac{i\sqrt{3}}{3} & \frac{2i\sqrt{3}}{3} \\ \frac{i\sqrt{3}}{3} & \frac{-i\sqrt{3}}{3} \end{bmatrix} \\ \frac{-i\sqrt{3}}{3} \begin{bmatrix} \frac{i\sqrt{3}}{3} & \frac{2i\sqrt{3}}{3} \\ \frac{i\sqrt{3}}{3} & \frac{-i\sqrt{3}}{3} \end{bmatrix} & \frac{i\sqrt{3}}{3} \begin{bmatrix} \frac{i\sqrt{3}}{3} & \frac{2i\sqrt{3}}{3} \\ \frac{i\sqrt{3}}{3} & \frac{-i\sqrt{3}}{3} \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} & \frac{4}{3} \\ \frac{1}{3} & \frac{-1}{3} & \frac{2}{3} & \frac{-2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{-1}{3} & \frac{-2}{3} \\ \frac{1}{3} & \frac{-1}{3} & \frac{-1}{3} & \frac{1}{3} \end{bmatrix}.$$

Now for $T = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$ (which corresponds to u in Reeder's character table).

$$(R_1(1,+) \otimes R_1(2,+))(T) = \begin{bmatrix} 1 & 0 \\ 0 & \frac{-1}{2} + i\frac{\sqrt{3}}{2} \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & \frac{-1}{2} - i\frac{\sqrt{3}}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 \cdot \begin{bmatrix} 1 & 0 \\ 0 & \frac{-1}{2} - i\frac{\sqrt{3}}{2} \end{bmatrix} & 0 \cdot \begin{bmatrix} 1 & 0 \\ 0 & \frac{-1}{2} - i\frac{\sqrt{3}}{2} \end{bmatrix} \\ 0 \cdot \begin{bmatrix} 1 & 0 \\ 0 & \frac{-1}{2} - i\frac{\sqrt{3}}{2} \end{bmatrix} & \frac{-1}{2} + i\frac{\sqrt{3}}{2} \cdot \begin{bmatrix} 1 & 0 \\ 0 & \frac{-1}{2} - i\frac{\sqrt{3}}{2} \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{-1-i\sqrt{3}}{2} & 0 & 0 \\ 0 & 0 & \frac{-1+\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{4}+\frac{3}{4} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{-1-i\sqrt{3}}{2} & 0 & 0 \\ 0 & 0 & \frac{-1+\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

It's not in block diagonal form so we need to perform a change of basis. For convenience, let $\rho_1 = R_1(1,+)$, $\rho_2 = R_1(2,+)$ and $\zeta = \exp(2\pi i/3)$. Recall that $V(\chi_1) = \{f_0, f_1\}$ and $f_0 = \delta_0$ and $f_1 = \delta_1 + \delta_2$. Then we have

$$\rho_1(T) = \begin{bmatrix} 1 & 0 \\ 0 & \zeta \end{bmatrix}$$

$$\rho_2(T) = \begin{bmatrix} 1 & 0 \\ 0 & \zeta^{-1} \end{bmatrix}$$

$$\rho_1 \otimes \rho_2(T) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \zeta^{-1} & 0 & 0 \\ 0 & 0 & \zeta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$\rho_1(S) = \frac{-i}{\sqrt{3}} \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$$

$$\rho_2(S) = \frac{i}{\sqrt{3}} \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$$

$$\rho_1 \otimes \rho_2(S) = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 & 4 \\ 1 & -1 & 2 & -2 \\ 1 & 2 & -1 & -2 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

$\rho_1 \otimes \rho_2(T)$ has an eigenvalue of 1 with multiplicity two. To determine the invariant subspace, we consider the eigenspace generated by eigenvalue of 1.

$$\begin{aligned} V_T &= \text{span}(\delta_0 \otimes \delta_0, (\delta_1 + \delta_2) \otimes (\delta_1 + \delta_2)) \\ &= \text{span}(\delta_0 \otimes \delta_0, \delta_1 \otimes \delta_1 + \delta_2 \otimes \delta_1 + \delta_1 \otimes \delta_2 + \delta_2 \otimes \delta_2) \\ &= \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} \\ &= \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}. \end{aligned}$$

Next recall that (the Fourier "expansion")

$$\begin{aligned} \rho_1(S)(\delta_0) &= \delta_1 + \delta_2 + \delta_3 \\ \rho_1(S)(\delta_1 + \delta_2) &= \delta_0 + \zeta_3\delta_1 + \zeta^2\delta_2 + \delta_0 + \zeta_3^2\delta_2 + \zeta_3\delta_3 \\ &= 2\delta_0 + (\zeta_3 + \zeta_3^3)\delta_1 + (\zeta_3^2 + \zeta_3)\delta_2 \\ &= 2\delta_0 - \delta_1 - \delta_2. \end{aligned}$$

Computing the Jordan form of $\rho_1 \otimes \rho_2(S)$ we obtain the Jordan matrix, J, and the D the matrix of the associated generalized eigenvectors:

$$J = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad D = \begin{bmatrix} 2 & 2 & 0 & -2 \\ 1 & 0 & -1 & 2 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

So the invariant subspace V_S is given by

$$V_S = \text{span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

So the invariant subspace spanned by the intersection is one-dimensional:

$$\begin{aligned} V_T \cap V_S &= \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} \cap \text{span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} \\ &= \text{span}\left\{ \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} \\ &= \text{span}(2\delta_0 \otimes \delta_0 + (\delta_1 + \delta_2) \otimes (\delta_1 + \delta_2)). \end{aligned}$$

Since this is one-dimensional, it corresponds to the trivial representation in the decomposition of $R_1(1,+) \otimes R_1(2,+)$. To find the Steinberg representation, we need to compute the basis of orthogonal complement of $V_T \cap V_S$. So,

$$\begin{aligned}
(V_T \cap V_S)^{\perp} &= \text{span } \{v \in V \mid v \cdot w = 0 \text{ for all } w \in (V_T \cap V_S)\} \\
&= \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\} \\
&= \text{span } \{\delta_0 \otimes \delta_0 - 2((\delta_1 + \delta_2) \otimes (\delta_1 + \delta_2)), \\
&\quad \delta_0 \otimes (\delta_1 + \delta_2), \\
&\quad (\delta_1 + \delta_2) \otimes \delta_0\}.
\end{aligned}$$

Lets try to put this in block diagonal form with respect to the new basis (the above set of linearly independent spanning vectors). Letting

$$M = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & -2 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned}
T_0 &= M^{-1}(\rho_1 \otimes \rho_2(T))M \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \zeta^2 & 0 \\ 0 & 0 & 0 & \zeta \end{bmatrix}
\end{aligned}$$

It's still block diagonal. So far so good.

$$S_0 = M^{-1}(\rho_1 \otimes \rho_2(S))M$$

$$= \begin{bmatrix} 1 & -1 & \frac{1}{5} & \frac{1}{5} \\ 0 & -\frac{1}{3} & \frac{4}{15} & \frac{4}{15} \\ 0 & \frac{5}{3} & -\frac{1}{3} & \frac{2}{3} \\ 0 & \frac{5}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix}$$

It's not block diagonal. Lets compute the others.

$$\begin{aligned} R_1(1,+) \otimes R_1(2,-)(S) &= \begin{bmatrix} \frac{-i\sqrt{3}}{3} & \frac{-2i\sqrt{3}}{3} \\ \frac{-i\sqrt{3}}{3} & \frac{i\sqrt{3}}{3} \end{bmatrix} \otimes (-1) \\ &= -R_1(1,+)(S). \end{aligned}$$

$$\begin{aligned} R_1(1,-) \otimes R_1(2,+)(S) &= 1 \otimes \begin{bmatrix} \frac{i\sqrt{3}}{3} & \frac{2i\sqrt{3}}{3} \\ \frac{i\sqrt{3}}{3} & \frac{-i\sqrt{3}}{3} \end{bmatrix} \\ &= R_1(2,+)(S). \end{aligned}$$

$$R_1(1,-) \otimes R_1(2,-)(S) = 1 \otimes (1) = 1.$$

$$\begin{aligned} R_1(1,+) \otimes R_1(2,-)(T) &= \begin{bmatrix} 1 & 0 \\ 0 & \frac{-1+i\sqrt{3}}{2} \end{bmatrix} \otimes \left(\frac{-1}{2} - i\frac{\sqrt{3}}{2} \right) \\ &= \left(\frac{-1}{2} - i\frac{\sqrt{3}}{2} \right) R_1(1,+)(T) \end{aligned}$$

$$\begin{aligned} R_1(1,-) \otimes R_1(2,+)(T) &= \left(\frac{-1}{2} + i\frac{\sqrt{3}}{2} \right) \otimes \begin{bmatrix} 1 & 0 \\ 0 & \frac{-1-i\sqrt{3}}{2} \end{bmatrix} \\ &= \left(\frac{-1}{2} + i\frac{\sqrt{3}}{2} \right) R_1(2,+)(T) \end{aligned}$$

$$R_1(1,-) \otimes R_1(2,-)(T) = \left(\frac{-1}{2} + i\frac{\sqrt{3}}{2}\right) \otimes \left(\frac{-1}{2} - i\frac{\sqrt{3}}{2}\right)$$
$$= 1.$$

Let's summarize what we calculated so far:

Representation	I	$-I$	u	u'	$-u$	$-u'$	s
$R_1(1,-) \otimes R_1(2,-)$ $= \text{Trivial}$	1	1	1	1	1	1	1
$R_1(1,+) \otimes R_1(2,+)$ $\cong 1 \oplus St$	4	4	1	1	1	1	0
$R_1(1,+) \cong \rho_0'$ $\cong R_1(1,-) \otimes R_1(2,+)$	2	-2	$\frac{1}{2} + i\frac{\sqrt{3}}{2}$	$\frac{1}{2} - i\frac{\sqrt{3}}{2}$	$-\frac{1}{2} - i\frac{\sqrt{3}}{2}$	$-\frac{1}{2} + i\frac{\sqrt{3}}{2}$	0
$R_1(2,+) \cong \rho_0''$ $\cong R_1(1,+) \otimes R_1(2,-)$	2	-2	$\frac{1}{2} - i\frac{\sqrt{3}}{2}$	$\frac{1}{2} + i\frac{\sqrt{3}}{2}$	$-\frac{1}{2} + i\frac{\sqrt{3}}{2}$	$-\frac{1}{2} - i\frac{\sqrt{3}}{2}$	0
$R_1(1,-) \cong \pi_0'$	1	1	$-\frac{1}{2} + i\frac{\sqrt{3}}{2}$	$-\frac{1}{2} - i\frac{\sqrt{3}}{2}$	$-\frac{1}{2} + i\frac{\sqrt{3}}{2}$	$-\frac{1}{2} - i\frac{\sqrt{3}}{2}$	1
$R_1(2,-) \cong \pi_0''$	1	1	$-\frac{1}{2} - i\frac{\sqrt{3}}{2}$	$-\frac{1}{2} + i\frac{\sqrt{3}}{2}$	$-\frac{1}{2} - i\frac{\sqrt{3}}{2}$	$-\frac{1}{2} + i\frac{\sqrt{3}}{2}$	1

Table 3.9.5: $R_1(1,\pm)$ and $R_1(2,\pm)$ for $\mathrm{SL}_2(\mathbb{F}_3)$

Nobs and Wolfart[24] tell us how to compute the Steinberg and the other two dimensional Weil representations.

3.10 $N_1(\chi)$ redux

In Nobs Part I[23] Theorem 3, we have another binary quadratic form for $M = \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$: $Q(x) = p^{-\lambda}(x_1^2 - ux_2^2)$. Its bilinear form is given by

$$\begin{aligned} B(x,y) &= Q(x+y) - Q(x) - Q(y) \\ &= \frac{(x_1+y_1)^2 - u(x_2+y_2)^2}{p^\lambda} - \frac{(x_1)^2 - u(x_2)^2}{p^\lambda} - \frac{(y_1)^2 - u(y_2)^2}{p^\lambda} \\ &= \frac{((x_1^2 + 2x_1y_1 + y_1^2) - u(x_2^2 + 2x_2y_2 + y_2^2)) - x_1^2 + ux_2^2 - y_1^2 + uy_2^2}{p^\lambda} \\ &= \frac{x_1^2 + 2x_1y_1 + y_1^2 - ux_2^2 - u2x_2y_2 - uy_2^2 - x_1^2 + ux_2^2 - y_1^2 + uy_2^2}{p^\lambda} \\ &= \frac{2x_1y_1 - 2ux_2y_2}{p^\lambda}. \end{aligned}$$

For our case, $p = 3$, $\lambda = 1$, and $u = 2$. Since we know the automorphism group of M will be $\mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})$, we will start there to compute $\mathrm{Aut}(M, Q)$.

The order of $\mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})$ is $(p^2-1)(p^2-p)$. So $|\mathrm{GL}_2(\mathbb{Z}/3\mathbb{Z})| = (9-1)(9-3) = 8 \cdot 6 = 48$. Lets list them:

$$\begin{bmatrix}1&0\\0&1\end{bmatrix}, \begin{bmatrix}2&0\\0&2\end{bmatrix}, \begin{bmatrix}0&1\\1&0\end{bmatrix}, \begin{bmatrix}0&2\\2&0\end{bmatrix}, \begin{bmatrix}0&2\\1&2\end{bmatrix}, \begin{bmatrix}0&1\\2&1\end{bmatrix}, \begin{bmatrix}0&1\\2&2\end{bmatrix}, \begin{bmatrix}0&2\\1&1\end{bmatrix},$$

$$\begin{bmatrix}1&0\\0&2\end{bmatrix}, \begin{bmatrix}2&0\\0&1\end{bmatrix}, \begin{bmatrix}0&2\\1&0\end{bmatrix}, \begin{bmatrix}0&1\\2&0\end{bmatrix}, \begin{bmatrix}2&2\\1&0\end{bmatrix}, \begin{bmatrix}1&1\\2&0\end{bmatrix}, \begin{bmatrix}2&1\\2&0\end{bmatrix}, \begin{bmatrix}1&2\\1&0\end{bmatrix},$$

$$\begin{bmatrix}2&2\\0&2\end{bmatrix}, \begin{bmatrix}1&1\\0&1\end{bmatrix}, \begin{bmatrix}2&0\\1&2\end{bmatrix}, \begin{bmatrix}1&0\\2&1\end{bmatrix}, \begin{bmatrix}2&0\\2&2\end{bmatrix}, \begin{bmatrix}1&0\\1&1\end{bmatrix}, \begin{bmatrix}2&1\\0&2\end{bmatrix}, \begin{bmatrix}1&2\\0&1\end{bmatrix},$$

$$\begin{bmatrix}1&1\\0&2\end{bmatrix}, \begin{bmatrix}2&2\\0&1\end{bmatrix}, \begin{bmatrix}2&0\\2&1\end{bmatrix}, \begin{bmatrix}1&0\\1&2\end{bmatrix}, \begin{bmatrix}2&0\\1&1\end{bmatrix}, \begin{bmatrix}1&0\\2&2\end{bmatrix}, \begin{bmatrix}1&2\\0&2\end{bmatrix}, \begin{bmatrix}2&1\\0&1\end{bmatrix},$$

$$\begin{bmatrix}1&2\\2&2\end{bmatrix},\ \begin{bmatrix}2&1\\1&1\end{bmatrix},\ \begin{bmatrix}1&1\\1&2\end{bmatrix},\ \begin{bmatrix}2&2\\2&1\end{bmatrix},\ \begin{bmatrix}2&1\\2&2\end{bmatrix},\ \begin{bmatrix}2&2\\1&2\end{bmatrix},\ \begin{bmatrix}1&2\\1&1\end{bmatrix},\ \begin{bmatrix}1&1\\2&1\end{bmatrix},$$

$$\begin{bmatrix}0&1\\1&1\end{bmatrix},\ \begin{bmatrix}1&2\\2&0\end{bmatrix},\ \begin{bmatrix}0&2\\2&1\end{bmatrix},\ \begin{bmatrix}1&1\\1&0\end{bmatrix},\ \begin{bmatrix}2&2\\2&0\end{bmatrix},\ \begin{bmatrix}0&1\\1&2\end{bmatrix},\ \begin{bmatrix}2&1\\1&0\end{bmatrix},\ \begin{bmatrix}0&2\\2&2\end{bmatrix}.$$

Lets write the domain and image of $Q(x)$.

	x_1	x_2	x_1^2	x_2^2	$x_1^2 - 2x_2^2$	$Q(x) = (x_1^2 - 2x_2^2)/3 \in \mathbb{Q}/\mathbb{Z}$
	0	0	0	0	0	0
	0	1	0	1	$0-2$	$-2/3 = 1/3$
	0	2	0	4	$0-8$	$-8/3 = 1/3$
	1	0	1	0	$1-0$	$1/3 = 1/3$
$*$	1	1	1	1	$1-2$	$-1/3 = 2/3$
$*$	1	2	1	4	$1-8$	$-7/3 = 2/3$
	2	0	4	0	$4-0$	$4/3 = 1/3$
$*$	2	1	4	1	$4-2$	$2/3 = 2/3$
$*$	2	2	4	4	$4-8$	$-4/3 = 2/3$

Now to determine the elements of $\mathrm{Aut}(M,Q)$. Since $\mathrm{Aut}(M,Q)$ is going to be a subgroup of $\mathrm{GL}_2(\mathbb{Z}/3\mathbb{Z})$, its order has to be $1, 2, 3, 4, 6, 8, 12, 16, 24$ or 48.

We have the identity element: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and its order is 1.

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$$

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is an element of order 2.

$$\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

$\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$ is an element of order 2.

$$\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

$\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$ is an element of order 4. So $|\operatorname{Aut}(M,Q)|$ is either $4, 8, 12, 16, 24$, or 48.

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix},$$

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ is an element of order 2.

$$\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

$\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$ is an element of order 4.

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix},$$

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix},$$

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ is an element of order 2.

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ is an element of order 2. The elements

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

map $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

The elements

$$\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}, \quad \begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix}, \quad \begin{bmatrix} 0 & 2 \\ 1 & 2 \end{bmatrix}, \quad \begin{bmatrix} 2 & 2 \\ 1 & 2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}, \quad \begin{bmatrix} 0 & 2 \\ 2 & 2 \end{bmatrix}$$

map $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$ to $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

The elements

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}, \quad \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$$

map $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ to $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

The elements

$$\begin{bmatrix} 2 & 0 \\ 2 & 1 \end{bmatrix}, \quad \begin{bmatrix} 2 & 1 \\ 2 & 0 \end{bmatrix}, \quad \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix}, \quad \begin{bmatrix} 2 & 2 \\ 2 & 1 \end{bmatrix}, \quad \begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix}, \quad \begin{bmatrix} 2 & 2 \\ 2 & 1 \end{bmatrix}$$

map $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ to $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

The elements

$$\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 2 & 2 \\ 1 & 2 \end{bmatrix}, \quad \begin{bmatrix} 2 & 2 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$$

map $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ to $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

The elements

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}$$

map $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ to $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$. $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ is not an element as it maps $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ to $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$.

Neither is $\begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix}$ as it maps $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ to $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$.

So since 37 elements are not in $\mathrm{Aut}(M,Q)$, the order (cardinality) of $\mathrm{Aut}(M,Q)$ is not greater than 8. So $\mathrm{Aut}(M,Q)$ will have to have order 8.

$$\mathrm{Aut}(M,Q) = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \right\}$$

Now to show that $\mathrm{Aut}(M,Q)$ is indeed a group. It inherits associativity from $\mathrm{GL}_2(\mathbb{Z}/3\mathbb{Z})$. We have the identity element. Each of the elements has an inverse. Each of the five elements of order two is an inverse to itself and the two elements of order four are inverses to each other:

$$\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Now to verify closure of the binary operation. First the order 2 elements.

$$\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}.$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

$$\begin{bmatrix}0&1\\2&0\end{bmatrix}\begin{bmatrix}0&2\\2&0\end{bmatrix}=\begin{bmatrix}2&0\\0&1\end{bmatrix},\begin{bmatrix}2&0\\0&1\end{bmatrix}\begin{bmatrix}0&2\\2&0\end{bmatrix}=\begin{bmatrix}0&1\\2&0\end{bmatrix},\begin{bmatrix}1&0\\0&2\end{bmatrix}\begin{bmatrix}0&2\\2&0\end{bmatrix}=\begin{bmatrix}0&2\\1&0\end{bmatrix},$$

$$\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}2&0\\0&2\end{bmatrix}=\begin{bmatrix}0&2\\1&0\end{bmatrix},\begin{bmatrix}0&2\\2&0\end{bmatrix}\begin{bmatrix}2&0\\0&2\end{bmatrix}=\begin{bmatrix}0&1\\1&0\end{bmatrix},\begin{bmatrix}0&2\\1&0\end{bmatrix}\begin{bmatrix}2&0\\0&2\end{bmatrix}=\begin{bmatrix}2&0\\0&1\end{bmatrix},$$

$$\begin{bmatrix}0&1\\2&0\end{bmatrix}\begin{bmatrix}2&0\\0&2\end{bmatrix}=\begin{bmatrix}0&2\\1&0\end{bmatrix},\begin{bmatrix}2&0\\0&1\end{bmatrix}\begin{bmatrix}2&0\\0&2\end{bmatrix}=\begin{bmatrix}1&0\\0&2\end{bmatrix},\begin{bmatrix}1&0\\0&2\end{bmatrix}\begin{bmatrix}2&0\\0&2\end{bmatrix}=\begin{bmatrix}2&0\\0&1\end{bmatrix},$$

$$\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}2&0\\0&1\end{bmatrix}=\begin{bmatrix}0&1\\2&0\end{bmatrix},\begin{bmatrix}0&2\\2&0\end{bmatrix}\begin{bmatrix}2&0\\0&1\end{bmatrix}=\begin{bmatrix}0&2\\1&0\end{bmatrix},\begin{bmatrix}0&2\\1&0\end{bmatrix}\begin{bmatrix}2&0\\0&1\end{bmatrix}=\begin{bmatrix}0&2\\2&0\end{bmatrix},$$

$$\begin{bmatrix}2&0\\0&2\end{bmatrix}\begin{bmatrix}2&0\\0&1\end{bmatrix}=\begin{bmatrix}1&0\\0&2\end{bmatrix},\begin{bmatrix}0&1\\2&0\end{bmatrix}\begin{bmatrix}2&0\\0&1\end{bmatrix}=\begin{bmatrix}0&1\\1&0\end{bmatrix},\begin{bmatrix}1&0\\0&2\end{bmatrix}\begin{bmatrix}2&0\\0&1\end{bmatrix}=\begin{bmatrix}2&0\\0&2\end{bmatrix},$$

$$\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}1&0\\0&2\end{bmatrix}=\begin{bmatrix}0&2\\1&0\end{bmatrix},\begin{bmatrix}0&2\\2&0\end{bmatrix}\begin{bmatrix}1&0\\0&2\end{bmatrix}=\begin{bmatrix}0&1\\2&0\end{bmatrix},\begin{bmatrix}0&2\\1&0\end{bmatrix}\begin{bmatrix}1&0\\0&2\end{bmatrix}=\begin{bmatrix}0&1\\1&0\end{bmatrix},$$

$$\begin{bmatrix}2&0\\0&2\end{bmatrix}\begin{bmatrix}1&0\\0&2\end{bmatrix}=\begin{bmatrix}2&0\\0&1\end{bmatrix},\begin{bmatrix}0&1\\2&0\end{bmatrix}\begin{bmatrix}1&0\\0&2\end{bmatrix}=\begin{bmatrix}0&2\\2&0\end{bmatrix},\begin{bmatrix}2&0\\0&1\end{bmatrix}\begin{bmatrix}1&0\\0&2\end{bmatrix}=\begin{bmatrix}2&0\\0&2\end{bmatrix}.$$

Now for the order 4 elements:

$$\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}0&1\\2&0\end{bmatrix}=\begin{bmatrix}2&0\\0&1\end{bmatrix},\begin{bmatrix}0&2\\2&0\end{bmatrix}\begin{bmatrix}0&1\\2&0\end{bmatrix}=\begin{bmatrix}2&0\\0&2\end{bmatrix},\begin{bmatrix}0&2\\1&0\end{bmatrix}\begin{bmatrix}0&1\\2&0\end{bmatrix}=\begin{bmatrix}1&0\\0&1\end{bmatrix},$$

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

We have closure under the binary operation (matrix multiplication). So $\mathrm{Aut}(M, Q)$ is a group. It is nonabelian. Letting $r = \left[\begin{smallmatrix} 0 & 1 \\ 2 & 0 \end{smallmatrix}\right]$ and $s = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$, we can generate the eight elements as follows:

$$s = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, r = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}, r^2 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, r^3 = srs = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}, r^4 = s^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$rs = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, r^2 s = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}, r^3 s = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}.$$

Since it has two elements of order 4, five elements of order 2, and the above relations with the two generators r and s, it is isomorphic to the dihedral group of order 8:

$$\mathrm{Aut}(M, Q) = \langle r, s \mid r^4 = s^2 = \mathbb{1}, srs = r^{-1} = r^3 \rangle.$$

It has an abelian subgroup of order 4 which is generated by either r or r^{-1}. That is, $\mathfrak{U} = \langle r \mid r^4 = \mathbb{1} \rangle$. So we will have characters of $\mathfrak{U}$ that are not 1 or -1.

Remark 3.10.1. *The orthogonal group $O^+(2,p)$ has a dihedral group of order $2(p+1)$ as its automorphism group. One can refer to Taylor[34] Theorem 11.4 for the details.*

Theorem 2 in Nobs[23],

Theorem 3.10.2 (Nobs, 2)**.** *Let χ, χ_1, χ_2 be primitive characters of $\mathfrak{U}$ with $\chi^2, \chi_1^2, \chi_2^2 \not\equiv 1$.*

a) $N_\lambda(\chi)$ is irreducible of level λ.

b) $N_\lambda(\chi_1) \cong N_\lambda(\chi_2)$ if and only if $\chi_1 = \chi_2$ or $\overline{\chi_2}$.

This gives $((p^2-1)/2)p^{\lambda-2}$ non-isomorphic irreducible representations of $\mathrm{SL}_2(A_\lambda)$ of degree $p^{\lambda-1}(p-1)$ for $\lambda > 1$, $p \neq 2$, and for $\lambda > 3$, $p = 2$. The corresponding numbers are $(p-1)/2$ for $p \neq 2$ and $\lambda = 1$, $\left\{ \begin{array}{l} 1 \text{ for } \lambda = 1,2 \\ 2 \text{ for } \lambda = 3 \end{array} \right\}$ and $p = 2$.

tells us that for $N_\lambda(\chi)$, there are $(p-1)/2$ non-isomorphic irreducible representations of $\mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})$ for $\lambda = 1$. So for $p = 3$, there is $(3-1)/2 = 1$ irreducible representation. Since $\mathfrak{U}$ is a cyclic group of order 4:

$$\mathfrak{U} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \right\},$$

so,

$$\chi\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 1, \quad \chi\left(\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}\right) = i, \quad \chi\left(\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\right) = -1, \quad \chi\left(\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}\right) = -i.$$

$\delta_\xi \in \mathbb{C}^M$ defined by

$$\delta_\xi(\eta) = \left\{ \begin{array}{ll} 1 & \text{for } \xi = \eta \in M \\ 0 & \text{otherwise} \end{array} \right\}.$$

Then the $f_\xi(\chi) := \sum_{\varepsilon \in \mathfrak{U}} \chi(\varepsilon)\delta_{\varepsilon\xi}$ form a system of generators of $\mathbb{C}^M$[24]. **However, as**

we will see, we only obtain two linearly independent functions.

$$f_{\begin{bmatrix}0\\0\end{bmatrix}}(\chi) = \chi\left(\begin{bmatrix}1&0\\0&1\end{bmatrix}\right)\delta_{\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}0\\0\end{bmatrix}} + \chi\left(\begin{bmatrix}0&1\\2&0\end{bmatrix}\right)\delta_{\begin{bmatrix}0&1\\1&2\end{bmatrix}\begin{bmatrix}0\\0\end{bmatrix}}$$
$$+\chi\left(\begin{bmatrix}2&0\\0&2\end{bmatrix}\right)\delta_{\begin{bmatrix}2&0\\0&2\end{bmatrix}\begin{bmatrix}0\\0\end{bmatrix}} + \chi\left(\begin{bmatrix}0&2\\1&0\end{bmatrix}\right)\delta_{\begin{bmatrix}0&2\\1&0\end{bmatrix}\begin{bmatrix}0\\0\end{bmatrix}}$$
$$= 1\cdot\delta_{\begin{bmatrix}0\\0\end{bmatrix}} + i\cdot\delta_{\begin{bmatrix}0\\0\end{bmatrix}} - 1\cdot\delta_{\begin{bmatrix}0\\0\end{bmatrix}} - i\cdot\delta_{\begin{bmatrix}0\\0\end{bmatrix}} = 0\cdot\delta_{\begin{bmatrix}0\\0\end{bmatrix}}$$
$$= 0.$$

This agrees with Nobs[24] ($f_\xi(\chi) = 0$ if χ is primitive and $\xi \in pM$, well, $pM = 0$ and since $\xi = 0$, it agrees).

$$f_{\begin{bmatrix}0\\1\end{bmatrix}}(\chi) = \chi\left(\begin{bmatrix}1&0\\0&1\end{bmatrix}\right)\delta_{\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}} + \chi\left(\begin{bmatrix}0&1\\2&0\end{bmatrix}\right)\delta_{\begin{bmatrix}0&1\\2&0\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}}$$
$$+\chi\left(\begin{bmatrix}2&0\\0&2\end{bmatrix}\right)\delta_{\begin{bmatrix}2&0\\0&2\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}} + \chi\left(\begin{bmatrix}0&2\\1&0\end{bmatrix}\right)\delta_{\begin{bmatrix}0&2\\1&0\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}}$$
$$= 1\cdot\delta_{\begin{bmatrix}0\\1\end{bmatrix}} + i\cdot\delta_{\begin{bmatrix}1\\0\end{bmatrix}} - 1\cdot\delta_{\begin{bmatrix}0\\2\end{bmatrix}} - i\cdot\delta_{\begin{bmatrix}2\\0\end{bmatrix}}.$$

$$f_{\begin{bmatrix}0\\2\end{bmatrix}}(\chi) = \chi\left(\begin{bmatrix}1&0\\0&1\end{bmatrix}\right)\delta_{\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}0\\2\end{bmatrix}} + \chi\left(\begin{bmatrix}0&1\\2&0\end{bmatrix}\right)\delta_{\begin{bmatrix}0&1\\2&0\end{bmatrix}\begin{bmatrix}0\\2\end{bmatrix}}$$
$$+ \chi\left(\begin{bmatrix}2&0\\0&2\end{bmatrix}\right)\delta_{\begin{bmatrix}2&0\\0&2\end{bmatrix}\begin{bmatrix}0\\2\end{bmatrix}} + \chi\left(\begin{bmatrix}0&2\\1&0\end{bmatrix}\right)\delta_{\begin{bmatrix}0&2\\1&0\end{bmatrix}\begin{bmatrix}0\\2\end{bmatrix}}$$
$$= 1\cdot\delta_{\begin{bmatrix}0\\2\end{bmatrix}} + i\cdot\delta_{\begin{bmatrix}2\\0\end{bmatrix}} - 1\cdot\delta_{\begin{bmatrix}0\\1\end{bmatrix}} - i\cdot\delta_{\begin{bmatrix}1\\0\end{bmatrix}}$$
$$= -1\cdot\delta_{\begin{bmatrix}0\\1\end{bmatrix}} - i\cdot\delta_{\begin{bmatrix}1\\0\end{bmatrix}} + 1\cdot\delta_{\begin{bmatrix}0\\2\end{bmatrix}} + i\cdot\delta_{\begin{bmatrix}2\\0\end{bmatrix}}.$$

So $f_{\begin{bmatrix}0\\2\end{bmatrix}}(\chi) = -f_{\begin{bmatrix}0\\1\end{bmatrix}}(\chi)$.

$$f_{\begin{bmatrix}1\\0\end{bmatrix}}(\chi) = \chi\left(\begin{bmatrix}1&0\\0&1\end{bmatrix}\right)\delta_{\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}} + \chi\left(\begin{bmatrix}0&1\\2&0\end{bmatrix}\right)\delta_{\begin{bmatrix}0&1\\2&0\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}}$$
$$+ \chi\left(\begin{bmatrix}2&0\\0&2\end{bmatrix}\right)\delta_{\begin{bmatrix}2&0\\0&2\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}} + \chi\left(\begin{bmatrix}0&2\\1&0\end{bmatrix}\right)\delta_{\begin{bmatrix}0&2\\1&0\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}}$$
$$= 1\cdot\delta_{\begin{bmatrix}1\\0\end{bmatrix}} + i\cdot\delta_{\begin{bmatrix}0\\2\end{bmatrix}} - 1\cdot\delta_{\begin{bmatrix}2\\0\end{bmatrix}} - i\cdot\delta_{\begin{bmatrix}0\\1\end{bmatrix}}$$

$$= -i \cdot \delta_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} + 1 \cdot \delta_{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} + i \cdot \delta_{\begin{bmatrix} 0 \\ 2 \end{bmatrix}} - 1 \cdot \delta_{\begin{bmatrix} 2 \\ 0 \end{bmatrix}}.$$

So $f_{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}(\chi) = i \cdot f_{\begin{bmatrix} 0 \\ 2 \end{bmatrix}}(\chi) = -i \cdot f_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}(\chi)$.

$$f_{\begin{bmatrix} 2 \\ 0 \end{bmatrix}}(\chi) = \chi\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\delta_{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix}} + \chi\left(\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}\right)\delta_{\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix}}$$

$$+ \chi\left(\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\right)\delta_{\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix}} + \chi\left(\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}\right)\delta_{\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix}}$$

$$= 1 \cdot \delta_{\begin{bmatrix} 2 \\ 0 \end{bmatrix}} + i \cdot \delta_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} - 1 \cdot \delta_{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} - i \cdot \delta_{\begin{bmatrix} 0 \\ 2 \end{bmatrix}}$$

$$= i \cdot \delta_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} - 1 \cdot \delta_{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} - i \cdot \delta_{\begin{bmatrix} 0 \\ 2 \end{bmatrix}} + 1 \cdot \delta_{\begin{bmatrix} 2 \\ 0 \end{bmatrix}}.$$

So $f_{\begin{bmatrix} 2 \\ 0 \end{bmatrix}}(\chi) = -i \cdot f_{\begin{bmatrix} 0 \\ 2 \end{bmatrix}}(\chi) = i \cdot f_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}(\chi)$.

$$f_{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}(\chi) = \chi\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\delta_{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}} + \chi\left(\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}\right)\delta_{\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

$$+\chi\left(\begin{bmatrix}2&0\\0&2\end{bmatrix}\right)\delta_{\begin{bmatrix}2&0\\0&2\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}}+\chi\left(\begin{bmatrix}0&2\\1&0\end{bmatrix}\right)\delta_{\begin{bmatrix}0&2\\1&0\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}}$$

$$=1\cdot\delta_{\begin{bmatrix}1\\1\end{bmatrix}}+i\cdot\delta_{\begin{bmatrix}1\\2\end{bmatrix}}-1\cdot\delta_{\begin{bmatrix}2\\2\end{bmatrix}}-i\cdot\delta_{\begin{bmatrix}2\\1\end{bmatrix}}.$$

So $f_{\begin{bmatrix}1\\1\end{bmatrix}}(\chi)$ is not a linear combination of the previous functions.

$$f_{\begin{bmatrix}1\\2\end{bmatrix}}(\chi)=\chi\left(\begin{bmatrix}1&0\\0&1\end{bmatrix}\right)\delta_{\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix}}+\chi\left(\begin{bmatrix}0&1\\2&0\end{bmatrix}\right)\delta_{\begin{bmatrix}0&1\\2&0\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix}}$$

$$+\chi\left(\begin{bmatrix}2&0\\0&2\end{bmatrix}\right)\delta_{\begin{bmatrix}2&0\\0&2\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix}}+\chi\left(\begin{bmatrix}0&2\\1&0\end{bmatrix}\right)\delta_{\begin{bmatrix}0&2\\1&0\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix}}$$

$$=1\cdot\delta_{\begin{bmatrix}1\\2\end{bmatrix}}+i\cdot\delta_{\begin{bmatrix}2\\2\end{bmatrix}}-1\cdot\delta_{\begin{bmatrix}2\\1\end{bmatrix}}-i\cdot\delta_{\begin{bmatrix}1\\1\end{bmatrix}}$$

$$=-i\cdot\delta_{\begin{bmatrix}1\\1\end{bmatrix}}+1\cdot\delta_{\begin{bmatrix}1\\2\end{bmatrix}}+i\cdot\delta_{\begin{bmatrix}2\\2\end{bmatrix}}-1\cdot\delta_{\begin{bmatrix}2\\1\end{bmatrix}}.$$

So $f_{\begin{bmatrix}1\\2\end{bmatrix}}(\chi)=-i\cdot f_{\begin{bmatrix}1\\1\end{bmatrix}}(\chi)$.

$$f_{\begin{bmatrix}2\\1\end{bmatrix}}(\chi) = \chi\left(\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\right)\delta_{\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix}} + \chi\left(\begin{bmatrix}0 & 1\\2 & 0\end{bmatrix}\right)\delta_{\begin{bmatrix}0 & 1\\2 & 0\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix}}$$
$$+ \chi\left(\begin{bmatrix}2 & 0\\0 & 2\end{bmatrix}\right)\delta_{\begin{bmatrix}2 & 0\\0 & 2\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix}} + \chi\left(\begin{bmatrix}0 & 2\\1 & 0\end{bmatrix}\right)\delta_{\begin{bmatrix}0 & 2\\1 & 0\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix}}$$
$$= 1\cdot\delta_{\begin{bmatrix}2\\1\end{bmatrix}} + i\cdot\delta_{\begin{bmatrix}1\\1\end{bmatrix}} - 1\cdot\delta_{\begin{bmatrix}1\\2\end{bmatrix}} - i\cdot\delta_{\begin{bmatrix}2\\2\end{bmatrix}}$$
$$= i\cdot\delta_{\begin{bmatrix}1\\1\end{bmatrix}} - 1\cdot\delta_{\begin{bmatrix}1\\2\end{bmatrix}} - i\cdot\delta_{\begin{bmatrix}2\\2\end{bmatrix}} + 1\cdot\delta_{\begin{bmatrix}2\\1\end{bmatrix}}.$$

So $f_{\begin{bmatrix}2\\1\end{bmatrix}}(\chi) = -f_{\begin{bmatrix}1\\2\end{bmatrix}}(\chi) = i\cdot f_{\begin{bmatrix}1\\1\end{bmatrix}}(\chi)$.

$$f_{\begin{bmatrix}2\\2\end{bmatrix}}(\chi) = \chi\left(\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\right)\delta_{\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}2\\2\end{bmatrix}} + \chi\left(\begin{bmatrix}0 & 1\\2 & 0\end{bmatrix}\right)\delta_{\begin{bmatrix}0 & 1\\2 & 0\end{bmatrix}\begin{bmatrix}2\\2\end{bmatrix}}$$
$$+ \chi\left(\begin{bmatrix}2 & 0\\0 & 2\end{bmatrix}\right)\delta_{\begin{bmatrix}2 & 0\\0 & 2\end{bmatrix}\begin{bmatrix}2\\2\end{bmatrix}} + \chi\left(\begin{bmatrix}0 & 2\\1 & 0\end{bmatrix}\right)\delta_{\begin{bmatrix}0 & 2\\1 & 0\end{bmatrix}\begin{bmatrix}2\\2\end{bmatrix}}$$
$$= 1\cdot\delta_{\begin{bmatrix}2\\2\end{bmatrix}} + i\cdot\delta_{\begin{bmatrix}2\\1\end{bmatrix}} - 1\cdot\delta_{\begin{bmatrix}1\\1\end{bmatrix}} - i\cdot\delta_{\begin{bmatrix}1\\2\end{bmatrix}}$$

$$= -1 \cdot \delta_{\begin{bmatrix}1\\1\end{bmatrix}} - i \cdot \delta_{\begin{bmatrix}1\\2\end{bmatrix}} + 1 \cdot \delta_{\begin{bmatrix}2\\2\end{bmatrix}} + i \cdot \delta_{\begin{bmatrix}2\\1\end{bmatrix}}.$$

So $f_{\begin{bmatrix}2\\2\end{bmatrix}}(\chi) = i \cdot f_{\begin{bmatrix}2\\1\end{bmatrix}}(\chi) = -f_{\begin{bmatrix}1\\1\end{bmatrix}}(\chi)$.

The two linearly independent generators are

$$f_{\begin{bmatrix}0\\1\end{bmatrix}}(\chi) \quad \text{and} \quad f_{\begin{bmatrix}1\\1\end{bmatrix}}(\chi).$$

Nobs[24] has another approach. For a fixed χ, $V(\chi)$ is generated by the f_ξ so that from these only one system can be selected linearly independent. From

$$f_{\gamma\xi} = \chi(\gamma)^{-1} f_\xi(\chi) \text{ for all } \gamma \in \mathfrak{U}, \xi \in M, \chi \in \text{Car } \mathfrak{U}$$

we compute and verify the above equation.

γ	ξ	$\gamma\xi$	$\chi(\gamma)$	$\chi(\gamma)^{-1}$	$\chi(\gamma)^{-1}f_\xi(\chi)$	$f_{\gamma\xi}$	$\chi(\gamma)^{-1}f_\xi(\chi)? = f_{\gamma\xi}$
$\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$	$\begin{bmatrix}0\\ 0\end{bmatrix}$	$\begin{bmatrix}0\\ 0\end{bmatrix}$	1	1	$f_{\begin{bmatrix}0\\ 0\end{bmatrix}}$	$f_{\begin{bmatrix}0\\ 0\end{bmatrix}}$	YES
$\begin{bmatrix}0 & 1\\ 2 & 0\end{bmatrix}$	$\begin{bmatrix}0\\ 0\end{bmatrix}$	$\begin{bmatrix}0\\ 0\end{bmatrix}$	i	$-i$	$-if_{\begin{bmatrix}0\\ 0\end{bmatrix}}$	$f_{\begin{bmatrix}0\\ 0\end{bmatrix}}$	YES
$\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}$	$\begin{bmatrix}0\\ 0\end{bmatrix}$	$\begin{bmatrix}0\\ 0\end{bmatrix}$	-1	-1	$-f_{\begin{bmatrix}0\\ 0\end{bmatrix}}$	$f_{\begin{bmatrix}0\\ 0\end{bmatrix}}$	YES
$\begin{bmatrix}0 & 2\\ 1 & 0\end{bmatrix}$	$\begin{bmatrix}0\\ 0\end{bmatrix}$	$\begin{bmatrix}0\\ 0\end{bmatrix}$	$-i$	i	$if_{\begin{bmatrix}0\\ 0\end{bmatrix}}$	$f_{\begin{bmatrix}0\\ 0\end{bmatrix}}$	YES

γ	ξ	$\gamma\xi$	$\chi(\gamma)$	$\chi(\gamma)^{-1}$	$\chi(\gamma)^{-1} f_\xi(\chi)$	$f_{\gamma\xi}$	$\chi(\gamma)^{-1} f_\xi(\chi) = f_{\gamma\xi}$?
$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$	1	1	$f_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}$	$f_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}$	YES
$\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	i	$-i$	$-if_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}$	$f_{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}$	YES
$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 2 \end{bmatrix}$	-1	-1	$-f_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}$	$f_{\begin{bmatrix} 0 \\ 2 \end{bmatrix}}$	YES
$\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 2 \\ 0 \end{bmatrix}$	$-i$	i	$if_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}$	$f_{\begin{bmatrix} 2 \\ 0 \end{bmatrix}}$	YES
$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 2 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 2 \end{bmatrix}$	1	1	$f_{\begin{bmatrix} 0 \\ 2 \end{bmatrix}}$	$f_{\begin{bmatrix} 0 \\ 2 \end{bmatrix}}$	YES
$\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 2 \end{bmatrix}$	$\begin{bmatrix} 2 \\ 0 \end{bmatrix}$	i	$-i$	$-if_{\begin{bmatrix} 0 \\ 2 \end{bmatrix}}$	$f_{\begin{bmatrix} 2 \\ 0 \end{bmatrix}}$	YES

γ	ξ	$\gamma\xi$	$\chi(\gamma)$	$\chi(\gamma)^{-1}$	$\chi(\gamma)^{-1}f_\xi(\chi)$	$f_{\gamma\xi}$	$\chi(\gamma)^{-1}f_\xi(\chi) = f_{\gamma\xi}$?
$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 2 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$	-1	-1	$-f_{\begin{bmatrix} 0 \\ 2 \end{bmatrix}}$	$f_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}$	YES
$\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 2 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$-i$	i	$if_{\begin{bmatrix} 0 \\ 2 \end{bmatrix}}$	$f_{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}$	YES
$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	1	1	$f_{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}$	$f_{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}$	YES
$\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 2 \end{bmatrix}$	i	$-i$	$-if_{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}$	$f_{\begin{bmatrix} 0 \\ 2 \end{bmatrix}}$	YES
$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 2 \\ 0 \end{bmatrix}$	-1	-1	$-f_{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}$	$f_{\begin{bmatrix} 2 \\ 0 \end{bmatrix}}$	YES
$\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$	$-i$	i	$if_{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}$	$f_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}$	YES

γ	ξ	$\gamma\xi$	$\chi(\gamma)$	$\chi(\gamma)^{-1}$	$\chi(\gamma)^{-1}f_\xi(\chi)$	$f_{\gamma\xi}$	$\chi(\gamma)^{-1}f_\xi(\chi) = f_{\gamma\xi}$?
$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$	$\begin{bmatrix}1\\1\end{bmatrix}$	$\begin{bmatrix}1\\1\end{bmatrix}$	1	1	$f_{\begin{bmatrix}1\\1\end{bmatrix}}$	$f_{\begin{bmatrix}1\\1\end{bmatrix}}$	YES
$\begin{bmatrix}0 & 1\\2 & 0\end{bmatrix}$	$\begin{bmatrix}1\\1\end{bmatrix}$	$\begin{bmatrix}1\\2\end{bmatrix}$	i	$-i$	$-if_{\begin{bmatrix}1\\1\end{bmatrix}}$	$f_{\begin{bmatrix}1\\2\end{bmatrix}}$	YES
$\begin{bmatrix}2 & 0\\0 & 2\end{bmatrix}$	$\begin{bmatrix}1\\1\end{bmatrix}$	$\begin{bmatrix}2\\2\end{bmatrix}$	-1	-1	$-f_{\begin{bmatrix}1\\1\end{bmatrix}}$	$f_{\begin{bmatrix}2\\2\end{bmatrix}}$	YES
$\begin{bmatrix}0 & 2\\1 & 0\end{bmatrix}$	$\begin{bmatrix}1\\1\end{bmatrix}$	$\begin{bmatrix}2\\1\end{bmatrix}$	$-i$	i	$if_{\begin{bmatrix}1\\1\end{bmatrix}}$	$f_{\begin{bmatrix}2\\1\end{bmatrix}}$	YES
$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$	$\begin{bmatrix}1\\2\end{bmatrix}$	$\begin{bmatrix}1\\2\end{bmatrix}$	1	1	$f_{\begin{bmatrix}1\\2\end{bmatrix}}$	$f_{\begin{bmatrix}1\\2\end{bmatrix}}$	YES
$\begin{bmatrix}0 & 1\\2 & 0\end{bmatrix}$	$\begin{bmatrix}1\\2\end{bmatrix}$	$\begin{bmatrix}2\\2\end{bmatrix}$	i	$-i$	$-if_{\begin{bmatrix}1\\2\end{bmatrix}}$	$f_{\begin{bmatrix}2\\2\end{bmatrix}}$	YES

γ	ξ	$\gamma\xi$	$\chi(\gamma)$	$\chi(\gamma)^{-1}$	$\chi(\gamma)^{-1}f_\xi(\chi)$	$f_{\gamma\xi}$	$\chi(\gamma)^{-1}f_\xi(\chi) = f_{\gamma\xi}$?
$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$	-1	-1	$-f_{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}$	$f_{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}$	YES
$\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	$-i$	i	$if_{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}$	$f_{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}$	YES
$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 2 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 2 \\ 0 \end{bmatrix}$	1	1	$f_{\begin{bmatrix} 2 \\ 0 \end{bmatrix}}$	$f_{\begin{bmatrix} 2 \\ 0 \end{bmatrix}}$	YES
$\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$	$\begin{bmatrix} 2 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$	i	$-i$	$-if_{\begin{bmatrix} 2 \\ 0 \end{bmatrix}}$	$f_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}$	YES
$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$	$\begin{bmatrix} 2 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	-1	-1	$-f_{\begin{bmatrix} 2 \\ 0 \end{bmatrix}}$	$f_{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}$	YES
$\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 2 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 2 \end{bmatrix}$	$-i$	i	$if_{\begin{bmatrix} 2 \\ 0 \end{bmatrix}}$	$f_{\begin{bmatrix} 0 \\ 2 \end{bmatrix}}$	YES

γ	ξ	$\gamma\xi$	$\chi(\gamma)$	$\chi(\gamma)^{-1}$	$\chi(\gamma)^{-1}f_\xi(\chi)$	$f_{\gamma\xi}$	$\chi(\gamma)^{-1}f_\xi(\chi) = f_{\gamma\xi}$?
$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$	$\begin{bmatrix}2\\1\end{bmatrix}$	$\begin{bmatrix}1\\1\end{bmatrix}$	1	1	$f_{\begin{bmatrix}1\\1\end{bmatrix}}$	$f_{\begin{bmatrix}1\\1\end{bmatrix}}$	YES
$\begin{bmatrix}0 & 1\\2 & 0\end{bmatrix}$	$\begin{bmatrix}2\\1\end{bmatrix}$	$\begin{bmatrix}2\\1\end{bmatrix}$	i	$-i$	$-if_{\begin{bmatrix}2\\1\end{bmatrix}}$	$f_{\begin{bmatrix}1\\1\end{bmatrix}}$	YES
$\begin{bmatrix}2 & 0\\0 & 2\end{bmatrix}$	$\begin{bmatrix}2\\1\end{bmatrix}$	$\begin{bmatrix}1\\2\end{bmatrix}$	-1	-1	$-f_{\begin{bmatrix}2\\1\end{bmatrix}}$	$f_{\begin{bmatrix}1\\2\end{bmatrix}}$	YES
$\begin{bmatrix}0 & 2\\1 & 0\end{bmatrix}$	$\begin{bmatrix}2\\1\end{bmatrix}$	$\begin{bmatrix}2\\2\end{bmatrix}$	$-i$	i	$if_{\begin{bmatrix}2\\1\end{bmatrix}}$	$f_{\begin{bmatrix}2\\2\end{bmatrix}}$	YES
$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$	$\begin{bmatrix}2\\2\end{bmatrix}$	$\begin{bmatrix}2\\2\end{bmatrix}$	1	1	$f_{\begin{bmatrix}2\\2\end{bmatrix}}$	$f_{\begin{bmatrix}2\\2\end{bmatrix}}$	YES

γ	ξ	$\gamma\xi$	$\chi(\gamma)$	$\chi(\gamma)^{-1}$	$\chi(\gamma)^{-1}f_\xi(\chi)$	$f_{\gamma\xi}$	$\chi(\gamma)^{-1}f_\xi(\chi) = f_{\gamma\xi}$?
$\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$	$\begin{bmatrix} 2 \\ 2 \end{bmatrix}$	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$	i	$-i$	$-if_{\begin{bmatrix} 2 \\ 2 \end{bmatrix}}$	$f_{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}$	YES
$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$	$\begin{bmatrix} 2 \\ 2 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	-1	-1	$-f_{\begin{bmatrix} 2 \\ 2 \end{bmatrix}}$	$f_{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}$	YES
$\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 2 \\ 2 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$	$-i$	i	$if_{\begin{bmatrix} 2 \\ 2 \end{bmatrix}}$	$f_{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}$	$YES.$

Again, we have that there are two linearly independent generators as before:

$$f_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}(\chi) \quad \text{and} \quad f_{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}(\chi).$$

3.10.1 The representation of the action by T

Letting $\xi = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $N(\xi) = x_1^2 - ux_2^2$.

With $u = 2$. Following Nobs[24], θ must contain exactly one ξ with $N(\xi) = a$ for every for every $a \in A_\lambda^\times$. So $a = 1$ or $a = 2$.

$\theta_1 = 1$ corresponds to $\xi = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\theta_2 = 2$ corresponds to $\xi = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Theorem 2 from Nobs[23][24] tells us that for $\xi = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} f_\xi(\chi) = \mathbf{e}(p^{-\lambda} N(\xi)) f_\xi(\chi)$$

$$= \exp\left(\frac{2\pi i}{3}(0^2 - 2 \cdot 1^2)\right) f_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}(\chi)$$

$$= \exp\left(\frac{-4\pi i}{3}\right) \cdot \left(1 \cdot \delta_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} + i \cdot \delta_{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} - 1 \cdot \delta_{\begin{bmatrix} 0 \\ 2 \end{bmatrix}} - i \cdot \delta_{\begin{bmatrix} 2 \\ 0 \end{bmatrix}}\right)$$

$$= \exp\left(\frac{2\pi i}{3}\right) \cdot \left(1 \cdot \delta_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} + i \cdot \delta_{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} - 1 \cdot \delta_{\begin{bmatrix} 0 \\ 2 \end{bmatrix}} - i \cdot \delta_{\begin{bmatrix} 2 \\ 0 \end{bmatrix}}\right)$$

$$= \exp\left(\frac{2\pi i}{3}\right) f_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}(\chi)$$

$$= \zeta_3 \cdot f_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}(\chi).$$

And for $\xi = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$,

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} f_\xi(\chi) = \mathbf{e}(p^{-\lambda} N(\xi)) f_\xi(\chi)$$

$$= \exp\left(\frac{2\pi i}{3}(1^2 - 2\cdot 1^2)\right) f_{\begin{bmatrix}1\\1\end{bmatrix}}(\chi)$$

$$= \exp\left(\frac{-2\pi i}{3}\right)\cdot\left(i\cdot\delta_{\begin{bmatrix}1\\1\end{bmatrix}} - 1\cdot\delta_{\begin{bmatrix}1\\2\end{bmatrix}} - i\cdot\delta_{\begin{bmatrix}2\\2\end{bmatrix}} + 1\cdot\delta_{\begin{bmatrix}2\\1\end{bmatrix}}\right)$$

$$= \exp\left(\frac{4\pi i}{3}\right)\cdot\left(i\cdot\delta_{\begin{bmatrix}1\\1\end{bmatrix}} - 1\cdot\delta_{\begin{bmatrix}1\\2\end{bmatrix}} - i\cdot\delta_{\begin{bmatrix}2\\2\end{bmatrix}} + 1\cdot\delta_{\begin{bmatrix}2\\1\end{bmatrix}}\right)$$

$$= \exp\left(\frac{4\pi i}{3}\right) f_{\begin{bmatrix}1\\1\end{bmatrix}}(\chi)$$

$$= \zeta_3^2\cdot f_{\begin{bmatrix}1\\1\end{bmatrix}}(\chi).$$

With respect to the basis vectors $f_{\begin{bmatrix}0\\1\end{bmatrix}}(\chi)$ and $f_{\begin{bmatrix}1\\1\end{bmatrix}}(\chi)$,

$$N_1(\chi)(T) = \begin{bmatrix}\zeta_3 & 0\\ 0 & \zeta_3^2\end{bmatrix}.$$

We verify that $\mathrm{Tr}(N_1(\chi)(T)) = \zeta_3 + \zeta_3^2 = -1$. It does agree with Reeder[25]. The representation $N_\lambda(\chi)$ is referred to as π_η and $\pi_\eta(u) = -1$ where u is the representative of the conjugacy class of $T = \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$.

3.10.2 The representation of the action by S using Nobs' method

Recall

$$B(x,y) = \frac{2x_1y_1 - 2ux_2y_2}{p^\lambda}.$$

So for our case,

$$B(x,y) = \frac{2x_1y_1 - 4x_2y_2}{3}.$$

Nobs and Wolfart[24] tells us that

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} f_\xi(\chi) = p^{-\lambda}(-1)^\lambda \sum_{\eta\in\theta} \left\{ \sum_{\varepsilon\in\mathfrak{U}} \chi(\varepsilon)\mathbf{e}(p^{-\lambda} \operatorname{Tr} \varepsilon\xi\overline{\eta}) \right\} f_\eta(\chi) \tag{5}$$

$$= p^{-\lambda}(-1)^\lambda \sum_{\eta\in\theta} \left\{ \sum_{\varepsilon\in\mathfrak{U}} \chi(\varepsilon)\mathbf{e}(B(\varepsilon\xi,\eta)) \right\} f_\eta(\chi).$$

With $\lambda = 1, p = 3$, we have

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} f_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}(\chi) = \frac{-1}{3} \left[\left\{ \chi\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \mathbf{e}\left(B\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)\right)\right.\right.$$

$$+ \chi\left(\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}\right) \mathbf{e}\left(B\left(\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)\right)$$

$$+ \chi\left(\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\right) \mathbf{e}\left(B\left(\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)\right)$$

$$\left. + \chi\left(\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}\right) \mathbf{e}\left(B\left(\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)\right) \right\} f_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}(\chi)$$

$$+ \left\{ \chi\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \mathbf{e}\left(B\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)\right)\right.$$

$$+ \chi\left(\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}\right)\mathbf{e}\left(B\left(\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)\right)$$

$$+ \chi\left(\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\right)\mathbf{e}\left(B\left(\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)\right)$$

$$\left. + \chi\left(\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}\right)\mathbf{e}\left(B\left(\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)\right)\right\} f_{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}(\chi)\Bigg]$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} f_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}(\chi) = \frac{-1}{3}\Bigg[\Bigg\{1\cdot\mathbf{e}\left(B\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)\right) + i\cdot\mathbf{e}\left(B\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)\right)$$

$$-1\cdot\mathbf{e}\left(B\left(\begin{bmatrix} 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)\right) - i\cdot\mathbf{e}\left(B\left(\begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)\right)\Bigg\} f_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}(\chi)$$

$$+ \Bigg\{1\cdot\mathbf{e}\left(B\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)\right) + i\cdot\mathbf{e}\left(B\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)\right)$$

$$-1\cdot\mathbf{e}\left(B\left(\begin{bmatrix} 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)\right) - i\cdot\mathbf{e}\left(B\left(\begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)\right)\Bigg\} f_{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}(\chi)\Bigg]$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} f_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}(\chi) = \frac{-1}{3}\Bigg[\Bigg\{1\cdot\mathbf{e}\left(\frac{2\cdot 0\cdot 0 - 4\cdot 1\cdot 1}{3}\right) + i\cdot\mathbf{e}\left(\frac{2\cdot 1\cdot 0 - 4\cdot 0\cdot 1}{3}\right)$$

$$+ \; -1 \cdot \mathbf{e}\left(\frac{2\cdot 0\cdot 0 - 4\cdot 2\cdot 1}{3}\right) \; + \; -i \cdot \mathbf{e}\left(\frac{2\cdot 2\cdot 0 - 4\cdot 0\cdot 1}{3}\right)\Bigg\} f_{\begin{bmatrix}0\\1\end{bmatrix}}(\chi)$$

$$+ \Bigg\{1 \cdot \mathbf{e}\left(\frac{2\cdot 0\cdot 1 - 4\cdot 1\cdot 1}{3}\right) \; + \; i \cdot \mathbf{e}\left(\frac{2\cdot 1\cdot 1 - 4\cdot 0\cdot 1}{3}\right)$$

$$-1 \cdot \mathbf{e}\left(\frac{2\cdot 0\cdot 1 - 4\cdot 2\cdot 1}{3}\right) \; -i \cdot \mathbf{e}\left(\frac{2\cdot 2\cdot 1 - 4\cdot 0\cdot 1}{3}\right)\Bigg\} f_{\begin{bmatrix}1\\1\end{bmatrix}}(\chi)\Bigg]$$

$$\begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix} f_{\begin{bmatrix}0\\1\end{bmatrix}}(\chi) = \frac{-1}{3}\Bigg[\Bigg\{1 \cdot \mathbf{e}\left(\frac{-4}{3}\right) + i \cdot \mathbf{e}\left(\frac{0}{3}\right) - 1 \cdot \mathbf{e}\left(\frac{-8}{3}\right) - i \cdot \mathbf{e}\left(\frac{0}{3}\right)\Bigg\} f_{\begin{bmatrix}0\\1\end{bmatrix}}(\chi)$$

$$+ \Bigg\{1 \cdot \mathbf{e}\left(\frac{-4}{3}\right) \; + \; i \cdot \mathbf{e}\left(\frac{2}{3}\right) \; -1 \cdot \mathbf{e}\left(\frac{-8}{3}\right) \; -i \cdot \mathbf{e}\left(\frac{4}{3}\right)\Bigg\} f_{\begin{bmatrix}1\\1\end{bmatrix}}(\chi)\Bigg]$$

$$\begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix} f_{\begin{bmatrix}0\\1\end{bmatrix}}(\chi) = \frac{-1}{3}\Bigg[\Bigg\{\exp\left(\frac{-8\pi i}{3}\right) - \exp\left(\frac{-16\pi i}{3}\right)\Bigg\} f_{\begin{bmatrix}0\\1\end{bmatrix}}(\chi)$$

$$+ \Bigg\{\exp\left(\frac{-8\pi i}{3}\right) \; + \; i \cdot \exp\left(\frac{4\pi i}{3}\right) \; -1 \cdot \exp\left(\frac{-16\pi i}{3}\right)$$

$$-i \cdot \exp\left(\frac{8\pi i}{3}\right)\Bigg\} f_{\begin{bmatrix}1\\1\end{bmatrix}}(\chi)\Bigg]$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} f_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}(\chi) = \frac{-1}{3} \left[\left\{ \exp\left(\frac{4\pi i}{3}\right) - \exp\left(\frac{2\pi i}{3}\right) \right\} f_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}(\chi) \right.$$
$$+ \left\{ \exp\left(\frac{4\pi i}{3}\right) + i \cdot \exp\left(\frac{4\pi i}{3}\right) - 1 \cdot \exp\left(\frac{2\pi i}{3}\right) \right.$$
$$\left. \left. -i \cdot \exp\left(\frac{2\pi i}{3}\right) \right\} f_{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}(\chi) \right]$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} f_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}(\chi) = \frac{-1}{3} \left[-i\sqrt{3} \cdot f_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}(\chi) + (\sqrt{3} - i\sqrt{3}) \cdot f_{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}(\chi) \right]$$
$$= \frac{i\sqrt{3}}{3} \cdot f_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}(\chi) + \frac{-\sqrt{3} + i\sqrt{3}}{3} \cdot f_{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}(\chi).$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} f_{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}(\chi) = \frac{-1}{3} \left[\left\{ \chi\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \mathbf{e}\left(B\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)\right) \right. \right.$$
$$+ \chi\left(\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}\right) \mathbf{e}\left(B\left(\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)\right)$$
$$+ \chi\left(\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\right) \mathbf{e}\left(B\left(\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)\right)$$

$$+\ \chi\left(\begin{bmatrix}0&2\\1&0\end{bmatrix}\right)\mathbf{e}\left(B\left(\begin{bmatrix}0&2\\1&0\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right)\right)\Bigg\}f_{\begin{bmatrix}0\\1\end{bmatrix}}(\chi)$$

$$+\Bigg\{\chi\left(\begin{bmatrix}1&0\\0&1\end{bmatrix}\right)\mathbf{e}\left(B\left(\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}1\\1\end{bmatrix}\right)\right)$$

$$+\ \chi\left(\begin{bmatrix}0&1\\2&0\end{bmatrix}\right)\mathbf{e}\left(B\left(\begin{bmatrix}0&1\\2&0\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}1\\1\end{bmatrix}\right)\right)$$

$$+\ \chi\left(\begin{bmatrix}2&0\\0&2\end{bmatrix}\right)\mathbf{e}\left(B\left(\begin{bmatrix}2&0\\0&2\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}1\\1\end{bmatrix}\right)\right)$$

$$+\ \chi\left(\begin{bmatrix}0&2\\1&0\end{bmatrix}\right)\mathbf{e}\left(B\left(\begin{bmatrix}0&2\\1&0\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}1\\1\end{bmatrix}\right)\right)\Bigg\}f_{\begin{bmatrix}1\\1\end{bmatrix}}(\chi)\Bigg]$$

$$\begin{bmatrix}0&1\\-1&0\end{bmatrix}f_{\begin{bmatrix}1\\1\end{bmatrix}}(\chi)=\frac{-1}{3}\Bigg[\Bigg\{\chi\left(\begin{bmatrix}1&0\\0&1\end{bmatrix}\right)\mathbf{e}\left(B\left(\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right)\right)$$

$$+\ \chi\left(\begin{bmatrix}0&1\\2&0\end{bmatrix}\right)\mathbf{e}\left(B\left(\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right)\right)$$

$$+\ \chi\left(\begin{bmatrix}2&0\\0&2\end{bmatrix}\right)\mathbf{e}\left(B\left(\begin{bmatrix}2\\2\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right)\right)$$

$$+\ \chi\left(\begin{bmatrix}0&2\\1&0\end{bmatrix}\right)\mathbf{e}\left(B\left(\begin{bmatrix}2\\1\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right)\right)\Bigg\}f_{\begin{bmatrix}0\\1\end{bmatrix}}(\chi)$$

$$+ \left\{ \chi\left(\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)\mathbf{e}\left(B\left(\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}1\\1\end{bmatrix}\right)\right)\right.$$

$$+ \chi\left(\begin{bmatrix}0 & 1\\ 2 & 0\end{bmatrix}\right)\mathbf{e}\left(B\left(\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}1\\1\end{bmatrix}\right)\right)$$

$$+ \chi\left(\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}\right)\mathbf{e}\left(B\left(\begin{bmatrix}2\\2\end{bmatrix},\begin{bmatrix}1\\1\end{bmatrix}\right)\right)$$

$$\left. + \chi\left(\begin{bmatrix}0 & 2\\ 1 & 0\end{bmatrix}\right)\mathbf{e}\left(B\left(\begin{bmatrix}2\\1\end{bmatrix},\begin{bmatrix}1\\1\end{bmatrix}\right)\right)\right\} f_{\begin{bmatrix}1\\1\end{bmatrix}}(\chi)\Bigg]$$

$$\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix} f_{\begin{bmatrix}1\\1\end{bmatrix}}(\chi) = \frac{-1}{3}\Bigg[\Bigg\{1\cdot\mathbf{e}\left(\frac{2\cdot 1\cdot 0 - 4\cdot 1\cdot 1}{3}\right) + i\cdot\mathbf{e}\left(\frac{2\cdot 1\cdot 0 - 4\cdot 2\cdot 1}{3}\right)$$

$$-1\cdot\mathbf{e}\left(\frac{2\cdot 2\cdot 0 - 4\cdot 2\cdot 1}{3}\right) - i\cdot\mathbf{e}\left(\frac{2\cdot 2\cdot 0 - 4\cdot 1\cdot 1}{3}\right)\Bigg\} f_{\begin{bmatrix}0\\1\end{bmatrix}}(\chi)$$

$$\Bigg\{1\cdot\mathbf{e}\left(\frac{2\cdot 1\cdot 1 - 4\cdot 1\cdot 1}{3}\right) + i\cdot\mathbf{e}\left(\frac{2\cdot 1\cdot 1 - 4\cdot 2\cdot 1}{3}\right)$$

$$-1\cdot\mathbf{e}\left(\frac{2\cdot 2\cdot 1 - 4\cdot 2\cdot 1}{3}\right) - i\cdot\mathbf{e}\left(\frac{2\cdot 2\cdot 1 - 4\cdot 1\cdot 1}{3}\right)\Bigg\} f_{\begin{bmatrix}1\\1\end{bmatrix}}(\chi)\Bigg]$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} f_{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}(\chi) = \frac{-1}{3} \left[\left\{ 1 \cdot \mathbf{e}\left(\frac{-4}{3}\right) + i \cdot \mathbf{e}\left(\frac{-8}{3}\right) - 1 \cdot \mathbf{e}\left(\frac{-8}{3}\right) \right. \right.$$

$$\left. - i \cdot \mathbf{e}\left(\frac{-4}{3}\right) \right\} f_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}(\chi) +$$

$$\left. + \left\{ 1 \cdot \mathbf{e}\left(\frac{-2}{3}\right) + i \cdot \mathbf{e}\left(\frac{-6}{3}\right) \ -1 \cdot \mathbf{e}\left(\frac{-4}{3}\right) \ -i \cdot \mathbf{e}\left(\frac{0}{3}\right) \right\} f_{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}(\chi) \right]$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} f_{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}(\chi) = \frac{-1}{3} \left[\left\{ 1 \cdot \exp\left(\frac{-8\pi i}{3}\right) + i \cdot \exp\left(\frac{-16\pi i}{3}\right) - 1 \cdot \exp\left(\frac{-16\pi i}{3}\right) \right. \right.$$

$$\left. - i \cdot \exp\left(\frac{-8\pi i}{3}\right) \right\} f_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}(\chi)$$

$$+ \left\{ 1 \cdot \exp\left(\frac{-4\pi i}{3}\right) + i \cdot \exp\left(\frac{-12\pi i}{3}\right) - 1 \cdot \exp\left(\frac{-8\pi i}{3}\right) \right.$$

$$\left. \left. -i \cdot \exp\left(\frac{0}{3}\right) \right\} f_{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}(\chi) \right]$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} f_{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}(\chi) = \frac{-1}{3}\left[\left\{1 \cdot \exp\left(\frac{4\pi i}{3}\right) + i \cdot \exp\left(\frac{2\pi i}{3}\right)\right.\right.$$

$$\left. -1 \cdot \exp\left(\frac{2\pi i}{3}\right) - i \cdot \exp\left(\frac{4\pi i}{3}\right)\right\} f_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}(\chi) +$$

$$+ \left\{1 \cdot \exp\left(\frac{2\pi i}{3}\right) + i \cdot \exp\left(\frac{0}{3}\right) + \right.$$

$$\left.\left. -1 \cdot \exp\left(\frac{4\pi i}{3}\right) - i \cdot \exp\left(\frac{0}{3}\right)\right\} f_{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}(\chi)\right].$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} f_{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}(\chi) = \frac{-1}{3}\left[(-i\sqrt{3} - \sqrt{3}) \cdot f_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}(\chi) + i\sqrt{3} \cdot f_{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}(\chi)\right]$$

$$= \frac{\sqrt{3} + i\sqrt{3}}{3} \cdot f_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}(\chi) - \frac{i\sqrt{3}}{3} \cdot f_{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}(\chi).$$

Recall,

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} f_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}(\chi) = \frac{i\sqrt{3}}{3} \cdot f_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}(\chi) + \frac{-\sqrt{3} + i\sqrt{3}}{3} \cdot f_{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}(\chi).$$

So with respect to the basis

$$\left\{ f_{\begin{bmatrix}0\\1\end{bmatrix}}(\chi), f_{\begin{bmatrix}1\\1\end{bmatrix}}(\chi) \right\}$$

The action of $\left[\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right]$ is given by

$$\begin{bmatrix} \frac{i\sqrt{3}}{3} & \frac{-\sqrt{3}+i\sqrt{3}}{3} \\ \frac{\sqrt{3}+i\sqrt{3}}{3} & \frac{-i\sqrt{3}}{3} \end{bmatrix} = \frac{\sqrt{3}}{3} \begin{bmatrix} i & -1+i \\ 1+i & -i \end{bmatrix}.$$

So the action by S, call it $N_1(\chi)(S)$, will be the conjugate transpose of the above matrix:

$$N_1(\chi)(S) = \begin{bmatrix} \frac{-i\sqrt{3}}{3} & \frac{\sqrt{3}-i\sqrt{3}}{3} \\ \frac{-\sqrt{3}-i\sqrt{3}}{3} & \frac{i\sqrt{3}}{3} \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} -i & -i+1 \\ -1-i & i \end{bmatrix}.$$

Its trace agrees with the character table in Bonnafé[4]. The representation in that text is called $R'(i^{\wedge})$ and the conjugacy class of S is $\mathbf{d}'(i)$. For T, the conjugacy class is u_+ and its character also agrees. We verify the relations $(N_1(\chi)(S))^4 = \mathbb{1}$ and $(N_1(\chi)(S)N_1(\chi)(T))^3 = (N_1(\chi)(S))^2$.

$$\begin{aligned}(N_1(\chi)(S))^2 &= \left[\frac{1}{\sqrt{3}} \begin{bmatrix} -i & -i+1 \\ -1-i & i \end{bmatrix} \right]^2 \\ &= \frac{1}{3} \begin{bmatrix} -i & -i+1 \\ -1-i & i \end{bmatrix} \begin{bmatrix} -i & -i+1 \\ -1-i & i \end{bmatrix} \\ &= \frac{1}{3} \begin{bmatrix} -1+-2 & -1-i+1+i \\ i-1-i+1 & -2-1 \end{bmatrix}\end{aligned}$$

$$= \frac{1}{3}\begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix}$$

$$= -\mathbb{1}.$$

So, $(N_1(\chi)(S))^4 = \mathbb{1}$. Recalling that

$$N_1(\chi)(T) = \begin{bmatrix} \zeta_3 & 0 \\ 0 & \zeta_3^2 \end{bmatrix},$$

we have

$$N_1(\chi)(S) \cdot N_1(\chi)(T) = \frac{1}{\sqrt{3}}\begin{bmatrix} -i & -i+1 \\ -1-i & i \end{bmatrix}\begin{bmatrix} \frac{-1+i\sqrt{3}}{2} & 0 \\ 0 & \frac{-1-i\sqrt{3}}{2} \end{bmatrix}$$

$$= \frac{1}{\sqrt{3}}\begin{bmatrix} \frac{i+\sqrt{3}}{2} & \frac{(i-\sqrt{3}-1-i\sqrt{3})}{2} \\ \frac{1-i\sqrt{3}+i+\sqrt{3}}{2} & \frac{-i+\sqrt{3}}{2} \end{bmatrix}$$

$$= \frac{1}{\sqrt{3}}\begin{bmatrix} \frac{\sqrt{3}+i}{2} & \frac{-(\sqrt{3}+1)+i(1-\sqrt{3})}{2} \\ \frac{(1+\sqrt{3})+i(1-\sqrt{3})}{2} & \frac{\sqrt{3}-i}{2} \end{bmatrix}$$

$$= \frac{1}{2\sqrt{3}}\begin{bmatrix} \sqrt{3}+i & -(\sqrt{3}+1)+i(1-\sqrt{3}) \\ (1+\sqrt{3})+i(1-\sqrt{3}) & \sqrt{3}-i \end{bmatrix},$$

$$(N_1(\chi)(S) \cdot N_1(\chi)(T))^2 = \frac{1}{12}\begin{bmatrix} \sqrt{3}+i & -[(\sqrt{3}+1)-i(1-\sqrt{3})] \\ (1+\sqrt{3})+i(1-\sqrt{3}) & \sqrt{3}-i \end{bmatrix}$$

$$\cdot \begin{bmatrix} \sqrt{3}+i & -[(\sqrt{3}+1)-i(1-\sqrt{3}]) \\ (1+\sqrt{3})+i(1-\sqrt{3}) & \sqrt{3}-i \end{bmatrix}$$

$$= \frac{1}{12}\begin{bmatrix} -6+2i\sqrt{3} & -(6+2\sqrt{3})+i(2\sqrt{3}-6) \\ (6+2\sqrt{3})+i(2\sqrt{3}-6) & -6-2i\sqrt{3} \end{bmatrix}$$

and finally

$$(N_1(\chi)(S)\cdot N_1(\chi)(T))^3 = \frac{1}{24\sqrt{3}}\begin{bmatrix} -6+2i\sqrt{3} & -(6+2\sqrt{3})+i(2\sqrt{3}-6) \\ (6+2\sqrt{3})+i(2\sqrt{3}-6) & -6-2i\sqrt{3} \end{bmatrix}$$

$$\cdot\begin{bmatrix} \sqrt{3}+i & -[(\sqrt{3}+1)-i(1-\sqrt{3}]) \\ (1+\sqrt{3})+i(1-\sqrt{3}) & \sqrt{3}-i \end{bmatrix}$$

$$= \frac{1}{24\sqrt{3}}\begin{bmatrix} -24\sqrt{3} & 0 \\ 0 & -24\sqrt{3} \end{bmatrix}$$

$$= -\mathbb{1}$$

$$= (N_1(\chi)(S))^2.$$

3.10.3 The representation of the action by S using the unreduced Nobs' method

Let

$$f_1 = f_{\begin{bmatrix}0\\1\end{bmatrix}}(\chi) = \left(1\cdot\delta_{\begin{bmatrix}0\\1\end{bmatrix}} + i\cdot\delta_{\begin{bmatrix}1\\0\end{bmatrix}} - 1\cdot\delta_{\begin{bmatrix}0\\2\end{bmatrix}} - i\cdot\delta_{\begin{bmatrix}2\\0\end{bmatrix}}\right)$$

and

$$f_2 = f_{\begin{bmatrix}1\\1\end{bmatrix}}(\chi) = \left(i\cdot\delta_{\begin{bmatrix}1\\1\end{bmatrix}} - 1\cdot\delta_{\begin{bmatrix}1\\2\end{bmatrix}} - i\cdot\delta_{\begin{bmatrix}2\\2\end{bmatrix}} + 1\cdot\delta_{\begin{bmatrix}2\\1\end{bmatrix}}\right)$$

as before. Using the action from Nobs[23][24] we will compute the action of S.

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} f_1(x) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \left(1 \cdot \delta_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} + i \cdot \delta_{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} - 1 \cdot \delta_{\begin{bmatrix} 0 \\ 2 \end{bmatrix}} - i \cdot \delta_{\begin{bmatrix} 2 \\ 0 \end{bmatrix}} \right)$$

$$\begin{aligned} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \cdot 1 \cdot \delta_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} &= S_Q(-1)|M|^{-1/2} \sum_{y \in M} \mathbf{e}(B(x,y)) \cdot \delta_y \\ &= \frac{-1}{3} \sum_{y \in M} \mathbf{e} \left(B \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, y \right) \right) \cdot \delta_y) \\ &= \frac{-1}{3} \sum_{y \in M} \mathbf{e} \left(\frac{2 \cdot 0 \cdot y_1 - 4 \cdot 1 \cdot y_2}{3} \right) \cdot \delta_y \\ &= \frac{-1}{3} \sum_{y \in M} \zeta_3^{-4y_2} \cdot \delta_y \\ &= \frac{-1}{3} \left(\zeta_3^0 \cdot \delta_{\begin{bmatrix} 0 \\ 0 \end{bmatrix}} + \zeta_3^{-4 \cdot 1} \cdot \delta_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} + \zeta_3^{-4 \cdot 2} \cdot \delta_{\begin{bmatrix} 0 \\ 2 \end{bmatrix}} \right. \\ &\quad + \zeta_3^{-4 \cdot 0} \cdot \delta_{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} + \zeta_3^{-4 \cdot 1} \cdot \delta_{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} + \zeta_3^{-4 \cdot 2} \cdot \delta_{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} \\ &\quad \left. + \zeta_3^{-4 \cdot 0} \cdot \delta_{\begin{bmatrix} 2 \\ 0 \end{bmatrix}} + \zeta_3^{-4 \cdot 1} \cdot \delta_{\begin{bmatrix} 2 \\ 1 \end{bmatrix}} + \zeta_3^{-4 \cdot 2} \cdot \delta_{\begin{bmatrix} 2 \\ 2 \end{bmatrix}} \right). \end{aligned}$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \cdot 1 \cdot \delta_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} = \frac{-1}{3} \left(\delta_{\begin{bmatrix} 0 \\ 0 \end{bmatrix}} + \zeta_3^2 \cdot \delta_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} + \zeta_3^1 \cdot \delta_{\begin{bmatrix} 0 \\ 2 \end{bmatrix}} \right.$$

$$+ \delta_{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} + \zeta_3^2 \cdot \delta_{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} + \zeta_3^1 \cdot \delta_{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}$$

$$\left. + \delta_{\begin{bmatrix} 2 \\ 0 \end{bmatrix}} + \zeta_3^2 \cdot \delta_{\begin{bmatrix} 2 \\ 1 \end{bmatrix}} + \zeta_3^1 \cdot \delta_{\begin{bmatrix} 2 \\ 2 \end{bmatrix}} \right).$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \cdot (-1) \cdot \delta_{\begin{bmatrix} 0 \\ 2 \end{bmatrix}} = (-1) \cdot \frac{-1}{3} \sum_{y \in M} \mathbf{e}\left(\frac{2 \cdot 0 \cdot y_1 - 4 \cdot 2 \cdot y_2}{3} \right) \cdot \delta_y$$

$$= \frac{1}{3} \sum_{y \in M} \zeta_3^{-8y_2} \cdot \delta_y$$

$$= \frac{1}{3} \sum_{y \in M} \zeta_3^{y_2} \cdot \delta_y$$

$$= \frac{1}{3} \left(\delta_{\begin{bmatrix} 0 \\ 0 \end{bmatrix}} + \zeta_3^1 \cdot \delta_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} + \zeta_3^2 \cdot \delta_{\begin{bmatrix} 0 \\ 2 \end{bmatrix}} \right.$$

$$+ \delta_{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} + \zeta_3 \cdot \delta_{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} + \zeta_3^2 \cdot \delta_{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}$$

$$+ \delta_{\begin{bmatrix}2\\0\end{bmatrix}} + \zeta_3^1 \cdot \delta_{\begin{bmatrix}2\\1\end{bmatrix}} + \zeta_3^2 \cdot \delta_{\begin{bmatrix}2\\2\end{bmatrix}} \Bigg).$$

$$\begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix} \cdot (i) \cdot \delta_{\begin{bmatrix}1\\0\end{bmatrix}} = (i) \cdot \frac{-1}{3} \sum_{y \in M} \mathbf{e}\left(\frac{2 \cdot 1 \cdot y_1 - 4 \cdot 0 \cdot y_2}{3}\right) \cdot \delta_y$$

$$= \frac{-i}{3} \sum_{y \in M} \zeta_3^{2y_1} \cdot \delta_y$$

$$= \frac{-i}{3} \Bigg(\delta_{\begin{bmatrix}0\\0\end{bmatrix}} + \zeta_3^0 \cdot \delta_{\begin{bmatrix}0\\1\end{bmatrix}} + \zeta_3^0 \cdot \delta_{\begin{bmatrix}0\\2\end{bmatrix}}$$

$$+ \zeta_3^2 \cdot \delta_{\begin{bmatrix}1\\0\end{bmatrix}} + \zeta_3^2 \cdot \delta_{\begin{bmatrix}1\\1\end{bmatrix}} + \zeta_3^2 \cdot \delta_{\begin{bmatrix}1\\2\end{bmatrix}}$$

$$+ \zeta_3^1 \cdot \delta_{\begin{bmatrix}2\\0\end{bmatrix}} + \zeta_3^1 \cdot \delta_{\begin{bmatrix}2\\1\end{bmatrix}} + \zeta_3^1 \cdot \delta_{\begin{bmatrix}2\\2\end{bmatrix}} \Bigg).$$

$$\begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix} \cdot (-i) \cdot \delta_{\begin{bmatrix}2\\0\end{bmatrix}} = (-i) \cdot \frac{-1}{3} \sum_{y \in M} \mathbf{e}\left(\frac{2 \cdot 2 \cdot y_1 - 4 \cdot 0 \cdot y_2}{3}\right) \cdot \delta_y$$

$$= \frac{i}{3} \sum_{y \in M} \zeta_3^{4y_1} \cdot \delta_y$$

$$= \frac{i}{3} \sum_{y \in M} \zeta_3^{y_1} \cdot \delta_y$$

$$= \frac{i}{3} \left(\delta_{\begin{bmatrix} 0 \\ 0 \end{bmatrix}} + \zeta_3^0 \cdot \delta_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} + \zeta_3^0 \cdot \delta_{\begin{bmatrix} 0 \\ 2 \end{bmatrix}} + \zeta_3^1 \cdot \delta_{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} + \zeta_3^1 \cdot \delta_{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} + \zeta_3^1 \cdot \delta_{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} + \zeta_3^2 \cdot \delta_{\begin{bmatrix} 2 \\ 0 \end{bmatrix}} + \zeta_3^2 \cdot \delta_{\begin{bmatrix} 2 \\ 1 \end{bmatrix}} + \zeta_3^2 \cdot \delta_{\begin{bmatrix} 2 \\ 2 \end{bmatrix}} \right).$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} f_1(x) = \frac{-1}{3} \left(\delta_{\begin{bmatrix} 0 \\ 0 \end{bmatrix}} + \zeta_3^2 \cdot \delta_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} + \zeta_3^1 \cdot \delta_{\begin{bmatrix} 0 \\ 2 \end{bmatrix}} + \delta_{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} + \zeta_3^2 \cdot \delta_{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} + \zeta_3^1 \cdot \delta_{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} + \delta_{\begin{bmatrix} 2 \\ 0 \end{bmatrix}} + \zeta_3^2 \cdot \delta_{\begin{bmatrix} 2 \\ 1 \end{bmatrix}} + \zeta_3^1 \cdot \delta_{\begin{bmatrix} 2 \\ 2 \end{bmatrix}} \right)$$

$$+\frac{1}{3}\left(\delta_{\begin{bmatrix}0\\0\end{bmatrix}}+\zeta_3^1\cdot\delta_{\begin{bmatrix}0\\1\end{bmatrix}}+\zeta_3^2\cdot\delta_{\begin{bmatrix}0\\2\end{bmatrix}}\right.$$

$$+\delta_{\begin{bmatrix}1\\0\end{bmatrix}}+\zeta_3^1\cdot\delta_{\begin{bmatrix}1\\1\end{bmatrix}}+\zeta_3^2\cdot\delta_{\begin{bmatrix}1\\2\end{bmatrix}}$$

$$\left.+\delta_{\begin{bmatrix}2\\0\end{bmatrix}}+\zeta_3^1\cdot\delta_{\begin{bmatrix}2\\1\end{bmatrix}}+\zeta_3^2\cdot\delta_{\begin{bmatrix}2\\2\end{bmatrix}}\right)$$

$$+\frac{-i}{3}\left(\delta_{\begin{bmatrix}0\\0\end{bmatrix}}+\zeta_3^0\cdot\delta_{\begin{bmatrix}0\\1\end{bmatrix}}+\zeta_3^0\cdot\delta_{\begin{bmatrix}0\\2\end{bmatrix}}\right.$$

$$+\zeta_3^2\cdot\delta_{\begin{bmatrix}1\\0\end{bmatrix}}+\zeta_3^2\cdot\delta_{\begin{bmatrix}1\\1\end{bmatrix}}+\zeta_3^2\cdot\delta_{\begin{bmatrix}1\\2\end{bmatrix}}$$

$$\left.+\zeta_3^1\cdot\delta_{\begin{bmatrix}2\\0\end{bmatrix}}+\zeta_3^1\cdot\delta_{\begin{bmatrix}2\\1\end{bmatrix}}+\zeta_3^1\cdot\delta_{\begin{bmatrix}2\\2\end{bmatrix}}\right)$$

$$+\frac{i}{3}\left(\delta_{\begin{bmatrix}0\\0\end{bmatrix}}+\zeta_3^0\cdot\delta_{\begin{bmatrix}0\\1\end{bmatrix}}+\zeta_3^0\cdot\delta_{\begin{bmatrix}0\\2\end{bmatrix}}\right.$$

$$+ \zeta_3^1 \cdot \delta_{\begin{bmatrix}1\\0\end{bmatrix}} + \zeta_3^1 \cdot \delta_{\begin{bmatrix}1\\1\end{bmatrix}} + \zeta_3^1 \cdot \delta_{\begin{bmatrix}1\\2\end{bmatrix}}$$

$$\left. + \zeta_3^2 \cdot \delta_{\begin{bmatrix}2\\0\end{bmatrix}} + \zeta_3^2 \cdot \delta_{\begin{bmatrix}2\\1\end{bmatrix}} + \zeta_3^2 \cdot \delta_{\begin{bmatrix}2\\2\end{bmatrix}} \right).$$

$$\begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix} f_1(x) = \left(\frac{-1}{3} + \frac{1}{3} + \frac{-i}{3} + \frac{i}{3}\right) \delta_{\begin{bmatrix}0\\0\end{bmatrix}}$$

$$+ \frac{1}{3}\left(-\zeta_3^2 + \zeta_3^1 - i + i\right) \delta_{\begin{bmatrix}0\\1\end{bmatrix}}$$

$$+ \frac{1}{3}\left(-\zeta_3^1 + \zeta_3^2 - i + i\right) \delta_{\begin{bmatrix}0\\2\end{bmatrix}}$$

$$+ \frac{1}{3}\left(-1 + 1 - i \cdot \zeta_3^2 + i \cdot \zeta_3^1\right) \delta_{\begin{bmatrix}1\\0\end{bmatrix}}$$

$$+ \frac{1}{3}\left(-\zeta_3^2 + \zeta_3^1 - i \cdot \zeta_3^2 + i \cdot \zeta_3^1\right) \delta_{\begin{bmatrix}1\\1\end{bmatrix}}$$

$$+ \frac{1}{3}\left(-\zeta_3^1 + \zeta_3^2 - i \cdot \zeta_3^2 + i \cdot \zeta_3^1\right) \delta_{\begin{bmatrix}1\\2\end{bmatrix}}$$

$$+\frac{1}{3}\left(-1+1-i\cdot\zeta_3^1+i\cdot\zeta_3^2\right)\delta_{\begin{bmatrix}2\\0\end{bmatrix}}$$

$$+\frac{1}{3}\left(-\zeta_3^2+\zeta_3^1-i\cdot\zeta_3^1+i\cdot\zeta_3^2\right)\delta_{\begin{bmatrix}2\\1\end{bmatrix}}$$

$$+\frac{1}{3}\left(-\zeta_3^1+\zeta_3^2-i\cdot\zeta_3^1+i\cdot\zeta_3^2\right)\delta_{\begin{bmatrix}2\\2\end{bmatrix}}$$

$$\begin{bmatrix}0&1\\-1&0\end{bmatrix}f_1(x)=(0)\,\delta_{\begin{bmatrix}0\\0\end{bmatrix}}+\frac{1}{3}\left(i\sqrt{3}\right)\delta_{\begin{bmatrix}0\\1\end{bmatrix}}+\frac{1}{3}\left(-i\sqrt{3}\right)\delta_{\begin{bmatrix}0\\2\end{bmatrix}}+\frac{1}{3}\left(-\sqrt{3}\right)\delta_{\begin{bmatrix}1\\0\end{bmatrix}}$$

$$+\frac{1}{3}\left(i\sqrt{3}-\sqrt{3}\right)\delta_{\begin{bmatrix}1\\1\end{bmatrix}}+\frac{1}{3}\left(-i\sqrt{3}-\sqrt{3}\right)\delta_{\begin{bmatrix}1\\2\end{bmatrix}}+\frac{1}{3}\left(\sqrt{3}\right)\delta_{\begin{bmatrix}2\\0\end{bmatrix}}$$

$$+\frac{1}{3}\left(i\sqrt{3}+\sqrt{3}\right)\delta_{\begin{bmatrix}2\\1\end{bmatrix}}+\frac{1}{3}\left(\sqrt{3}-i\sqrt{3}\right)\delta_{\begin{bmatrix}2\\2\end{bmatrix}}$$

$$=\frac{i\sqrt{3}}{3}\cdot\delta_{\begin{bmatrix}0\\1\end{bmatrix}}+\frac{-\sqrt{3}}{3}\cdot\delta_{\begin{bmatrix}1\\0\end{bmatrix}}+\frac{-i\sqrt{3}}{3}\cdot\delta_{\begin{bmatrix}0\\2\end{bmatrix}}+\frac{\sqrt{3}}{3}\cdot\delta_{\begin{bmatrix}2\\0\end{bmatrix}}$$

$$+\frac{i\sqrt{3}-\sqrt{3}}{3}\delta_{\begin{bmatrix}1\\1\end{bmatrix}}+\frac{-i\sqrt{3}-\sqrt{3}}{3}\delta_{\begin{bmatrix}1\\2\end{bmatrix}}+\frac{\sqrt{3}-i\sqrt{3}}{3}\delta_{\begin{bmatrix}2\\2\end{bmatrix}}$$

$$+\frac{i\sqrt{3}+i\sqrt{3}}{3}\delta_{\begin{bmatrix}2\\1\end{bmatrix}}$$

$$= \frac{i\sqrt{3}}{3}\left(\delta_{\begin{bmatrix}0\\1\end{bmatrix}} + i\cdot\delta_{\begin{bmatrix}1\\0\end{bmatrix}} - \delta_{\begin{bmatrix}0\\2\end{bmatrix}} - i\cdot\delta_{\begin{bmatrix}2\\0\end{bmatrix}}\right)$$

$$+ \frac{i\sqrt{3}+\sqrt{3}}{3}\left(\delta_{\begin{bmatrix}1\\1\end{bmatrix}} - \delta_{\begin{bmatrix}1\\2\end{bmatrix}} - i\delta_{\begin{bmatrix}2\\2\end{bmatrix}} + \delta_{\begin{bmatrix}2\\1\end{bmatrix}}\right).$$

So,

$$\begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix} f_1(x) = \frac{i\sqrt{3}}{3}f_1 + \left(\frac{i\sqrt{3}+\sqrt{3}}{3}\right) f_2.$$

Next, we work on

$$\begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix} f_2(x) = \begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}\left(i\cdot\delta_{\begin{bmatrix}1\\1\end{bmatrix}} - 1\cdot\delta_{\begin{bmatrix}1\\2\end{bmatrix}} - i\cdot\delta_{\begin{bmatrix}2\\2\end{bmatrix}} + 1\cdot\delta_{\begin{bmatrix}2\\1\end{bmatrix}}\right).$$

$$\begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}\cdot(i)\cdot\delta_{\begin{bmatrix}1\\1\end{bmatrix}} = \frac{-i}{3}\sum_{y\in M}\mathbf{e}\left(\frac{2\cdot 1\cdot y_1 - 4\cdot 1\cdot y_2}{3}\right)\cdot\delta_y$$

$$= \frac{-i}{3}\sum_{y\in M}\zeta_3^{2y_1-4y_2}\cdot\delta_y = \frac{-i}{3}\sum_{y\in M}\zeta_3^{2y_1+2y_2}\cdot\delta_y = \frac{-i}{3}\sum_{y\in M}\zeta_3^{2(y_1+y_2)}\cdot\delta_y$$

$$= \frac{-i}{3}\left(\zeta_3^0\cdot\delta_{\begin{bmatrix}0\\0\end{bmatrix}} + \zeta_3^{2(0+1)}\cdot\delta_{\begin{bmatrix}0\\1\end{bmatrix}} + \zeta_3^{2(0+2)}\cdot\delta_{\begin{bmatrix}0\\2\end{bmatrix}}\right.$$

$$+ \zeta_3^{2(1+0)} \cdot \delta_{\begin{bmatrix}1\\0\end{bmatrix}} + \zeta_3^{2(1+1)} \cdot \delta_{\begin{bmatrix}1\\1\end{bmatrix}} + \zeta_3^{2(1+2)} \cdot \delta_{\begin{bmatrix}1\\2\end{bmatrix}}$$

$$\left. + \zeta_3^{2(2+0)} \cdot \delta_{\begin{bmatrix}2\\0\end{bmatrix}} + \zeta_3^{2(2+1)} \cdot \delta_{\begin{bmatrix}2\\1\end{bmatrix}} + \zeta_3^{2(2+2)} \cdot \delta_{\begin{bmatrix}2\\2\end{bmatrix}} \right)$$

$$= \frac{-i}{3} \left(\delta_{\begin{bmatrix}0\\0\end{bmatrix}} + \zeta_3^{2} \cdot \delta_{\begin{bmatrix}0\\1\end{bmatrix}} + \zeta_3^{1} \cdot \delta_{\begin{bmatrix}0\\2\end{bmatrix}} \right.$$

$$+ \zeta_3^{2} \cdot \delta_{\begin{bmatrix}1\\0\end{bmatrix}} + \zeta_3^{1} \cdot \delta_{\begin{bmatrix}1\\1\end{bmatrix}} + \delta_{\begin{bmatrix}1\\2\end{bmatrix}}$$

$$\left. + \zeta_3^{1} \cdot \delta_{\begin{bmatrix}2\\0\end{bmatrix}} + \delta_{\begin{bmatrix}2\\1\end{bmatrix}} + \zeta_3^{2} \cdot \delta_{\begin{bmatrix}2\\2\end{bmatrix}} \right).$$

$$\begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix} \cdot (-1) \cdot \delta_{\begin{bmatrix}1\\2\end{bmatrix}} = \frac{1}{3} \sum_{y \in M} \mathbf{e}\left(\frac{2 \cdot 1 \cdot y_1 - 4 \cdot 2 \cdot y_2}{3}\right) \cdot \delta_y$$

$$= \frac{1}{3} \sum_{y \in M} \zeta_3^{2y_1 - 8y_2} \cdot \delta_y$$

$$= \frac{1}{3} \sum_{y \in M} \zeta_3^{2y_1 + y_2} \cdot \delta_y$$

$$= \frac{1}{3}\Bigg(\zeta_3^0 \cdot \delta_{\begin{bmatrix}0\\0\end{bmatrix}} + \zeta_3^{2(0)+1} \cdot \delta_{\begin{bmatrix}0\\1\end{bmatrix}} + \zeta_3^{2(0)+2} \cdot \delta_{\begin{bmatrix}0\\2\end{bmatrix}}$$

$$+ \zeta_3^{2(1)+0} \cdot \delta_{\begin{bmatrix}1\\0\end{bmatrix}} + \zeta_3^{2(1)+1} \cdot \delta_{\begin{bmatrix}1\\1\end{bmatrix}} + \zeta_3^{2(1)+2} \cdot \delta_{\begin{bmatrix}1\\2\end{bmatrix}}$$

$$+ \zeta_3^{2(2)+0)} \cdot \delta_{\begin{bmatrix}2\\0\end{bmatrix}} + \zeta_3^{2(2)+1} \cdot \delta_{\begin{bmatrix}2\\1\end{bmatrix}} + \zeta_3^{2(2)+2} \cdot \delta_{\begin{bmatrix}2\\2\end{bmatrix}} \Bigg)$$

$$= \frac{1}{3}\Bigg(\delta_{\begin{bmatrix}0\\0\end{bmatrix}} + \zeta_3^1 \cdot \delta_{\begin{bmatrix}0\\1\end{bmatrix}} + \zeta_3^2 \cdot \delta_{\begin{bmatrix}0\\2\end{bmatrix}}$$

$$+ \zeta_3^2 \cdot \delta_{\begin{bmatrix}1\\0\end{bmatrix}} + \delta_{\begin{bmatrix}1\\1\end{bmatrix}} + \zeta_3^1 \cdot \delta_{\begin{bmatrix}1\\2\end{bmatrix}}$$

$$+ \zeta_3^1 \cdot \delta_{\begin{bmatrix}2\\0\end{bmatrix}} + \zeta_3^2 \cdot \delta_{\begin{bmatrix}2\\1\end{bmatrix}} + \delta_{\begin{bmatrix}2\\2\end{bmatrix}} \Bigg).$$

$$\begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix} \cdot (-i) \cdot \delta_{\begin{bmatrix}2\\2\end{bmatrix}} = \frac{i}{3}\sum_{y \in M} \mathbf{e}\left(\frac{2 \cdot 2 \cdot y_1 - 4 \cdot 2 \cdot y_2}{3}\right) \cdot \delta_y$$

$$= \frac{i}{3} \sum_{y \in M} \zeta_3^{4y_1 - 8y_2} \cdot \delta_y$$

$$= \frac{i}{3} \sum_{y \in M} \zeta_3^{y_1 + y_2} \cdot \delta_y$$

$$= \frac{i}{3} \Bigg(\zeta_3^0 \cdot \delta_{\begin{bmatrix} 0 \\ 0 \end{bmatrix}} + \zeta_3^{0+1} \cdot \delta_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} + \zeta_3^{0+2} \cdot \delta_{\begin{bmatrix} 0 \\ 2 \end{bmatrix}}$$

$$+ \zeta_3^{1+0} \cdot \delta_{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} + \zeta_3^{1+1} \cdot \delta_{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} + \zeta_3^{1+2} \cdot \delta_{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}$$

$$+ \zeta_3^{2+0} \cdot \delta_{\begin{bmatrix} 2 \\ 0 \end{bmatrix}} + \zeta_3^{2+1} \cdot \delta_{\begin{bmatrix} 2 \\ 1 \end{bmatrix}} + \zeta_3^{2+2} \cdot \delta_{\begin{bmatrix} 2 \\ 2 \end{bmatrix}} \Bigg)$$

$$= \frac{i}{3} \Bigg(\delta_{\begin{bmatrix} 0 \\ 0 \end{bmatrix}} + \zeta_3^1 \cdot \delta_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} + \zeta_3^2 \cdot \delta_{\begin{bmatrix} 0 \\ 2 \end{bmatrix}}$$

$$+ \zeta_3^1 \cdot \delta_{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} + \zeta_3^2 \delta_{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} + \delta_{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}$$

$$+ \zeta_3^2 \cdot \delta_{\begin{bmatrix} 2 \\ 0 \end{bmatrix}} + \delta_{\begin{bmatrix} 2 \\ 1 \end{bmatrix}} + \zeta_3^1 \cdot \delta_{\begin{bmatrix} 2 \\ 2 \end{bmatrix}} \Bigg).$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \cdot (1) \cdot \delta_{\begin{bmatrix} 2 \\ 1 \end{bmatrix}} = \frac{-1}{3} \sum_{y \in M} \mathbf{e}\left(\frac{2 \cdot 2 \cdot y_1 - 4 \cdot 1 \cdot y_2}{3}\right) \cdot \delta_y$$

$$= \frac{-1}{3} \sum_{y \in M} \zeta_3^{4y_1 - 4y_2} \cdot \delta_y$$

$$= \frac{-1}{3} \sum_{y \in M} \zeta_3^{y_1 + 2y_2} \cdot \delta_y$$

$$= \frac{-1}{3} \left(\zeta_3^0 \cdot \delta_{\begin{bmatrix} 0 \\ 0 \end{bmatrix}} + \zeta_3^{0+2(1)} \cdot \delta_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} + \zeta_3^{0+2(2)} \cdot \delta_{\begin{bmatrix} 0 \\ 2 \end{bmatrix}} \right.$$

$$+ \zeta_3^{1+2(0)} \cdot \delta_{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} + \zeta_3^{1+2(1)} \cdot \delta_{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} + \zeta_3^{1+2(2)} \cdot \delta_{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}$$

$$\left. + \zeta_3^{2+2(0)} \cdot \delta_{\begin{bmatrix} 2 \\ 0 \end{bmatrix}} + \zeta_3^{2+2(1)} \cdot \delta_{\begin{bmatrix} 2 \\ 1 \end{bmatrix}} + \zeta_3^{2+2(2)} \cdot \delta_{\begin{bmatrix} 2 \\ 2 \end{bmatrix}} \right)$$

$$= \frac{-1}{3} \left(\delta_{\begin{bmatrix} 0 \\ 0 \end{bmatrix}} + \zeta_3^2 \cdot \delta_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} + \zeta_3^1 \cdot \delta_{\begin{bmatrix} 0 \\ 2 \end{bmatrix}} \right.$$

$$+ \zeta_3^1 \cdot \delta_{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} + \delta_{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} + \zeta_3^2 \delta_{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}$$

$$+ \zeta_3^2 \cdot \delta_{\begin{bmatrix} 2 \\ 0 \end{bmatrix}} + \zeta_3^1 \delta_{\begin{bmatrix} 2 \\ 1 \end{bmatrix}} + \delta_{\begin{bmatrix} 2 \\ 2 \end{bmatrix}} \Bigg).$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} f_2(x) = \frac{-i}{3} \Bigg(\delta_{\begin{bmatrix} 0 \\ 0 \end{bmatrix}} + \zeta_3^2 \cdot \delta_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} + \zeta_3^1 \cdot \delta_{\begin{bmatrix} 0 \\ 2 \end{bmatrix}}$$

$$+ \zeta_3^2 \cdot \delta_{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} + \zeta_3^1 \cdot \delta_{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} + \delta_{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}$$

$$+ \zeta_3^1 \cdot \delta_{\begin{bmatrix} 2 \\ 0 \end{bmatrix}} + \delta_{\begin{bmatrix} 2 \\ 1 \end{bmatrix}} + \zeta_3^2 \cdot \delta_{\begin{bmatrix} 2 \\ 2 \end{bmatrix}} \Bigg)$$

$$+ \frac{1}{3} \Bigg(\delta_{\begin{bmatrix} 0 \\ 0 \end{bmatrix}} + \zeta_3^1 \cdot \delta_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} + \zeta_3^2 \cdot \delta_{\begin{bmatrix} 0 \\ 2 \end{bmatrix}}$$

$$+ \zeta_3^2 \cdot \delta_{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} + \delta_{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} + \zeta_3^1 \cdot \delta_{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}$$

$$+ \zeta_3^1 \cdot \delta_{\begin{bmatrix}2\\0\end{bmatrix}} + \zeta_3^2 \delta_{\begin{bmatrix}2\\1\end{bmatrix}} + \delta_{\begin{bmatrix}2\\2\end{bmatrix}} \Bigg)$$

$$= \frac{i}{3} \Bigg(\delta_{\begin{bmatrix}0\\0\end{bmatrix}} + \zeta_3^1 \cdot \delta_{\begin{bmatrix}0\\1\end{bmatrix}} + \zeta_3^2 \cdot \delta_{\begin{bmatrix}0\\2\end{bmatrix}}$$

$$+ \zeta_3^1 \cdot \delta_{\begin{bmatrix}1\\0\end{bmatrix}} + \zeta_3^2 \delta_{\begin{bmatrix}1\\1\end{bmatrix}} + \delta_{\begin{bmatrix}1\\2\end{bmatrix}}$$

$$+ \zeta_3^2 \cdot \delta_{\begin{bmatrix}2\\0\end{bmatrix}} + \delta_{\begin{bmatrix}2\\1\end{bmatrix}} + \zeta_3^1 \cdot \delta_{\begin{bmatrix}2\\2\end{bmatrix}} \Bigg)$$

$$= \frac{-1}{3} \Bigg(\delta_{\begin{bmatrix}0\\0\end{bmatrix}} + \zeta_3^2 \cdot \delta_{\begin{bmatrix}0\\1\end{bmatrix}} + \zeta_3^1 \cdot \delta_{\begin{bmatrix}0\\2\end{bmatrix}}$$

$$+ \zeta_3^1 \cdot \delta_{\begin{bmatrix}1\\0\end{bmatrix}} + \delta_{\begin{bmatrix}1\\1\end{bmatrix}} + \zeta_3^2 \delta_{\begin{bmatrix}1\\2\end{bmatrix}}$$

$$+ \zeta_3^2 \cdot \delta_{\begin{bmatrix}2\\0\end{bmatrix}} + \zeta_3^1 \delta_{\begin{bmatrix}2\\1\end{bmatrix}} + \delta_{\begin{bmatrix}2\\2\end{bmatrix}} \Bigg).$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} f_2(x) = \frac{1}{3}\,(0) \cdot \delta_{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}$$

$$+ \frac{1}{3}\left(-i \cdot \zeta_3^2 + \zeta_3^1 + i \cdot \zeta_3 - \zeta_3^2\right) \cdot \delta_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}$$

$$+ \frac{1}{3}\left(-i \cdot \zeta_3 + \zeta_3^2 + i \cdot \zeta_3^2 - \zeta_3\right) \cdot \delta_{\begin{bmatrix} 0 \\ 2 \end{bmatrix}}$$

$$+ \frac{1}{3}\left(-i \cdot \zeta_3^2 + \zeta_3^2 + i \cdot \zeta_3^1 - \zeta_3^1\right) \cdot \delta_{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}$$

$$+ \frac{1}{3}\left(-i \cdot \zeta_3^1 + 1 + i \cdot \zeta_3^2 - 1\right) \cdot \delta_{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

$$+ \frac{1}{3}\left(-i + \zeta_3^1 + i - \zeta_3^2\right) \cdot \delta_{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}$$

$$+ \frac{1}{3}\left(-i \cdot \zeta_3^1 + \zeta_3^1 + i \cdot \zeta_3^2 - \zeta_3^2\right) \cdot \delta_{\begin{bmatrix} 2 \\ 0 \end{bmatrix}}$$

$$+ \frac{1}{3}\left(-i + \zeta_3^2 + i - \zeta_3^1\right) \cdot \delta_{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}$$

$$+ \frac{1}{3}\left(-i \cdot \zeta_3^2 + 1 + i \cdot \zeta_3^1 - 1\right) \cdot \delta_{\begin{bmatrix} 2 \\ 2 \end{bmatrix}}.$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} f_2(x) = \left(\frac{-\sqrt{3}+i\sqrt{3}}{3}\right)\delta_{\begin{bmatrix}0\\1\end{bmatrix}} + \left(\frac{-\sqrt{3}-i\sqrt{3}}{3}\right)\delta_{\begin{bmatrix}1\\0\end{bmatrix}} + \left(\frac{\sqrt{3}-i\sqrt{3}}{3}\right)\delta_{\begin{bmatrix}0\\2\end{bmatrix}}$$
$$+ \left(\frac{\sqrt{3}+i\sqrt{3}}{3}\right)\delta_{\begin{bmatrix}2\\0\end{bmatrix}} + \sqrt{3}\cdot\delta_{\begin{bmatrix}1\\1\end{bmatrix}} + i\sqrt{3}\cdot\delta_{\begin{bmatrix}1\\2\end{bmatrix}} - \sqrt{3}\cdot\delta_{\begin{bmatrix}2\\2\end{bmatrix}} - i\sqrt{3}\cdot\delta_{\begin{bmatrix}2\\1\end{bmatrix}}$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} f_2(x) = \left(\frac{-\sqrt{3}+i\sqrt{3}}{3}\right)\left[\delta_{\begin{bmatrix}0\\1\end{bmatrix}} + i\cdot\delta_{\begin{bmatrix}1\\0\end{bmatrix}} - \delta_{\begin{bmatrix}0\\2\end{bmatrix}} - i\cdot\delta_{\begin{bmatrix}2\\0\end{bmatrix}}\right]$$
$$- i\sqrt{3}\left[i\cdot\delta_{\begin{bmatrix}1\\1\end{bmatrix}} - \delta_{\begin{bmatrix}1\\2\end{bmatrix}} - i\cdot\delta_{\begin{bmatrix}2\\2\end{bmatrix}} + \delta_{\begin{bmatrix}2\\1\end{bmatrix}}\right]$$
$$= \left(\frac{-\sqrt{3}+i\sqrt{3}}{3}\right) f_1(x) - i\sqrt{3} f_2(x).$$

So the action by $\left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$ is given by

$$\begin{bmatrix} \frac{i\sqrt{3}}{3} & \frac{-\sqrt{3}+i\sqrt{3}}{3} \\ \frac{\sqrt{3}+i\sqrt{3}}{3} & \frac{-i\sqrt{3}}{3} \end{bmatrix} = \frac{\sqrt{3}}{3}\begin{bmatrix} i & -1+i \\ i+1 & -i \end{bmatrix}$$

So by our lemma the action of $\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$ will be the conjugate transpose of the action by $\left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$:

$$N_1(\chi)(S) = \frac{\sqrt{3}}{3}\begin{bmatrix} -i & -i+1 \\ -1-i & i \end{bmatrix}$$

Then

$$\left(\frac{\sqrt{3}}{3}\begin{bmatrix} -i & -i+1 \\ -1-i & i \end{bmatrix}\right)^4 = \mathbb{1}.$$

Recalling that

$$N_1(\chi)(T) = \begin{bmatrix} \zeta_3 & 0 \\ 0 & \zeta_3^2 \end{bmatrix}.$$

We have that

$$(N_1(\chi)(S) \cdot N_1(\chi)(T))^3 = (N_1(\chi)(S))^2.$$

It agrees with the previous section.

3.11 $N_1(\chi_1)$: the Steinberg Representation by Nobs and Wolfart

Recall

$$\mathrm{Aut}(M, Q) = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \right\}.$$

Let χ_1 denote the trivial character.

$$\begin{aligned} V(\chi) &= \{f \in \mathbb{C}^M \mid f(\varepsilon x) = \chi_1(\varepsilon) f(x) \ \forall \varepsilon \in \mathrm{Aut}(M, Q), \ \forall x \in M\} \\ &= \{f \in \mathbb{C}^M \mid f(\varepsilon x) = f(x) \ \forall \varepsilon \in \mathrm{Aut}(M, Q), \ \forall x \in M\}. \end{aligned}$$

Lets try the delta functions.

$$f_0(x) = \delta_{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}(x)$$

works.

$f(x) = \delta_{\begin{bmatrix}0\\1\end{bmatrix}}(x)$ does not work for $\varepsilon = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$ and $x = \begin{bmatrix}0\\1\end{bmatrix}$.

$f(x) = \delta_{\begin{bmatrix}0\\2\end{bmatrix}}(x)$ does not work for $\varepsilon = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$ and $x = \begin{bmatrix}0\\2\end{bmatrix}$.

$f(x) = \delta_{\begin{bmatrix}1\\0\end{bmatrix}}(x)$ does not work for $\varepsilon = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$ and $x = \begin{bmatrix}1\\0\end{bmatrix}$.

$f(x) = \delta_{\begin{bmatrix}2\\0\end{bmatrix}}(x)$ does not work for $\varepsilon = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$ and $x = \begin{bmatrix}2\\0\end{bmatrix}$.

$$f_1(x) = \left(\delta_{\begin{bmatrix}1\\0\end{bmatrix}} + \delta_{\begin{bmatrix}0\\1\end{bmatrix}} + \delta_{\begin{bmatrix}2\\0\end{bmatrix}} + \delta_{\begin{bmatrix}0\\2\end{bmatrix}}\right)(x)$$

works.

$$f_2(x) = \left(\delta_{\begin{bmatrix}1\\2\end{bmatrix}} + \delta_{\begin{bmatrix}2\\1\end{bmatrix}} + \delta_{\begin{bmatrix}1\\1\end{bmatrix}} + \delta_{\begin{bmatrix}2\\2\end{bmatrix}}\right)(x)$$

also works.

Brute force computations show they work. ε sends elements of the form $\begin{bmatrix}0\\a\end{bmatrix}$ to either

$\begin{bmatrix} 0 \\ c \end{bmatrix}$ or $\begin{bmatrix} d \\ 0 \end{bmatrix}$ where a, c and $d \in \{1,2\}$. And ε sends elements of the form $\begin{bmatrix} a \\ e \end{bmatrix}$ to $\begin{bmatrix} b \\ c \end{bmatrix}$ where $a, b, c, e \in \{1,2\}$. We also exploit the property of the delta function. Now it is clear that f_0, f_1 and f_2 are linearly independent and orthogonal.

So we have an orthogonal basis. Letting $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $j = 0, 1$, and 2,

$$\begin{aligned}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} f_j(x) &= \mathbf{e}(Q(x)) f_0(x) \\ &= \exp\left(\frac{2\pi i(x_1^2 - 2x_2^2)}{3}\right) f_j(x) \\ &= \zeta_3^{x_1^2 - 2x_2^2} \cdot f_j(x).\end{aligned}$$

So,

$$\begin{aligned}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} f_0(x) &= \mathbf{e}(Q(x)) f_0(x) \\ &= \exp\left(\frac{2\pi i(x_1^2 - 2x_2^2)}{3}\right) f_0(x) \\ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \delta_{\begin{bmatrix} 0 \\ 0 \end{bmatrix}} &= \zeta_3^{0^2 - 2\cdot 0^2} \cdot \delta_{\begin{bmatrix} 0 \\ 0 \end{bmatrix}} \\ &= \delta_{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}.\end{aligned}$$

and that is equivalent to

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} f_0(x) = \cdot f_0(x).$$

Now for the action on the second basis function.

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} f_1(x) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \left(\delta_{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} + \delta_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} + \delta_{\begin{bmatrix} 2 \\ 0 \end{bmatrix}} + \delta_{\begin{bmatrix} 0 \\ 2 \end{bmatrix}} \right).$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \delta_{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} = \zeta_3^{1^2 - 2 \cdot 0^2} \cdot \delta_{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} = \zeta_3^1 \cdot \delta_{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}.$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \delta_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} = \zeta_3^{0^2 - 2 \cdot 1^2} \cdot \delta_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} = \zeta_3^1 \cdot \delta_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}.$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \delta_{\begin{bmatrix} 2 \\ 0 \end{bmatrix}} = \zeta_3^{2^2 - 2 \cdot 0^2} \cdot \delta_{\begin{bmatrix} 2 \\ 0 \end{bmatrix}} = \zeta_3^1 \cdot \delta_{\begin{bmatrix} 2 \\ 0 \end{bmatrix}}.$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \delta_{\begin{bmatrix} 0 \\ 2 \end{bmatrix}} = \zeta_3^{0^2 - 2 \cdot 2^2} \cdot \delta_{\begin{bmatrix} 0 \\ 2 \end{bmatrix}} = \zeta_3^1 \cdot \delta_{\begin{bmatrix} 0 \\ 2 \end{bmatrix}}.$$

So,

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} f_1(x) = \zeta_3 \cdot f_1(x).$$

Now for the action on the third basis function.

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} f_2(x) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \left(\delta_{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} + \delta_{\begin{bmatrix} 2 \\ 1 \end{bmatrix}} + \delta_{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} + \delta_{\begin{bmatrix} 2 \\ 2 \end{bmatrix}} \right).$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \delta_{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} = \zeta_3^{1^2 - 2 \cdot 2^2} \cdot \delta_{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} = \zeta_3^2 \cdot \delta_{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}.$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \delta_{\begin{bmatrix} 2 \\ 1 \end{bmatrix}} = \zeta_3^{2^2 - 2 \cdot 1^2} \cdot \delta_{\begin{bmatrix} 2 \\ 1 \end{bmatrix}} = \zeta_3^2 \cdot \delta_{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}.$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \delta_{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} = \zeta_3^{1^2 - 2 \cdot 1^2} \cdot \delta_{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} = \zeta_3^2 \cdot \delta_{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}.$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \delta_{\begin{bmatrix} 2 \\ 2 \end{bmatrix}} = \zeta_3^{2^2 - 2 \cdot 2^2} \cdot \delta_{\begin{bmatrix} 2 \\ 2 \end{bmatrix}} = \zeta_3^2 \cdot \delta_{\begin{bmatrix} 2 \\ 2 \end{bmatrix}}.$$

So,

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} f_2(x) = \zeta_3^2 \cdot f_2(x).$$

Hence the action of T with respect to the basis $\{f_1(x), f_2(x), f_3(x)\}$ is given by (using the notation from Nobs and Wolfart[24]):

$$N_1(\chi_1)(T) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \zeta_3 & 0 \\ 0 & 0 & \zeta_3^2 \end{bmatrix}.$$

Now to compute the action by S by computing the action of $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ first and then taking the conjugate transpose.

Theorems 2[23] and 3[24] tell us that

$$\begin{aligned}
\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} f(x) &= S_Q(-1)|M|^{-1/2} \sum_{y \in M} \mathbf{e}(B(x, y)) \cdot f(y) \\
&= (-1)^{\lambda} |M|^{-1/2} \sum_{y \in M} \mathbf{e}(B(x, y)) \cdot f(y) \\
&= \frac{-1}{3} \sum_{y \in M} \mathbf{e}\left(\frac{2x_1y_1 - 4x_2y_2}{3}\right) \cdot f(y) \\
&= \frac{-1}{3} \sum_{y \in M} \zeta_3^{2x_1y_1 - 4x_2y_2} \cdot f(y).
\end{aligned}$$

Next,

$$\begin{aligned}
\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} f_0(x) &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \delta_{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}(x) \\
&= \frac{-1}{3} \sum_{y \in M} \zeta_3^{2x_1y_1 - 4x_2y_2} \cdot \delta_y \\
&= \frac{-1}{3} \sum_{y \in M} \zeta_3^{2 \cdot 0 y_1 - 4 \cdot 0 \cdot y_2} \cdot \delta_y \\
&= \frac{-1}{3} \sum_{y \in M} \zeta_3^{0} \cdot \delta_{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}(y)
\end{aligned}$$

$$= \frac{-1}{3} \sum_{y \in M} \delta_y$$

$$= \frac{-1}{3} \left(\delta_{\begin{bmatrix} 0 \\ 0 \end{bmatrix}} + \delta_{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} + \delta_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} + \delta_{\begin{bmatrix} 2 \\ 0 \end{bmatrix}} + \delta_{\begin{bmatrix} 0 \\ 2 \end{bmatrix}} + \delta_{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} + \delta_{\begin{bmatrix} 2 \\ 1 \end{bmatrix}} + \delta_{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} + \delta_{\begin{bmatrix} 2 \\ 2 \end{bmatrix}} \right)$$

$$= \frac{-1}{3} \left(f_0 + f_1 + f_2 \right).$$

Next, we compute the action on $f_1(x)$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} f_1(x) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \left(\delta_{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}(x) + \delta_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}(x) + \delta_{\begin{bmatrix} 2 \\ 0 \end{bmatrix}}(x) + \delta_{\begin{bmatrix} 0 \\ 2 \end{bmatrix}}(x) \right).$$

piecemeal as before:

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \delta_{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}(x) = \frac{-1}{3} \sum_{y \in M} \zeta_3^{(2x_1y_1 - 4x_2y_2)} \cdot \delta_y$$

$$= \frac{-1}{3} \sum_{y \in M} \zeta_3^{2 \cdot 1 \cdot y_1 - 4 \cdot 0 \cdot y_2} \cdot \delta_y$$

$$= \frac{-1}{3} \sum_{y \in M} \zeta_3^{2y_1} \cdot \delta_y$$

$$= \frac{-1}{3} \left(\zeta_3^0 \cdot \delta_{\begin{bmatrix} 0 \\ 0 \end{bmatrix}} + \zeta_3^2 \cdot \delta_{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} + \zeta_3^0 \cdot \delta_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} + \zeta_3^1 \cdot \delta_{\begin{bmatrix} 2 \\ 0 \end{bmatrix}} + \zeta_3^0 \cdot \delta_{\begin{bmatrix} 0 \\ 2 \end{bmatrix}} \right.$$

$$+\zeta_3^2\cdot\delta_{\begin{bmatrix}1\\2\end{bmatrix}}+\zeta_3^1\cdot\delta_{\begin{bmatrix}2\\1\end{bmatrix}}+\zeta_3^2\cdot\delta_{\begin{bmatrix}1\\1\end{bmatrix}}+\zeta_3^1\cdot\delta_{\begin{bmatrix}2\\2\end{bmatrix}}\Bigg).$$

$$\begin{bmatrix}0&1\\-1&0\end{bmatrix}\delta_{\begin{bmatrix}0\\1\end{bmatrix}}(x)=\frac{-1}{3}\sum_{y\in M}\zeta_3^{(2x_1y_1-4x_2y_2)}\cdot\delta_y$$

$$=\frac{-1}{3}\sum_{y\in M}\zeta_3^{2\cdot 0\cdot y_1-4\cdot 1\cdot y_2}\cdot\delta_y$$

$$=\frac{-1}{3}\sum_{y\in M}\zeta_3^{2y_2}\cdot\delta_y$$

$$=\frac{-1}{3}\Bigg(\zeta_3^0\cdot\delta_{\begin{bmatrix}0\\0\end{bmatrix}}+\zeta_3^0\cdot\delta_{\begin{bmatrix}1\\0\end{bmatrix}}+\zeta_3^2\cdot\delta_{\begin{bmatrix}0\\1\end{bmatrix}}+\zeta_3^0\cdot\delta_{\begin{bmatrix}2\\0\end{bmatrix}}+\zeta_3^1\cdot\delta_{\begin{bmatrix}0\\2\end{bmatrix}}$$

$$+\zeta_3^1\cdot\delta_{\begin{bmatrix}1\\2\end{bmatrix}}+\zeta_3^2\cdot\delta_{\begin{bmatrix}2\\1\end{bmatrix}}+\zeta_3^2\cdot\delta_{\begin{bmatrix}1\\1\end{bmatrix}}+\zeta_3^1\cdot\delta_{\begin{bmatrix}2\\2\end{bmatrix}}\Bigg).$$

$$\begin{bmatrix}0&1\\-1&0\end{bmatrix}\delta_{\begin{bmatrix}2\\0\end{bmatrix}}(x)=\frac{-1}{3}\sum_{y\in M}\zeta_3^{(2x_1y_1-4x_2y_2)}\cdot\delta_y$$

$$=\frac{-1}{3}\sum_{y\in M}\zeta_3^{2\cdot 2\cdot y_1-4\cdot 0\cdot y_2}\cdot\delta_y$$

$$=\frac{-1}{3}\sum_{y\in M}\zeta_3^{y_1}\cdot\delta_y$$

$$= \frac{-1}{3}\left(\zeta_3^0 \cdot \delta_{\begin{bmatrix}0\\0\end{bmatrix}} + \zeta_3^1 \cdot \delta_{\begin{bmatrix}1\\0\end{bmatrix}} + \zeta_3^0 \cdot \delta_{\begin{bmatrix}0\\1\end{bmatrix}} + \zeta_3^2 \cdot \delta_{\begin{bmatrix}2\\0\end{bmatrix}} + \zeta_3^0 \cdot \delta_{\begin{bmatrix}0\\2\end{bmatrix}}\right.$$

$$\left. + \zeta_3^1 \cdot \delta_{\begin{bmatrix}1\\2\end{bmatrix}} + \zeta_3^2 \cdot \delta_{\begin{bmatrix}2\\1\end{bmatrix}} + \zeta_3^1 \cdot \delta_{\begin{bmatrix}1\\1\end{bmatrix}} + \zeta_3^2 \cdot \delta_{\begin{bmatrix}2\\2\end{bmatrix}}\right).$$

$$\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\delta_{\begin{bmatrix}0\\2\end{bmatrix}}(x) = \frac{-1}{3}\sum_{y\in M}\zeta_3^{(2x_1y_1-4x_2y_2)} \cdot \delta_y$$

$$= \frac{-1}{3}\sum_{y\in M}\zeta_3^{2\cdot 0\cdot y_1-4\cdot 2\cdot y_2} \cdot \delta_y$$

$$= \frac{-1}{3}\sum_{y\in M}\zeta_3^{y_2} \cdot \delta_y$$

$$= \frac{-1}{3}\left(\zeta_3^0 \cdot \delta_{\begin{bmatrix}0\\0\end{bmatrix}} + \zeta_3^0 \cdot \delta_{\begin{bmatrix}1\\0\end{bmatrix}} + \zeta_3^1 \cdot \delta_{\begin{bmatrix}0\\1\end{bmatrix}} + \zeta_3^0 \cdot \delta_{\begin{bmatrix}2\\0\end{bmatrix}} + \zeta_3^2 \cdot \delta_{\begin{bmatrix}0\\2\end{bmatrix}}\right.$$

$$\left. + \zeta_3^2 \cdot \delta_{\begin{bmatrix}1\\2\end{bmatrix}} + \zeta_3^1 \cdot \delta_{\begin{bmatrix}2\\1\end{bmatrix}} + \zeta_3^1 \cdot \delta_{\begin{bmatrix}1\\1\end{bmatrix}} + \zeta_3^2 \cdot \delta_{\begin{bmatrix}2\\2\end{bmatrix}}\right).$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} f_1(x) = \frac{-1}{3}\Bigg(\zeta_3^0 \cdot \delta_{\begin{bmatrix} 0 \\ 0 \end{bmatrix}} + \zeta_3^2 \cdot \delta_{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} + \zeta_3^0 \cdot \delta_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} + \zeta_3^1 \cdot \delta_{\begin{bmatrix} 2 \\ 0 \end{bmatrix}} + \zeta_3^0 \cdot \delta_{\begin{bmatrix} 0 \\ 2 \end{bmatrix}}$$

$$+ \zeta_3^2 \cdot \delta_{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} + \zeta_3^1 \cdot \delta_{\begin{bmatrix} 2 \\ 1 \end{bmatrix}} + \zeta_3^2 \cdot \delta_{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} + \zeta_3^1 \cdot \delta_{\begin{bmatrix} 2 \\ 2 \end{bmatrix}}$$

$$+ \zeta_3^0 \cdot \delta_{\begin{bmatrix} 0 \\ 0 \end{bmatrix}} + \zeta_3^0 \cdot \delta_{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} + \zeta_3^2 \cdot \delta_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} + \zeta_3^0 \cdot \delta_{\begin{bmatrix} 2 \\ 0 \end{bmatrix}} + \zeta_3^1 \cdot \delta_{\begin{bmatrix} 0 \\ 2 \end{bmatrix}}$$

$$+ \zeta_3^1 \cdot \delta_{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} + \zeta_3^2 \cdot \delta_{\begin{bmatrix} 2 \\ 1 \end{bmatrix}} + \zeta_3^2 \cdot \delta_{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} + \zeta_3^1 \cdot \delta_{\begin{bmatrix} 2 \\ 2 \end{bmatrix}}$$

$$+ \zeta_3^0 \cdot \delta_{\begin{bmatrix} 0 \\ 0 \end{bmatrix}} + \zeta_3^1 \cdot \delta_{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} + \zeta_3^0 \cdot \delta_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} + \zeta_3^2 \cdot \delta_{\begin{bmatrix} 2 \\ 0 \end{bmatrix}} + \zeta_3^0 \cdot \delta_{\begin{bmatrix} 0 \\ 2 \end{bmatrix}}$$

$$+ \zeta_3^1 \cdot \delta_{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} + \zeta_3^2 \cdot \delta_{\begin{bmatrix} 2 \\ 1 \end{bmatrix}} + \zeta_3^1 \cdot \delta_{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} + \zeta_3^2 \cdot \delta_{\begin{bmatrix} 2 \\ 2 \end{bmatrix}}$$

$$+ \zeta_3^0 \cdot \delta_{\begin{bmatrix} 0 \\ 0 \end{bmatrix}} + \zeta_3^0 \cdot \delta_{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} + \zeta_3^1 \cdot \delta_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} + \zeta_3^0 \cdot \delta_{\begin{bmatrix} 2 \\ 0 \end{bmatrix}} + \zeta_3^2 \cdot \delta_{\begin{bmatrix} 0 \\ 2 \end{bmatrix}}$$

$$+\zeta_3^2 \cdot \delta_{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} + \zeta_3^1 \cdot \delta_{\begin{bmatrix} 2 \\ 1 \end{bmatrix}} + \zeta_3^1 \cdot \delta_{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} + \zeta_3^2 \cdot \delta_{\begin{bmatrix} 2 \\ 2 \end{bmatrix}} \Bigg).$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} f_1(x) = \frac{-1}{3} \Bigg(4 \cdot \delta_{\begin{bmatrix} 0 \\ 0 \end{bmatrix}} + (\zeta_3^2 + 1 + \zeta_3 + 1) \cdot \delta_{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} + (1 + \zeta_3^2 + 1 + \zeta_3) \cdot \delta_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}$$

$$+ (\zeta_3 + 1 + \zeta_3^2 + 1) \cdot \delta_{\begin{bmatrix} 2 \\ 0 \end{bmatrix}} + (1 + \zeta_3 + 1 + \zeta_3^2) \cdot \delta_{\begin{bmatrix} 0 \\ 2 \end{bmatrix}}$$

$$+ (\zeta_3^2 + \zeta_3 + \zeta_3 + \zeta_3^2) \cdot \delta_{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} + (\zeta_3 + \zeta_3^2 + \zeta_3^2 + \zeta_3) \cdot \delta_{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}$$

$$+ (\zeta_3^2 + \zeta_3^2 + \zeta_3 + \zeta_3) \cdot \delta_{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} + (\zeta_3 + \zeta_3 + \zeta_3^2 + \zeta_3^2) \cdot \delta_{\begin{bmatrix} 2 \\ 2 \end{bmatrix}} \Bigg)$$

which simplifies to

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} f_1(x) = \frac{-1}{3} \Bigg(4 \cdot \delta_{\begin{bmatrix} 0 \\ 0 \end{bmatrix}} + (1) \cdot \delta_{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} + (1) \cdot \delta_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}$$

$$+ (1) \cdot \delta_{\begin{bmatrix} 2 \\ 0 \end{bmatrix}} + (1) \cdot \delta_{\begin{bmatrix} 0 \\ 2 \end{bmatrix}} + (-2) \cdot \delta_{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}$$

$$+ (-2)\cdot\delta_{\begin{bmatrix}2\\1\end{bmatrix}} + (-2)\cdot\delta_{\begin{bmatrix}1\\1\end{bmatrix}} + (-2)\cdot\delta_{\begin{bmatrix}2\\2\end{bmatrix}}\Bigg)$$

$$= \frac{-1}{3}(4f_0(x) + f_1(x) - 2f_2(x))$$

Next, we compute the action on $f_2(x)$

$$\begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix} f_2(x) = \begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}\left(\delta_{\begin{bmatrix}1\\2\end{bmatrix}}(x) + \delta_{\begin{bmatrix}2\\1\end{bmatrix}}(x) + \delta_{\begin{bmatrix}1\\1\end{bmatrix}}(x) + \delta_{\begin{bmatrix}2\\2\end{bmatrix}}(x)\right).$$

piecemeal as before:

$$\begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}\delta_{\begin{bmatrix}1\\2\end{bmatrix}}(x) = \frac{-1}{3}\sum_{y\in M}\zeta_3^{(2x_1y_1-4x_2y_2)}\cdot\delta_y$$

$$= \frac{-1}{3}\sum_{y\in M}\zeta_3^{2\cdot1\cdot y_1-4\cdot2\cdot y_2}\cdot\delta_y$$

$$= \frac{-1}{3}\sum_{y\in M}\zeta_3^{2y_1+y_2}\cdot\delta_y$$

$$= \frac{-1}{3}\Bigg(\zeta_3^0\cdot\delta_{\begin{bmatrix}0\\0\end{bmatrix}} + \zeta_3^2\cdot\delta_{\begin{bmatrix}1\\0\end{bmatrix}} + \zeta_3^1\cdot\delta_{\begin{bmatrix}0\\1\end{bmatrix}} + \zeta_3^1\cdot\delta_{\begin{bmatrix}2\\0\end{bmatrix}} + \zeta_3^2\cdot\delta_{\begin{bmatrix}0\\2\end{bmatrix}}$$

$$+\zeta_3^1\cdot\delta_{\begin{bmatrix}1\\2\end{bmatrix}} + \zeta_3^2\cdot\delta_{\begin{bmatrix}2\\1\end{bmatrix}} + \zeta_3^0\cdot\delta_{\begin{bmatrix}1\\1\end{bmatrix}} + \zeta_3^0\cdot\delta_{\begin{bmatrix}2\\2\end{bmatrix}}\Bigg).$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \delta_{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}(x) = \frac{-1}{3} \sum_{y \in M} \zeta_3^{(2x_1y_1 - 4x_2y_2)} \cdot \delta_y$$

$$= \frac{-1}{3} \sum_{y \in M} \zeta_3^{2 \cdot 2 \cdot y_1 - 4 \cdot 1 \cdot y_2} \cdot \delta_y$$

$$= \frac{-1}{3} \sum_{y \in M} \zeta_3^{y_1 + 2y_2} \cdot \delta_y$$

$$= \frac{-1}{3} \left(\zeta_3^0 \cdot \delta_{\begin{bmatrix} 0 \\ 0 \end{bmatrix}} + \zeta_3^1 \cdot \delta_{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} + \zeta_3^2 \cdot \delta_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} + \zeta_3^2 \cdot \delta_{\begin{bmatrix} 2 \\ 0 \end{bmatrix}} + \zeta_3^1 \cdot \delta_{\begin{bmatrix} 0 \\ 2 \end{bmatrix}} \right.$$

$$\left. + \zeta_3^2 \cdot \delta_{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} + \zeta_3^1 \cdot \delta_{\begin{bmatrix} 2 \\ 1 \end{bmatrix}} + \zeta_3^0 \cdot \delta_{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} + \zeta_3^0 \cdot \delta_{\begin{bmatrix} 2 \\ 2 \end{bmatrix}} \right).$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \delta_{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}(x) = \frac{-1}{3} \sum_{y \in M} \zeta_3^{(2x_1y_1 - 4x_2y_2)} \cdot \delta_y$$

$$= \frac{-1}{3} \sum_{y \in M} \zeta_3^{2 \cdot 1 \cdot y_1 - 4 \cdot 1 \cdot y_2} \cdot \delta_y$$

$$= \frac{-1}{3} \sum_{y \in M} \zeta_3^{2(y_1 + y_2)} \cdot \delta_y$$

$$= \frac{-1}{3} \left(\zeta_3^0 \cdot \delta_{\begin{bmatrix} 0 \\ 0 \end{bmatrix}} + \zeta_3^2 \cdot \delta_{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} + \zeta_3^2 \cdot \delta_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} + \zeta_3^1 \cdot \delta_{\begin{bmatrix} 2 \\ 0 \end{bmatrix}} + \zeta_3^1 \cdot \delta_{\begin{bmatrix} 0 \\ 2 \end{bmatrix}} \right.$$

$$+\zeta_3^0\cdot\delta_{\begin{bmatrix}1\\2\end{bmatrix}}+\zeta_3^0\cdot\delta_{\begin{bmatrix}2\\1\end{bmatrix}}+\zeta_3^1\cdot\delta_{\begin{bmatrix}1\\1\end{bmatrix}}+\zeta_3^2\cdot\delta_{\begin{bmatrix}2\\2\end{bmatrix}}\Bigg).$$

$$\begin{bmatrix}0&1\\-1&0\end{bmatrix}\delta_{\begin{bmatrix}2\\2\end{bmatrix}}(x)=\frac{-1}{3}\sum_{y\in M}\zeta_3^{(2x_1y_1-4x_2y_2)}\cdot\delta_y$$

$$=\frac{-1}{3}\sum_{y\in M}\zeta_3^{2\cdot2\cdot y_1-4\cdot2\cdot y_2}\cdot\delta_y$$

$$=\frac{-1}{3}\sum_{y\in M}\zeta_3^{y_1+y_2}\cdot\delta_y$$

$$=\frac{-1}{3}\Bigg(\zeta_3^0\cdot\delta_{\begin{bmatrix}0\\0\end{bmatrix}}+\zeta_3^1\cdot\delta_{\begin{bmatrix}1\\0\end{bmatrix}}+\zeta_3^1\cdot\delta_{\begin{bmatrix}0\\1\end{bmatrix}}+\zeta_3^2\cdot\delta_{\begin{bmatrix}2\\0\end{bmatrix}}+\zeta_3^2\cdot\delta_{\begin{bmatrix}0\\2\end{bmatrix}}$$

$$+\zeta_3^0\cdot\delta_{\begin{bmatrix}1\\2\end{bmatrix}}+\zeta_3^0\cdot\delta_{\begin{bmatrix}2\\1\end{bmatrix}}+\zeta_3^2\cdot\delta_{\begin{bmatrix}1\\1\end{bmatrix}}+\zeta_3^1\cdot\delta_{\begin{bmatrix}2\\2\end{bmatrix}}\Bigg).$$

$$\begin{bmatrix}0&1\\-1&0\end{bmatrix}f_2(x)=\frac{-1}{3}\Bigg(\zeta_3^0\cdot\delta_{\begin{bmatrix}0\\0\end{bmatrix}}+\zeta_3^2\cdot\delta_{\begin{bmatrix}1\\0\end{bmatrix}}+\zeta_3^1\cdot\delta_{\begin{bmatrix}0\\1\end{bmatrix}}+\zeta_3^1\cdot\delta_{\begin{bmatrix}2\\0\end{bmatrix}}+\zeta_3^2\cdot\delta_{\begin{bmatrix}0\\2\end{bmatrix}}$$

$$+ \zeta_3^1 \cdot \delta_{\begin{bmatrix}1\\2\end{bmatrix}} + \zeta_3^2 \cdot \delta_{\begin{bmatrix}2\\1\end{bmatrix}} + \zeta_3^0 \cdot \delta_{\begin{bmatrix}1\\1\end{bmatrix}} + \zeta_3^0 \cdot \delta_{\begin{bmatrix}2\\2\end{bmatrix}}$$

$$+ \zeta_3^0 \cdot \delta_{\begin{bmatrix}0\\0\end{bmatrix}} + \zeta_3^1 \cdot \delta_{\begin{bmatrix}1\\0\end{bmatrix}} + \zeta_3^2 \cdot \delta_{\begin{bmatrix}0\\1\end{bmatrix}} + \zeta_3^2 \cdot \delta_{\begin{bmatrix}2\\0\end{bmatrix}} + \zeta_3^1 \cdot \delta_{\begin{bmatrix}0\\2\end{bmatrix}}$$

$$+ \zeta_3^2 \cdot \delta_{\begin{bmatrix}1\\2\end{bmatrix}} + \zeta_3^1 \cdot \delta_{\begin{bmatrix}2\\1\end{bmatrix}} + \zeta_3^0 \cdot \delta_{\begin{bmatrix}1\\1\end{bmatrix}} + \zeta_3^0 \cdot \delta_{\begin{bmatrix}2\\2\end{bmatrix}}$$

$$+ \zeta_3^0 \cdot \delta_{\begin{bmatrix}0\\0\end{bmatrix}} + \zeta_3^2 \cdot \delta_{\begin{bmatrix}1\\0\end{bmatrix}} + \zeta_3^2 \cdot \delta_{\begin{bmatrix}0\\1\end{bmatrix}} + \zeta_3^1 \cdot \delta_{\begin{bmatrix}2\\0\end{bmatrix}} + \zeta_3^1 \cdot \delta_{\begin{bmatrix}0\\2\end{bmatrix}}$$

$$+ \zeta_3^0 \cdot \delta_{\begin{bmatrix}1\\2\end{bmatrix}} + \zeta_3^0 \cdot \delta_{\begin{bmatrix}2\\1\end{bmatrix}} + \zeta_3^1 \cdot \delta_{\begin{bmatrix}1\\1\end{bmatrix}} + \zeta_3^2 \cdot \delta_{\begin{bmatrix}2\\2\end{bmatrix}}$$

$$+ \zeta_3^0 \cdot \delta_{\begin{bmatrix}0\\0\end{bmatrix}} + \zeta_3^1 \cdot \delta_{\begin{bmatrix}1\\0\end{bmatrix}} + \zeta_3^1 \cdot \delta_{\begin{bmatrix}0\\1\end{bmatrix}} + \zeta_3^2 \cdot \delta_{\begin{bmatrix}2\\0\end{bmatrix}} + \zeta_3^2 \cdot \delta_{\begin{bmatrix}0\\2\end{bmatrix}}$$

$$\left. + \zeta_3^0 \cdot \delta_{\begin{bmatrix}1\\2\end{bmatrix}} + \zeta_3^0 \cdot \delta_{\begin{bmatrix}2\\1\end{bmatrix}} + \zeta_3^2 \cdot \delta_{\begin{bmatrix}1\\1\end{bmatrix}} + \zeta_3^1 \cdot \delta_{\begin{bmatrix}2\\2\end{bmatrix}} \right).$$

$$\begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix} f_2(x) = \frac{-1}{3} \left(4 \cdot \delta_{\begin{bmatrix}0\\0\end{bmatrix}} + (\zeta_3^2 + \zeta_3 + \zeta_3^2 + \zeta_3) \cdot \delta_{\begin{bmatrix}1\\0\end{bmatrix}} + (\zeta_3 + \zeta_3^2 + \zeta_3^2 + \zeta_3) \cdot \delta_{\begin{bmatrix}0\\1\end{bmatrix}} \right.$$

$$+ (\zeta_3 + \zeta_3^2 + \zeta_3 + \zeta_3^2) \cdot \delta_{\begin{bmatrix} 2 \\ 0 \end{bmatrix}} + (\zeta_3^2 + \zeta_3 + \zeta_3 + \zeta_3^2) \cdot \delta_{\begin{bmatrix} 0 \\ 2 \end{bmatrix}}$$

$$+ (\zeta_3 + \zeta_3^2 + 1 + 1) \cdot \delta_{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} + (\zeta_3^2 + \zeta_3 + 1 + 1) \cdot \delta_{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}$$

$$\left.(1 + 1 + \zeta_3^2 + \zeta_3^2) \cdot \delta_{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} + (1 + 1 + \zeta_3^2 + \zeta_3) \cdot \delta_{\begin{bmatrix} 2 \\ 2 \end{bmatrix}}\right)$$

which simplifies to

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} f_1(x) = \frac{-1}{3} \left(4 \cdot \delta_{\begin{bmatrix} 0 \\ 0 \end{bmatrix}} + (-2) \cdot \delta_{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} + (-2) \cdot \delta_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} \right.$$

$$+ (-2) \cdot \delta_{\begin{bmatrix} 2 \\ 0 \end{bmatrix}} + (-2) \cdot \delta_{\begin{bmatrix} 0 \\ 2 \end{bmatrix}} + (1) \cdot \delta_{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}$$

$$\left. + (1) \cdot \delta_{\begin{bmatrix} 2 \\ 1 \end{bmatrix}} + (1) \cdot \delta_{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} + (1) \cdot \delta_{\begin{bmatrix} 2 \\ 2 \end{bmatrix}} \right)$$

$$= \frac{-1}{3}(4f_0(x) - 2f_1(x) + f_2(x)).$$

So the action of $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ with respect to the basis $\{f_0, f_1, f_2\}$ is given by

$$\frac{-1}{3}\begin{bmatrix} 1 & 4 & 4 \\ 1 & 1 & -2 \\ 1 & -2 & 1 \end{bmatrix}.$$

So the action by S is the conjugate transpose of the matrix above:

$$N_1(\chi_1)(S) = \frac{-1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 4 & 1 & -2 \\ 4 & -2 & 1 \end{bmatrix}.$$

Its trace is -1. We verify the relations: $(N_1(\chi_1)(S))^4 = \mathbb{1}$ and $(N_1(\chi_1)(S_1)(N_1(\chi_1)(T)))^3 = (N_1(\chi_1)(S))^2$. For typesetting convenience $X = (N_1(\chi_1)(S_1)(N_1(\chi_1)(T)))^3$. Since

$$\begin{aligned}(N_1(\chi_1)(S)(N_1(\chi_1)(T)) &= \frac{-1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 4 & 1 & -2 \\ 4 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \zeta_3 & 0 \\ 0 & 0 & \zeta_3^2 \end{bmatrix} \\ &= \frac{-1}{3}\begin{bmatrix} 1 & \zeta_3 & \zeta_3^2 \\ 4 & \zeta_3 & -2\zeta_3^2 \\ 4 & -2\zeta_3 & \zeta_3^2 \end{bmatrix},\end{aligned}$$

we have

$$\begin{aligned}A &= (N_1(\chi_1)(S)(N_1(\chi_1)(T))^3 \\ &= \frac{-1}{27}\begin{bmatrix} 1 & \zeta_3 & \zeta_3^2 \\ 4 & \zeta_3 & -2\zeta_3^2 \\ 4 & -2\zeta_3 & \zeta_3^2 \end{bmatrix}\cdot\begin{bmatrix} 1 & \zeta_3 & \zeta_3^2 \\ 4 & \zeta_3 & -2\zeta_3^2 \\ 4 & -2\zeta_3 & \zeta_3^2 \end{bmatrix}\cdot\begin{bmatrix} 1 & \zeta_3 & \zeta_3^2 \\ 4 & \zeta_3 & -2\zeta_3^2 \\ 4 & -2\zeta_3 & \zeta_3^2 \end{bmatrix}\end{aligned}$$

$$= \frac{-1}{27}\begin{bmatrix} 1 & \zeta_3 & \zeta_3^2 \\ 4 & \zeta_3 & -2\zeta_3^2 \\ 4 & -2\zeta_3 & \zeta_3^2 \end{bmatrix} \cdot \begin{bmatrix} -3 & -3 & -3 \\ 4+4\zeta_3-8\zeta_3^2 & 4\zeta_3+\zeta_3^2+4 & 4\zeta_3^2-2-2\zeta_3 \\ 4-8\zeta_3+4\zeta_3^2 & 4\zeta_3-2\zeta_3^2-2 & 4\zeta_3^2+4+\zeta_3 \end{bmatrix}$$

$$= \frac{-1}{27}\begin{bmatrix} 1 & \zeta_3 & \zeta_3^2 \\ 4 & \zeta_3 & -2\zeta_3^2 \\ 4 & -2\zeta_3 & \zeta_3^2 \end{bmatrix} \cdot \begin{bmatrix} -3 & -3 & -3 \\ -12\zeta_3^2 & -3\zeta_3^2 & 6\zeta_3^2 \\ -12\zeta_3 & 6\zeta_3 & -3\zeta_3 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 1 & \zeta_3 & \zeta_3^2 \\ 4 & \zeta_3 & -2\zeta_3^2 \\ 4 & -2\zeta_3 & \zeta_3^2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 \\ 4\zeta_3^2 & \zeta_3^2 & -2\zeta_3^2 \\ 4\zeta_3 & -2\zeta_3 & \zeta_3 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 1+4+4 & 1+1-2 & 1-2+1 \\ 4+4-8 & 4+1+4 & 4-2-2 \\ 4-8+4 & 4-2-2 & 4+4+1 \end{bmatrix}$$

$$= \mathbb{1}.$$

Since

$$(N_1(\chi_1)(S))^2 = \frac{1}{9}\begin{bmatrix} 1 & 1 & 1 \\ 4 & 1 & -2 \\ 4 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 4 & 1 & -2 \\ 4 & -2 & 1 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \mathbb{1},$$

our verification is complete.

Note that the using this method, the Steingber representation will not be realizable over $\mathbb{Z}$. It is a well-known fact that the Steinberg representation is defined over $\mathbb{Z}$. So how can we realize it over $\mathbb{Z}$?

3.12 $N_1(\chi_1)$, the Steinberg representation realized over the integers

Inducing the trivial character of the Borel subgroup gives us the four dimensional reducible representation $1 + St$ for S and T:

$$(1+St)(S) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \qquad (1+St)(T) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Then using the basis

$$\{f_1 = \delta_0 - \delta_1,\ f_2 = \delta_1 - \delta_2,\ f_3 = \delta_2 - \delta_3\},$$

and SAGE, we find the three dimensional irreducible Steinberg representations of S and T:

$$St(S) = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \qquad St(T) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}.$$

3.13 The character table for $\mathrm{SL}_2(\mathbb{F}_3)$

The following table lists the characters of $\mathrm{SL}_2(\mathbb{F}_3)$. The choice of representatives are as follows: $u = T = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right]$, $u' = T^2 = \left[\begin{smallmatrix} 1 & 2 \\ 1 & 0 \end{smallmatrix}\right]$ and $s = S = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$.

Representation	I	$-I$	u	u'	$-u$	$-u'$	s
$R_1(1,-) \otimes R_1(2,-)$ $= \text{Trivial}$	1	1	1	1	1	1	1
$R_1(1,+) \cong \rho_0'$ $\cong R_+(\alpha_0)$ $\cong R_1(1,-) \otimes R_1(2,+)$	2	-2	$\frac{1}{2} + i\frac{\sqrt{3}}{2}$	$\frac{1}{2} - i\frac{\sqrt{3}}{2}$	$-\frac{1}{2} - i\frac{\sqrt{3}}{2}$	$-\frac{1}{2} + i\frac{\sqrt{3}}{2}$	0
$R_1(2,+) \cong \rho_0'' \cong R_-(\alpha_0)$ $\cong R_1(1,+) \otimes R_1(2,-)$	2	-2	$\frac{1}{2} - i\frac{\sqrt{3}}{2}$	$\frac{1}{2} + i\frac{\sqrt{3}}{2}$	$-\frac{1}{2} + i\frac{\sqrt{3}}{2}$	$-\frac{1}{2} - i\frac{\sqrt{3}}{2}$	0
$R_1(1,-) \cong \pi_0' \cong R_+'(\theta_0)$	1	1	$-\frac{1}{2} + i\frac{\sqrt{3}}{2}$	$-\frac{1}{2} - i\frac{\sqrt{3}}{2}$	$-\frac{1}{2} + i\frac{\sqrt{3}}{2}$	$-\frac{1}{2} - i\frac{\sqrt{3}}{2}$	1
$R_1(2,-) \cong \pi_0'' \cong R_-'(\theta_0)$	1	1	$-\frac{1}{2} - i\frac{\sqrt{3}}{2}$	$-\frac{1}{2} + i\frac{\sqrt{3}}{2}$	$-\frac{1}{2} - i\frac{\sqrt{3}}{2}$	$-\frac{1}{2} + i\frac{\sqrt{3}}{2}$	1
$N_1(\chi) \cong \pi_\eta \cong R'(i^\wedge)$	2	-2	-1	-1	1	1	0
$N_1(\chi_1) = St_G$	3	3	0	0	0	0	-1

Table 3.13.1: Character Table for $\mathrm{SL}_2(\mathbb{F}_3)$

This agrees with Reeder[25] and Bonnafé[4]. We combined the notations of Nobs, Reeder and Bonnafé.

CHAPTER 4 The Irreducible Representations Of SL(2,5)

4.1 The 6-dim. Irred. Principal Series Representation

We want to compute the six-dimensional irreducible representation for $\mathrm{SL}_2(\mathbb{Z}/5\mathbb{Z})$. This is a principal series representation. Since $|SL_2(\mathbb{Z}/p\mathbb{Z})| = p(p^2-1)$, $\mathrm{SL}_2(\mathbb{Z}/5\mathbb{Z})| = 5(5^2-1) = 5 \cdot 24 = 120$. Using the fact that the order of the group is equal to the sum of squares of the dimensions of the irreducible representations, we have

$$120 = 1 + 36 + 32 + 18 + 8 + 25 = 1 + 6^2 + 4^2 + 4^2 + 3^2 + 3^2 + 2^2 + 2^2 + 5^2.$$

giving us one irreducible representations of degree 1, two of degree 2, two of degree 3, two of degree four, one of degree 5, and one of degree 6. It agrees with Nobs II[24] (Nobs does not list the trivial representation). So there exists a six-dimensional irreducible representation for $\mathrm{SL}_2(\mathbb{Z}/5\mathbb{Z})$.

Representations of Level 1, $p = 5$		Degree	Number	Remarks
$D_1(\chi)$	$\chi \in \mathfrak{B}$	$p+1=6$	$\frac{1}{2}(p-3)=1$	Theorem 1
$N_1(\chi)$	$\chi \in \mathfrak{B}$	$p-1=4$	$\frac{1}{2}(p-1)=2$	Theorem 2
$R_1(1,\pm), R_1(n,\pm)$	$\left(\frac{n}{p}\right) = -1$	$\frac{p \pm 1}{2} = 3,\ 2$	4	Theorem 4
$N_1(\chi_1)$		$p = 5$	1	"Steinberg" Representation"

$D_1(\chi) \cong R_1(1,+) \oplus R_1(n,+)$ for $\chi \not\equiv 1, \chi^2 \equiv 1$,

$N_1(\chi) \cong R_1(1,-) \oplus R_1(n,-)$ for $\chi \not\equiv 1, \chi^2 \equiv 1$,

$D_1(\chi_1) \cong N_1(\chi_1) \oplus C_1 \oplus C_1$.

We are interested in $D_1(\chi)$. According to Nobs and Wolfart[24] (see §3 The Disassembled Row), D_λ is computed using $M = \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$, $Q((x_1, x_2)) = p^{-\lambda}x_1x_2$ for $\lambda \geq 1$. For our case, $\lambda = 1$, $A_\lambda = \mathbb{Z}/5\mathbb{Z}$, $M = \mathbb{Z}/5\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}$, $Q((x_1, x_2)) = 5^{-1}x_1x_2$. The bilinear form is given by

$$\begin{aligned} B(x, y) &= Q(x + y) - Q(x) - Q(y) \\ &= 5^{-1}((x_1 + y_1)(x_2 + y_2) - (x_1x_2) - (y_1y_2)) \\ &= 5^{-1}(x_1x_2 + y_1x_2 + x_1y_2 + y_1y_2 - (x_1x_2) - (y_1y_2)) \\ &= 5^{-1}(y_1x_2 + x_1y_2). \end{aligned}$$

We can use the same procedure as we used for computing $N_1(\chi)$. $\text{Aut}(M, Q)$ is the group of automorphisms of M invariant under Q, i.e., for every $\varphi \in \text{Aut}(M, Q)$, $Q(\varphi(x)) = Q(a)$ for all $a \in M$. $\text{Aut}(M, Q)$ will consist of the identity, the inverse maps, and $\kappa : (x, y) \mapsto (y, x)$, and the action by $c \in A_\lambda^\times$. The effect of $c \in A_\lambda^\times$ on M will be defined by $c : (x, y) \mapsto (c^{-1}x, cy)$. Lets enumerate all four values of c:

$c = 1$. This is the identity map. So $\mathbb{1} : (x, y) \mapsto (x, y)$. That is, $c_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

$c = 2$. $c^{-1} = 3$. So $c_2 : (x, y) \mapsto (3x, 2y)$.

$Q(c_2(x, y)) = Q((3 \cdot 2)xy/5)) = Q(xy/5) = Q(x, y)$. $c_2 = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$.

$c = 3$. $c^{-1} = 2$. So $c_3 : (x, y) \mapsto (2x, 3y)$.

$Q(c_3(x, y)) = Q((2 \cdot 3)xy/5)) = Q(xy/5) = Q(x, y)$. $c_3 = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$.

$c = 4$. $c^{-1} = 4$. So $c_4 : (x, y) \mapsto (4x, 4y)$.

$$Q(c_4(x,y)) = Q((4\cdot 4)xy/5)) = Q(xy/5) = Q(x,y).\ c_4 = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}.$$

Now to find the relations.

(1) $c_2c_3 = c_3c_2 = (6x, 6y) = (x, y)$ so $c_2 = c_3^{-1} = \mathbb{1}$.

(2) $c_2^2 : (x, y) \mapsto (9x, 4y) = (4x, 4y)$. So $c_2^2 = c_4$.

(3) $c_3^2 : (x, y) \mapsto (4x, 9y) = (4x, 4y)$. So $c_3^2 = c_4$.

(4) $c_4^2 : (x, y) \mapsto (16x, 16y) = (x, y)$. So $c_4^2 = \mathbb{1}$.

(5) $c_4^2 = \kappa^2 = \mathbb{1}$ and since $c_2^4 = c_3^2 = c_4$, $c_2^8 = c_3^4 = \mathbb{1}$.

(6) We have $(\kappa \circ c_i)(x, y) = \kappa(c_i(x, y)) = \kappa(c_i{}^{-1}x, c_iy) = (c_iy, c_i{}^{-1}x)$ and $(c_i \circ \kappa)(x, y) = c_i(y, x) = (c_i{}^{-1}y, c_ix)$, so κ does not commute with c_i.

We now rearrange the relations (so that we can use GAP or SAGE):

$$c_2^4 = c_3^4 = c_4^2 = c_2c_4c_3 = c_3c_4c_2 = c_2c_3 = c_3c_2 = \kappa^2 = c_3\kappa c_3\kappa = c_2\kappa c_2\kappa = c_4\kappa c_4\kappa = \mathbb{1}.$$

Using the following script in SAGE,

```
F.<c2,c3,c4, k>=FreeGroup()
G = F / [c2^4, c3^4, c4^2, c2*c4*c3,  c3*c4*c2, c2*c3, c3*c2, k*k,
 c3*k*c3*k, c2*k*c2*k, c4*k*c4*k ]
G.order()
G.list()
```

we see that the group is of order 8. Thus, $\mathrm{Aut}(M, Q) \cong D_8$, the dihedral group of order 8. The abelian subgroup $\mathfrak{U}$ is cyclic of order four. So $\mathfrak{U} \cong A_1^\times = (\mathbb{Z}/5\mathbb{Z})^\times \cong C_4$. $A_1^\times = \{1, 2, 3, 4\}$. The invariant subspace is given by

$$V(\chi) := \{f \in \mathbb{C}^M \mid f(ax) = \chi(a)\cdot f(x)\ \ \forall a \in A_1^\times,\ x \in M\}.$$

We define D_λ by the following operation of $\mathrm{SL}_2(A_\lambda)$ on $\mathbb{C}^M$[24]:

$$(S \cdot f)(X) := f(XS) \text{ for } S \in \mathrm{SL}_2(A_\lambda),\ X \in M$$

with X written as a row vector and XS is matrix multiplication. A basis of $V_\lambda(\chi)$ is obtained through functions f_Y[24] with

$$f_Y(X) = \begin{cases} \chi(a) & \text{for } X = aY,\ a \in A_\lambda^\times, \\ \\ 0 & \text{for } X \notin [Y], \end{cases}$$

where Y runs through a system of all straight lines in M. Note that

$$a \in \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \right\}.$$

Since $(\mathbb{Z}/5\mathbb{Z})^\times \cong \mathbb{Z}/4\mathbb{Z}$, $a \in (\mathbb{Z}/5\mathbb{Z})^\times$, we compute $\chi(a)$ as follows (a is represented by a matrix and not a scalar):

$$\chi\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 1, \quad \chi\left(\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}\right) = i, \quad \chi\left(\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}\right) = -1, \text{ and } \chi\left(\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\right) = -i.$$

We list the equivalence classes of the elements of $M = (\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z})$ below.

Verification follows.

$$(1)\ \left[\begin{bmatrix}0\\0\end{bmatrix}\right] = \left\{\begin{bmatrix}0\\0\end{bmatrix}\right\}$$

$$(2)\ \left[\begin{bmatrix}0\\1\end{bmatrix}\right] = \left\{\begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}0\\2\end{bmatrix}, \begin{bmatrix}0\\4\end{bmatrix}, \begin{bmatrix}0\\3\end{bmatrix}\right\}$$

$$(3)\ \left[\begin{bmatrix}1\\0\end{bmatrix}\right] = \left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}3\\0\end{bmatrix}, \begin{bmatrix}4\\0\end{bmatrix}, \begin{bmatrix}2\\0\end{bmatrix}\right\}$$

$$(4)\ \left[\begin{bmatrix}1\\1\end{bmatrix}\right] = \left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}3\\2\end{bmatrix}, \begin{bmatrix}4\\4\end{bmatrix}, \begin{bmatrix}2\\3\end{bmatrix}\right\}$$

$$(5)\ \left[\begin{bmatrix}1\\2\end{bmatrix}\right] = \left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}3\\4\end{bmatrix}, \begin{bmatrix}4\\3\end{bmatrix}, \begin{bmatrix}2\\1\end{bmatrix}\right\}$$

$$(6)\ \left[\begin{bmatrix}1\\3\end{bmatrix}\right] = \left\{\begin{bmatrix}1\\3\end{bmatrix}, \begin{bmatrix}3\\1\end{bmatrix}, \begin{bmatrix}4\\2\end{bmatrix}, \begin{bmatrix}2\\4\end{bmatrix}\right\}$$

$$(7)\ \left[\begin{bmatrix}1\\4\end{bmatrix}\right] = \left\{\begin{bmatrix}1\\4\end{bmatrix}, \begin{bmatrix}3\\3\end{bmatrix}, \begin{bmatrix}4\\1\end{bmatrix}, \begin{bmatrix}2\\2\end{bmatrix}\right\}.$$

Next, explicit calculations of the equivalence classes are given.

Y	$\begin{bmatrix}1&0\\0&1\end{bmatrix}\cdot Y$	$\begin{bmatrix}3&0\\0&2\end{bmatrix}\cdot Y$	$\begin{bmatrix}4&0\\0&4\end{bmatrix}\cdot Y$	$\begin{bmatrix}2&0\\0&3\end{bmatrix}\cdot Y$
$\begin{bmatrix}0\\1\end{bmatrix}$	$\begin{bmatrix}0\\1\end{bmatrix}$	$\begin{bmatrix}3&0\\0&2\end{bmatrix}\cdot\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}0\\2\end{bmatrix}$	$\begin{bmatrix}4&0\\0&4\end{bmatrix}\cdot\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}0\\4\end{bmatrix}$	$\begin{bmatrix}2&0\\0&3\end{bmatrix}\cdot\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}0\\3\end{bmatrix}$
$\begin{bmatrix}0\\2\end{bmatrix}$	$\begin{bmatrix}0\\2\end{bmatrix}$	$\begin{bmatrix}3&0\\0&2\end{bmatrix}\cdot\begin{bmatrix}0\\2\end{bmatrix}=\begin{bmatrix}0\\4\end{bmatrix}$	$\begin{bmatrix}4&0\\0&4\end{bmatrix}\cdot\begin{bmatrix}0\\2\end{bmatrix}=\begin{bmatrix}0\\3\end{bmatrix}$	$\begin{bmatrix}2&0\\0&3\end{bmatrix}\cdot\begin{bmatrix}0\\2\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix}$
$\begin{bmatrix}0\\3\end{bmatrix}$	$\begin{bmatrix}0\\3\end{bmatrix}$	$\begin{bmatrix}3&0\\0&2\end{bmatrix}\cdot\begin{bmatrix}0\\3\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix}$	$\begin{bmatrix}4&0\\0&4\end{bmatrix}\cdot\begin{bmatrix}0\\3\end{bmatrix}=\begin{bmatrix}0\\2\end{bmatrix}$	$\begin{bmatrix}2&0\\0&3\end{bmatrix}\cdot\begin{bmatrix}0\\3\end{bmatrix}=\begin{bmatrix}0\\4\end{bmatrix}$
$\begin{bmatrix}0\\4\end{bmatrix}$	$\begin{bmatrix}0\\4\end{bmatrix}$	$\begin{bmatrix}3&0\\0&2\end{bmatrix}\cdot\begin{bmatrix}0\\4\end{bmatrix}=\begin{bmatrix}0\\3\end{bmatrix}$	$\begin{bmatrix}4&0\\0&4\end{bmatrix}\cdot\begin{bmatrix}0\\4\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix}$	$\begin{bmatrix}2&0\\0&3\end{bmatrix}\cdot\begin{bmatrix}0\\4\end{bmatrix}=\begin{bmatrix}0\\2\end{bmatrix}$
$\begin{bmatrix}1\\0\end{bmatrix}$	$\begin{bmatrix}1\\0\end{bmatrix}$	$\begin{bmatrix}3&0\\0&2\end{bmatrix}\cdot\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}3\\0\end{bmatrix}$	$\begin{bmatrix}4&0\\0&4\end{bmatrix}\cdot\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}4\\0\end{bmatrix}$	$\begin{bmatrix}2&0\\0&3\end{bmatrix}\cdot\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}2\\0\end{bmatrix}$
$\begin{bmatrix}1\\1\end{bmatrix}$	$\begin{bmatrix}1\\1\end{bmatrix}$	$\begin{bmatrix}3&0\\0&2\end{bmatrix}\cdot\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}3\\2\end{bmatrix}$	$\begin{bmatrix}4&0\\0&4\end{bmatrix}\cdot\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}4\\4\end{bmatrix}$	$\begin{bmatrix}2&0\\0&3\end{bmatrix}\cdot\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}2\\3\end{bmatrix}$
$\begin{bmatrix}1\\2\end{bmatrix}$	$\begin{bmatrix}1\\2\end{bmatrix}$	$\begin{bmatrix}3&0\\0&2\end{bmatrix}\cdot\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}3\\4\end{bmatrix}$	$\begin{bmatrix}4&0\\0&4\end{bmatrix}\cdot\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}4\\3\end{bmatrix}$	$\begin{bmatrix}2&0\\0&3\end{bmatrix}\cdot\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}2\\1\end{bmatrix}$
$\begin{bmatrix}1\\3\end{bmatrix}$	$\begin{bmatrix}1\\3\end{bmatrix}$	$\begin{bmatrix}3&0\\0&2\end{bmatrix}\cdot\begin{bmatrix}1\\3\end{bmatrix}=\begin{bmatrix}3\\1\end{bmatrix}$	$\begin{bmatrix}4&0\\0&4\end{bmatrix}\cdot\begin{bmatrix}1\\3\end{bmatrix}=\begin{bmatrix}4\\2\end{bmatrix}$	$\begin{bmatrix}2&0\\0&3\end{bmatrix}\cdot\begin{bmatrix}1\\3\end{bmatrix}=\begin{bmatrix}2\\4\end{bmatrix}$

Y	$\begin{bmatrix}1&0\\0&1\end{bmatrix}\cdot Y$	$\begin{bmatrix}3&0\\0&2\end{bmatrix}\cdot Y$	$\begin{bmatrix}4&0\\0&4\end{bmatrix}\cdot Y$	$\begin{bmatrix}2&0\\0&3\end{bmatrix}\cdot Y$
$\begin{bmatrix}1\\4\end{bmatrix}$	$\begin{bmatrix}1\\4\end{bmatrix}$	$\begin{bmatrix}3&0\\0&2\end{bmatrix}\cdot\begin{bmatrix}1\\4\end{bmatrix}=\begin{bmatrix}3\\3\end{bmatrix}$	$\begin{bmatrix}4&0\\0&4\end{bmatrix}\cdot\begin{bmatrix}1\\4\end{bmatrix}=\begin{bmatrix}4\\1\end{bmatrix}$	$\begin{bmatrix}2&0\\0&3\end{bmatrix}\cdot\begin{bmatrix}1\\4\end{bmatrix}=\begin{bmatrix}2\\2\end{bmatrix}$
$\begin{bmatrix}2\\0\end{bmatrix}$	$\begin{bmatrix}2\\0\end{bmatrix}$	$\begin{bmatrix}3&0\\0&2\end{bmatrix}\cdot\begin{bmatrix}2\\0\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}$	$\begin{bmatrix}4&0\\0&4\end{bmatrix}\cdot\begin{bmatrix}2\\0\end{bmatrix}=\begin{bmatrix}3\\0\end{bmatrix}$	$\begin{bmatrix}2&0\\0&3\end{bmatrix}\cdot\begin{bmatrix}2\\0\end{bmatrix}=\begin{bmatrix}4\\0\end{bmatrix}$
$\begin{bmatrix}2\\1\end{bmatrix}$	$\begin{bmatrix}2\\1\end{bmatrix}$	$\begin{bmatrix}3&0\\0&2\end{bmatrix}\cdot\begin{bmatrix}2\\1\end{bmatrix}=\begin{bmatrix}1\\2\end{bmatrix}$	$\begin{bmatrix}4&0\\0&4\end{bmatrix}\cdot\begin{bmatrix}2\\1\end{bmatrix}=\begin{bmatrix}3\\4\end{bmatrix}$	$\begin{bmatrix}2&0\\0&3\end{bmatrix}\cdot\begin{bmatrix}2\\1\end{bmatrix}=\begin{bmatrix}4\\3\end{bmatrix}$
$\begin{bmatrix}2\\2\end{bmatrix}$	$\begin{bmatrix}2\\2\end{bmatrix}$	$\begin{bmatrix}3&0\\0&2\end{bmatrix}\cdot\begin{bmatrix}2\\2\end{bmatrix}=\begin{bmatrix}1\\4\end{bmatrix}$	$\begin{bmatrix}4&0\\0&4\end{bmatrix}\cdot\begin{bmatrix}2\\2\end{bmatrix}=\begin{bmatrix}3\\3\end{bmatrix}$	$\begin{bmatrix}2&0\\0&3\end{bmatrix}\cdot\begin{bmatrix}2\\2\end{bmatrix}=\begin{bmatrix}4\\1\end{bmatrix}$
$\begin{bmatrix}2\\3\end{bmatrix}$	$\begin{bmatrix}2\\3\end{bmatrix}$	$\begin{bmatrix}3&0\\0&2\end{bmatrix}\cdot\begin{bmatrix}2\\3\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix}$	$\begin{bmatrix}4&0\\0&4\end{bmatrix}\cdot\begin{bmatrix}2\\3\end{bmatrix}=\begin{bmatrix}3\\2\end{bmatrix}$	$\begin{bmatrix}2&0\\0&3\end{bmatrix}\cdot\begin{bmatrix}2\\3\end{bmatrix}=\begin{bmatrix}4\\4\end{bmatrix}$
$\begin{bmatrix}2\\4\end{bmatrix}$	$\begin{bmatrix}2\\4\end{bmatrix}$	$\begin{bmatrix}3&0\\0&2\end{bmatrix}\cdot\begin{bmatrix}2\\4\end{bmatrix}=\begin{bmatrix}1\\3\end{bmatrix}$	$\begin{bmatrix}4&0\\0&4\end{bmatrix}\cdot\begin{bmatrix}2\\4\end{bmatrix}=\begin{bmatrix}3\\1\end{bmatrix}$	$\begin{bmatrix}2&0\\0&3\end{bmatrix}\cdot\begin{bmatrix}2\\4\end{bmatrix}=\begin{bmatrix}4\\2\end{bmatrix}$
$\begin{bmatrix}3\\0\end{bmatrix}$	$\begin{bmatrix}3\\0\end{bmatrix}$	$\begin{bmatrix}3&0\\0&2\end{bmatrix}\cdot\begin{bmatrix}3\\0\end{bmatrix}=\begin{bmatrix}4\\0\end{bmatrix}$	$\begin{bmatrix}4&0\\0&4\end{bmatrix}\cdot\begin{bmatrix}3\\0\end{bmatrix}=\begin{bmatrix}2\\0\end{bmatrix}$	$\begin{bmatrix}2&0\\0&3\end{bmatrix}\cdot\begin{bmatrix}3\\0\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}$
$\begin{bmatrix}3\\1\end{bmatrix}$	$\begin{bmatrix}3\\1\end{bmatrix}$	$\begin{bmatrix}3&0\\0&2\end{bmatrix}\cdot\begin{bmatrix}3\\1\end{bmatrix}=\begin{bmatrix}4\\2\end{bmatrix}$	$\begin{bmatrix}4&0\\0&4\end{bmatrix}\cdot\begin{bmatrix}3\\1\end{bmatrix}=\begin{bmatrix}2\\4\end{bmatrix}$	$\begin{bmatrix}2&0\\0&3\end{bmatrix}\cdot\begin{bmatrix}3\\1\end{bmatrix}=\begin{bmatrix}1\\3\end{bmatrix}$

Y	$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\cdot Y$	$\begin{bmatrix}3 & 0\\0 & 2\end{bmatrix}\cdot Y$	$\begin{bmatrix}4 & 0\\0 & 4\end{bmatrix}\cdot Y$	$\begin{bmatrix}2 & 0\\0 & 3\end{bmatrix}\cdot Y$
$\begin{bmatrix}3\\2\end{bmatrix}$	$\begin{bmatrix}3\\2\end{bmatrix}$	$\begin{bmatrix}3 & 0\\0 & 2\end{bmatrix}\cdot\begin{bmatrix}3\\2\end{bmatrix}=\begin{bmatrix}4\\4\end{bmatrix}$	$\begin{bmatrix}4 & 0\\0 & 4\end{bmatrix}\cdot\begin{bmatrix}3\\2\end{bmatrix}=\begin{bmatrix}2\\3\end{bmatrix}$	$\begin{bmatrix}2 & 0\\0 & 3\end{bmatrix}\cdot\begin{bmatrix}3\\2\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix}$
$\begin{bmatrix}3\\3\end{bmatrix}$	$\begin{bmatrix}3\\3\end{bmatrix}$	$\begin{bmatrix}3 & 0\\0 & 2\end{bmatrix}\cdot\begin{bmatrix}3\\3\end{bmatrix}=\begin{bmatrix}4\\1\end{bmatrix}$	$\begin{bmatrix}4 & 0\\0 & 4\end{bmatrix}\cdot\begin{bmatrix}3\\3\end{bmatrix}=\begin{bmatrix}2\\2\end{bmatrix}$	$\begin{bmatrix}2 & 0\\0 & 3\end{bmatrix}\cdot\begin{bmatrix}3\\3\end{bmatrix}=\begin{bmatrix}1\\4\end{bmatrix}$
$\begin{bmatrix}3\\4\end{bmatrix}$	$\begin{bmatrix}3\\4\end{bmatrix}$	$\begin{bmatrix}3 & 0\\0 & 2\end{bmatrix}\cdot\begin{bmatrix}3\\4\end{bmatrix}=\begin{bmatrix}4\\3\end{bmatrix}$	$\begin{bmatrix}4 & 0\\0 & 4\end{bmatrix}\cdot\begin{bmatrix}3\\4\end{bmatrix}=\begin{bmatrix}2\\1\end{bmatrix}$	$\begin{bmatrix}2 & 0\\0 & 3\end{bmatrix}\cdot\begin{bmatrix}3\\4\end{bmatrix}=\begin{bmatrix}1\\2\end{bmatrix}$
$\begin{bmatrix}4\\0\end{bmatrix}$	$\begin{bmatrix}4\\0\end{bmatrix}$	$\begin{bmatrix}3 & 0\\0 & 2\end{bmatrix}\cdot\begin{bmatrix}4\\0\end{bmatrix}=\begin{bmatrix}2\\0\end{bmatrix}$	$\begin{bmatrix}4 & 0\\0 & 4\end{bmatrix}\cdot\begin{bmatrix}4\\0\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}$	$\begin{bmatrix}2 & 0\\0 & 3\end{bmatrix}\cdot\begin{bmatrix}4\\0\end{bmatrix}=\begin{bmatrix}3\\0\end{bmatrix}$
$\begin{bmatrix}4\\1\end{bmatrix}$	$\begin{bmatrix}4\\1\end{bmatrix}$	$\begin{bmatrix}3 & 0\\0 & 2\end{bmatrix}\cdot\begin{bmatrix}4\\1\end{bmatrix}=\begin{bmatrix}2\\2\end{bmatrix}$	$\begin{bmatrix}4 & 0\\0 & 4\end{bmatrix}\cdot\begin{bmatrix}4\\1\end{bmatrix}=\begin{bmatrix}1\\4\end{bmatrix}$	$\begin{bmatrix}2 & 0\\0 & 3\end{bmatrix}\cdot\begin{bmatrix}4\\1\end{bmatrix}=\begin{bmatrix}3\\3\end{bmatrix}$
$\begin{bmatrix}4\\2\end{bmatrix}$	$\begin{bmatrix}4\\2\end{bmatrix}$	$\begin{bmatrix}3 & 0\\0 & 2\end{bmatrix}\cdot\begin{bmatrix}4\\2\end{bmatrix}=\begin{bmatrix}2\\4\end{bmatrix}$	$\begin{bmatrix}4 & 0\\0 & 4\end{bmatrix}\cdot\begin{bmatrix}4\\2\end{bmatrix}=\begin{bmatrix}1\\3\end{bmatrix}$	$\begin{bmatrix}2 & 0\\0 & 3\end{bmatrix}\cdot\begin{bmatrix}4\\2\end{bmatrix}=\begin{bmatrix}3\\1\end{bmatrix}$
$\begin{bmatrix}4\\3\end{bmatrix}$	$\begin{bmatrix}4\\3\end{bmatrix}$	$\begin{bmatrix}3 & 0\\0 & 2\end{bmatrix}\cdot\begin{bmatrix}4\\3\end{bmatrix}=\begin{bmatrix}2\\1\end{bmatrix}$	$\begin{bmatrix}4 & 0\\0 & 4\end{bmatrix}\cdot\begin{bmatrix}4\\3\end{bmatrix}=\begin{bmatrix}1\\2\end{bmatrix}$	$\begin{bmatrix}2 & 0\\0 & 3\end{bmatrix}\cdot\begin{bmatrix}4\\3\end{bmatrix}=\begin{bmatrix}3\\4\end{bmatrix}$
$\begin{bmatrix}4\\4\end{bmatrix}$	$\begin{bmatrix}4\\4\end{bmatrix}$	$\begin{bmatrix}3 & 0\\0 & 2\end{bmatrix}\cdot\begin{bmatrix}4\\4\end{bmatrix}=\begin{bmatrix}2\\3\end{bmatrix}$	$\begin{bmatrix}4 & 0\\0 & 4\end{bmatrix}\cdot\begin{bmatrix}4\\4\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix}$	$\begin{bmatrix}2 & 0\\0 & 3\end{bmatrix}\cdot\begin{bmatrix}4\\4\end{bmatrix}=\begin{bmatrix}3\\2\end{bmatrix}$

4.1.1 Basis for $D_1(\lambda)$

Thus, our basis consists of functions f_Y where

$$Y \in \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \end{bmatrix} \right\}.$$

We state the functions in terms of delta functions using our tables of equivalence class calculations

$$f_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} = \delta_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} + i \cdot \delta_{\begin{bmatrix} 0 \\ 2 \end{bmatrix}} + (-1) \cdot \delta_{\begin{bmatrix} 0 \\ 4 \end{bmatrix}} + (-i) \cdot \delta_{\begin{bmatrix} 0 \\ 3 \end{bmatrix}}$$

$$f_{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} = \delta_{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} + i \cdot \delta_{\begin{bmatrix} 3 \\ 0 \end{bmatrix}} + (-1) \cdot \delta_{\begin{bmatrix} 4 \\ 0 \end{bmatrix}} + (-i) \cdot \delta_{\begin{bmatrix} 2 \\ 0 \end{bmatrix}}$$

$$f_{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} = \delta_{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} + i \cdot \delta_{\begin{bmatrix} 3 \\ 2 \end{bmatrix}} + (-1) \cdot \delta_{\begin{bmatrix} 4 \\ 4 \end{bmatrix}} + (-i) \cdot \delta_{\begin{bmatrix} 2 \\ 3 \end{bmatrix}}$$

$$f_{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} = \delta_{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} + i \cdot \delta_{\begin{bmatrix} 3 \\ 4 \end{bmatrix}} + (-1) \cdot \delta_{\begin{bmatrix} 4 \\ 3 \end{bmatrix}} + (-i) \cdot \delta_{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}$$

$$f_{\begin{bmatrix} 1 \\ 3 \end{bmatrix}} = \delta_{\begin{bmatrix} 1 \\ 3 \end{bmatrix}} + i \cdot \delta_{\begin{bmatrix} 3 \\ 1 \end{bmatrix}} + (-1) \cdot \delta_{\begin{bmatrix} 4 \\ 2 \end{bmatrix}} + (-i) \cdot \delta_{\begin{bmatrix} 2 \\ 4 \end{bmatrix}}$$

$$f_{\begin{bmatrix} 1 \\ 4 \end{bmatrix}} = \delta_{\begin{bmatrix} 1 \\ 4 \end{bmatrix}} + i \cdot \delta_{\begin{bmatrix} 3 \\ 3 \end{bmatrix}} + (-1) \cdot \delta_{\begin{bmatrix} 4 \\ 1 \end{bmatrix}} + (-i) \cdot \delta_{\begin{bmatrix} 2 \\ 2 \end{bmatrix}}.$$

4.1.2 Action of S on the delta functions

Since $|M| = 25$ and

$$\begin{aligned}
S_Q(a) &= |M|^{-1/2} \sum_{x \in M} \mathbf{e}(-aQ(x)) \\
S_Q(-1) &= (25)^{-1/2} \sum_{x \in M} \mathbf{e}(-(-1)Q(x)) \\
&= 5^{-1} \sum_{x \in M} \mathbf{e}(Q(x)) \\
&= 5^{-1}(\exp(2\pi i(0\cdot 0)/5)) + \exp(2\pi i(0\cdot 1)/5)) + \exp(2\pi i(0\cdot 2)/5)+ \\
&\quad + \exp(2\pi i(0\cdot 3)/5) + \exp(2\pi i(0\cdot 4)/5)) + \exp(2\pi i(1\cdot 0)/5)+ \\
&\quad + \exp(2\pi i(1\cdot 1)/5)) + \exp(2\pi i(1\cdot 2)/5)) + \exp(2\pi i(1\cdot 3)/5)+ \\
&\quad + \exp(2\pi i(1\cdot 4)/5)) + \exp(2\pi i(2\cdot 0)/5) + \exp(2\pi i(2\cdot 1)/5)) \\
&\quad + \exp(2\pi i(2\cdot 2)/5)) + \exp(2\pi i(2\cdot 3)/5)+ \\
&\quad + \exp(2\pi i(2\cdot 4)/5)) + \exp(2\pi i(3\cdot 0)/5) + \exp(2\pi i(3\cdot 1)/5))+ \\
&\quad + \exp(2\pi i(3\cdot 2)/5)) + \exp(2\pi i(3\cdot 3)/5)) + \exp(2\pi i(3\cdot 4)/5) \\
&\quad + \exp(2\pi i(4\cdot 0)/5) + \exp(2\pi i(4\cdot 1)/5)) + \exp(2\pi i(4\cdot 2)/5)+ \\
&\quad + \exp(2\pi i(4\cdot 3)/5) + \exp(2\pi i(4\cdot 4)/5) \\
&= 5^{-1}(1+1+1+1+1+1+\zeta_5+\zeta_5^2+\zeta_5^3+\zeta_5^4+1+\zeta_5^2+\zeta_5^4+\zeta_5+\zeta_5^3 \\
&\quad + 1+\zeta_5^3+\zeta_5+\zeta_5^4+\zeta_5^2+1+\zeta_5^4+\zeta_5^3+\zeta_5^2+\zeta_5) \\
&= 5^{-1}(5) = 1,
\end{aligned}$$

we have

$$\begin{aligned}
\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \cdot f(x) &= S_Q(-1)|M|^{-1/2} \cdot \sum_{y \in M} \mathbf{e}((B(x,y)) \cdot f(y) \\
&= 5^{-1} \cdot \sum_{y \in M} \left(\exp(2\pi i \cdot (y_1x_2 + x_1y_2)/5) \cdot f(x)\right).
\end{aligned}$$

We compute the action of $S = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$ on the delta functions.

$$\begin{aligned}
\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \cdot \delta_{\begin{bmatrix} 0 \\ 0 \end{bmatrix}} &= 5^{-1} \cdot \sum_{y \in M} \left(\exp(2\pi i (y_1 \cdot 0 + 0 \cdot y_2)/5) \cdot \delta_{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}} \right) \\
&= 5^{-1} \cdot \sum_{y \in M} \left(\delta_{\left[\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix}\right]} \right) \\
&= 5^{-1} \cdot \left(\delta_{\begin{bmatrix} 0 \\ 0 \end{bmatrix}} + \delta_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} + \delta_{\begin{bmatrix} 0 \\ 2 \end{bmatrix}} + \delta_{\begin{bmatrix} 0 \\ 3 \end{bmatrix}} + \delta_{\begin{bmatrix} 0 \\ 4 \end{bmatrix}} + \right. \\
&\qquad + \delta_{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} + \delta_{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} + \delta_{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} + \delta_{\begin{bmatrix} 1 \\ 3 \end{bmatrix}} + \delta_{\begin{bmatrix} 1 \\ 4 \end{bmatrix}} + \\
&\qquad + \delta_{\begin{bmatrix} 2 \\ 0 \end{bmatrix}} + \delta_{\begin{bmatrix} 2 \\ 1 \end{bmatrix}} + \delta_{\begin{bmatrix} 2 \\ 2 \end{bmatrix}} + \delta_{\begin{bmatrix} 2 \\ 3 \end{bmatrix}} + \delta_{\begin{bmatrix} 2 \\ 4 \end{bmatrix}} + \\
&\qquad + \delta_{\begin{bmatrix} 3 \\ 0 \end{bmatrix}} + \delta_{\begin{bmatrix} 3 \\ 1 \end{bmatrix}} + \delta_{\begin{bmatrix} 3 \\ 2 \end{bmatrix}} + \delta_{\begin{bmatrix} 3 \\ 3 \end{bmatrix}} + \delta_{\begin{bmatrix} 3 \\ 4 \end{bmatrix}} + \\
&\qquad \left. + \delta_{\begin{bmatrix} 4 \\ 0 \end{bmatrix}} + \delta_{\begin{bmatrix} 4 \\ 1 \end{bmatrix}} + \delta_{\begin{bmatrix} 4 \\ 2 \end{bmatrix}} + \delta_{\begin{bmatrix} 4 \\ 3 \end{bmatrix}} + \delta_{\begin{bmatrix} 4 \\ 4 \end{bmatrix}} \right).
\end{aligned}$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \cdot \delta_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} = 5^{-1} \cdot \sum_{y \in M} \left(\exp(2\pi i (y_1 \cdot 1 + 0 \cdot y_2)/5) \cdot \delta_{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}} \right)$$

$$= 5^{-1} \cdot \sum_{y \in M} \left(\zeta_5^{y_1} \cdot \delta_{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}} \right)$$

$$= 5^{-1} \cdot \Bigg(\zeta_5^0 \cdot \delta_{\begin{bmatrix} 0 \\ 0 \end{bmatrix}} + \zeta_5^0 \cdot \delta_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} + \zeta_5^0 \cdot \delta_{\begin{bmatrix} 0 \\ 2 \end{bmatrix}} + \zeta_5^0 \cdot \delta_{\begin{bmatrix} 0 \\ 3 \end{bmatrix}} + \zeta_5^0 \cdot \delta_{\begin{bmatrix} 0 \\ 4 \end{bmatrix}} +$$

$$+ \zeta_5^1 \cdot \delta_{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} + \zeta_5^1 \cdot \delta_{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} + \zeta_5^1 \cdot \delta_{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} + \zeta_5^1 \cdot \delta_{\begin{bmatrix} 1 \\ 3 \end{bmatrix}} + \zeta_5^1 \cdot \delta_{\begin{bmatrix} 1 \\ 4 \end{bmatrix}} +$$

$$+ \zeta_5^2 \cdot \delta_{\begin{bmatrix} 2 \\ 0 \end{bmatrix}} + \zeta_5^2 \cdot \delta_{\begin{bmatrix} 2 \\ 1 \end{bmatrix}} + \zeta_5^2 \cdot \delta_{\begin{bmatrix} 2 \\ 2 \end{bmatrix}} + \zeta_5^2 \cdot \delta_{\begin{bmatrix} 2 \\ 3 \end{bmatrix}} + \zeta_5^2 \cdot \delta_{\begin{bmatrix} 2 \\ 4 \end{bmatrix}} +$$

$$+ \zeta_5^3 \cdot \delta_{\begin{bmatrix} 3 \\ 0 \end{bmatrix}} + \zeta_5^3 \cdot \delta_{\begin{bmatrix} 3 \\ 1 \end{bmatrix}} + \zeta_5^3 \cdot \delta_{\begin{bmatrix} 3 \\ 2 \end{bmatrix}} + \zeta_5^3 \cdot \delta_{\begin{bmatrix} 3 \\ 3 \end{bmatrix}} + \zeta_5^3 \cdot \delta_{\begin{bmatrix} 3 \\ 4 \end{bmatrix}} +$$

$$+ \zeta_5^4 \cdot \delta_{\begin{bmatrix} 4 \\ 0 \end{bmatrix}} + \zeta_5^4 \cdot \delta_{\begin{bmatrix} 4 \\ 1 \end{bmatrix}} + \zeta_5^4 \cdot \delta_{\begin{bmatrix} 4 \\ 2 \end{bmatrix}} + \zeta_5^4 \cdot \delta_{\begin{bmatrix} 4 \\ 3 \end{bmatrix}} + \zeta_5^4 \cdot \delta_{\begin{bmatrix} 4 \\ 4 \end{bmatrix}} \Bigg).$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \cdot \delta_{\begin{bmatrix} 0 \\ 2 \end{bmatrix}} = 5^{-1} \cdot \sum_{y \in M} \left(\exp(2\pi i (y_1 \cdot 2 + 0 \cdot y_2)/5) \cdot \delta_{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}} \right)$$

$$= 5^{-1} \cdot \sum_{y \in M} \left(\zeta_5^{2y_1} \cdot \delta_{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}} \right)$$

$$= 5^{-1} \cdot \left(\zeta_5^0 \cdot \delta_{\begin{bmatrix} 0 \\ 0 \end{bmatrix}} + \zeta_5^0 \cdot \delta_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} + \zeta_5^0 \cdot \delta_{\begin{bmatrix} 0 \\ 2 \end{bmatrix}} + \zeta_5^0 \cdot \delta_{\begin{bmatrix} 0 \\ 3 \end{bmatrix}} + \zeta_5^0 \cdot \delta_{\begin{bmatrix} 0 \\ 4 \end{bmatrix}} + \right.$$

$$+ \zeta_5^2 \cdot \delta_{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} + \zeta_5^2 \cdot \delta_{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} + \zeta_5^2 \cdot \delta_{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} + \zeta_5^2 \cdot \delta_{\begin{bmatrix} 1 \\ 3 \end{bmatrix}} + \zeta_5^2 \cdot \delta_{\begin{bmatrix} 1 \\ 4 \end{bmatrix}} +$$

$$+ \zeta_5^4 \cdot \delta_{\begin{bmatrix} 2 \\ 0 \end{bmatrix}} + \zeta_5^4 \cdot \delta_{\begin{bmatrix} 2 \\ 1 \end{bmatrix}} + \zeta_5^4 \cdot \delta_{\begin{bmatrix} 2 \\ 2 \end{bmatrix}} + \zeta_5^4 \cdot \delta_{\begin{bmatrix} 2 \\ 3 \end{bmatrix}} + \zeta_5^4 \cdot \delta_{\begin{bmatrix} 2 \\ 4 \end{bmatrix}} +$$

$$+ \zeta_5^1 \cdot \delta_{\begin{bmatrix} 3 \\ 0 \end{bmatrix}} + \zeta_5^1 \cdot \delta_{\begin{bmatrix} 3 \\ 1 \end{bmatrix}} + \zeta_5^1 \cdot \delta_{\begin{bmatrix} 3 \\ 2 \end{bmatrix}} + \zeta_5^1 \cdot \delta_{\begin{bmatrix} 3 \\ 3 \end{bmatrix}} + \zeta_5^1 \cdot \delta_{\begin{bmatrix} 3 \\ 4 \end{bmatrix}} +$$

$$\left. + \zeta_5^3 \cdot \delta_{\begin{bmatrix} 4 \\ 0 \end{bmatrix}} + \zeta_5^3 \cdot \delta_{\begin{bmatrix} 4 \\ 1 \end{bmatrix}} + \zeta_5^3 \cdot \delta_{\begin{bmatrix} 4 \\ 2 \end{bmatrix}} + \zeta_5^3 \cdot \delta_{\begin{bmatrix} 4 \\ 3 \end{bmatrix}} + \zeta_5^3 \cdot \delta_{\begin{bmatrix} 4 \\ 4 \end{bmatrix}} \right).$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \cdot \delta_{\begin{bmatrix} 0 \\ 3 \end{bmatrix}} = 5^{-1} \cdot \sum_{y \in M} \left(\exp(2\pi i(y_1 \cdot 3 + 0 \cdot y_2)/5) \cdot \delta_{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}} \right)$$

$$= 5^{-1} \cdot \sum_{y \in M} \left(\zeta_5^{3y_1} \cdot \delta_{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}} \right)$$

$$= 5^{-1} \cdot \Bigg(\zeta_5^0 \cdot \delta_{\begin{bmatrix} 0 \\ 0 \end{bmatrix}} + \zeta_5^0 \cdot \delta_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} + \zeta_5^0 \cdot \delta_{\begin{bmatrix} 0 \\ 2 \end{bmatrix}} + \zeta_5^0 \cdot \delta_{\begin{bmatrix} 0 \\ 3 \end{bmatrix}} + \zeta_5^0 \cdot \delta_{\begin{bmatrix} 0 \\ 4 \end{bmatrix}} +$$

$$+ \zeta_5^3 \cdot \delta_{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} + \zeta_5^3 \cdot \delta_{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} + \zeta_5^3 \cdot \delta_{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} + \zeta_5^3 \cdot \delta_{\begin{bmatrix} 1 \\ 3 \end{bmatrix}} + \zeta_5^3 \cdot \delta_{\begin{bmatrix} 1 \\ 4 \end{bmatrix}} +$$

$$+ \zeta_5^1 \cdot \delta_{\begin{bmatrix} 2 \\ 0 \end{bmatrix}} + \zeta_5^1 \cdot \delta_{\begin{bmatrix} 2 \\ 1 \end{bmatrix}} + \zeta_5^1 \cdot \delta_{\begin{bmatrix} 2 \\ 2 \end{bmatrix}} + \zeta_5^1 \cdot \delta_{\begin{bmatrix} 2 \\ 3 \end{bmatrix}} + \zeta_5^1 \cdot \delta_{\begin{bmatrix} 2 \\ 4 \end{bmatrix}} +$$

$$+ \zeta_5^4 \cdot \delta_{\begin{bmatrix} 3 \\ 0 \end{bmatrix}} + \zeta_5^4 \cdot \delta_{\begin{bmatrix} 3 \\ 1 \end{bmatrix}} + \zeta_5^4 \cdot \delta_{\begin{bmatrix} 3 \\ 2 \end{bmatrix}} + \zeta_5^4 \cdot \delta_{\begin{bmatrix} 3 \\ 3 \end{bmatrix}} + \zeta_5^4 \cdot \delta_{\begin{bmatrix} 3 \\ 4 \end{bmatrix}} +$$

$$+ \zeta_5^2 \cdot \delta_{\begin{bmatrix} 4 \\ 0 \end{bmatrix}} + \zeta_5^2 \cdot \delta_{\begin{bmatrix} 4 \\ 1 \end{bmatrix}} + \zeta_5^2 \cdot \delta_{\begin{bmatrix} 4 \\ 2 \end{bmatrix}} + \zeta_5^2 \cdot \delta_{\begin{bmatrix} 4 \\ 3 \end{bmatrix}} + \zeta_5^2 \cdot \delta_{\begin{bmatrix} 4 \\ 4 \end{bmatrix}} \Bigg).$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \cdot \delta_{\begin{bmatrix} 0 \\ 4 \end{bmatrix}} = 5^{-1} \cdot \sum_{y \in M} \left(\exp(2\pi i(y_1 \cdot 4 + 0 \cdot y_2)/5) \cdot \delta_{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}} \right)$$

$$= 5^{-1} \cdot \sum_{y \in M} \left(\zeta_5^{4y_1} \cdot \delta_{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}} \right)$$

$$= 5^{-1} \cdot \Bigg(\zeta_5^{0} \cdot \delta_{\begin{bmatrix} 0 \\ 0 \end{bmatrix}} + \zeta_5^{0} \cdot \delta_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} + \zeta_5^{0} \cdot \delta_{\begin{bmatrix} 0 \\ 2 \end{bmatrix}} + \zeta_5^{0} \cdot \delta_{\begin{bmatrix} 0 \\ 3 \end{bmatrix}} + \zeta_5^{0} \cdot \delta_{\begin{bmatrix} 0 \\ 4 \end{bmatrix}} +$$

$$+ \zeta_5^{4} \cdot \delta_{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} + \zeta_5^{4} \cdot \delta_{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} + \zeta_5^{4} \cdot \delta_{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} + \zeta_5^{4} \cdot \delta_{\begin{bmatrix} 1 \\ 3 \end{bmatrix}} + \zeta_5^{4} \cdot \delta_{\begin{bmatrix} 1 \\ 4 \end{bmatrix}} +$$

$$+ \zeta_5^{3} \cdot \delta_{\begin{bmatrix} 2 \\ 0 \end{bmatrix}} + \zeta_5^{3} \cdot \delta_{\begin{bmatrix} 2 \\ 1 \end{bmatrix}} + \zeta_5^{3} \cdot \delta_{\begin{bmatrix} 2 \\ 2 \end{bmatrix}} + \zeta_5^{3} \cdot \delta_{\begin{bmatrix} 2 \\ 3 \end{bmatrix}} + \zeta_5^{3} \cdot \delta_{\begin{bmatrix} 2 \\ 4 \end{bmatrix}} +$$

$$+ \zeta_5^{2} \cdot \delta_{\begin{bmatrix} 3 \\ 0 \end{bmatrix}} + \zeta_5^{2} \cdot \delta_{\begin{bmatrix} 3 \\ 1 \end{bmatrix}} + \zeta_5^{2} \cdot \delta_{\begin{bmatrix} 3 \\ 2 \end{bmatrix}} + \zeta_5^{2} \cdot \delta_{\begin{bmatrix} 3 \\ 3 \end{bmatrix}} + \zeta_5^{2} \cdot \delta_{\begin{bmatrix} 3 \\ 4 \end{bmatrix}} +$$

$$+ \zeta_5^{1} \cdot \delta_{\begin{bmatrix} 4 \\ 0 \end{bmatrix}} + \zeta_5^{1} \cdot \delta_{\begin{bmatrix} 4 \\ 1 \end{bmatrix}} + \zeta_5^{1} \cdot \delta_{\begin{bmatrix} 4 \\ 2 \end{bmatrix}} + \zeta_5^{1} \cdot \delta_{\begin{bmatrix} 4 \\ 3 \end{bmatrix}} + \zeta_5^{1} \cdot \delta_{\begin{bmatrix} 4 \\ 4 \end{bmatrix}} \Bigg).$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \cdot \delta_{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} = 5^{-1} \cdot \sum_{y \in M} \left(\exp(2\pi i (y_1 \cdot 0 + 1 \cdot y_2)/5) \cdot \delta_{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}} \right)$$

$$= 5^{-1} \cdot \sum_{y \in M} \left(\zeta_5^{y_2} \cdot \delta_{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}} \right)$$

$$= 5^{-1} \cdot \left(\zeta_5^0 \cdot \delta_{\begin{bmatrix} 0 \\ 0 \end{bmatrix}} + \zeta_5^1 \cdot \delta_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} + \zeta_5^2 \cdot \delta_{\begin{bmatrix} 0 \\ 2 \end{bmatrix}} + \zeta_5^3 \cdot \delta_{\begin{bmatrix} 0 \\ 3 \end{bmatrix}} + \zeta_5^4 \cdot \delta_{\begin{bmatrix} 0 \\ 4 \end{bmatrix}} + \right.$$

$$+ \zeta_5^0 \cdot \delta_{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} + \zeta_5^1 \cdot \delta_{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} + \zeta_5^2 \cdot \delta_{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} + \zeta_5^3 \cdot \delta_{\begin{bmatrix} 1 \\ 3 \end{bmatrix}} + \zeta_5^4 \cdot \delta_{\begin{bmatrix} 1 \\ 4 \end{bmatrix}} +$$

$$+ \zeta_5^0 \cdot \delta_{\begin{bmatrix} 2 \\ 0 \end{bmatrix}} + \zeta_5^1 \cdot \delta_{\begin{bmatrix} 2 \\ 1 \end{bmatrix}} + \zeta_5^2 \cdot \delta_{\begin{bmatrix} 2 \\ 2 \end{bmatrix}} + \zeta_5^3 \cdot \delta_{\begin{bmatrix} 2 \\ 3 \end{bmatrix}} + \zeta_5^4 \cdot \delta_{\begin{bmatrix} 2 \\ 4 \end{bmatrix}} +$$

$$+ \zeta_5^0 \cdot \delta_{\begin{bmatrix} 3 \\ 0 \end{bmatrix}} + \zeta_5^1 \cdot \delta_{\begin{bmatrix} 3 \\ 1 \end{bmatrix}} + \zeta_5^2 \cdot \delta_{\begin{bmatrix} 3 \\ 2 \end{bmatrix}} + \zeta_5^3 \cdot \delta_{\begin{bmatrix} 3 \\ 3 \end{bmatrix}} + \zeta_5^4 \cdot \delta_{\begin{bmatrix} 3 \\ 4 \end{bmatrix}} +$$

$$\left. + \zeta_5^0 \cdot \delta_{\begin{bmatrix} 4 \\ 0 \end{bmatrix}} + \zeta_5^1 \cdot \delta_{\begin{bmatrix} 4 \\ 1 \end{bmatrix}} + \zeta_5^2 \cdot \delta_{\begin{bmatrix} 4 \\ 2 \end{bmatrix}} + \zeta_5^3 \cdot \delta_{\begin{bmatrix} 4 \\ 3 \end{bmatrix}} + \zeta_5^4 \cdot \delta_{\begin{bmatrix} 4 \\ 4 \end{bmatrix}} \right).$$

$$\begin{aligned}
\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \cdot \delta_{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} &= 5^{-1} \cdot \sum_{y \in M} \left(\exp(2\pi i (y_1 \cdot 1 + 1 \cdot y_2)/5) \cdot \delta_{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}} \right) \\
&= 5^{-1} \cdot \sum_{y \in M} \left(\zeta_5^{y_1 + y_2} \cdot \delta_{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}} \right) \\
&= 5^{-1} \cdot \Bigg(\zeta_5^0 \cdot \delta_{\begin{bmatrix} 0 \\ 0 \end{bmatrix}} + \zeta_5^1 \cdot \delta_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} + \zeta_5^2 \cdot \delta_{\begin{bmatrix} 0 \\ 2 \end{bmatrix}} + \zeta_5^3 \cdot \delta_{\begin{bmatrix} 0 \\ 3 \end{bmatrix}} + \zeta_5^4 \cdot \delta_{\begin{bmatrix} 0 \\ 4 \end{bmatrix}} + \\
&\qquad + \zeta_5^1 \cdot \delta_{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} + \zeta_5^2 \cdot \delta_{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} + \zeta_5^3 \cdot \delta_{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} + \zeta_5^4 \cdot \delta_{\begin{bmatrix} 1 \\ 3 \end{bmatrix}} + \zeta_5^0 \cdot \delta_{\begin{bmatrix} 1 \\ 4 \end{bmatrix}} + \\
&\qquad + \zeta_5^2 \cdot \delta_{\begin{bmatrix} 2 \\ 0 \end{bmatrix}} + \zeta_5^3 \cdot \delta_{\begin{bmatrix} 2 \\ 1 \end{bmatrix}} + \zeta_5^4 \cdot \delta_{\begin{bmatrix} 2 \\ 2 \end{bmatrix}} + \zeta_5^5 \cdot \delta_{\begin{bmatrix} 2 \\ 3 \end{bmatrix}} + \zeta_5^0 \cdot \delta_{\begin{bmatrix} 2 \\ 4 \end{bmatrix}} + \\
&\qquad + \zeta_5^3 \cdot \delta_{\begin{bmatrix} 3 \\ 0 \end{bmatrix}} + \zeta_5^4 \cdot \delta_{\begin{bmatrix} 3 \\ 1 \end{bmatrix}} + \zeta_5^5 \cdot \delta_{\begin{bmatrix} 3 \\ 2 \end{bmatrix}} + \zeta_5^0 \cdot \delta_{\begin{bmatrix} 3 \\ 3 \end{bmatrix}} + \zeta_5^1 \cdot \delta_{\begin{bmatrix} 3 \\ 4 \end{bmatrix}} + \\
&\qquad + \zeta_5^4 \cdot \delta_{\begin{bmatrix} 4 \\ 0 \end{bmatrix}} + \zeta_5^5 \cdot \delta_{\begin{bmatrix} 4 \\ 1 \end{bmatrix}} + \zeta_5^0 \cdot \delta_{\begin{bmatrix} 4 \\ 2 \end{bmatrix}} + \zeta_5^1 \cdot \delta_{\begin{bmatrix} 4 \\ 3 \end{bmatrix}} + \zeta_5^2 \cdot \delta_{\begin{bmatrix} 4 \\ 4 \end{bmatrix}} \Bigg).
\end{aligned}$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \cdot \delta_{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} = 5^{-1} \cdot \sum_{y \in M} \left(\exp(2\pi i (y_1 \cdot 2 + 1 \cdot y_2)/5) \cdot \delta_{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}} \right)$$

$$= 5^{-1} \cdot \sum_{y \in M} \left(\zeta_5^{2y_1 + y_2} \cdot \delta_{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}} \right)$$

$$= 5^{-1} \cdot \left(\zeta_5^0 \cdot \delta_{\begin{bmatrix} 0 \\ 0 \end{bmatrix}} + \zeta_5^1 \cdot \delta_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} + \zeta_5^2 \cdot \delta_{\begin{bmatrix} 0 \\ 2 \end{bmatrix}} + \zeta_5^3 \cdot \delta_{\begin{bmatrix} 0 \\ 3 \end{bmatrix}} + \zeta_5^4 \cdot \delta_{\begin{bmatrix} 0 \\ 4 \end{bmatrix}} + \right.$$

$$+ \zeta_5^2 \cdot \delta_{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} + \zeta_5^3 \cdot \delta_{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} + \zeta_5^4 \cdot \delta_{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} + \zeta_5^0 \cdot \delta_{\begin{bmatrix} 1 \\ 3 \end{bmatrix}} + \zeta_5^1 \cdot \delta_{\begin{bmatrix} 1 \\ 4 \end{bmatrix}} +$$

$$+ \zeta_5^4 \cdot \delta_{\begin{bmatrix} 2 \\ 0 \end{bmatrix}} + \zeta_5^0 \cdot \delta_{\begin{bmatrix} 2 \\ 1 \end{bmatrix}} + \zeta_5^1 \cdot \delta_{\begin{bmatrix} 2 \\ 2 \end{bmatrix}} + \zeta_5^2 \cdot \delta_{\begin{bmatrix} 2 \\ 3 \end{bmatrix}} + \zeta_5^3 \cdot \delta_{\begin{bmatrix} 2 \\ 4 \end{bmatrix}} +$$

$$+ \zeta_5^1 \cdot \delta_{\begin{bmatrix} 3 \\ 0 \end{bmatrix}} + \zeta_5^2 \cdot \delta_{\begin{bmatrix} 3 \\ 1 \end{bmatrix}} + \zeta_5^3 \cdot \delta_{\begin{bmatrix} 3 \\ 2 \end{bmatrix}} + \zeta_5^4 \cdot \delta_{\begin{bmatrix} 3 \\ 3 \end{bmatrix}} + \zeta_5^0 \cdot \delta_{\begin{bmatrix} 3 \\ 4 \end{bmatrix}} +$$

$$\left. + \zeta_5^3 \cdot \delta_{\begin{bmatrix} 4 \\ 0 \end{bmatrix}} + \zeta_5^4 \cdot \delta_{\begin{bmatrix} 4 \\ 1 \end{bmatrix}} + \zeta_5^0 \cdot \delta_{\begin{bmatrix} 4 \\ 2 \end{bmatrix}} + \zeta_5^1 \cdot \delta_{\begin{bmatrix} 4 \\ 3 \end{bmatrix}} + \zeta_5^2 \cdot \delta_{\begin{bmatrix} 4 \\ 4 \end{bmatrix}} \right).$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \cdot \delta_{\begin{bmatrix} 1 \\ 3 \end{bmatrix}} = 5^{-1} \cdot \sum_{y \in M} \left(\exp(2\pi i (y_1 \cdot 3 + 1 \cdot y_2)/5) \cdot \delta_{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}} \right)$$

$$= 5^{-1} \cdot \sum_{y \in M} \left(\zeta_5^{3y_1 + y_2} \cdot \delta_{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}} \right)$$

$$= 5^{-1} \cdot \Bigg(\zeta_5^0 \cdot \delta_{\begin{bmatrix} 0 \\ 0 \end{bmatrix}} + \zeta_5^1 \cdot \delta_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} + \zeta_5^2 \cdot \delta_{\begin{bmatrix} 0 \\ 2 \end{bmatrix}} + \zeta_5^3 \cdot \delta_{\begin{bmatrix} 0 \\ 3 \end{bmatrix}} + \zeta_5^4 \cdot \delta_{\begin{bmatrix} 0 \\ 4 \end{bmatrix}} +$$

$$+ \zeta_5^3 \cdot \delta_{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} + \zeta_5^4 \cdot \delta_{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} + \zeta_5^0 \cdot \delta_{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} + \zeta_5^1 \cdot \delta_{\begin{bmatrix} 1 \\ 3 \end{bmatrix}} + \zeta_5^2 \cdot \delta_{\begin{bmatrix} 1 \\ 4 \end{bmatrix}} +$$

$$+ \zeta_5^1 \cdot \delta_{\begin{bmatrix} 2 \\ 0 \end{bmatrix}} + \zeta_5^1 \cdot \delta_{\begin{bmatrix} 2 \\ 1 \end{bmatrix}} + \zeta_5^2 \cdot \delta_{\begin{bmatrix} 2 \\ 2 \end{bmatrix}} + \zeta_5^3 \cdot \delta_{\begin{bmatrix} 2 \\ 3 \end{bmatrix}} + \zeta_5^4 \cdot \delta_{\begin{bmatrix} 2 \\ 4 \end{bmatrix}} +$$

$$+ \zeta_5^4 \cdot \delta_{\begin{bmatrix} 3 \\ 0 \end{bmatrix}} + \zeta_5^0 \cdot \delta_{\begin{bmatrix} 3 \\ 1 \end{bmatrix}} + \zeta_5^1 \cdot \delta_{\begin{bmatrix} 3 \\ 2 \end{bmatrix}} + \zeta_5^2 \cdot \delta_{\begin{bmatrix} 3 \\ 3 \end{bmatrix}} + \zeta_5^3 \cdot \delta_{\begin{bmatrix} 3 \\ 4 \end{bmatrix}} +$$

$$+ \zeta_5^2 \cdot \delta_{\begin{bmatrix} 4 \\ 0 \end{bmatrix}} + \zeta_5^3 \cdot \delta_{\begin{bmatrix} 4 \\ 1 \end{bmatrix}} + \zeta_5^4 \cdot \delta_{\begin{bmatrix} 4 \\ 2 \end{bmatrix}} + \zeta_5^0 \cdot \delta_{\begin{bmatrix} 4 \\ 3 \end{bmatrix}} + \zeta_5^1 \cdot \delta_{\begin{bmatrix} 4 \\ 4 \end{bmatrix}} \Bigg).$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \cdot \delta_{\begin{bmatrix} 1 \\ 4 \end{bmatrix}} = 5^{-1} \cdot \sum_{y \in M} \left(\exp(2\pi i (y_1 \cdot 4 + 1 \cdot y_2)/5) \cdot \delta_{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}} \right)$$

$$= 5^{-1} \cdot \sum_{y \in M} \left(\zeta_5^{4y_1 + y_2} \cdot \delta_{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}} \right)$$

$$= 5^{-1} \cdot \left(\zeta_5^{0} \cdot \delta_{\begin{bmatrix} 0 \\ 0 \end{bmatrix}} + \zeta_5^{1} \cdot \delta_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} + \zeta_5^{2} \cdot \delta_{\begin{bmatrix} 0 \\ 2 \end{bmatrix}} + \zeta_5^{3} \cdot \delta_{\begin{bmatrix} 0 \\ 3 \end{bmatrix}} + \zeta_5^{4} \cdot \delta_{\begin{bmatrix} 0 \\ 4 \end{bmatrix}} + \right.$$

$$+ \zeta_5^{4} \cdot \delta_{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} + \zeta_5^{0} \cdot \delta_{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} + \zeta_5^{1} \cdot \delta_{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} + \zeta_5^{2} \cdot \delta_{\begin{bmatrix} 1 \\ 3 \end{bmatrix}} + \zeta_5^{3} \cdot \delta_{\begin{bmatrix} 1 \\ 4 \end{bmatrix}} +$$

$$+ \zeta_5^{3} \cdot \delta_{\begin{bmatrix} 2 \\ 0 \end{bmatrix}} + \zeta_5^{4} \cdot \delta_{\begin{bmatrix} 2 \\ 1 \end{bmatrix}} + \zeta_5^{0} \cdot \delta_{\begin{bmatrix} 2 \\ 2 \end{bmatrix}} + \zeta_5^{1} \cdot \delta_{\begin{bmatrix} 2 \\ 3 \end{bmatrix}} + \zeta_5^{2} \cdot \delta_{\begin{bmatrix} 2 \\ 4 \end{bmatrix}} +$$

$$+ \zeta_5^{2} \cdot \delta_{\begin{bmatrix} 3 \\ 0 \end{bmatrix}} + \zeta_5^{3} \cdot \delta_{\begin{bmatrix} 3 \\ 1 \end{bmatrix}} + \zeta_5^{4} \cdot \delta_{\begin{bmatrix} 3 \\ 2 \end{bmatrix}} + \zeta_5^{0} \cdot \delta_{\begin{bmatrix} 3 \\ 3 \end{bmatrix}} + \zeta_5^{1} \cdot \delta_{\begin{bmatrix} 3 \\ 4 \end{bmatrix}} +$$

$$\left. + \zeta_5^{1} \cdot \delta_{\begin{bmatrix} 4 \\ 0 \end{bmatrix}} + \zeta_5^{2} \cdot \delta_{\begin{bmatrix} 4 \\ 1 \end{bmatrix}} + \zeta_5^{3} \cdot \delta_{\begin{bmatrix} 4 \\ 2 \end{bmatrix}} + \zeta_5^{4} \cdot \delta_{\begin{bmatrix} 4 \\ 3 \end{bmatrix}} + \zeta_5^{0} \cdot \delta_{\begin{bmatrix} 4 \\ 4 \end{bmatrix}} \right).$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \cdot \delta_{\begin{bmatrix} 2 \\ 0 \end{bmatrix}} = 5^{-1} \cdot \sum_{y \in M} \left(\exp(2\pi i(y_1 \cdot 0 + 2 \cdot y_2)/5) \cdot \delta_{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}} \right)$$

$$= 5^{-1} \cdot \sum_{y \in M} \left(\zeta_5^{2y_2} \cdot \delta_{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}} \right)$$

$$= 5^{-1} \cdot \Bigg(\zeta_5^0 \cdot \delta_{\begin{bmatrix} 0 \\ 0 \end{bmatrix}} + \zeta_5^2 \cdot \delta_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} + \zeta_5^4 \cdot \delta_{\begin{bmatrix} 0 \\ 2 \end{bmatrix}} + \zeta_5^1 \cdot \delta_{\begin{bmatrix} 0 \\ 3 \end{bmatrix}} + \zeta_5^3 \cdot \delta_{\begin{bmatrix} 0 \\ 4 \end{bmatrix}} +$$

$$+ \zeta_5^0 \cdot \delta_{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} + \zeta_5^2 \cdot \delta_{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} + \zeta_5^4 \cdot \delta_{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} + \zeta_5^1 \cdot \delta_{\begin{bmatrix} 1 \\ 3 \end{bmatrix}} + \zeta_5^3 \cdot \delta_{\begin{bmatrix} 1 \\ 4 \end{bmatrix}} +$$

$$+ \zeta_5^0 \cdot \delta_{\begin{bmatrix} 2 \\ 0 \end{bmatrix}} + \zeta_5^2 \cdot \delta_{\begin{bmatrix} 2 \\ 1 \end{bmatrix}} + \zeta_5^4 \cdot \delta_{\begin{bmatrix} 2 \\ 2 \end{bmatrix}} + \zeta_5^1 \cdot \delta_{\begin{bmatrix} 2 \\ 3 \end{bmatrix}} + \zeta_5^3 \cdot \delta_{\begin{bmatrix} 2 \\ 4 \end{bmatrix}} +$$

$$+ \zeta_5^0 \cdot \delta_{\begin{bmatrix} 3 \\ 0 \end{bmatrix}} + \zeta_5^2 \cdot \delta_{\begin{bmatrix} 3 \\ 1 \end{bmatrix}} + \zeta_5^4 \cdot \delta_{\begin{bmatrix} 3 \\ 2 \end{bmatrix}} + \zeta_5^1 \cdot \delta_{\begin{bmatrix} 3 \\ 3 \end{bmatrix}} + \zeta_5^3 \cdot \delta_{\begin{bmatrix} 3 \\ 4 \end{bmatrix}} +$$

$$+ \zeta_5^0 \cdot \delta_{\begin{bmatrix} 4 \\ 0 \end{bmatrix}} + \zeta_5^2 \cdot \delta_{\begin{bmatrix} 4 \\ 1 \end{bmatrix}} + \zeta_5^4 \cdot \delta_{\begin{bmatrix} 4 \\ 2 \end{bmatrix}} + \zeta_5^1 \cdot \delta_{\begin{bmatrix} 4 \\ 3 \end{bmatrix}} + \zeta_5^3 \cdot \delta_{\begin{bmatrix} 4 \\ 4 \end{bmatrix}} \Bigg).$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \cdot \delta_{\begin{bmatrix} 2 \\ 1 \end{bmatrix}} = 5^{-1} \cdot \sum_{y \in M} \left(\exp(2\pi i(y_1 \cdot 1 + 2 \cdot y_2)/5) \cdot \delta_{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}} \right)$$

$$= 5^{-1} \cdot \sum_{y \in M} \left(\zeta_5^{y_1+2y_2} \cdot \delta_{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}} \right)$$

$$= 5^{-1} \cdot \Bigg(\zeta_5^0 \cdot \delta_{\begin{bmatrix} 0 \\ 0 \end{bmatrix}} + \zeta_5^2 \cdot \delta_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} + \zeta_5^4 \cdot \delta_{\begin{bmatrix} 0 \\ 2 \end{bmatrix}} + \zeta_5^1 \cdot \delta_{\begin{bmatrix} 0 \\ 3 \end{bmatrix}} + \zeta_5^3 \cdot \delta_{\begin{bmatrix} 0 \\ 4 \end{bmatrix}} +$$

$$+ \zeta_5^1 \cdot \delta_{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} + \zeta_5^3 \cdot \delta_{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} + \zeta_5^0 \cdot \delta_{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} + \zeta_5^2 \cdot \delta_{\begin{bmatrix} 1 \\ 3 \end{bmatrix}} + \zeta_5^4 \cdot \delta_{\begin{bmatrix} 1 \\ 4 \end{bmatrix}} +$$

$$+ \zeta_5^2 \cdot \delta_{\begin{bmatrix} 2 \\ 0 \end{bmatrix}} + \zeta_5^4 \cdot \delta_{\begin{bmatrix} 2 \\ 1 \end{bmatrix}} + \zeta_5^1 \cdot \delta_{\begin{bmatrix} 2 \\ 2 \end{bmatrix}} + \zeta_5^3 \cdot \delta_{\begin{bmatrix} 2 \\ 3 \end{bmatrix}} + \zeta_5^0 \cdot \delta_{\begin{bmatrix} 2 \\ 4 \end{bmatrix}} +$$

$$+ \zeta_5^3 \cdot \delta_{\begin{bmatrix} 3 \\ 0 \end{bmatrix}} + \zeta_5^0 \cdot \delta_{\begin{bmatrix} 3 \\ 1 \end{bmatrix}} + \zeta_5^2 \cdot \delta_{\begin{bmatrix} 3 \\ 2 \end{bmatrix}} + \zeta_5^4 \cdot \delta_{\begin{bmatrix} 3 \\ 3 \end{bmatrix}} + \zeta_5^1 \cdot \delta_{\begin{bmatrix} 3 \\ 4 \end{bmatrix}} +$$

$$+ \zeta_5^4 \cdot \delta_{\begin{bmatrix} 4 \\ 0 \end{bmatrix}} + \zeta_5^1 \cdot \delta_{\begin{bmatrix} 4 \\ 1 \end{bmatrix}} + \zeta_5^3 \cdot \delta_{\begin{bmatrix} 4 \\ 2 \end{bmatrix}} + \zeta_5^0 \cdot \delta_{\begin{bmatrix} 4 \\ 3 \end{bmatrix}} + \zeta_5^2 \cdot \delta_{\begin{bmatrix} 4 \\ 4 \end{bmatrix}} \Bigg).$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \cdot \delta_{\begin{bmatrix} 2 \\ 2 \end{bmatrix}} = 5^{-1} \cdot \sum_{y \in M} \left(\exp(2\pi i(y_1 \cdot 2 + 2 \cdot y_2)/5) \cdot \delta_{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}} \right)$$

$$= 5^{-1} \cdot \sum_{y \in M} \left(\zeta_5^{2y_1+2y_2} \cdot \delta_{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}} \right)$$

$$= 5^{-1} \cdot \Bigg(\zeta_5^0 \cdot \delta_{\begin{bmatrix} 0 \\ 0 \end{bmatrix}} + \zeta_5^2 \cdot \delta_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} + \zeta_5^4 \cdot \delta_{\begin{bmatrix} 0 \\ 2 \end{bmatrix}} + \zeta_5^1 \cdot \delta_{\begin{bmatrix} 0 \\ 3 \end{bmatrix}} + \zeta_5^3 \cdot \delta_{\begin{bmatrix} 0 \\ 4 \end{bmatrix}} +$$

$$+ \zeta_5^2 \cdot \delta_{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} + \zeta_5^4 \cdot \delta_{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} + \zeta_5^1 \cdot \delta_{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} + \zeta_5^3 \cdot \delta_{\begin{bmatrix} 1 \\ 3 \end{bmatrix}} + \zeta_5^0 \cdot \delta_{\begin{bmatrix} 1 \\ 4 \end{bmatrix}} +$$

$$+ \zeta_5^4 \cdot \delta_{\begin{bmatrix} 2 \\ 0 \end{bmatrix}} + \zeta_5^1 \cdot \delta_{\begin{bmatrix} 2 \\ 1 \end{bmatrix}} + \zeta_5^3 \cdot \delta_{\begin{bmatrix} 2 \\ 2 \end{bmatrix}} + \zeta_5^0 \cdot \delta_{\begin{bmatrix} 2 \\ 3 \end{bmatrix}} + \zeta_5^2 \cdot \delta_{\begin{bmatrix} 2 \\ 4 \end{bmatrix}} +$$

$$+ \zeta_5^1 \cdot \delta_{\begin{bmatrix} 3 \\ 0 \end{bmatrix}} + \zeta_5^3 \cdot \delta_{\begin{bmatrix} 3 \\ 1 \end{bmatrix}} + \zeta_5^0 \cdot \delta_{\begin{bmatrix} 3 \\ 2 \end{bmatrix}} + \zeta_5^2 \cdot \delta_{\begin{bmatrix} 3 \\ 3 \end{bmatrix}} + \zeta_5^4 \cdot \delta_{\begin{bmatrix} 3 \\ 4 \end{bmatrix}} +$$

$$+ \zeta_5^3 \cdot \delta_{\begin{bmatrix} 4 \\ 0 \end{bmatrix}} + \zeta_5^0 \cdot \delta_{\begin{bmatrix} 4 \\ 1 \end{bmatrix}} + \zeta_5^2 \cdot \delta_{\begin{bmatrix} 4 \\ 2 \end{bmatrix}} + \zeta_5^4 \cdot \delta_{\begin{bmatrix} 4 \\ 3 \end{bmatrix}} + \zeta_5^1 \cdot \delta_{\begin{bmatrix} 4 \\ 4 \end{bmatrix}} \Bigg).$$

$$
\begin{aligned}
\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \cdot \delta_{\begin{bmatrix} 2 \\ 3 \end{bmatrix}} &= 5^{-1} \cdot \sum_{y \in M} \left(\exp(2\pi i (y_1 \cdot 3 + 2 \cdot y_2)/5) \cdot \delta_{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}} \right) \\
&= 5^{-1} \cdot \sum_{y \in M} \left(\zeta_5^{3y_1 + 2y_2} \cdot \delta_{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}} \right) \\
&= 5^{-1} \cdot \Bigg(\zeta_5^0 \cdot \delta_{\begin{bmatrix} 0 \\ 0 \end{bmatrix}} + \zeta_5^2 \cdot \delta_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} + \zeta_5^4 \cdot \delta_{\begin{bmatrix} 0 \\ 2 \end{bmatrix}} + \zeta_5^1 \cdot \delta_{\begin{bmatrix} 0 \\ 3 \end{bmatrix}} + \zeta_5^3 \cdot \delta_{\begin{bmatrix} 0 \\ 4 \end{bmatrix}} + \\
&\quad + \zeta_5^3 \cdot \delta_{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} + \zeta_5^0 \cdot \delta_{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} + \zeta_5^2 \cdot \delta_{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} + \zeta_5^4 \cdot \delta_{\begin{bmatrix} 1 \\ 3 \end{bmatrix}} + \zeta_5^1 \cdot \delta_{\begin{bmatrix} 1 \\ 4 \end{bmatrix}} + \\
&\quad + \zeta_5^1 \cdot \delta_{\begin{bmatrix} 2 \\ 0 \end{bmatrix}} + \zeta_5^3 \cdot \delta_{\begin{bmatrix} 2 \\ 1 \end{bmatrix}} + \zeta_5^0 \cdot \delta_{\begin{bmatrix} 2 \\ 2 \end{bmatrix}} + \zeta_5^2 \cdot \delta_{\begin{bmatrix} 2 \\ 3 \end{bmatrix}} + \zeta_5^4 \cdot \delta_{\begin{bmatrix} 2 \\ 4 \end{bmatrix}} + \\
&\quad + \zeta_5^4 \cdot \delta_{\begin{bmatrix} 3 \\ 0 \end{bmatrix}} + \zeta_5^1 \cdot \delta_{\begin{bmatrix} 3 \\ 1 \end{bmatrix}} + \zeta_5^3 \cdot \delta_{\begin{bmatrix} 3 \\ 2 \end{bmatrix}} + \zeta_5^0 \cdot \delta_{\begin{bmatrix} 3 \\ 3 \end{bmatrix}} + \zeta_5^2 \cdot \delta_{\begin{bmatrix} 3 \\ 4 \end{bmatrix}} + \\
&\quad + \zeta_5^2 \cdot \delta_{\begin{bmatrix} 4 \\ 0 \end{bmatrix}} + \zeta_5^4 \cdot \delta_{\begin{bmatrix} 4 \\ 1 \end{bmatrix}} + \zeta_5^1 \cdot \delta_{\begin{bmatrix} 4 \\ 2 \end{bmatrix}} + \zeta_5^3 \cdot \delta_{\begin{bmatrix} 4 \\ 3 \end{bmatrix}} + \zeta_5^0 \cdot \delta_{\begin{bmatrix} 4 \\ 4 \end{bmatrix}} \Bigg).
\end{aligned}
$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \cdot \delta_{\begin{bmatrix} 2 \\ 4 \end{bmatrix}} = 5^{-1} \cdot \sum_{y \in M} \left(\exp(2\pi i (y_1 \cdot 4 + 2 \cdot y_2)/5) \cdot \delta_{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}} \right)$$

$$= 5^{-1} \cdot \sum_{y \in M} \left(\zeta_5^{4y_1+2y_2} \cdot \delta_{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}} \right)$$

$$= 5^{-1} \cdot \left(\zeta_5^0 \cdot \delta_{\begin{bmatrix} 0 \\ 0 \end{bmatrix}} + \zeta_5^2 \cdot \delta_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} + \zeta_5^4 \cdot \delta_{\begin{bmatrix} 0 \\ 2 \end{bmatrix}} + \zeta_5^1 \cdot \delta_{\begin{bmatrix} 0 \\ 3 \end{bmatrix}} + \zeta_5^3 \cdot \delta_{\begin{bmatrix} 0 \\ 4 \end{bmatrix}} + \right.$$

$$+ \zeta_5^4 \cdot \delta_{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} + \zeta_5^1 \cdot \delta_{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} + \zeta_5^3 \cdot \delta_{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} + \zeta_5^0 \cdot \delta_{\begin{bmatrix} 1 \\ 3 \end{bmatrix}} + \zeta_5^2 \cdot \delta_{\begin{bmatrix} 1 \\ 4 \end{bmatrix}} +$$

$$+ \zeta_5^3 \cdot \delta_{\begin{bmatrix} 2 \\ 0 \end{bmatrix}} + \zeta_5^0 \cdot \delta_{\begin{bmatrix} 2 \\ 1 \end{bmatrix}} + \zeta_5^2 \cdot \delta_{\begin{bmatrix} 2 \\ 2 \end{bmatrix}} + \zeta_5^4 \cdot \delta_{\begin{bmatrix} 2 \\ 3 \end{bmatrix}} + \zeta_5^1 \cdot \delta_{\begin{bmatrix} 2 \\ 4 \end{bmatrix}} +$$

$$+ \zeta_5^2 \cdot \delta_{\begin{bmatrix} 3 \\ 0 \end{bmatrix}} + \zeta_5^4 \cdot \delta_{\begin{bmatrix} 3 \\ 1 \end{bmatrix}} + \zeta_5^1 \cdot \delta_{\begin{bmatrix} 3 \\ 2 \end{bmatrix}} + \zeta_5^3 \cdot \delta_{\begin{bmatrix} 3 \\ 3 \end{bmatrix}} + \zeta_5^0 \cdot \delta_{\begin{bmatrix} 3 \\ 4 \end{bmatrix}} +$$

$$\left. + \zeta_5^1 \cdot \delta_{\begin{bmatrix} 4 \\ 0 \end{bmatrix}} + \zeta_5^3 \cdot \delta_{\begin{bmatrix} 4 \\ 1 \end{bmatrix}} + \zeta_5^0 \cdot \delta_{\begin{bmatrix} 4 \\ 2 \end{bmatrix}} + \zeta_5^2 \cdot \delta_{\begin{bmatrix} 4 \\ 3 \end{bmatrix}} + \zeta_5^4 \cdot \delta_{\begin{bmatrix} 4 \\ 4 \end{bmatrix}} \right).$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \cdot \delta_{\begin{bmatrix} 3 \\ 0 \end{bmatrix}} = 5^{-1} \cdot \sum_{y \in M} \left(\exp(2\pi i(y_1 \cdot 0 + 3 \cdot y_2)/5) \cdot \delta_{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}} \right)$$

$$= 5^{-1} \cdot \sum_{y \in M} \left(\zeta_5^{3y_2} \cdot \delta_{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}} \right)$$

$$= 5^{-1} \cdot \left(\zeta_5^0 \cdot \delta_{\begin{bmatrix} 0 \\ 0 \end{bmatrix}} + \zeta_5^3 \cdot \delta_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} + \zeta_5^1 \cdot \delta_{\begin{bmatrix} 0 \\ 2 \end{bmatrix}} + \zeta_5^4 \cdot \delta_{\begin{bmatrix} 0 \\ 3 \end{bmatrix}} + \zeta_5^2 \cdot \delta_{\begin{bmatrix} 0 \\ 4 \end{bmatrix}} + \right.$$

$$+ \zeta_5^0 \cdot \delta_{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} + \zeta_5^3 \cdot \delta_{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} + \zeta_5^1 \cdot \delta_{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} + \zeta_5^4 \cdot \delta_{\begin{bmatrix} 1 \\ 3 \end{bmatrix}} + \zeta_5^2 \cdot \delta_{\begin{bmatrix} 1 \\ 4 \end{bmatrix}} +$$

$$+ \zeta_5^0 \cdot \delta_{\begin{bmatrix} 2 \\ 0 \end{bmatrix}} + \zeta_5^3 \cdot \delta_{\begin{bmatrix} 2 \\ 1 \end{bmatrix}} + \zeta_5^1 \cdot \delta_{\begin{bmatrix} 2 \\ 2 \end{bmatrix}} + \zeta_5^4 \cdot \delta_{\begin{bmatrix} 2 \\ 3 \end{bmatrix}} + \zeta_5^2 \cdot \delta_{\begin{bmatrix} 2 \\ 4 \end{bmatrix}} +$$

$$+ \zeta_5^0 \cdot \delta_{\begin{bmatrix} 3 \\ 0 \end{bmatrix}} + \zeta_5^3 \cdot \delta_{\begin{bmatrix} 3 \\ 1 \end{bmatrix}} + \zeta_5^1 \cdot \delta_{\begin{bmatrix} 3 \\ 2 \end{bmatrix}} + \zeta_5^4 \cdot \delta_{\begin{bmatrix} 3 \\ 3 \end{bmatrix}} + \zeta_5^2 \cdot \delta_{\begin{bmatrix} 3 \\ 4 \end{bmatrix}} +$$

$$\left. + \zeta_5^0 \cdot \delta_{\begin{bmatrix} 4 \\ 0 \end{bmatrix}} + \zeta_5^3 \cdot \delta_{\begin{bmatrix} 4 \\ 1 \end{bmatrix}} + \zeta_5^1 \cdot \delta_{\begin{bmatrix} 4 \\ 2 \end{bmatrix}} + \zeta_5^4 \cdot \delta_{\begin{bmatrix} 4 \\ 3 \end{bmatrix}} + \zeta_5^2 \cdot \delta_{\begin{bmatrix} 4 \\ 4 \end{bmatrix}} \right).$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \cdot \delta_{\begin{bmatrix} 3 \\ 1 \end{bmatrix}} = 5^{-1} \cdot \sum_{y \in M} \left(\exp(2\pi i(y_1 \cdot 1 + 3 \cdot y_2)/5) \cdot \delta_{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}} \right)$$

$$= 5^{-1} \cdot \sum_{y \in M} \left(\zeta_5^{y_1 + 3y_2} \cdot \delta_{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}} \right)$$

$$= 5^{-1} \cdot \left(\zeta_5^0 \cdot \delta_{\begin{bmatrix} 0 \\ 0 \end{bmatrix}} + \zeta_5^3 \cdot \delta_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} + \zeta_5^1 \cdot \delta_{\begin{bmatrix} 0 \\ 2 \end{bmatrix}} + \zeta_5^4 \cdot \delta_{\begin{bmatrix} 0 \\ 3 \end{bmatrix}} + \zeta_5^2 \cdot \delta_{\begin{bmatrix} 0 \\ 4 \end{bmatrix}} + \right.$$

$$+ \zeta_5^1 \cdot \delta_{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} + \zeta_5^4 \cdot \delta_{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} + \zeta_5^2 \cdot \delta_{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} + \zeta_5^0 \cdot \delta_{\begin{bmatrix} 1 \\ 3 \end{bmatrix}} + \zeta_5^3 \cdot \delta_{\begin{bmatrix} 1 \\ 4 \end{bmatrix}} +$$

$$+ \zeta_5^2 \cdot \delta_{\begin{bmatrix} 2 \\ 0 \end{bmatrix}} + \zeta_5^0 \cdot \delta_{\begin{bmatrix} 2 \\ 1 \end{bmatrix}} + \zeta_5^3 \cdot \delta_{\begin{bmatrix} 2 \\ 2 \end{bmatrix}} + \zeta_5^1 \cdot \delta_{\begin{bmatrix} 2 \\ 3 \end{bmatrix}} + \zeta_5^4 \cdot \delta_{\begin{bmatrix} 2 \\ 4 \end{bmatrix}} +$$

$$+ \zeta_5^3 \cdot \delta_{\begin{bmatrix} 3 \\ 0 \end{bmatrix}} + \zeta_5^1 \cdot \delta_{\begin{bmatrix} 3 \\ 1 \end{bmatrix}} + \zeta_5^4 \cdot \delta_{\begin{bmatrix} 3 \\ 2 \end{bmatrix}} + \zeta_5^2 \cdot \delta_{\begin{bmatrix} 3 \\ 3 \end{bmatrix}} + \zeta_5^0 \cdot \delta_{\begin{bmatrix} 3 \\ 4 \end{bmatrix}} +$$

$$\left. + \zeta_5^4 \cdot \delta_{\begin{bmatrix} 4 \\ 0 \end{bmatrix}} + \zeta_5^2 \cdot \delta_{\begin{bmatrix} 4 \\ 1 \end{bmatrix}} + \zeta_5^0 \cdot \delta_{\begin{bmatrix} 4 \\ 2 \end{bmatrix}} + \zeta_5^3 \cdot \delta_{\begin{bmatrix} 4 \\ 3 \end{bmatrix}} + \zeta_5^1 \cdot \delta_{\begin{bmatrix} 4 \\ 4 \end{bmatrix}} \right).$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \cdot \delta_{\begin{bmatrix} 3 \\ 2 \end{bmatrix}} = 5^{-1} \cdot \sum_{y \in M} \left(\exp(2\pi i (y_1 \cdot 2 + 3 \cdot y_2)/5) \cdot \delta_{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}} \right)$$

$$= 5^{-1} \cdot \sum_{y \in M} \left(\zeta_5^{2y_1+3y_2} \cdot \delta_{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}} \right)$$

$$= 5^{-1} \cdot \left(\zeta_5^0 \cdot \delta_{\begin{bmatrix} 0 \\ 0 \end{bmatrix}} + \zeta_5^3 \cdot \delta_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} + \zeta_5^1 \cdot \delta_{\begin{bmatrix} 0 \\ 2 \end{bmatrix}} + \zeta_5^4 \cdot \delta_{\begin{bmatrix} 0 \\ 3 \end{bmatrix}} + \zeta_5^2 \cdot \delta_{\begin{bmatrix} 0 \\ 4 \end{bmatrix}} + \right.$$

$$+ \zeta_5^2 \cdot \delta_{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} + \zeta_5^0 \cdot \delta_{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} + \zeta_5^3 \cdot \delta_{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} + \zeta_5^1 \cdot \delta_{\begin{bmatrix} 1 \\ 3 \end{bmatrix}} + \zeta_5^4 \cdot \delta_{\begin{bmatrix} 1 \\ 4 \end{bmatrix}} +$$

$$+ \zeta_5^4 \cdot \delta_{\begin{bmatrix} 2 \\ 0 \end{bmatrix}} + \zeta_5^2 \cdot \delta_{\begin{bmatrix} 2 \\ 1 \end{bmatrix}} + \zeta_5^0 \cdot \delta_{\begin{bmatrix} 2 \\ 2 \end{bmatrix}} + \zeta_5^3 \cdot \delta_{\begin{bmatrix} 2 \\ 3 \end{bmatrix}} + \zeta_5^1 \cdot \delta_{\begin{bmatrix} 2 \\ 4 \end{bmatrix}} +$$

$$+ \zeta_5^1 \cdot \delta_{\begin{bmatrix} 3 \\ 0 \end{bmatrix}} + \zeta_5^4 \cdot \delta_{\begin{bmatrix} 3 \\ 1 \end{bmatrix}} + \zeta_5^2 \cdot \delta_{\begin{bmatrix} 3 \\ 2 \end{bmatrix}} + \zeta_5^0 \cdot \delta_{\begin{bmatrix} 3 \\ 3 \end{bmatrix}} + \zeta_5^3 \cdot \delta_{\begin{bmatrix} 3 \\ 4 \end{bmatrix}} +$$

$$\left. + \zeta_5^3 \cdot \delta_{\begin{bmatrix} 4 \\ 0 \end{bmatrix}} + \zeta_5^1 \cdot \delta_{\begin{bmatrix} 4 \\ 1 \end{bmatrix}} + \zeta_5^4 \cdot \delta_{\begin{bmatrix} 4 \\ 2 \end{bmatrix}} + \zeta_5^2 \cdot \delta_{\begin{bmatrix} 4 \\ 3 \end{bmatrix}} + \zeta_5^0 \cdot \delta_{\begin{bmatrix} 4 \\ 4 \end{bmatrix}} \right).$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \cdot \delta_{\begin{bmatrix} 3 \\ 3 \end{bmatrix}} = 5^{-1} \cdot \sum_{y \in M} \left(\exp(2\pi i (y_1 \cdot 3 + 3 \cdot y_2)/5) \cdot \delta_{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}} \right)$$

$$= 5^{-1} \cdot \sum_{y \in M} \left(\zeta_5^{3y_1 + 3y_2} \cdot \delta_{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}} \right)$$

$$= 5^{-1} \cdot \Bigg(\zeta_5^0 \cdot \delta_{\begin{bmatrix} 0 \\ 0 \end{bmatrix}} + \zeta_5^3 \cdot \delta_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} + \zeta_5^1 \cdot \delta_{\begin{bmatrix} 0 \\ 2 \end{bmatrix}} + \zeta_5^4 \cdot \delta_{\begin{bmatrix} 0 \\ 3 \end{bmatrix}} + \zeta_5^2 \cdot \delta_{\begin{bmatrix} 0 \\ 4 \end{bmatrix}} +$$

$$+ \zeta_5^3 \cdot \delta_{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} + \zeta_5^1 \cdot \delta_{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} + \zeta_5^4 \cdot \delta_{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} + \zeta_5^2 \cdot \delta_{\begin{bmatrix} 1 \\ 3 \end{bmatrix}} + \zeta_5^0 \cdot \delta_{\begin{bmatrix} 1 \\ 4 \end{bmatrix}} +$$

$$+ \zeta_5^1 \cdot \delta_{\begin{bmatrix} 2 \\ 0 \end{bmatrix}} + \zeta_5^4 \cdot \delta_{\begin{bmatrix} 2 \\ 1 \end{bmatrix}} + \zeta_5^2 \cdot \delta_{\begin{bmatrix} 2 \\ 2 \end{bmatrix}} + \zeta_5^0 \cdot \delta_{\begin{bmatrix} 2 \\ 3 \end{bmatrix}} + \zeta_5^3 \cdot \delta_{\begin{bmatrix} 2 \\ 4 \end{bmatrix}} +$$

$$+ \zeta_5^4 \cdot \delta_{\begin{bmatrix} 3 \\ 0 \end{bmatrix}} + \zeta_5^2 \cdot \delta_{\begin{bmatrix} 3 \\ 1 \end{bmatrix}} + \zeta_5^0 \cdot \delta_{\begin{bmatrix} 3 \\ 2 \end{bmatrix}} + \zeta_5^3 \cdot \delta_{\begin{bmatrix} 3 \\ 3 \end{bmatrix}} + \zeta_5^1 \cdot \delta_{\begin{bmatrix} 3 \\ 4 \end{bmatrix}} +$$

$$+ \zeta_5^2 \cdot \delta_{\begin{bmatrix} 4 \\ 0 \end{bmatrix}} + \zeta_5^0 \cdot \delta_{\begin{bmatrix} 4 \\ 1 \end{bmatrix}} + \zeta_5^3 \cdot \delta_{\begin{bmatrix} 4 \\ 2 \end{bmatrix}} + \zeta_5^1 \cdot \delta_{\begin{bmatrix} 4 \\ 3 \end{bmatrix}} + \zeta_5^4 \cdot \delta_{\begin{bmatrix} 4 \\ 4 \end{bmatrix}} \Bigg).$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \cdot \delta_{\begin{bmatrix} 3 \\ 4 \end{bmatrix}} = 5^{-1} \cdot \sum_{y \in M} \left(\exp(2\pi i (y_1 \cdot 4 + 3 \cdot y_2)/5) \cdot \delta_{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}} \right)$$

$$= 5^{-1} \cdot \sum_{y \in M} \left(\zeta_5^{4y_1 + 3y_2} \cdot \delta_{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}} \right)$$

$$= 5^{-1} \cdot \Bigg(\zeta_5^0 \cdot \delta_{\begin{bmatrix} 0 \\ 0 \end{bmatrix}} + \zeta_5^3 \cdot \delta_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} + \zeta_5^1 \cdot \delta_{\begin{bmatrix} 0 \\ 2 \end{bmatrix}} + \zeta_5^4 \cdot \delta_{\begin{bmatrix} 0 \\ 3 \end{bmatrix}} + \zeta_5^2 \cdot \delta_{\begin{bmatrix} 0 \\ 4 \end{bmatrix}} +$$

$$+ \zeta_5^4 \cdot \delta_{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} + \zeta_5^2 \cdot \delta_{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} + \zeta_5^0 \cdot \delta_{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} + \zeta_5^3 \cdot \delta_{\begin{bmatrix} 1 \\ 3 \end{bmatrix}} + \zeta_5^1 \cdot \delta_{\begin{bmatrix} 1 \\ 4 \end{bmatrix}} +$$

$$+ \zeta_5^3 \cdot \delta_{\begin{bmatrix} 2 \\ 0 \end{bmatrix}} + \zeta_5^1 \cdot \delta_{\begin{bmatrix} 2 \\ 1 \end{bmatrix}} + \zeta_5^4 \cdot \delta_{\begin{bmatrix} 2 \\ 2 \end{bmatrix}} + \zeta_5^2 \cdot \delta_{\begin{bmatrix} 2 \\ 3 \end{bmatrix}} + \zeta_5^0 \cdot \delta_{\begin{bmatrix} 2 \\ 4 \end{bmatrix}} +$$

$$+ \zeta_5^2 \cdot \delta_{\begin{bmatrix} 3 \\ 0 \end{bmatrix}} + \zeta_5^0 \cdot \delta_{\begin{bmatrix} 3 \\ 1 \end{bmatrix}} + \zeta_5^3 \cdot \delta_{\begin{bmatrix} 3 \\ 2 \end{bmatrix}} + \zeta_5^1 \cdot \delta_{\begin{bmatrix} 3 \\ 3 \end{bmatrix}} + \zeta_5^4 \cdot \delta_{\begin{bmatrix} 3 \\ 4 \end{bmatrix}} +$$

$$+ \zeta_5^1 \cdot \delta_{\begin{bmatrix} 4 \\ 0 \end{bmatrix}} + \zeta_5^4 \cdot \delta_{\begin{bmatrix} 4 \\ 1 \end{bmatrix}} + \zeta_5^2 \cdot \delta_{\begin{bmatrix} 4 \\ 2 \end{bmatrix}} + \zeta_5^0 \cdot \delta_{\begin{bmatrix} 4 \\ 3 \end{bmatrix}} + \zeta_5^3 \cdot \delta_{\begin{bmatrix} 4 \\ 4 \end{bmatrix}} \Bigg).$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \cdot \delta_{\begin{bmatrix} 4 \\ 0 \end{bmatrix}} = 5^{-1} \cdot \sum_{y \in M} \left(\exp(2\pi i (y_1 \cdot 0 + 4 \cdot y_2)/5) \cdot \delta_{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}} \right)$$

$$= 5^{-1} \cdot \sum_{y \in M} \left(\zeta_5^{4y_2} \cdot \delta_{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}} \right)$$

$$= 5^{-1} \cdot \Bigg(\zeta_5^0 \cdot \delta_{\begin{bmatrix} 0 \\ 0 \end{bmatrix}} + \zeta_5^4 \cdot \delta_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} + \zeta_5^3 \cdot \delta_{\begin{bmatrix} 0 \\ 2 \end{bmatrix}} + \zeta_5^2 \cdot \delta_{\begin{bmatrix} 0 \\ 3 \end{bmatrix}} + \zeta_5^1 \cdot \delta_{\begin{bmatrix} 0 \\ 4 \end{bmatrix}} +$$

$$+ \zeta_5^0 \cdot \delta_{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} + \zeta_5^4 \cdot \delta_{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} + \zeta_5^3 \cdot \delta_{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} + \zeta_5^2 \cdot \delta_{\begin{bmatrix} 1 \\ 3 \end{bmatrix}} + \zeta_5^1 \cdot \delta_{\begin{bmatrix} 1 \\ 4 \end{bmatrix}} +$$

$$+ \zeta_5^0 \cdot \delta_{\begin{bmatrix} 2 \\ 0 \end{bmatrix}} + \zeta_5^4 \cdot \delta_{\begin{bmatrix} 2 \\ 1 \end{bmatrix}} + \zeta_5^3 \cdot \delta_{\begin{bmatrix} 2 \\ 2 \end{bmatrix}} + \zeta_5^2 \cdot \delta_{\begin{bmatrix} 2 \\ 3 \end{bmatrix}} + \zeta_5^1 \cdot \delta_{\begin{bmatrix} 2 \\ 4 \end{bmatrix}} +$$

$$+ \zeta_5^0 \cdot \delta_{\begin{bmatrix} 3 \\ 0 \end{bmatrix}} + \zeta_5^4 \cdot \delta_{\begin{bmatrix} 3 \\ 1 \end{bmatrix}} + \zeta_5^3 \cdot \delta_{\begin{bmatrix} 3 \\ 2 \end{bmatrix}} + \zeta_5^2 \cdot \delta_{\begin{bmatrix} 3 \\ 3 \end{bmatrix}} + \zeta_5^1 \cdot \delta_{\begin{bmatrix} 3 \\ 4 \end{bmatrix}} +$$

$$+ \zeta_5^0 \cdot \delta_{\begin{bmatrix} 4 \\ 0 \end{bmatrix}} + \zeta_5^4 \cdot \delta_{\begin{bmatrix} 4 \\ 1 \end{bmatrix}} + \zeta_5^3 \cdot \delta_{\begin{bmatrix} 4 \\ 2 \end{bmatrix}} + \zeta_5^2 \cdot \delta_{\begin{bmatrix} 4 \\ 3 \end{bmatrix}} + \zeta_5^1 \cdot \delta_{\begin{bmatrix} 4 \\ 4 \end{bmatrix}} \Bigg).$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \cdot \delta_{\begin{bmatrix} 4 \\ 1 \end{bmatrix}} = 5^{-1} \cdot \sum_{y \in M} \left(\exp(2\pi i (y_1 \cdot 1 + 4 \cdot y_2)/5) \cdot \delta_{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}} \right)$$

$$= 5^{-1} \cdot \sum_{y \in M} \left(\zeta_5^{y_1 + 4y_2} \cdot \delta_{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}} \right)$$

$$= 5^{-1} \cdot \left(\zeta_5^0 \cdot \delta_{\begin{bmatrix} 0 \\ 0 \end{bmatrix}} + \zeta_5^4 \cdot \delta_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} + \zeta_5^3 \cdot \delta_{\begin{bmatrix} 0 \\ 2 \end{bmatrix}} + \zeta_5^2 \cdot \delta_{\begin{bmatrix} 0 \\ 3 \end{bmatrix}} + \zeta_5^1 \cdot \delta_{\begin{bmatrix} 0 \\ 4 \end{bmatrix}} + \right.$$

$$+ \zeta_5^1 \cdot \delta_{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} + \zeta_5^0 \cdot \delta_{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} + \zeta_5^4 \cdot \delta_{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} + \zeta_5^3 \cdot \delta_{\begin{bmatrix} 1 \\ 3 \end{bmatrix}} + \zeta_5^2 \cdot \delta_{\begin{bmatrix} 1 \\ 4 \end{bmatrix}} +$$

$$+ \zeta_5^2 \cdot \delta_{\begin{bmatrix} 2 \\ 0 \end{bmatrix}} + \zeta_5^1 \cdot \delta_{\begin{bmatrix} 2 \\ 1 \end{bmatrix}} + \zeta_5^0 \cdot \delta_{\begin{bmatrix} 2 \\ 2 \end{bmatrix}} + \zeta_5^4 \cdot \delta_{\begin{bmatrix} 2 \\ 3 \end{bmatrix}} + \zeta_5^3 \cdot \delta_{\begin{bmatrix} 2 \\ 4 \end{bmatrix}} +$$

$$+ \zeta_5^3 \cdot \delta_{\begin{bmatrix} 3 \\ 0 \end{bmatrix}} + \zeta_5^2 \cdot \delta_{\begin{bmatrix} 3 \\ 1 \end{bmatrix}} + \zeta_5^1 \cdot \delta_{\begin{bmatrix} 3 \\ 2 \end{bmatrix}} + \zeta_5^0 \cdot \delta_{\begin{bmatrix} 3 \\ 3 \end{bmatrix}} + \zeta_5^4 \cdot \delta_{\begin{bmatrix} 3 \\ 4 \end{bmatrix}} +$$

$$\left. + \zeta_5^4 \cdot \delta_{\begin{bmatrix} 4 \\ 0 \end{bmatrix}} + \zeta_5^3 \cdot \delta_{\begin{bmatrix} 4 \\ 1 \end{bmatrix}} + \zeta_5^2 \cdot \delta_{\begin{bmatrix} 4 \\ 2 \end{bmatrix}} + \zeta_5^1 \cdot \delta_{\begin{bmatrix} 4 \\ 3 \end{bmatrix}} + \zeta_5^0 \cdot \delta_{\begin{bmatrix} 4 \\ 4 \end{bmatrix}} \right).$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \cdot \delta_{\begin{bmatrix} 4 \\ 2 \end{bmatrix}} = 5^{-1} \cdot \sum_{y \in M} \left(\exp(2\pi i (y_1 \cdot 2 + 4 \cdot y_2)/5) \cdot \delta_{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}} \right)$$

$$= 5^{-1} \cdot \sum_{y \in M} \left(\zeta_5^{2y_1 + 4y_2} \cdot \delta_{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}} \right)$$

$$\begin{aligned} = 5^{-1} \cdot \Bigg(& \zeta_5^0 \cdot \delta_{\begin{bmatrix} 0 \\ 0 \end{bmatrix}} + \zeta_5^4 \cdot \delta_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} + \zeta_5^3 \cdot \delta_{\begin{bmatrix} 0 \\ 2 \end{bmatrix}} + \zeta_5^2 \cdot \delta_{\begin{bmatrix} 0 \\ 3 \end{bmatrix}} + \zeta_5^1 \cdot \delta_{\begin{bmatrix} 0 \\ 4 \end{bmatrix}} + \\ & + \zeta_5^2 \cdot \delta_{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} + \zeta_5^1 \cdot \delta_{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} + \zeta_5^0 \cdot \delta_{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} + \zeta_5^4 \cdot \delta_{\begin{bmatrix} 1 \\ 3 \end{bmatrix}} + \zeta_5^3 \cdot \delta_{\begin{bmatrix} 1 \\ 4 \end{bmatrix}} + \\ & + \zeta_5^4 \cdot \delta_{\begin{bmatrix} 2 \\ 0 \end{bmatrix}} + \zeta_5^3 \cdot \delta_{\begin{bmatrix} 2 \\ 1 \end{bmatrix}} + \zeta_5^2 \cdot \delta_{\begin{bmatrix} 2 \\ 2 \end{bmatrix}} + \zeta_5^1 \cdot \delta_{\begin{bmatrix} 2 \\ 3 \end{bmatrix}} + \zeta_5^0 \cdot \delta_{\begin{bmatrix} 2 \\ 4 \end{bmatrix}} + \\ & + \zeta_5^1 \cdot \delta_{\begin{bmatrix} 3 \\ 0 \end{bmatrix}} + \zeta_5^0 \cdot \delta_{\begin{bmatrix} 3 \\ 1 \end{bmatrix}} + \zeta_5^4 \cdot \delta_{\begin{bmatrix} 3 \\ 2 \end{bmatrix}} + \zeta_5^3 \cdot \delta_{\begin{bmatrix} 3 \\ 3 \end{bmatrix}} + \zeta_5^2 \cdot \delta_{\begin{bmatrix} 3 \\ 4 \end{bmatrix}} + \\ & + \zeta_5^3 \cdot \delta_{\begin{bmatrix} 4 \\ 0 \end{bmatrix}} + \zeta_5^2 \cdot \delta_{\begin{bmatrix} 4 \\ 1 \end{bmatrix}} + \zeta_5^1 \cdot \delta_{\begin{bmatrix} 4 \\ 2 \end{bmatrix}} + \zeta_5^0 \cdot \delta_{\begin{bmatrix} 4 \\ 3 \end{bmatrix}} + \zeta_5^4 \cdot \delta_{\begin{bmatrix} 4 \\ 4 \end{bmatrix}} \Bigg). \end{aligned}$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \cdot \delta_{\begin{bmatrix} 4 \\ 3 \end{bmatrix}} = 5^{-1} \cdot \sum_{y \in M} \left(\exp(2\pi i (y_1 \cdot 3 + 4 \cdot y_2)/5) \cdot \delta_{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}} \right)$$

$$= 5^{-1} \cdot \sum_{y \in M} \left(\zeta_5^{3y_1 + 4y_2} \cdot \delta_{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}} \right)$$

$$= 5^{-1} \cdot \Bigg(\zeta_5^0 \cdot \delta_{\begin{bmatrix} 0 \\ 0 \end{bmatrix}} + \zeta_5^4 \cdot \delta_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} + \zeta_5^3 \cdot \delta_{\begin{bmatrix} 0 \\ 2 \end{bmatrix}} + \zeta_5^2 \cdot \delta_{\begin{bmatrix} 0 \\ 3 \end{bmatrix}} + \zeta_5^1 \cdot \delta_{\begin{bmatrix} 0 \\ 4 \end{bmatrix}} +$$

$$+ \zeta_5^3 \cdot \delta_{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} + \zeta_5^2 \cdot \delta_{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} + \zeta_5^1 \cdot \delta_{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} + \zeta_5^0 \cdot \delta_{\begin{bmatrix} 1 \\ 3 \end{bmatrix}} + \zeta_5^4 \cdot \delta_{\begin{bmatrix} 1 \\ 4 \end{bmatrix}} +$$

$$+ \zeta_5^1 \cdot \delta_{\begin{bmatrix} 2 \\ 0 \end{bmatrix}} + \zeta_5^0 \cdot \delta_{\begin{bmatrix} 2 \\ 1 \end{bmatrix}} + \zeta_5^4 \cdot \delta_{\begin{bmatrix} 2 \\ 2 \end{bmatrix}} + \zeta_5^3 \cdot \delta_{\begin{bmatrix} 2 \\ 3 \end{bmatrix}} + \zeta_5^2 \cdot \delta_{\begin{bmatrix} 2 \\ 4 \end{bmatrix}} +$$

$$+ \zeta_5^4 \cdot \delta_{\begin{bmatrix} 3 \\ 0 \end{bmatrix}} + \zeta_5^3 \cdot \delta_{\begin{bmatrix} 3 \\ 1 \end{bmatrix}} + \zeta_5^2 \cdot \delta_{\begin{bmatrix} 3 \\ 2 \end{bmatrix}} + \zeta_5^1 \cdot \delta_{\begin{bmatrix} 3 \\ 3 \end{bmatrix}} + \zeta_5^0 \cdot \delta_{\begin{bmatrix} 3 \\ 4 \end{bmatrix}} +$$

$$+ \zeta_5^2 \cdot \delta_{\begin{bmatrix} 4 \\ 0 \end{bmatrix}} + \zeta_5^1 \cdot \delta_{\begin{bmatrix} 4 \\ 1 \end{bmatrix}} + \zeta_5^0 \cdot \delta_{\begin{bmatrix} 4 \\ 2 \end{bmatrix}} + \zeta_5^4 \cdot \delta_{\begin{bmatrix} 4 \\ 3 \end{bmatrix}} + \zeta_5^3 \cdot \delta_{\begin{bmatrix} 4 \\ 4 \end{bmatrix}} \Bigg).$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \cdot \delta_{\begin{bmatrix} 4 \\ 4 \end{bmatrix}} = 5^{-1} \cdot \sum_{y \in M} \left(\exp(2\pi i(y_1 \cdot 4 + 4 \cdot y_2)/5) \cdot \delta_{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}} \right)$$

$$= 5^{-1} \cdot \sum_{y \in M} \left(\zeta_5^{4y_1+4y_2} \cdot \delta_{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}} \right)$$

$$= 5^{-1} \cdot \left(\zeta_5^{0} \cdot \delta_{\begin{bmatrix} 0 \\ 0 \end{bmatrix}} + \zeta_5^{4} \cdot \delta_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} + \zeta_5^{3} \cdot \delta_{\begin{bmatrix} 0 \\ 2 \end{bmatrix}} + \zeta_5^{2} \cdot \delta_{\begin{bmatrix} 0 \\ 3 \end{bmatrix}} + \zeta_5^{1} \cdot \delta_{\begin{bmatrix} 0 \\ 4 \end{bmatrix}} + \right.$$

$$+ \zeta_5^{4} \cdot \delta_{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} + \zeta_5^{3} \cdot \delta_{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} + \zeta_5^{2} \cdot \delta_{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} + \zeta_5^{1} \cdot \delta_{\begin{bmatrix} 1 \\ 3 \end{bmatrix}} + \zeta_5^{0} \cdot \delta_{\begin{bmatrix} 1 \\ 4 \end{bmatrix}} +$$

$$+ \zeta_5^{3} \cdot \delta_{\begin{bmatrix} 2 \\ 0 \end{bmatrix}} + \zeta_5^{2} \cdot \delta_{\begin{bmatrix} 2 \\ 1 \end{bmatrix}} + \zeta_5^{1} \cdot \delta_{\begin{bmatrix} 2 \\ 2 \end{bmatrix}} + \zeta_5^{0} \cdot \delta_{\begin{bmatrix} 2 \\ 3 \end{bmatrix}} + \zeta_5^{4} \cdot \delta_{\begin{bmatrix} 2 \\ 4 \end{bmatrix}} +$$

$$+ \zeta_5^{2} \cdot \delta_{\begin{bmatrix} 3 \\ 0 \end{bmatrix}} + \zeta_5^{1} \cdot \delta_{\begin{bmatrix} 3 \\ 1 \end{bmatrix}} + \zeta_5^{0} \cdot \delta_{\begin{bmatrix} 3 \\ 2 \end{bmatrix}} + \zeta_5^{4} \cdot \delta_{\begin{bmatrix} 3 \\ 3 \end{bmatrix}} + \zeta_5^{3} \cdot \delta_{\begin{bmatrix} 3 \\ 4 \end{bmatrix}} +$$

$$\left. + \zeta_5^{1} \cdot \delta_{\begin{bmatrix} 4 \\ 0 \end{bmatrix}} + \zeta_5^{0} \cdot \delta_{\begin{bmatrix} 4 \\ 1 \end{bmatrix}} + \zeta_5^{4} \cdot \delta_{\begin{bmatrix} 4 \\ 2 \end{bmatrix}} + \zeta_5^{3} \cdot \delta_{\begin{bmatrix} 4 \\ 3 \end{bmatrix}} + \zeta_5^{2} \cdot \delta_{\begin{bmatrix} 4 \\ 4 \end{bmatrix}} \right).$$

4.1.3 Representation of S with respect to the basis of delta functions.

Combining the calculations gives the following matrix for S.

$$S = \frac{1}{5} \cdot \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & \zeta_5^4 & \zeta_5^4 & \zeta_5^4 & \zeta_5^4 & \zeta_5^4 & \zeta_5^3 & \zeta_5^3 & \zeta_5^3 & \zeta_5^3 & \zeta_5^3 & \zeta_5^2 & \zeta_5^2 & \zeta_5^2 & \zeta_5^2 & \zeta_5^2 & \zeta_5 & \zeta_5 & \zeta_5 & \zeta_5 & \zeta_5 \\
1 & 1 & 1 & 1 & 1 & \zeta_5^3 & \zeta_5^3 & \zeta_5^3 & \zeta_5^3 & \zeta_5^3 & \zeta_5 & \zeta_5 & \zeta_5 & \zeta_5 & \zeta_5 & \zeta_5^4 & \zeta_5^4 & \zeta_5^4 & \zeta_5^4 & \zeta_5^4 & \zeta_5^2 & \zeta_5^2 & \zeta_5^2 & \zeta_5^2 & \zeta_5^2 \\
1 & 1 & 1 & 1 & 1 & \zeta_5^2 & \zeta_5^2 & \zeta_5^2 & \zeta_5^2 & \zeta_5^2 & \zeta_5^4 & \zeta_5^4 & \zeta_5^4 & \zeta_5^4 & \zeta_5^4 & \zeta_5 & \zeta_5 & \zeta_5 & \zeta_5 & \zeta_5 & \zeta_5^3 & \zeta_5^3 & \zeta_5^3 & \zeta_5^3 & \zeta_5^3 \\
1 & 1 & 1 & 1 & 1 & \zeta_5 & \zeta_5 & \zeta_5 & \zeta_5 & \zeta_5 & \zeta_5^2 & \zeta_5^2 & \zeta_5^2 & \zeta_5^2 & \zeta_5^2 & \zeta_5^3 & \zeta_5^3 & \zeta_5^3 & \zeta_5^3 & \zeta_5^3 & \zeta_5^4 & \zeta_5^4 & \zeta_5^4 & \zeta_5^4 & \zeta_5^4 \\
1 & \zeta_5^4 & \zeta_5^3 & \zeta_5^2 & \zeta_5 & 1 & \zeta_5^4 & \zeta_5^3 & \zeta_5^2 & \zeta_5 & 1 & \zeta_5^4 & \zeta_5^3 & \zeta_5^2 & \zeta_5 & 1 & \zeta_5^4 & \zeta_5^3 & \zeta_5^2 & \zeta_5 & 1 & \zeta_5^4 & \zeta_5^3 & \zeta_5^2 & \zeta_5 \\
1 & \zeta_5^4 & \zeta_5^3 & \zeta_5^2 & \zeta_5 & \zeta_5^4 & \zeta_5^3 & \zeta_5^2 & \zeta_5 & 1 & \zeta_5^3 & \zeta_5^2 & \zeta_5 & 1 & \zeta_5^4 & \zeta_5^2 & \zeta_5 & 1 & \zeta_5^4 & \zeta_5^3 & \zeta_5 & 1 & \zeta_5^4 & \zeta_5^3 & \zeta_5^2 \\
1 & \zeta_5^4 & \zeta_5^3 & \zeta_5^2 & \zeta_5 & \zeta_5^3 & \zeta_5^2 & \zeta_5 & 1 & \zeta_5^4 & \zeta_5 & 1 & \zeta_5^4 & \zeta_5^3 & \zeta_5^2 & \zeta_5^4 & \zeta_5^3 & \zeta_5^2 & \zeta_5 & 1 & \zeta_5^2 & \zeta_5 & 1 & \zeta_5^4 & \zeta_5^3 \\
1 & \zeta_5^4 & \zeta_5^3 & \zeta_5^2 & \zeta_5 & \zeta_5^2 & \zeta_5 & 1 & \zeta_5^4 & \zeta_5^3 & \zeta_5^4 & \zeta_5^3 & \zeta_5^2 & \zeta_5 & 1 & \zeta_5 & 1 & \zeta_5^4 & \zeta_5^3 & \zeta_5^2 & \zeta_5^3 & \zeta_5^2 & \zeta_5 & 1 & \zeta_5^4 \\
1 & \zeta_5^4 & \zeta_5^3 & \zeta_5^2 & \zeta_5 & \zeta_5 & 1 & \zeta_5^4 & \zeta_5^3 & \zeta_5^2 & \zeta_5^2 & \zeta_5 & 1 & \zeta_5^4 & \zeta_5^3 & \zeta_5^3 & \zeta_5^2 & \zeta_5 & 1 & \zeta_5^4 & \zeta_5^4 & \zeta_5^3 & \zeta_5^2 & \zeta_5 & 1 \\
1 & \zeta_5^3 & \zeta_5 & \zeta_5^4 & \zeta_5^2 & 1 & \zeta_5^3 & \zeta_5 & \zeta_5^4 & \zeta_5^2 & 1 & \zeta_5^3 & \zeta_5 & \zeta_5^4 & \zeta_5^2 & 1 & \zeta_5^3 & \zeta_5 & \zeta_5^4 & \zeta_5^2 & 1 & \zeta_5^3 & \zeta_5 & \zeta_5^4 & \zeta_5^2 \\
1 & \zeta_5^3 & \zeta_5 & \zeta_5^4 & \zeta_5^2 & \zeta_5^4 & \zeta_5^2 & 1 & \zeta_5^3 & \zeta_5 & \zeta_5^3 & \zeta_5 & \zeta_5^4 & \zeta_5^2 & 1 & \zeta_5^2 & 1 & \zeta_5^3 & \zeta_5 & \zeta_5^4 & \zeta_5 & \zeta_5^4 & \zeta_5^2 & 1 & \zeta_5^3 \\
1 & \zeta_5^3 & \zeta_5 & \zeta_5^4 & \zeta_5^2 & \zeta_5^3 & \zeta_5 & \zeta_5^4 & \zeta_5^2 & 1 & \zeta_5 & \zeta_5^4 & \zeta_5^2 & 1 & \zeta_5^3 & \zeta_5^4 & \zeta_5^2 & 1 & \zeta_5^3 & \zeta_5 & \zeta_5^2 & 1 & \zeta_5^3 & \zeta_5 & \zeta_5^4 \\
1 & \zeta_5^3 & \zeta_5 & \zeta_5^4 & \zeta_5^2 & \zeta_5^2 & 1 & \zeta_5^3 & \zeta_5 & \zeta_5^4 & \zeta_5^4 & \zeta_5^2 & 1 & \zeta_5^3 & \zeta_5 & \zeta_5 & \zeta_5^4 & \zeta_5^2 & 1 & \zeta_5^3 & \zeta_5^3 & \zeta_5 & \zeta_5^4 & \zeta_5^2 & 1 \\
1 & \zeta_5^3 & \zeta_5 & \zeta_5^4 & \zeta_5^2 & \zeta_5 & \zeta_5^4 & \zeta_5^2 & 1 & \zeta_5^3 & \zeta_5^2 & 1 & \zeta_5^3 & \zeta_5 & \zeta_5^4 & \zeta_5^3 & \zeta_5 & \zeta_5^4 & \zeta_5^2 & 1 & \zeta_5^4 & \zeta_5^2 & 1 & \zeta_5^3 & \zeta_5 \\
1 & \zeta_5^2 & \zeta_5^4 & \zeta_5 & \zeta_5^3 & 1 & \zeta_5^2 & \zeta_5^4 & \zeta_5 & \zeta_5^3 & 1 & \zeta_5^2 & \zeta_5^4 & \zeta_5 & \zeta_5^3 & 1 & \zeta_5^2 & \zeta_5^4 & \zeta_5 & \zeta_5^3 & 1 & \zeta_5^2 & \zeta_5^4 & \zeta_5 & \zeta_5^3 \\
1 & \zeta_5^2 & \zeta_5^4 & \zeta_5 & \zeta_5^3 & \zeta_5^4 & \zeta_5 & \zeta_5^3 & 1 & \zeta_5^2 & \zeta_5^3 & 1 & \zeta_5^2 & \zeta_5^4 & \zeta_5 & \zeta_5^2 & \zeta_5^4 & \zeta_5 & \zeta_5^3 & 1 & \zeta_5 & \zeta_5^3 & 1 & \zeta_5^2 & \zeta_5^4 \\
1 & \zeta_5^2 & \zeta_5^4 & \zeta_5 & \zeta_5^3 & \zeta_5^3 & 1 & \zeta_5^2 & \zeta_5^4 & \zeta_5 & \zeta_5 & \zeta_5^3 & 1 & \zeta_5^2 & \zeta_5^4 & \zeta_5^4 & \zeta_5 & \zeta_5^3 & 1 & \zeta_5^2 & \zeta_5^2 & \zeta_5^4 & \zeta_5 & \zeta_5^3 & 1 \\
1 & \zeta_5^2 & \zeta_5^4 & \zeta_5 & \zeta_5^3 & \zeta_5^2 & \zeta_5^4 & \zeta_5 & \zeta_5^3 & 1 & \zeta_5^4 & \zeta_5 & \zeta_5^3 & 1 & \zeta_5^2 & \zeta_5 & \zeta_5^3 & 1 & \zeta_5^2 & \zeta_5^4 & \zeta_5^3 & 1 & \zeta_5^2 & \zeta_5^4 & \zeta_5 \\
1 & \zeta_5^2 & \zeta_5^4 & \zeta_5 & \zeta_5^3 & \zeta_5 & \zeta_5^3 & 1 & \zeta_5^2 & \zeta_5^4 & \zeta_5^2 & \zeta_5^4 & \zeta_5 & \zeta_5^3 & 1 & \zeta_5^3 & 1 & \zeta_5^2 & \zeta_5^4 & \zeta_5 & \zeta_5^4 & \zeta_5 & \zeta_5^3 & 1 & \zeta_5^2 \\
1 & \zeta_5 & \zeta_5^2 & \zeta_5^3 & \zeta_5^4 & 1 & \zeta_5 & \zeta_5^2 & \zeta_5^3 & \zeta_5^4 & 1 & \zeta_5 & \zeta_5^2 & \zeta_5^3 & \zeta_5^4 & 1 & \zeta_5 & \zeta_5^2 & \zeta_5^3 & \zeta_5^4 & 1 & \zeta_5 & \zeta_5^2 & \zeta_5^3 & \zeta_5^4 \\
1 & \zeta_5 & \zeta_5^2 & \zeta_5^3 & \zeta_5^4 & \zeta_5^4 & 1 & \zeta_5 & \zeta_5^2 & \zeta_5^3 & \zeta_5^3 & \zeta_5^4 & 1 & \zeta_5 & \zeta_5^2 & \zeta_5^2 & \zeta_5^3 & \zeta_5^4 & 1 & \zeta_5 & \zeta_5 & \zeta_5^2 & \zeta_5^3 & \zeta_5^4 & 1 \\
1 & \zeta_5 & \zeta_5^2 & \zeta_5^3 & \zeta_5^4 & \zeta_5^3 & \zeta_5^4 & 1 & \zeta_5 & \zeta_5^2 & \zeta_5 & \zeta_5^2 & \zeta_5^3 & \zeta_5^4 & 1 & \zeta_5^4 & 1 & \zeta_5 & \zeta_5^2 & \zeta_5^3 & \zeta_5^2 & \zeta_5^3 & \zeta_5^4 & 1 & \zeta_5 \\
1 & \zeta_5 & \zeta_5^2 & \zeta_5^3 & \zeta_5^4 & \zeta_5^2 & \zeta_5^3 & \zeta_5^4 & 1 & \zeta_5 & \zeta_5^4 & 1 & \zeta_5 & \zeta_5^2 & \zeta_5^3 & \zeta_5 & \zeta_5^2 & \zeta_5^3 & \zeta_5^4 & 1 & \zeta_5^3 & \zeta_5^4 & 1 & \zeta_5 & \zeta_5^2 \\
1 & \zeta_5 & \zeta_5^2 & \zeta_5^3 & \zeta_5^4 & \zeta_5 & \zeta_5^2 & \zeta_5^3 & \zeta_5^4 & 1 & \zeta_5^2 & \zeta_5^3 & \zeta_5^4 & 1 & \zeta_5 & \zeta_5^3 & \zeta_5^4 & 1 & \zeta_5 & \zeta_5^2 & \zeta_5^4 & 1 & \zeta_5 & \zeta_5^2 & \zeta_5^3
\end{bmatrix}$$

4.1.4 Representation of T with respect to the basis of delta functions.

With $b = 1$, Theorem 2[23] tell us that

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \cdot \delta_{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} = \exp(Q(x)) \cdot \delta_{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} = \exp(2\pi i \cdot x_1 x_2/5) \cdot \delta_{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} = \zeta_5^{x_1 x_2} \cdot \delta_{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}.$$

Applying the formula gives the following matrix. Due to the size of the matrix, it is displayed on the next page.

$$T = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \zeta_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \zeta_5^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \zeta_5^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \zeta_5^4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \zeta_5^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \zeta_5^4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \zeta_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \zeta_5^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \zeta_5^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \zeta_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \zeta_5^4 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \zeta_5^2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \zeta_5^4 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \zeta_5^3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \zeta_5^2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \zeta_5
\end{bmatrix}$$

4.1.5 Actions of S and T on the basis functions of $V(\chi)$

The actions of S and T on the basis of $V(\chi)$ are given by

$$S \cdot f_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} = S \cdot \delta_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} + S \cdot i \cdot \delta_{\begin{bmatrix} 0 \\ 2 \end{bmatrix}} + S \cdot (-1) \cdot \delta_{\begin{bmatrix} 0 \\ 4 \end{bmatrix}} + S \cdot (-i) \cdot \delta_{\begin{bmatrix} 0 \\ 3 \end{bmatrix}}$$

$$S \cdot f_{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} = S \cdot \delta_{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} + S \cdot i \cdot \delta_{\begin{bmatrix} 3 \\ 0 \end{bmatrix}} + S \cdot (-1) \cdot \delta_{\begin{bmatrix} 4 \\ 0 \end{bmatrix}} + S \cdot (-i) \cdot \delta_{\begin{bmatrix} 2 \\ 0 \end{bmatrix}}$$

$$S \cdot f_{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} = S \cdot \delta_{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} + S \cdot i \cdot \delta_{\begin{bmatrix} 3 \\ 2 \end{bmatrix}} + S \cdot (-1) \cdot \delta_{\begin{bmatrix} 4 \\ 4 \end{bmatrix}} + S \cdot (-i) \cdot \delta_{\begin{bmatrix} 2 \\ 3 \end{bmatrix}}$$

$$S \cdot f_{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} = S \cdot \delta_{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} + S \cdot i \cdot \delta_{\begin{bmatrix} 3 \\ 4 \end{bmatrix}} + S \cdot (-1) \cdot \delta_{\begin{bmatrix} 4 \\ 3 \end{bmatrix}} + S \cdot (-i) \cdot \delta_{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}$$

$$S \cdot f_{\begin{bmatrix} 1 \\ 3 \end{bmatrix}} = S \cdot \delta_{\begin{bmatrix} 1 \\ 3 \end{bmatrix}} + S \cdot i \cdot \delta_{\begin{bmatrix} 3 \\ 1 \end{bmatrix}} + S \cdot (-1) \cdot \delta_{\begin{bmatrix} 4 \\ 2 \end{bmatrix}} + S \cdot (-i) \cdot \delta_{\begin{bmatrix} 2 \\ 4 \end{bmatrix}}$$

$$S \cdot f_{\begin{bmatrix} 1 \\ 4 \end{bmatrix}} = S \cdot \delta_{\begin{bmatrix} 1 \\ 4 \end{bmatrix}} + S \cdot i \cdot \delta_{\begin{bmatrix} 3 \\ 3 \end{bmatrix}} + S \cdot (-1) \cdot \delta_{\begin{bmatrix} 4 \\ 1 \end{bmatrix}} + S \cdot (-i) \cdot \delta_{\begin{bmatrix} 2 \\ 2 \end{bmatrix}}$$

and

$$T \cdot f_{\begin{bmatrix}0\\1\end{bmatrix}} = T \cdot \delta_{\begin{bmatrix}0\\1\end{bmatrix}} + T \cdot i \cdot \delta_{\begin{bmatrix}0\\2\end{bmatrix}} + T \cdot (-1) \cdot \delta_{\begin{bmatrix}0\\4\end{bmatrix}} + T \cdot (-i) \cdot \delta_{\begin{bmatrix}0\\3\end{bmatrix}}$$

$$T \cdot f_{\begin{bmatrix}1\\0\end{bmatrix}} = T \cdot \delta_{\begin{bmatrix}1\\0\end{bmatrix}} + T \cdot i \cdot \delta_{\begin{bmatrix}3\\0\end{bmatrix}} + T \cdot (-1) \cdot \delta_{\begin{bmatrix}4\\0\end{bmatrix}} + T \cdot (-i) \cdot \delta_{\begin{bmatrix}2\\0\end{bmatrix}}$$

$$T \cdot f_{\begin{bmatrix}1\\1\end{bmatrix}} = T \cdot \delta_{\begin{bmatrix}1\\1\end{bmatrix}} + T \cdot i \cdot \delta_{\begin{bmatrix}3\\2\end{bmatrix}} + T \cdot (-1) \cdot \delta_{\begin{bmatrix}4\\4\end{bmatrix}} + T \cdot (-i) \cdot \delta_{\begin{bmatrix}2\\3\end{bmatrix}}$$

$$T \cdot f_{\begin{bmatrix}1\\2\end{bmatrix}} = T \cdot \delta_{\begin{bmatrix}1\\2\end{bmatrix}} + T \cdot i \cdot \delta_{\begin{bmatrix}3\\4\end{bmatrix}} + T \cdot (-1) \cdot \delta_{\begin{bmatrix}4\\3\end{bmatrix}} + T \cdot (-i) \cdot \delta_{\begin{bmatrix}2\\1\end{bmatrix}}$$

$$T \cdot f_{\begin{bmatrix}1\\3\end{bmatrix}} = T \cdot \delta_{\begin{bmatrix}1\\3\end{bmatrix}} + T \cdot i \cdot \delta_{\begin{bmatrix}3\\1\end{bmatrix}} + T \cdot (-1) \cdot \delta_{\begin{bmatrix}4\\2\end{bmatrix}} + T \cdot (-i) \cdot \delta_{\begin{bmatrix}2\\4\end{bmatrix}}$$

$$T \cdot f_{\begin{bmatrix}1\\4\end{bmatrix}} = T \cdot \delta_{\begin{bmatrix}1\\4\end{bmatrix}} + T \cdot i \cdot \delta_{\begin{bmatrix}3\\3\end{bmatrix}} + T \cdot (-1) \cdot \delta_{\begin{bmatrix}4\\1\end{bmatrix}} + T \cdot (-i) \cdot \delta_{\begin{bmatrix}2\\2\end{bmatrix}}.$$

Determining the coordinates of six by six matrices with respect to the basis of the invariant subspace is rather tedious and will not properly fit on this paper size. Since we have a product of $i = \sqrt{-1}$ with ζ_5, the resulting matrix entries will not be in $\mathbb{Q}[\zeta_5]$. So we will employ SAGE to compute it for us. We will enlarge our ring so that it contains $\sqrt{-1}$.

$$D_1(\chi)(S) = \frac{1}{5} \cdot \begin{bmatrix} 0 & -\zeta_{20}^{7} - \zeta_{20}^{6} - \zeta_{20}^{4} + \zeta_{20}^{3} & -\zeta_{20}^{7} - \zeta_{20}^{6} - \zeta_{20}^{4} + \zeta_{20}^{3} & -\zeta_{20}^{7} - \zeta_{20}^{6} - \zeta_{20}^{4} + \zeta_{20}^{3} & -\zeta_{20}^{7} - \zeta_{20}^{6} - \zeta_{20}^{4} + \zeta_{20}^{3} & -\zeta_{20}^{7} - \zeta_{20}^{6} - \zeta_{20}^{4} + \zeta_{20}^{3} \\ \zeta_{20}^{7} - \zeta_{20}^{6} - \zeta_{20}^{4} - \zeta_{20}^{3} & 0 & \zeta_{20}^{7} - \zeta_{20}^{6} - \zeta_{20}^{4} - \zeta_{20}^{3} & -\zeta_{20}^{7} - \zeta_{20}^{6} + \zeta_{20}^{5} + \zeta_{20}^{4} - \zeta_{20}^{3} - \frac{2}{5}\zeta_{20}^{2} + \frac{2}{5}\zeta_{20} + 1 & \zeta_{20}^{7} + \zeta_{20}^{6} - \zeta_{20}^{5} - \zeta_{20}^{4} + \zeta_{20}^{3} + \frac{2}{5}\zeta_{20}^{2} - \frac{2}{5}\zeta_{20} - 1 & -\zeta_{20}^{7} + \zeta_{20}^{6} + \zeta_{20}^{4} + \zeta_{20}^{3} \\ \zeta_{20}^{7} - \zeta_{20}^{6} - \zeta_{20}^{4} - \zeta_{20}^{3} & -\zeta_{20}^{7} - \zeta_{20}^{6} - \zeta_{20}^{4} + \zeta_{20}^{3} & -\zeta_{20}^{6} + \zeta_{20}^{4} - \frac{2}{5}\zeta_{20}^{2} + 1 & \zeta_{20}^{7} + \zeta_{20}^{6} - \zeta_{20}^{4} - \zeta_{20}^{3} + \frac{2}{5}\zeta_{20}^{2} - 1 & -\zeta_{20}^{7} + \zeta_{20}^{6} + \zeta_{20}^{5} + \zeta_{20}^{4} - \zeta_{20}^{3} + \frac{2}{5}\zeta_{20} & \zeta_{20}^{7} - \zeta_{20}^{5} + \zeta_{20}^{3} - \frac{2}{5}\zeta_{20} \\ \zeta_{20}^{7} - \zeta_{20}^{6} - \zeta_{20}^{4} - \zeta_{20}^{3} & \zeta_{20}^{7} - \zeta_{20}^{6} - \zeta_{20}^{5} + \zeta_{20}^{4} + \zeta_{20}^{3} - \frac{2}{5}\zeta_{20}^{2} - \frac{2}{5}\zeta_{20} + 1 & -\zeta_{20}^{7} + \zeta_{20}^{6} - \zeta_{20}^{4} + \zeta_{20}^{3} + \frac{2}{5}\zeta_{20}^{2} - 1 & \zeta_{20}^{6} + \zeta_{20}^{4} & \zeta_{20}^{7} - \zeta_{20}^{3} & -\zeta_{20}^{7} - \zeta_{20}^{6} + \zeta_{20}^{5} - \zeta_{20}^{4} - \zeta_{20}^{3} + \frac{2}{5}\zeta_{20} \\ \zeta_{20}^{7} - \zeta_{20}^{6} - \zeta_{20}^{4} - \zeta_{20}^{3} & -\zeta_{20}^{7} + \zeta_{20}^{6} + \zeta_{20}^{5} - \zeta_{20}^{4} - \zeta_{20}^{3} + \frac{2}{5}\zeta_{20}^{2} + \frac{2}{5}\zeta_{20} - 1 & \zeta_{20}^{7} + \zeta_{20}^{6} - \zeta_{20}^{5} + \zeta_{20}^{4} + \zeta_{20}^{3} - \frac{2}{5}\zeta_{20} & -\zeta_{20}^{7} + \zeta_{20}^{3} & -\zeta_{20}^{6} - \zeta_{20}^{4} & \zeta_{20}^{7} - \zeta_{20}^{6} + \zeta_{20}^{4} - \zeta_{20}^{3} - \frac{2}{5}\zeta_{20}^{2} + 1 \\ \zeta_{20}^{7} - \zeta_{20}^{6} - \zeta_{20}^{4} - \zeta_{20}^{3} & \zeta_{20}^{7} + \zeta_{20}^{6} + \zeta_{20}^{4} - \zeta_{20}^{3} & -\zeta_{20}^{7} + \zeta_{20}^{5} - \zeta_{20}^{3} + \frac{2}{5}\zeta_{20} & \zeta_{20}^{7} - \zeta_{20}^{6} - \zeta_{20}^{5} - \zeta_{20}^{4} + \zeta_{20}^{3} - \frac{2}{5}\zeta_{20} & -\zeta_{20}^{7} - \zeta_{20}^{6} + \zeta_{20}^{4} + \zeta_{20}^{3} - \frac{2}{5}\zeta_{20}^{2} + 1 & \zeta_{20}^{6} - \zeta_{20}^{4} + \frac{2}{5}\zeta_{20}^{2} - 1 \end{bmatrix} \tag{4.1.1}$$

For $T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$,

$$D_1(\chi)(T) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \zeta_{20}^4 & 0 & 0 & 0 \\ 0 & 0 & 0 & \zeta_{20}^6 - \zeta_{20}^4 + \zeta_{20}^2 - 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\zeta_{20}^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\zeta_{20}^6 \end{bmatrix}.$$

We have our six-dimensional irreducible representations for the generators of $\mathrm{SL}_2(\mathbb{Z}/5\mathbb{Z})$. Its entries are over $\mathbb{Z}[\zeta_{20}]$. Can we find a smaller ring that contains the entries? Yes, as stated in the previous chapter, Riese[26] gives a much tighter bound. He proves that the ring is $\mathbb{Z}[\zeta_4]$ (the Gaussian integers). How can we can construct it?

4.2 The 6-dim Irred. Principal Series Rep. that is integral over $\mathbb{Z}[\zeta_4] = \mathbb{Z}[i]$.

We will employ the procedure from Chapter 2 Section 5. We will also employ the Frobenius Reciprocity Theorem. Let's state it for our application.

Let $G = \mathrm{SL}_2(\mathbb{F}_p)$. Fix the non-trivial one-dimensional character $\chi = \chi_j$ of B, inflated from T, so that

$$\chi \begin{bmatrix} a^i & x \\ 0 & a^{-1} \end{bmatrix} = \xi^{ij}$$

for some j. Consider the representation (space)

$$V = \mathrm{Ind}_B^G(\chi) = \{f : G \to \mathbb{C} : f(bg) = \chi(b)f(g), \forall b \in B, g \in G\}.$$

Let ρ be the representation of G on V. If H is subgroup of G and and $\sigma : H \to \mathbb{C}^\times$ is a one dimensional character (representation) of H, then the **induced representation**,

$\mathrm{Ind}_H^G(\sigma)$, has vector space

$$V = \mathrm{Ind}_H^G(\sigma) = \{f : G \to \mathbb{C} : f(hg) = \sigma(h)f(g),\ \forall g \in G,\ h \in H\}$$

and representation $\rho : G \to \mathrm{GL}(V)$, given by the G-action $[\rho(g)f](x) = f(xg)$ for all $x, g \in G$. The dimension of the induced representation is given by the index of G and H, i.e.,

$$\dim(\mathrm{Ind}_H^G(\sigma)) = [G : H].)$$

Since $\chi_j = \chi$ is a one-dimensional character of B,

$$\dim(V) = [G : B] \cdot 1 = p + 1.$$

Is (ρ, V) an irreducible representation? Yes, for the detailed proofs, see Kirby[16].

In Chapter 2, we learned the character table for B (Table 2.5.1), how to obtain the cosets of $\mathrm{SL}_2(\mathbb{F}_p)/B$, and that the induced six-dimensional representation is irreducible, we can construct it. The six coset representatives are

$$X_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad X_1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad X_2 = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$$

$$X_3 = \begin{bmatrix} 2 & 0 \\ 3 & 3 \end{bmatrix} \quad X_4 = \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix} \quad X_5 = \begin{bmatrix} 0 & 4 \\ 1 & 4 \end{bmatrix},$$

Let $S = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$. Then the induced six-dimensional representation of S is

$$D_1(\chi)(S) = \begin{bmatrix} \chi(X_0^{-1}SX_0) & \chi(X_0^{-1}SX_1) & \cdots & \chi(X_0^{-1}SX_4) & \chi(X_0^{-1}SX_5) \\ \chi(X_1^{-1}SX_0) & \chi(X_1^{-1}SX_1) & \cdots & \chi(X_1^{-1}SX_4) & \chi(X_1^{-1}SX_5) \\ \chi(X_2^{-1}SX_0) & \chi(X_2^{-1}SX_1) & \cdots & \chi(X_2^{-1}SX_4) & \chi(X_2^{-1}SX_5) \\ \chi(X_3^{-1}SX_0) & \chi(X_3^{-1}SX_1) & \cdots & \chi(X_3^{-1}SX_4) & \chi(X_3^{-1}SX_5) \\ \chi(X_4^{-1}SX_0) & \chi(X_4^{-1}SX_1) & \cdots & \chi(X_4^{-1}SX_4) & \chi(X_4^{-1}SX_5) \\ \chi(X_5^{-1}SX_0) & \chi(X_5^{-1}SX_1) & \cdots & \chi(X_5^{-1}SX_4) & \chi(X_5^{-1}SX_5) \end{bmatrix}$$

$$= \begin{bmatrix} \chi\left(\left[\begin{smallmatrix} 0 & 4 \\ 1 & 0 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 4 & 4 \\ 1 & 0 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 4 & 2 \\ 2 & 0 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 2 & 2 \\ 2 & 0 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 1 & 2 \\ 2 & 0 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 4 & 1 \\ 0 & 4 \end{smallmatrix}\right]\right) \\ \chi\left(\left[\begin{smallmatrix} 0 & 4 \\ 1 & 1 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 4 & 4 \\ 2 & 1 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 4 & 2 \\ 3 & 3 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 2 & 2 \\ 0 & 3 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 1 & 2 \\ 1 & 3 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 4 & 1 \\ 1 & 3 \end{smallmatrix}\right]\right) \\ \chi\left(\left[\begin{smallmatrix} 0 & 2 \\ 2 & 1 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 2 & 2 \\ 3 & 1 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 2 & 1 \\ 0 & 3 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 1 & 1 \\ 2 & 3 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 3 & 1 \\ 3 & 3 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 2 & 3 \\ 1 & 2 \end{smallmatrix}\right]\right) \\ \chi\left(\left[\begin{smallmatrix} 0 & 2 \\ 2 & 3 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 2 & 2 \\ 0 & 3 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 2 & 1 \\ 2 & 4 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 1 & 1 \\ 3 & 4 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 3 & 1 \\ 1 & 4 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 2 & 3 \\ 3 & 0 \end{smallmatrix}\right]\right) \\ \chi\left(\left[\begin{smallmatrix} 0 & 2 \\ 2 & 4 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 2 & 2 \\ 1 & 4 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 2 & 1 \\ 3 & 2 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 1 & 1 \\ 1 & 2 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 3 & 1 \\ 0 & 2 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 2 & 3 \\ 4 & 4 \end{smallmatrix}\right]\right) \\ \chi\left(\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 3 & 3 \\ 1 & 3 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 0 & 3 \\ 3 & 3 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 1 & 3 \\ 4 & 3 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 1 & 3 \\ 1 & 4 \end{smallmatrix}\right]\right) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -\zeta_4 & 0 & 0 \\ 0 & 0 & -\zeta_4 & 0 & 0 & 0 \\ 0 & -\zeta_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \zeta_4 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -i & 0 & 0 \\ 0 & 0 & -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Let $T = \left[\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right]$. Then the induced six-dimensional representation of T is

$$D_1(\chi)(T) = \begin{bmatrix} \chi(X_0^{-1}TX_0) & \chi(X_0^{-1}TX_1) & \cdots & \chi(X_0^{-1}TX_4) & \chi(X_0^{-1}TX_5) \\ \chi(X_1^{-1}TX_0) & \chi(X_1^{-1}TX_1) & \cdots & \chi(X_1^{-1}TX_4) & \chi(X_1^{-1}TX_5) \\ \chi(X_2^{-1}TX_0) & \chi(X_2^{-1}TX_1) & \cdots & \chi(X_2^{-1}TX_4) & \chi(X_2^{-1}TX_5) \\ \chi(X_3^{-1}TX_0) & \chi(X_3^{-1}TX_1) & \cdots & \chi(X_3^{-1}TX_4) & \chi(X_3^{-1}TX_5) \\ \chi(X_4^{-1}TX_0) & \chi(X_4^{-1}TX_1) & \cdots & \chi(X_4^{-1}TX_4) & \chi(X_4^{-1}TX_5) \\ \chi(X_5^{-1}TX_0) & \chi(X_5^{-1}TX_1) & \cdots & \chi(X_5^{-1}TX_4) & \chi(X_5^{-1}TX_5) \end{bmatrix}$$

$$= \begin{bmatrix} \chi\left(\left[\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix}2&1\\1&1\end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix}3&3\\1&3\end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix}0&3\\3&3\end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix}1&3\\4&3\end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix}1&3\\1&4\end{smallmatrix}\right]\right) \\ \chi\left(\left[\begin{smallmatrix}1&1\\4&0\end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix}2&1\\4&0\end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix}3&3\\3&0\end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix}0&3\\3&0\end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix}1&3\\3&0\end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix}1&3\\0&1\end{smallmatrix}\right]\right) \\ \chi\left(\left[\begin{smallmatrix}3&3\\4&1\end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix}1&3\\0&1\end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix}4&4\\4&3\end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix}0&4\\1&3\end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix}3&4\\2&3\end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix}3&4\\1&0\end{smallmatrix}\right]\right) \\ \chi\left(\left[\begin{smallmatrix}3&3\\2&4\end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix}1&3\\1&4\end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix}4&4\\3&2\end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix}0&4\\1&2\end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix}3&4\\0&2\end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix}3&4\\4&4\end{smallmatrix}\right]\right) \\ \chi\left(\left[\begin{smallmatrix}3&3\\1&3\end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix}1&3\\4&3\end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix}4&4\\0&4\end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix}0&4\\1&4\end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix}3&4\\4&4\end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix}3&4\\3&1\end{smallmatrix}\right]\right) \\ \chi\left(\left[\begin{smallmatrix}4&0\\4&4\end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix}4&0\\3&4\end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix}3&0\\2&2\end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix}3&0\\0&2\end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix}3&0\\4&2\end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix}0&1\\4&2\end{smallmatrix}\right]\right) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \zeta_4 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \zeta_4 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 0 & 0 \end{bmatrix}.$$

Lets verify the relations $(D_1(\chi)(S) \cdot D_1(\chi)(T))^3 = (D_1(\chi)(S))^2$ and $(D_1(\chi)(S))^4 = \mathbb{1}$. Let $A = (D_1(\chi)(S) \cdot D_1(\chi)(T))^3$. Then

$$A = \left(\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -i & 0 & 0 \\ 0 & 0 & -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 0 & 0 \end{bmatrix} \right)^3$$

$$= \left(\begin{bmatrix} 0 & 0 & 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right)^3$$

$$= \begin{bmatrix} 0 & 0 & 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -i & 0 \\ -i & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}.$$

$$\begin{aligned} (D_1(\chi)(S))^2 &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -\zeta_4 & 0 & 0 \\ 0 & 0 & -\zeta_4 & 0 & 0 & 0 \\ 0 & -\zeta_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \zeta_4 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -\zeta_4 & 0 & 0 \\ 0 & 0 & -\zeta_4 & 0 & 0 & 0 \\ 0 & -\zeta_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \zeta_4 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \\ &= A \\ &= (D_1(\chi)(S) \cdot D_1(\chi)(T))^3. \end{aligned}$$

$$
\begin{aligned}
(D_1(\chi)(S))^4 &= \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}^2 \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \mathbb{1}.
\end{aligned}
$$

The relations hold. So the integral representations are over the the Gaussian integers. This agrees with Riese's Proposition 1[26]. Can we induce the other principal series representations? The answer is yes. We can easily induce the (reducible principal series representations): the 1+St representation and $R_1(1,+) \oplus R_1(2,+)$ (also referred to as $\rho_0' \oplus \rho_0''$). To directly compute the irreducible ones, Nobs' methods are the easiest to compute (as we did for $\mathrm{SL}_2(\mathbb{Z}/3\mathbb{Z})$). These reducible ones are calculated for completeness.

4.3 The Steinberg representation, $N_1(\chi_1)$

Recall that when we computed the Steinberg representation of $\mathrm{SL}_2(\mathbb{Z}/3\mathbb{Z})$, we learned that the $\mathrm{Aut}(M,Q)$ is isomorphic to the dihedral group of order $2(p+1)$ ([34] Theorem 11.4). We have $M=\mathbb{Z}/5\mathbb{Z}\oplus\mathbb{Z}/5\mathbb{Z}$ and $Q(x)=5^{-1}(x_1^2-ux_2^2)=5^{-1}(x_1^2-2x_2^2)$ from [23] Theorem 3. The bilinear form is $B(x,y)=5^{-1}(2x_1y_1-4x_2y_2)=5^{-1}(2x_1y_1+x_2y_2)$. We proceed as before. Lets write down the domain and image of $Q(x)$.

x	x_1	x_2	x_1^2	x_2^2	$x_1^2-2x_2^2$	$Q(x)=(x_1^2-2x_2^2)/3\in\mathbb{Q}/\mathbb{Z}$
0	0	0	0	0	0	0
1	0	1	0	1	$0-2$	$-2/5=3/5$
2	0	2	0	4	$0-8$	$-8/5=2/5$
3	0	3	0	9	$0-18$	$-18/5=2/5$
4	0	4	0	16	$0-32$	$-32/5=3/5$
5	1	0	1	0	$1-0$	$1/5$
6	1	1	1	1	$1-2$	$-1/5=4/5$
7	1	2	1	4	$1-8$	$-7/5=3/5$
8	1	3	1	9	$1-18$	$-17/5=3/5$
9	1	4	1	16	$1-32$	$-31/5=4/5$
10	2	0	4	0	$4-0$	$4/5$
11	2	1	4	1	$4-2$	$2/5$
12	2	2	4	4	$4-8$	$-4/5=1/5$
13	2	3	4	9	$4-18$	$-14/5=1/5$
14	2	4	4	16	$4-32$	$-28/5=2/5$
15	3	0	9	0	$9-0$	$9/5=4/5$

x	x_1	x_2	x_1^2	x_2^2	$x_1^2 - 2x_2^2$	$Q(x) = (x_1^2 - 2x_2^2)/3 \in \mathbb{Q}/\mathbb{Z}$
16	3	1	9	1	$9-2$	$7/5 = 2/5$
17	3	2	9	4	$9-8$	$1/5$
18	3	3	9	9	$9-18$	$-9/5 = 1/5$
19	3	4	9	16	$9-32$	$-23/5 = 2/5$
20	4	0	16	0	$16-0$	$16/5 = 1/5$
21	4	1	16	1	$16-2$	$14/5 = 4/5$
22	4	2	16	4	$16-8$	$8/5 = 3/5$
23	4	3	16	9	$16-18$	$-2/5 = 3/5$
24	4	4	16	16	$16-32$	$-16/5 = 4/5$

Now grouping the images gives us the basis vectors δ_x.

						$Q(x) \in \mathbb{Q}/\mathbb{Z}$
x	x_1	x_2	x_1^2	x_2^2	$x_1^2 - 2x_2^2$	$(x_1^2 - 2x_2^2)/5$
0	0	0	0	0	0	0
5	1	0	1	0	$1-0$	$1/5$
12	2	2	4	4	$4-8$	$-4/5 = 1/5$
13	2	3	4	9	$4-18$	$-14/5 = 1/5$
17	3	2	9	4	$9-8$	$1/5$
18	3	3	9	9	$9-18$	$-9/5 = 1/5$
20	4	0	16	0	$16-0$	$16/5 = 1/5$

x	x_1	x_2	x_1^2	x_2^2	$x_1^2 - 2x_2^2$	$Q(x) \in \mathbb{Q}/\mathbb{Z}$ $(x_1^2 - 2x_2^2)/5$
2	0	2	0	4	$0 - 8$	$-8/5 = 2/5$
3	0	3	0	9	$0 - 18$	$-18/5 = 2/5$
11	2	1	4	1	$4 - 2$	$2/5$
14	2	4	4	16	$4 - 32$	$-28/5 = 2/5$
16	3	1	9	1	$9 - 2$	$7/5 = 2/5$
19	3	4	9	16	$9 - 32$	$-23/5 = 2/5$
1	0	1	0	1	$0 - 2$	$-2/5 = 3/5$
4	0	4	0	16	$0 - 32$	$-32/5 = 3/5$
7	1	2	1	4	$1 - 8$	$-7/5 = 3/5$
8	1	3	1	9	$1 - 18$	$-17/5 = 3/5$
22	4	2	16	4	$16 - 8$	$8/5 = 3/5$
23	4	3	16	9	$16 - 18$	$-2/5 = 3/5$
6	1	1	1	1	$1 - 2$	$-1/5 = 4/5$
9	1	4	1	16	$1 - 32$	$-31/5 = 4/5$
10	2	0	4	0	$4 - 0$	$4/5$
15	3	0	9	0	$9 - 0$	$9/5 = 4/5$
21	4	1	16	1	$16 - 2$	$14/5 = 4/5$
24	4	4	16	16	$16 - 32$	$-16/5 = 4/5$

That is,

$$f_0(x) = \delta_{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}(x),$$

$$f_1(x) = \left(\delta_{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} + \delta_{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} + \delta_{\begin{bmatrix} 2 \\ 3 \end{bmatrix}} + \delta_{\begin{bmatrix} 3 \\ 2 \end{bmatrix}} + \delta_{\begin{bmatrix} 3 \\ 3 \end{bmatrix}} + \delta_{\begin{bmatrix} 4 \\ 0 \end{bmatrix}} \right)(x),$$

$$f_2(x) = \left(\delta_{\begin{bmatrix} 0 \\ 2 \end{bmatrix}} + \delta_{\begin{bmatrix} 0 \\ 3 \end{bmatrix}} + \delta_{\begin{bmatrix} 2 \\ 1 \end{bmatrix}} + \delta_{\begin{bmatrix} 2 \\ 4 \end{bmatrix}} + \delta_{\begin{bmatrix} 3 \\ 1 \end{bmatrix}} + \delta_{\begin{bmatrix} 3 \\ 4 \end{bmatrix}} \right)(x),$$

$$f_3(x) = \left(\delta_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} + \delta_{\begin{bmatrix} 0 \\ 4 \end{bmatrix}} + \delta_{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} + \delta_{\begin{bmatrix} 1 \\ 3 \end{bmatrix}} + \delta_{\begin{bmatrix} 4 \\ 2 \end{bmatrix}} + \delta_{\begin{bmatrix} 4 \\ 3 \end{bmatrix}} \right)(x),$$

and

$$f_4(x) = \left(\delta_{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} + \delta_{\begin{bmatrix} 1 \\ 4 \end{bmatrix}} + \delta_{\begin{bmatrix} 2 \\ 0 \end{bmatrix}} + \delta_{\begin{bmatrix} 3 \\ 0 \end{bmatrix}} + \delta_{\begin{bmatrix} 4 \\ 1 \end{bmatrix}} + \delta_{\begin{bmatrix} 4 \\ 4 \end{bmatrix}} \right)(x).$$

We have our five basis vectors. Using SAGE, we determine the representations to be

$$N_1(\chi_1)(S) = \frac{1}{5}\begin{bmatrix} -1 & -1 & -1 & -1 & -1 \\ -6 & \zeta_5^3+\zeta_5^2+2 & 2\zeta_5^3+2\zeta_5^2 & -2\zeta_5^3-2\zeta_5^2-2 & -\zeta_5^3-\zeta_5^2+1 \\ -6 & 2\zeta_5^3+2\zeta_5^2 & -\zeta_5^3-\zeta_5^2+1 & \zeta_5^3+\zeta_5^2+2 & -2\zeta_5^3-2\zeta_5^2-2 \\ -6 & -2\zeta_5^3-2\zeta_5^2-2 & \zeta_5^3+\zeta_5^2+2 & -\zeta_5^3-\zeta_5^2+1 & 2\zeta_5^3+2\zeta_5^2 \\ -6 & -\zeta_5^3-\zeta_5^2+1 & -2\zeta_5^3-2\zeta_5^2-2 & 2\zeta_5^3+2\zeta_5^2 & \zeta_5^3+\zeta_5^2+2 \end{bmatrix}$$

and

$$N_1(\chi_1)(T) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \zeta_5 & 0 & 0 & 0 \\ 0 & 0 & \zeta_5^2 & 0 & 0 \\ 0 & 0 & 0 & \zeta_5^3 & 0 \\ 0 & 0 & 0 & 0 & \zeta_5^4 \end{bmatrix}.$$

Using SAGE we confirm that $(N_1(\chi_1)(S))^2 = (N_1(\chi_1)(S))^4 = \mathbb{1}$ and $(N_1(\chi_1)(S) \cdot N_1(\chi_1)(T))^3 = (N_1(\chi_1)(S)^2)$. We also confirm that characters agree with Reeder[25]: $\mathrm{Tr}(N_1(\chi_1)(S)) = 1$ and $\mathrm{Tr}(N_1(\chi_1)(T)) = 0$. Using Nobs and Wolfart's[24] method, we can realize the Steinberg representation over $\mathbb{Z}[\zeta_5]$. **However**, as Riese[26] and others state that the Steinberg representation can be realized over $\mathbb{Z}$!

4.4 1+St, the 6-dim. reducible principal series representation of SL(2,5)

We can compute the six dimensional reducible principal series representation of $\mathrm{SL}_2(\mathbb{Z}/5\mathbb{Z})$ that is the direct sum of the trivial and Steinberg representations: $1 + \mathrm{St}$. We can do this by inducing the trivial representation[16].

Let $S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Then the induced six-dimensional representation of S is

$$(1+\mathrm{St})(S) = \begin{bmatrix} \chi(X_0^{-1}SX_0) & \chi(X_0^{-1}SX_1) & \cdots & \chi(X_0^{-1}SX_5) & \\ \chi(X_1^{-1}SX_0) & \chi(X_1^{-1}SX_1) & \cdots & \chi(X_1^{-1}SX_4) & \chi(X_1^{-1}SX_5) \\ \chi(X_2^{-1}SX_0) & \chi(X_2^{-1}SX_1) & \cdots & \chi(X_2^{-1}SX_4) & \chi(X_2^{-1}SX_5) \\ \chi(X_3^{-1}SX_0) & \chi(X_3^{-1}SX_1) & \cdots & \chi(X_3^{-1}SX_4) & \chi(X_3^{-1}SX_5) \\ \chi(X_4^{-1}SX_0) & \chi(X_4^{-1}SX_1) & \cdots & \chi(X_4^{-1}SX_4) & \chi(X_4^{-1}SX_5) \\ \chi(X_5^{-1}SX_0) & \chi(X_5^{-1}SX_1) & \cdots & \chi(X_5^{-1}SX_4) & \chi(X_5^{-1}SX_5) \end{bmatrix}$$

$$= \begin{bmatrix} \chi\left(\left[\begin{smallmatrix} 0 & 4 \\ 1 & 0 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 4 & 4 \\ 1 & 0 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 4 & 2 \\ 2 & 0 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 2 & 2 \\ 2 & 0 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 1 & 2 \\ 2 & 0 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 4 & 1 \\ 0 & 4 \end{smallmatrix}\right]\right) \\ \chi\left(\left[\begin{smallmatrix} 0 & 4 \\ 1 & 1 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 4 & 4 \\ 2 & 1 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 4 & 2 \\ 3 & 3 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 2 & 2 \\ 0 & 3 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 1 & 2 \\ 1 & 3 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 4 & 1 \\ 1 & 3 \end{smallmatrix}\right]\right) \\ \chi\left(\left[\begin{smallmatrix} 0 & 2 \\ 2 & 1 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 2 & 2 \\ 3 & 1 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 2 & 1 \\ 0 & 3 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 1 & 1 \\ 2 & 3 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 3 & 1 \\ 3 & 3 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 2 & 3 \\ 1 & 2 \end{smallmatrix}\right]\right) \\ \chi\left(\left[\begin{smallmatrix} 0 & 2 \\ 2 & 3 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 2 & 2 \\ 0 & 3 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 2 & 1 \\ 2 & 4 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 1 & 1 \\ 3 & 4 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 3 & 1 \\ 1 & 4 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 2 & 3 \\ 3 & 0 \end{smallmatrix}\right]\right) \\ \chi\left(\left[\begin{smallmatrix} 0 & 2 \\ 2 & 4 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 2 & 2 \\ 1 & 4 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 2 & 1 \\ 3 & 2 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 1 & 1 \\ 1 & 2 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 3 & 1 \\ 0 & 2 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 2 & 3 \\ 4 & 4 \end{smallmatrix}\right]\right) \\ \chi\left(\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 3 & 3 \\ 1 & 3 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 0 & 3 \\ 3 & 3 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 1 & 3 \\ 4 & 3 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 1 & 3 \\ 1 & 4 \end{smallmatrix}\right]\right) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Let $T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Then the induced six-dimensional representation of T is

$$(1+\mathrm{St})(T) = \begin{bmatrix} \chi(X_0^{-1}TX_0) & \chi(X_0^{-1}TX_1) & \cdots & \chi(X_0^{-1}TX_4) & \chi(X_0^{-1}TX_5) \\ \chi(X_1^{-1}TX_0) & \chi(X_1^{-1}TX_1) & \cdots & \chi(X_1^{-1}TX_4) & \chi(X_1^{-1}TX_5) \\ \chi(X_2^{-1}TX_0) & \chi(X_2^{-1}TX_1) & \cdots & \chi(X_2^{-1}TX_4) & \chi(X_2^{-1}TX_5) \\ \chi(X_3^{-1}TX_0) & \chi(X_3^{-1}TX_1) & \cdots & \chi(X_3^{-1}TX_4) & \chi(X_3^{-1}TX_5) \\ \chi(X_4^{-1}TX_0) & \chi(X_4^{-1}TX_1) & \cdots & \chi(X_4^{-1}TX_4) & \chi(X_4^{-1}TX_5) \\ \chi(X_5^{-1}TX_0) & \chi(X_5^{-1}TX_1) & \cdots & \chi(X_5^{-1}TX_4) & \chi(X_5^{-1}TX_5) \end{bmatrix}$$

$$= \begin{bmatrix} \chi\left(\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 3 & 3 \\ 1 & 3 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 0 & 3 \\ 3 & 3 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 1 & 3 \\ 4 & 3 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 1 & 3 \\ 1 & 4 \end{smallmatrix}\right]\right) \\ \chi\left(\left[\begin{smallmatrix} 1 & 1 \\ 4 & 0 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 2 & 1 \\ 4 & 0 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 3 & 3 \\ 3 & 0 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 0 & 3 \\ 3 & 0 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 1 & 3 \\ 3 & 0 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 1 & 3 \\ 0 & 1 \end{smallmatrix}\right]\right) \\ \chi\left(\left[\begin{smallmatrix} 3 & 3 \\ 4 & 1 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 1 & 3 \\ 0 & 1 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 4 & 4 \\ 4 & 3 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 0 & 4 \\ 1 & 3 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 3 & 4 \\ 2 & 3 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 3 & 4 \\ 1 & 0 \end{smallmatrix}\right]\right) \\ \chi\left(\left[\begin{smallmatrix} 3 & 3 \\ 2 & 4 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 1 & 3 \\ 1 & 4 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 4 & 4 \\ 3 & 2 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 0 & 4 \\ 1 & 2 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 3 & 4 \\ 0 & 2 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 3 & 4 \\ 4 & 4 \end{smallmatrix}\right]\right) \\ \chi\left(\left[\begin{smallmatrix} 3 & 3 \\ 1 & 3 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 1 & 3 \\ 4 & 3 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 4 & 4 \\ 0 & 4 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 0 & 4 \\ 1 & 4 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 3 & 4 \\ 4 & 4 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 3 & 4 \\ 3 & 1 \end{smallmatrix}\right]\right) \\ \chi\left(\left[\begin{smallmatrix} 4 & 0 \\ 4 & 4 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 4 & 0 \\ 3 & 4 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 3 & 0 \\ 2 & 2 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 3 & 0 \\ 0 & 2 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 3 & 0 \\ 4 & 2 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 0 & 1 \\ 4 & 2 \end{smallmatrix}\right]\right) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

$$((1+\mathrm{St})(S)\cdot(1+\mathrm{St})(T))^3 = \left(\begin{bmatrix} 0&0&0&0&0&1\\ 0&0&0&1&0&0\\ 0&0&1&0&0&0\\ 0&1&0&0&0&0\\ 0&0&0&0&1&0\\ 1&0&0&0&0&0 \end{bmatrix}\begin{bmatrix} 1&0&0&0&0&0\\ 0&0&0&0&0&1\\ 0&1&0&0&0&0\\ 0&0&0&0&1&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0 \end{bmatrix}\right)^3$$

$$= \begin{bmatrix} 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&1&0&0&0&0\\ 0&0&0&0&0&1\\ 0&0&1&0&0&0\\ 1&0&0&0&0&0 \end{bmatrix}^3$$

$$= \begin{bmatrix} 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&1&0&0&0&0\\ 0&0&0&0&0&1\\ 0&0&1&0&0&0\\ 1&0&0&0&0&0 \end{bmatrix}^2 \begin{bmatrix} 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&1&0&0&0&0\\ 0&0&0&0&0&1\\ 0&0&1&0&0&0\\ 1&0&0&0&0&0 \end{bmatrix}$$

$$= \begin{bmatrix} 0&0&0&0&0&1\\ 0&0&1&0&0&0\\ 0&0&0&0&1&0\\ 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&0&1&0&0 \end{bmatrix} \begin{bmatrix} 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&1&0&0&0&0\\ 0&0&0&0&0&1\\ 0&0&1&0&0&0\\ 1&0&0&0&0&0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

$$((1+\mathrm{St})(S))^2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

So,

$$((1+\mathrm{St})(S))^4 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The representations are not in block diagonal form but the characters agree (see calculations in Reeder[25]). That is, $1+St$ is realized over $\mathbb{Z}$ but can we realize the Steinberg representation over $\mathbb{Z}$?

4.5 The Steinberg representation is realized over $\mathbb{Z}$

Since we computed $1+St$ for the prime $p=5$, we can extract the five-dimensional Steinberg representation St out of $1+St$. We need the appropriate basis. Reeder[25] states that $1+St$ is just the permutation representation of G on $\mathbb{P}^1(\mathbb{Z}/p\mathbb{Z})$, and 1 is the subspace of constant functions, the vector space of St is given by

$$St = \left\{ f : \mathbb{P}^1(\mathbb{Z}/p\mathbb{Z})(f) \to \mathbb{C} \qquad \text{such that} \qquad \sum_{\ell \in \mathbb{P}^1(\mathbb{Z}/p\mathbb{Z})(f)} f(\ell) = 0 \right\}.$$

So what is a basis for this space St?

Proposition 4.5.1. *For* $\mathrm{SL}(2,5)$*, the five dimensional Steinberg representation can be realized over* $\mathbb{Z}$ *with the basis*

$$\{(\delta_0-\delta_1),(\delta_1-\delta_2),(\delta_2-\delta_3),(\delta_3-\delta_4),(\delta_4-\delta_5)\}.$$

Proof. Using SAGE, we explicitly compute the Steinberg representations of the generators $S = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$ and $T = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$. They are

$$St(S) = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 1 & -1 \\ 0 & -1 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } St(T) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{bmatrix},$$

and $(St(S) \cdot (St(T))^3 = (St(S))^2$ and $(St(S))^4 = \mathbb{1}$. □

4.6 The 6-dim. rep. that is a direct sum of two 3-dim. irred. rep.

Let a be a generator of $\mathbb{Z}/p\mathbb{Z}$. Let φ denote the Legendre character of the diagonal subgroup of $\mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})$:

$$\varphi\left(\begin{bmatrix} a^i & 0 \\ 0 & a^{-i} \end{bmatrix}\right) = \begin{cases} 1 & \text{if } i \text{ is even} \\ -1 & \text{if } i \text{ is odd} \end{cases}.$$

We can induce a six-dimensional representation that is a direct sum of two three-dimensional representations[16].

Let $S = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$. Then the induced six-dimensional representation of S is

$$\rho(S) = \begin{bmatrix} \chi(X_0^{-1}SX_0) & \chi(X_0^{-1}SX_1) & \cdots & \chi(X_0^{-1}SX_4) & \chi(X_0^{-1}SX_5) \\ \chi(X_1^{-1}SX_0) & \chi(X_1^{-1}SX_1) & \cdots & \chi(X_1^{-1}SX_4) & \chi(X_1^{-1}SX_5) \\ \chi(X_2^{-1}SX_0) & \chi(X_2^{-1}SX_1) & \cdots & \chi(X_2^{-1}SX_4) & \chi(X_2^{-1}SX_5) \\ \chi(X_3^{-1}SX_0) & \chi(X_3^{-1}SX_1) & \cdots & \chi(X_3^{-1}SX_4) & \chi(X_3^{-1}SX_5) \\ \chi(X_4^{-1}SX_0) & \chi(X_4^{-1}SX_1) & \cdots & \chi(X_4^{-1}SX_4) & \chi(X_4^{-1}SX_5) \\ \chi(X_5^{-1}SX_0) & \chi(X_5^{-1}SX_1) & \cdots & \chi(X_5^{-1}SX_4) & \chi(X_5^{-1}SX_5) \end{bmatrix}$$

$$= \begin{bmatrix} \chi\left(\left[\begin{smallmatrix} 0 & 4 \\ 1 & 0 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 4 & 4 \\ 1 & 0 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 4 & 2 \\ 2 & 0 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 2 & 2 \\ 2 & 0 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 1 & 2 \\ 2 & 0 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 4 & 1 \\ 0 & 4 \end{smallmatrix}\right]\right) \\ \chi\left(\left[\begin{smallmatrix} 0 & 4 \\ 1 & 1 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 4 & 4 \\ 2 & 1 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 4 & 2 \\ 3 & 3 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 2 & 2 \\ 0 & 3 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 1 & 2 \\ 1 & 3 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 4 & 1 \\ 1 & 3 \end{smallmatrix}\right]\right) \\ \chi\left(\left[\begin{smallmatrix} 0 & 2 \\ 2 & 1 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 2 & 2 \\ 3 & 1 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 2 & 1 \\ 0 & 3 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 1 & 1 \\ 2 & 3 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 3 & 1 \\ 3 & 3 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 2 & 3 \\ 1 & 2 \end{smallmatrix}\right]\right) \\ \chi\left(\left[\begin{smallmatrix} 0 & 2 \\ 2 & 3 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 2 & 2 \\ 0 & 3 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 2 & 1 \\ 2 & 4 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 1 & 1 \\ 3 & 4 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 3 & 1 \\ 1 & 4 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 2 & 3 \\ 3 & 0 \end{smallmatrix}\right]\right) \\ \chi\left(\left[\begin{smallmatrix} 0 & 2 \\ 2 & 4 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 2 & 2 \\ 1 & 4 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 2 & 1 \\ 3 & 2 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 1 & 1 \\ 1 & 2 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 3 & 1 \\ 0 & 2 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 2 & 3 \\ 4 & 4 \end{smallmatrix}\right]\right) \\ \chi\left(\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 3 & 3 \\ 1 & 3 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 0 & 3 \\ 3 & 3 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 1 & 3 \\ 4 & 3 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 1 & 3 \\ 1 & 4 \end{smallmatrix}\right]\right) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Let $T = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$. Then the induced six-dimensional representation of T is

$$\rho(T) = \begin{bmatrix} \chi(X_0^{-1}TX_0) & \chi(X_0^{-1}TX_1) & \cdots & \chi(X_0^{-1}TX_4) & \chi(X_0^{-1}TX_5) \\ \chi(X_1^{-1}TX_0) & \chi(X_1^{-1}TX_1) & \cdots & \chi(X_1^{-1}TX_4) & \chi(X_1^{-1}TX_5) \\ \chi(X_2^{-1}TX_0) & \chi(X_2^{-1}TX_1) & \cdots & \chi(X_2^{-1}TX_4) & \chi(X_2^{-1}TX_5) \\ \chi(X_3^{-1}TX_0) & \chi(X_3^{-1}TX_1) & \cdots & \chi(X_3^{-1}TX_4) & \chi(X_3^{-1}TX_5) \\ \chi(X_4^{-1}TX_0) & \chi(X_4^{-1}TX_1) & \cdots & \chi(X_4^{-1}TX_4) & \chi(X_4^{-1}TX_5) \\ \chi(X_5^{-1}TX_0) & \chi(X_5^{-1}TX_1) & \cdots & \chi(X_5^{-1}TX_4) & \chi(X_5^{-1}TX_5) \end{bmatrix}$$

$$= \begin{bmatrix} \chi\left(\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 3 & 3 \\ 1 & 3 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 0 & 3 \\ 3 & 3 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 1 & 3 \\ 4 & 3 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 1 & 3 \\ 1 & 4 \end{smallmatrix}\right]\right) \\ \chi\left(\left[\begin{smallmatrix} 1 & 1 \\ 4 & 0 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 2 & 1 \\ 4 & 0 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 3 & 3 \\ 3 & 0 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 0 & 3 \\ 3 & 0 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 1 & 3 \\ 3 & 0 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 1 & 3 \\ 0 & 1 \end{smallmatrix}\right]\right) \\ \chi\left(\left[\begin{smallmatrix} 3 & 3 \\ 4 & 1 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 1 & 3 \\ 0 & 1 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 4 & 4 \\ 4 & 3 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 0 & 4 \\ 1 & 3 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 3 & 4 \\ 2 & 3 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 3 & 4 \\ 1 & 0 \end{smallmatrix}\right]\right) \\ \chi\left(\left[\begin{smallmatrix} 3 & 3 \\ 2 & 4 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 1 & 3 \\ 1 & 4 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 4 & 4 \\ 3 & 2 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 0 & 4 \\ 1 & 2 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 3 & 4 \\ 0 & 2 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 3 & 4 \\ 4 & 4 \end{smallmatrix}\right]\right) \\ \chi\left(\left[\begin{smallmatrix} 3 & 3 \\ 1 & 3 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 1 & 3 \\ 4 & 3 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 4 & 4 \\ 0 & 4 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 0 & 4 \\ 1 & 4 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 3 & 4 \\ 4 & 4 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 3 & 4 \\ 3 & 1 \end{smallmatrix}\right]\right) \\ \chi\left(\left[\begin{smallmatrix} 4 & 0 \\ 4 & 4 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 4 & 0 \\ 3 & 4 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 3 & 0 \\ 2 & 2 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 3 & 0 \\ 0 & 2 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 3 & 0 \\ 4 & 2 \end{smallmatrix}\right]\right) & \chi\left(\left[\begin{smallmatrix} 0 & 1 \\ 4 & 2 \end{smallmatrix}\right]\right) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{bmatrix}.$$

Lets check the relations $(\rho(S) \cdot \rho(T))^3 = (\rho(S))^2$ and $(\rho(S))^4 = \mathbb{1}$.

$$(\rho(S)\cdot\rho(T))^3 = \left(\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{bmatrix}\right)^3$$

$$= \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^3$$

$$= \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

We also have

$$(\rho(S))^2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

so that $(\rho(S))^4 = (\rho(S))^2 = \mathbb{1}$. The relations are verified. They are not in block diagonal form but the traces agree (see Reeder[25]).

CHAPTER 5 Realization Of The Weil Characters For SL(2,p)

In this chapter we state a conjecture of Candelori, briefly discuss Zemel's result [44], introduce the Gauss sum, and attempt to show Wang's basis[36] realizes $R(1,+)$ and $R(n,+)$ over $\mathbb{Z}[(1+\Omega_p)/2]$ via a direct method. This direct method fails.

5.1 Candelori's Basis (Conjectural)

Let $\mathfrak{t} = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$ and $\mathfrak{s} = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$. Let $\rho(\mathfrak{s})$ and $\rho(\mathfrak{t})$ be the Weil representations of $\mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})$ as before.

Let $U = \rho(\mathfrak{s}) \cdot \rho(\mathfrak{t}) \cdot \rho(\mathfrak{s})$, $v_+ = \delta_1 + \delta_{p-1}$ and $v_- = \delta_1 - \delta_{p-1}$. Then, Candelori conjectures that a basis for $R(1,+)$ and $R(n,+)$ is given by

$$\left\{v_+,\ U \cdot v_+,\ U^2 \cdot v_+, \cdots,\ U^{p-1} \cdot v_+\right\}, \tag{5.1.1}$$

and a basis for $R(1,-)$ and $R(n,-)$ is given by

$$\left\{v_-,\ U \cdot v_-,\ U^2 \cdot v_-, \cdots,\ U^{p-1} \cdot v_-\right\}. \tag{5.1.2}$$

For $p = 3, 5, 7, 11, 13, 17, 19, 23$ and 29, we verified that these bases realize the representations over $\mathbb{Z}[\zeta_p]$ but not for smaller rings.

We further conjecture that for $1 \leq a \leq p-1$ and if $v_+ = \delta_a + \delta_{p-a}$ and $v_- = \delta_a - \delta_{p-a}$, then we also obtain an integral basis. The entries for the representations of $R(1,\pm)$ and $R(n,\pm)$ restricted to their respective basis lie in $\mathbb{Z}[\zeta_p]$.

5.2 Zemel's Basis

Zemel[44] gives an integral basis for the $(p-1)/2$ dimensional representations $R(1,-)$ and $R(n,-)$. Let

$$v := \sum_{m=0}^{(p-1)} \frac{\zeta_p^{2m} - \zeta_p^{-2m}}{\sqrt{p}} \delta_m.$$

Then the integral basis is given by

$$\left\{v,\ \rho(\mathfrak{t})\cdot v,\ (\rho(\mathfrak{t}))^2\cdot v,\cdots\ (\rho(\mathfrak{t}))^{\frac{p-3}{2}}\cdot v\right\}.$$

Using SAGE we verified that $R(1,-)$ cannot be realized over a smaller ring.

Calculations with SAGE show that the multiplicative factor $\sqrt{p}$ is not necessary. So let us see if we can prove integrality over $\mathbb{Z}[\zeta_p]$ without it. Let p be an odd prime. We set the first basis vector as

$$v := \sum_{m=0}^{(p-1)}(\zeta_p^{2m}-\zeta_p^{-2m})\delta_m = \begin{bmatrix} 0 \\ \zeta_p^2-\zeta_p^{-2} \\ \zeta_p^4-\zeta_p^{-4} \\ \zeta_p^6-\zeta_p^{-6} \\ \vdots \\ \zeta_p^{p-2}-\zeta_p^{-(p-2)} \\ \zeta_p^{p-1}-\zeta_p^{-(p-1)}. \end{bmatrix}$$

It is clear that the j-th coordinate of v is given by $(\zeta_p^{2j}-\zeta_p^{-2j})$. Recalling $\rho(\mathfrak{t})$ is a diagonal matrix and that $\rho(\mathfrak{t})_{jj}=\zeta_p^{j^2}$ for $0\le j\le p-1$. Routine calculations show that

$$(\rho(\mathfrak{t})^n\cdot v)_j = (\zeta_p^{j^{2n}}\cdot(\zeta_p^{2j}-\zeta_p^{-2j}) = \zeta_p^{2nj+2j}-\zeta_p^{2nj-2j} = \zeta_p^{2j(n+1)}-\zeta_p^{2j(n-1)},$$

That is, for $0\le n\le p-1$,

$$\rho(\mathfrak{t})^n\cdot v = \begin{bmatrix} 0 \\ \zeta_p^{2(n+1)}-\zeta_p^{2(n-1)} \\ \zeta_p^{4(n+1)}-\zeta_p^{4(n-1)} \\ \vdots \\ \zeta_p^{2(p-2)(n+1)}-\zeta_p^{2(p-2)(n-1)} \\ \zeta_p^{2(p-1)(n+1)}-\zeta_p^{2(p-1)(n-1)} \end{bmatrix}$$

Let B be the matrix of the basis vectors,

$$B := \left\{v,\ \rho(\mathfrak{t})\cdot v,\ (\rho(\mathfrak{t}))^2\cdot v, \cdots\ (\rho(\mathfrak{t}))^{\frac{p-3}{2}}\cdot v\right\}.$$

The characteristic polynomial of $\rho(\mathfrak{t})$ is $x^p - 1 = (x-1)(x^{p-1} + x^{p-2} + \cdots + x + 1) = (x-1)\Phi_p(x)$, where $\Phi_p(x)$ is the p-th cyclotomic polynomial. Let $R(1,-)(\mathfrak{t})$ denote the $(p-1)/2$-dimensional irreducible representation of $\mathfrak{t}$ with respect to the basis B. Let $m(x)$ denote the minimal polynomial of $R(1,-)(\mathfrak{t})$. Then $m(x)$ divides the characteristic polynomial of $\rho(\mathfrak{t})$. $R(1,-)(\mathfrak{t})$ has the following form (after making the necessary substitutions):

$$R(1,-)(\mathfrak{t}) = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & -a_{(p-1)/2} \\ 1 & 0 & 0 & \cdots & 0 & -a_{(p-3)/2} \\ 0 & 1 & 0 & \cdots & 0 & -a_{(p-5)/2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & \cdots & 1 & -a_1 \end{bmatrix}. \tag{5.2.1}$$

Let $r = (p-3)/2$. Using the fact that a matrix and its transpose have the same mimimal polynomial and Wang's Lemma 4[36], the minimal polynomial $m(x)$ is given by

$$m(x) = x^{r+1} + a_1 x^r + a_2 x^{r-1} + \cdots + a_r x + a_{r+1}. \tag{5.2.2}$$

Since both polynomials are completely reducible over $\mathbb{Q}[\zeta_p]$, $m(x)$ factors as $(x-1)(x-\theta_1)(x-\theta_2)\cdots(x-\theta_r)$. The θ_i are roots of $x^p - 1$ as well. ζ_p^i where $0 \le i \le p-1$ are roots of $x^p - 1$. There are $(p-1)/2$ roots of $m(x)$. By a theorem of Viète (or Vieta)[35] we can write the coefficients of the polynomial in terms of its roots:

$$\begin{aligned}
a_1 &= -(1 + \theta_1 + \theta_2 + \theta_3 + \cdots + \theta_{(p-1)/2}), \\
a_2 &= (1 \cdot \theta_1 + 1 \cdot \theta_2 + \cdots + 1 \cdot \theta_{(p-1)/2}) \\
&\quad + (\theta_1 \cdot \theta_2 + \theta_1 \cdot \theta_3 + \cdots \theta_1 \cdot \theta_{(p-1)/2}) \\
&\quad + (\theta_2 \cdot \theta_3 + \theta_2 \cdot \theta_4 + \cdots \theta_2 \cdot \theta_{(p-1)/2}) \\
&\qquad\qquad \vdots \\
&\quad + \theta_{(p-3)/2} \cdot \theta_{(p-1)/2}, \\
\vdots &\qquad\qquad \vdots \\
a_{(p-1)/2} &= (-1)^{(p-1)/2} \cdot 1 \cdot \theta_1 \cdot \theta_2 \cdot \cdots \cdot \theta_{(p-3)/2}.
\end{aligned} \tag{5.2.3}$$

Hence the representation $R(1,-)(\mathfrak{t})$ is integral over $\mathbb{Z}[\zeta_p]$.

5.3 The Gauss Sum

Riese's Proposition 4[26] stated that the Weil character ξ can be realized over $R = \mathbb{Z}\left[\frac{1+\sqrt{p}}{2}\right]$ for $p \equiv 5 \mod 8$. We will prove a stronger result that covers all odd primes.

For a prime p, the quadratic Gauss sum can be expressed as follows:

$$\epsilon = \sum_{x \in \mathbb{Z}/p\mathbb{Z}} \zeta_p^{x^2} = \begin{cases} \sqrt{p} & \text{for } p \equiv 1 \mod 4 \\ \\ \sqrt{-p} & \text{for } p \equiv 3 \mod 4 \end{cases} \tag{5.3.1}$$

Remark 5.3.1. *Murthy and Pathak[22] give a proof of this for any natural number n that uses only elementary methods.*

The Kronecker-Weber theorem tells us that every finite abelian extension of $\mathbb{Q}$ is contained in a cyclotomic extension. So $\mathbb{Q}(\sqrt{p}) \subset \mathbb{Q}(\zeta_p)$ and $\mathbb{Q}(\sqrt{-p}) \subset \mathbb{Q}(\zeta_p)$ since quadratic extensions are abelian. The fundamental theorem of Galois theory tells us that for $p \geq 5$ we have

where $\langle \sigma \rangle = \mathrm{Gal}(\mathbb{Q}(\zeta_p)) = \mathbb{Z}/(p-1)\mathbb{Z}$. The generator σ is the automorphism $\sigma : \zeta_p \to \zeta_p^i$

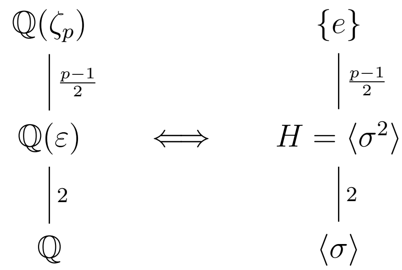

Figure 5.3.1: Quadratic Extension of $\mathbb{Q}(\zeta_p)$

where $\gcd(i, p-1) = 1$. Then $\sigma^2 : \zeta_p \to \zeta_p^{i^2}$. Since 1 and i^2 are squares, σ^2 sends squares to squares. That is, if j is a square, then $\sigma^2 : \zeta^j \to \zeta^{j \cdot i^2}$. Since a product of squares is a square, σ^2 maps squares to squares. If j is not a square, then $\sigma^2 : \zeta_p^j \to \zeta_p^{j \cdot i^2}$. Since ji^2 is not a square, σ^2 sends non-squares to non-squares. σ^2 fixes ζ_p^0. For an odd prime p and a an integer coprime to p, Euler's criterion states

$$\left(\frac{a}{p}\right) \equiv a^{\frac{p-1}{2}} \mod p. \tag{5.3.2}$$

So modulo an odd prime, there are $(p-1)/2$ quadratic residues (excluding 0) and $(p-1)/2$ quadratic non-residues. We can write the quadratic Gauss sum strictly as a sum of powers of ζ_p that are squares:

$$\begin{aligned} \varepsilon &= 1 + \sum_{i \in (\mathbb{Z}/p\mathbb{Z})^\times} \zeta_p^{i^2} \\ &= 1 + 2 \sum_{k \in (\mathbb{Z}/p\mathbb{Z})^\times \ k \ a \ square} \zeta_p^k. \end{aligned} \tag{5.3.3}$$

Then

$$\sum_{k \in (\mathbb{Z}/p\mathbb{Z})^\times \ k \ a \ square} \zeta_p^k = \frac{-1 + \varepsilon}{2}. \tag{5.3.4}$$

We can also write the quadratic Gauss sum strictly as powers of ζ_p that are NOT squares:

$$\begin{aligned}
\varepsilon &= \sum_{i\in(\mathbb{Z}/p\mathbb{Z})} \zeta_p^{i^2} \\
&= 1 + 2 \sum_{k\in(\mathbb{Z}/p\mathbb{Z})^\times \ k \ a \ square} \zeta_p^k \\
&= 1 + 2\left(-1 - \sum_{j\in(\mathbb{Z}/p\mathbb{Z})^\times \ j \ not \ a \ square} \zeta_p^j\right) \\
&= -1 - 2\left(\sum_{j\in(\mathbb{Z}/p\mathbb{Z})^\times \ j \ not \ a \ square} \zeta_p^j\right).
\end{aligned} \tag{5.3.5}$$

Then

$$\sum_{j\in(\mathbb{Z}/p\mathbb{Z})^\times \ j \ not \ a \ square} \zeta_p^j = \frac{-1-\varepsilon}{2}. \tag{5.3.6}$$

Since

$$-\left(\frac{-1-\varepsilon}{2}\right) = \frac{1+\varepsilon}{2} \text{ and } -\left(\frac{-1-\varepsilon}{2}\right) - 1 = \frac{-1+\varepsilon}{2} \tag{5.3.7}$$

we see that they are contained in $\mathbb{Z}[\frac{1+\varepsilon}{2}]$. Since $\mathbb{Z}[\frac{1+\varepsilon}{2}] \subset \mathbb{Q}(\varepsilon)$, we have proved the following lemma:

Lemma 5.3.2. *Let $a, b, c, d \in \mathbb{Z}$. The $\mathbb{Z}$-linear combinations*

$$a \cdot \left(\sum_{i \ a \ square} \zeta_p^i\right) + b \quad \textit{and} \quad c \cdot \left(\sum_{j \ not \ a \ square} \zeta_p^j\right) + d$$

lie in $\mathbb{Q}(\varepsilon)$.

We explore the cases $p = 5, 7, 11, 13, 17, 19, 23$, and 29. The maps in red correspond to exponents of ζ_p that are squares modulo p and those in blue to exponents of ζ_p that are NOT squares module p.

5.3.1 Primes congruent to 1 modulo 4

Case $p = 5$

Since $\gcd(3,4) = 1$, $i = 3$. We have

$$\begin{aligned} \sigma : & 1 \mapsto 1 & \quad \zeta_5^3 &\to \zeta_5^9 = \zeta_5^4 \\ & \zeta_5^1 \to \zeta_5^3 & \quad \zeta_5^4 &\to \zeta_5^{12} = \zeta_5^2 \\ & \zeta_5^2 \to \zeta_5^6 = \zeta_5^1 & & \end{aligned} \tag{5.3.8}$$

with $|\sigma| = 4$. Then

$$\begin{aligned} \sigma^2 : & 1 \mapsto 1 & \quad \zeta_5^3 &\to \zeta_5^2 \\ & \zeta_5^1 \to \zeta_5^4 & \quad \zeta_5^4 &\to \zeta_5 \\ & \zeta_5^2 \to \zeta_5^3 & & \end{aligned}, \tag{5.3.9}$$

and $|\sigma^2| = (5-1)/2 = 2$. The $\mathbb{Z}$-linear combinations of ζ_p^j that belong to $\mathbb{Q}(\varepsilon)$ are those that are fixed by σ^2. The ones that are fixed are $a(\zeta_5 + \zeta_5^4) + b$ and $c(\zeta_5^2 + \zeta_5^3) + d$ where $a, b, c, d \in \mathbb{Z}$. For $p = 5$, $\varepsilon = \sqrt{5}$ since $5 \equiv 1 \mod 4$. We have

$$\begin{aligned} \varepsilon &= \sqrt{5} \\ &= \sum_{x \in \mathbb{Z}/5\mathbb{Z}} \zeta_p^{x^2} \\ &= \zeta_5^{0^2} + \zeta_5^{1^2} + \zeta_5^{2^2} + \zeta_5^{3^2} + \zeta_5^{4^2} \\ &= 1 + \zeta_5^1 + \zeta_5^4 + \zeta_5^9 + \zeta_5^{16} \\ &= 1 + \zeta_5^1 + \zeta_5^4 + \zeta_5^4 + \zeta_5^1 \\ &= 1 + 2\zeta_5^1 + 2\zeta_5^4 \\ &= 1 + 2(-1 - \zeta_5^3 - 2\zeta_5^2) \\ &= -1 + -2(\zeta_5^3 + 2\zeta_5^2). \end{aligned} \tag{5.3.10}$$

This implies

$$\zeta_5^2 + \zeta_5^3 = -\frac{1+\sqrt{5}}{2}. \tag{5.3.11}$$

From our computation of $R(1,+)$, we see that the entries are of the form $c(\zeta_5^2 + \zeta_5^3) + d$. So $R(1,+)$ and $R(n,+)$ are realized over $\mathbb{Z}[\frac{1}{2}(1+\sqrt{5})]$.

Case $p = 13$

Since $\gcd(7,12) = 1$, $i = 7$. (5 does not work). We have

$$\begin{aligned}
\sigma :& 1 \mapsto 1 & \zeta_{13}^{7} &\to \zeta_{13}^{49} = \zeta_{13}^{10}\\
& \zeta_{13}^{1} \to \zeta_{13}^{7} & \zeta_{13}^{8} &\to \zeta_{13}^{56} = \zeta_{13}^{4}\\
& \zeta_{13}^{2} \to \zeta_{13}^{14} = \zeta_{13}^{1} & \zeta_{13}^{9} &\to \zeta_{13}^{63} = \zeta_{13}^{11}\\
& \zeta_{13}^{3} \to \zeta_{15}^{21} = \zeta_{13}^{8} & \zeta_{13}^{10} &\to \zeta_{13}^{70} = \zeta_{13}^{5}\\
& \zeta_{13}^{4} \to \zeta_{13}^{28} = \zeta_{13}^{2} & \zeta_{13}^{11} &\to \zeta_{13}^{77} = \zeta_{13}^{12}\\
& \zeta_{13}^{5} \to \zeta_{13}^{35} = \zeta_{13}^{9} & \zeta_{13}^{12} &\to \zeta_{13}^{84} = \zeta_{13}^{6}\\
& \zeta_{13}^{6} \to \zeta_{13}^{42} = \zeta_{13}^{3} & &
\end{aligned} \tag{5.3.12}$$

with $|\sigma| = 12$. Then

$$\begin{aligned}
\sigma^2 :& 1 \mapsto 1 & \zeta_{13}^{7} &\to \zeta_{13}^{5}\\
& \zeta_{13}^{1} \to \zeta_{13}^{10} & \zeta_{13}^{8} &\to \zeta_{13}^{2}\\
& \zeta_{13}^{2} \to \zeta_{13}^{7} & \zeta_{13}^{9} &\to \zeta_{13}^{12}\\
& \zeta_{13}^{3} \to \zeta_{13}^{4} & \zeta_{13}^{10} &\to \zeta_{13}^{9}\\
& \zeta_{13}^{4} \to \zeta_{13}^{1} & \zeta_{13}^{11} &\to \zeta_{13}^{6}\\
& \zeta_{13}^{5} \to \zeta_{13}^{11} & \zeta_{13}^{12} &\to \zeta_{13}^{3}\\
& \zeta_{13}^{6} \to \zeta_{13}^{8} & &
\end{aligned} \tag{5.3.13}$$

with $|\sigma^2| = (13-1)/2 = 6$. The $\mathbb{Z}$-linear combinations of ζ_p^j that belong to $\mathbb{Q}(\varepsilon)$ are those that are fixed by σ^2. The ones that are fixed are $a(\zeta_{13}^1 + \zeta_{13}^3 + \zeta_{13}^4 + \zeta_{13}^9 + \zeta_{13}^{10} + \zeta_{13}^{12}) + b$

and $c(\zeta_{13}^2+\zeta_{13}^5+\zeta_{13}^6+\zeta_{13}^7+\zeta_{13}^8+\zeta_{13}^{11})+d$. For $p=13$, $\varepsilon=\sqrt{13}$ since $13\equiv 1 \mod 4$. We have

$$\begin{aligned}\varepsilon &= \sqrt{13}\\ &= \sum_{x\in\mathbb{Z}/13\mathbb{Z}} \zeta_p^{x^2} \\ &= 1+2(\zeta_{13}^2+\zeta_{13}^5+\zeta_{13}^6+\zeta_{13}^7+\zeta_{13}^8+\zeta_{13}^{11}).\end{aligned} \tag{5.3.14}$$

This implies

$$\zeta_{13}^2+\zeta_{13}^5+\zeta_{13}^6+\zeta_{13}^7+\zeta_{13}^8+\zeta_{13}^{11}=\frac{-1+\sqrt{13}}{2}. \tag{5.3.15}$$

From our computation of $R(1,+)$, we see that the entries are of the form $c(\zeta_{13}^2+\zeta_{13}^5+\zeta_{13}^6+\zeta_{13}^7+\zeta_{13}^8+\zeta_{13}^{11})+d$. So $R(1,+)$ and $R(n,+)$ are realized over $\mathbb{Z}[\frac{1}{2}(1+\sqrt{13})]$.

Case $p=17$

Since $\gcd(3,16)=1$, $i=3$. We have

$$\begin{aligned}
\sigma :&1 \mapsto 1 & \sigma :&\zeta_{17}^9 \to \zeta_{17}^{27}=\zeta_{17}^{10}\\
&\zeta_{17}^1 \to \zeta_{17}^3 & &\zeta_{17}^{10} \to \zeta_{17}^{30}=\zeta_{17}^{13}\\
&\zeta_{17}^2 \to \zeta_{17}^6 & &\zeta_{17}^{11} \to \zeta_{17}^{33}=\zeta_{17}^{16}\\
&\zeta_{17}^3 \to \zeta_{15}^9 & &\zeta_{17}^{12} \to \zeta_{17}^{36}=\zeta_{17}^{2}\\
&\zeta_{17}^4 \to \zeta_{17}^{12} & &\zeta_{17}^{13} \to \zeta_{17}^{39}=\zeta_{17}^{5}\\
&\zeta_{17}^5 \to \zeta_{17}^{15} & &\zeta_{17}^{14} \to \zeta_{17}^{42}=\zeta_{17}^{8}\\
&\zeta_{17}^6 \to \zeta_{17}^{18}=\zeta_{17}^{1} & &\zeta_{17}^{15} \to \zeta_{17}^{45}=\zeta_{17}^{11}\\
&\zeta_{17}^7 \to \zeta_{17}^{21}=\zeta_{17}^{4} & &\zeta_{17}^{16} \to \zeta_{17}^{48}=\zeta_{17}^{14}\\
&\zeta_{17}^8 \to \zeta_{17}^{24}=\zeta_{17}^{7} & &
\end{aligned} \tag{5.3.16}$$

with $|\sigma| = 16$. Then

$$\begin{aligned}
\sigma^2 :& 1 \mapsto 1 & \sigma^2 :& \zeta_{17}^{9} \to \zeta_{17}^{13} \\
& \zeta_{17}^{1} \to \zeta_{17}^{9} & & \zeta_{17}^{10} \to \zeta_{17}^{5} \\
& \zeta_{17}^{2} \to \zeta_{17}^{1} & & \zeta_{17}^{11} \to \zeta_{17}^{14} \\
& \zeta_{17}^{3} \to \zeta_{17}^{10} & & \zeta_{17}^{12} \to \zeta_{17}^{6} \\
& \zeta_{17}^{4} \to \zeta_{17}^{2} & & \zeta_{17}^{13} \to \zeta_{17}^{15} \\
& \zeta_{17}^{5} \to \zeta_{17}^{11} & & \zeta_{17}^{14} \to \zeta_{17}^{7} \\
& \zeta_{17}^{6} \to \zeta_{17}^{3} & & \zeta_{17}^{15} \to \zeta_{17}^{16} \\
& \zeta_{17}^{7} \to \zeta_{17}^{12} & & \zeta_{17}^{16} \to \zeta_{17}^{8} \\
& \zeta_{17}^{8} \to \zeta_{17}^{4} & &
\end{aligned} \tag{5.3.17}$$

with $|\sigma^2| = (17-1)/2 = 8$. The $\mathbb{Z}$-linear combinations of ζ_p^j that belong to $\mathbb{Q}(\varepsilon)$ are those that are fixed by σ^2. The ones that are fixed are $a(\zeta_{17}^1+\zeta_{17}^2+\zeta_{17}^4+\zeta_{17}^8+\zeta_{17}^9+\zeta_{17}^{13}+\zeta_{17}^{15}+\zeta_{17}^{16})+b$ and $c(\zeta_{17}^3+\zeta_{17}^5+\zeta_{17}^6+\zeta_{17}^7+\zeta_{17}^{10}+\zeta_{17}^{11}+\zeta_{17}^{12}+\zeta_{17}^{14})+d$. For $p = 17$, $\varepsilon = \sqrt{17}$ since $17 \equiv 1 \mod 4$. We have

$$\begin{aligned}
\varepsilon &= \sqrt{17} \\
&= \sum_{x \in \mathbb{Z}/17\mathbb{Z}} \zeta_p^{x^2} \\
&= 1 + 2(\zeta_{17}^1+\zeta_{17}^2+\zeta_{17}^4+\zeta_{17}^8+\zeta_{17}^9+\zeta_{17}^{13}+\zeta_{17}^{15}+\zeta_{17}^{16}) \\
&= 1 + 2(-1-\zeta_{17}^3-\zeta_{17}^5-\zeta_{17}^6-\zeta_{17}^7-\zeta_{17}^{10}+\zeta_{17}^{11}+\zeta_{17}^{12}+\zeta_{17}^{14}).
\end{aligned} \tag{5.3.18}$$

This implies

$$\zeta_{17}^3+\zeta_{17}^5+\zeta_{17}^6+\zeta_{17}^7+\zeta_{17}^{10}+\zeta_{17}^{11}+\zeta_{17}^{12}+\zeta_{17}^{14} = \frac{1-\sqrt{17}}{2}. \tag{5.3.19}$$

From our computation of $R(1,+)$, we see that the entries are of the form $c(\zeta_{17}^3+\zeta_{17}^5+\zeta_{17}^6+\zeta_{17}^7+\zeta_{17}^{10}+\zeta_{17}^{11}+\zeta_{17}^{12}+\zeta_{17}^{14})+d$. So $R(1,+)$ and $R(n,+)$ are realized in $\mathbb{Z}[\frac{1}{2}(1+\sqrt{17})]$.

Case $p = 29$

Since $\gcd(19, 28) = 1$, $i = 19$. We have $|\sigma| = 28$ and $|\sigma^2 = (29-1)/2 = 14$. The $\mathbb{Z}$-linear combinations of ζ_p^j that belong to $\mathbb{Q}(\varepsilon)$ are those that are fixed by σ^2. For $p = 29$, $\varepsilon = \sqrt{29}$ since $29 \equiv 1 \mod 4$. We have

$$\begin{aligned}
\varepsilon &= \sqrt{29} \\
&= \sum_{x \in \mathbb{Z}/29\mathbb{Z}} \zeta_p^{x^2} \\
&= 1 + 2(\zeta_{29}^1 + \zeta_{29}^4 + \zeta_{29}^9 + \zeta_{29}^5 + \zeta_{29}^6 + \zeta_{29}^7 + \zeta_{29}^9 + \zeta_{29}^{13} + \zeta_{29}^{16} + \zeta_{29}^{20} + \zeta_{29}^{22} + \zeta_{29}^{23} \\
&\qquad + \zeta_{29}^{24}) + \zeta_{29}^{25} + \zeta_{29}^{28}) \\
&= 1 + 2(-1 - \zeta_{29}^2 - \zeta_{29}^3 - \zeta_{29}^8 - \zeta_{29}^{10} - \zeta_{29}^{11} - \zeta_{29}^{12} - \zeta_{29}^{14} - \zeta_{29}^{15} - \zeta_{29}^{17} - \zeta_{29}^{18} \\
&\qquad - \zeta_{29}^{19} - \zeta_{29}^{21} - \zeta_{29}^{26} - \zeta_{29}^{27}) \\
&= -1 - \zeta_{29}^2 - \zeta_{29}^3 - \zeta_{29}^8 - \zeta_{29}^{10} - \zeta_{29}^{11} - \zeta_{29}^{12} - \zeta_{29}^{14} - \zeta_{29}^{15} \\
&\qquad - \zeta_{29}^{17} - \zeta_{29}^{18} - \zeta_{29}^{19} - \zeta_{29}^{21} - \zeta_{29}^{26} - \zeta_{29}^{27}).
\end{aligned} \tag{5.3.20}$$

This implies

$$\begin{aligned}
\frac{-1+\sqrt{29}}{2} = \zeta_{29}^2 + \zeta_{29}^3 + \zeta_{29}^8 + \zeta_{29}^{10} + \zeta_{29}^{11} + \zeta_{29}^{12} + \zeta_{29}^{14} + \zeta_{29}^{15} \\
+ \zeta_{29}^{17} + \zeta_{29}^{18} + \zeta_{29}^{19} + \zeta_{29}^{21} + \zeta_{29}^{26} + \zeta_{29}^{27}).
\end{aligned} \tag{5.3.21}$$

From our computation of $R(1, +)$, we see that the entries are of the form

$$\begin{aligned}
a(\zeta_{29}^2 + \zeta_{29}^3 + \zeta_{29}^8 + \zeta_{29}^{10} + \zeta_{29}^{11} + \zeta_{29}^{12} + \zeta_{29}^{14} + \zeta_{29}^{15} \\
+ \zeta_{29}^{17} + \zeta_{29}^{18} + \zeta_{29}^{19} + \zeta_{29}^{21} + \zeta_{29}^{26} + \zeta_{29}^{27}) + b.
\end{aligned} \tag{5.3.22}$$

So $R(1, +)$ and $R(n, +)$ are realized in $\mathbb{Z}[\frac{1}{2}(1+\sqrt{29})]$.

5.3.2 Primes congruent to 3 modulo 4

Case $p = 7$

Since $\gcd(5,6) = 1$, $i = 5$. We have

$$\begin{aligned} \sigma :& 1 \mapsto 1 \\ & \zeta_7^1 \to \zeta_7^5 \\ & \zeta_7^2 \to \zeta_7^{10} = \zeta_7^3 \\ & \zeta_7^3 \to \zeta_7^{15} = \zeta_7^1 \end{aligned} \qquad \begin{aligned} & \zeta_7^4 \to \zeta_7^{20} = \zeta_7^6 \\ & \zeta_7^5 \to \zeta_7^{25} = \zeta_7^4 \\ & \zeta_7^6 \to \zeta_7^{30} = \zeta_7^2 \end{aligned} \tag{5.3.23}$$

with $|\sigma| = 6$. Then

$$\begin{aligned} \sigma^2 :& 1 \mapsto 1 \\ & \zeta_7^1 \to \zeta_7^4 \\ & \zeta_7^2 \to \zeta_7^1 \\ & \zeta_7^3 \to \zeta_7^5 \end{aligned} \qquad \begin{aligned} & \zeta_7^4 \to \zeta_7^2 \\ & \zeta_7^5 \to \zeta_7^6 \\ & \zeta_7^6 \to \zeta_7^3 \end{aligned} \tag{5.3.24}$$

and $|\sigma^2| = (7-1)/2 = 3$. The $\mathbb{Z}$-linear combinations of ζ_p^j that belong to $\mathbb{Q}(\varepsilon)$ are those that are fixed by σ^2. The ones that are fixed are $a(\zeta_7 + \zeta_7^2 + \zeta_7^4) + b$ and $c(\zeta_7^3 + \zeta_7^5 + \zeta_7^6) + d$ where $a, b, c, d \in \mathbb{Z}$. For $p = 7$, $\varepsilon = \sqrt{-7}$ since $7 \equiv 3 \mod 4$. We have

$$\begin{aligned} \varepsilon &= \sqrt{-7} \\ &= \sum_{x \in \mathbb{Z}/7\mathbb{Z}} \zeta_p^{x^2} \\ &= \zeta_7^{0^2} + \zeta_7^{1^2} + \zeta_7^{2^2} + \zeta_7^{3^2} + \zeta_7^{4^2} + \zeta_7^{5^2} + \zeta_7^{6^2} \\ &= 1 + \zeta_7^1 + \zeta_7^4 + \zeta_7^9 + \zeta_7^{16} + \zeta_7^{25} + \zeta_7^{36} \\ &= 1 + \zeta_7^1 + \zeta_7^4 + \zeta_7^2 + \zeta_7^2 + \zeta_7^4 + \zeta_7^1 \\ &= 1 + 2(\zeta_7^1 + \zeta_7^2 + \zeta_7^4). \end{aligned} \tag{5.3.25}$$

This implies

$$\zeta_7^1 + \zeta_7^2 + \zeta_7^4 = \frac{-1+\sqrt{-7}}{2}. \tag{5.3.26}$$

From our computation of $R(1,+)$, we see that the entries are of the form $a(\zeta_7+\zeta_7^2+\zeta_7^4)+b$. So $R(1,+)$ and $R(n,+)$ are realized over $\mathbb{Z}[\frac{1}{2}(1+\sqrt{-7})]$.

Case $p = 11$

Case $p = 11$. Since $\gcd(3,10) = 1$, $i = 3$. We have

$$\begin{aligned}
\sigma :& 1 \mapsto 1 & \zeta_{11}^6 &\to \zeta_{11}^{18} = \zeta_{11}^7 \\
& \zeta_{11}^1 \to \zeta_{11}^3 & \zeta_{11}^7 &\to \zeta_{11}^{21} = \zeta_{11}^{10} \\
& \zeta_{11}^2 \to \zeta_{11}^6 & \zeta_{11}^8 &\to \zeta_{11}^{24} = \zeta_{11}^2 \\
& \zeta_{11}^3 \to \zeta_{11}^9 & \zeta_{11}^9 &\to \zeta_{11}^{27} = \zeta_{11}^5 \\
& \zeta_{11}^4 \to \zeta_{11}^{12} = \zeta_{11}^1 & \zeta_{11}^{10} &\to \zeta_{11}^{30} = \zeta_{11}^8 \\
& \zeta_{11}^5 \to \zeta_{11}^{15} = \zeta_{11}^4 & &
\end{aligned} \tag{5.3.27}$$

with $|\sigma| = 5$. This does not generate the entire Galois group. With $i = 7$, $\gcd(7,10) = 1$ and we have

$$\begin{aligned}
\sigma :& 1 \mapsto 1 & \zeta_{11}^6 &\to \zeta_{11}^{42} = \zeta_{11}^9 \\
& \zeta_{11}^1 \to \zeta_{11}^7 & \zeta_{11}^7 &\to \zeta_{11}^{49} = \zeta_{11}^5 \\
& \zeta_{11}^2 \to \zeta_{11}^{14} = \zeta_{11}^3 & \zeta_{11}^8 &\to \zeta_{11}^{56} = \zeta_{11}^1 \\
& \zeta_{11}^3 \to \zeta_{11}^{21} = \zeta_{11}^{10} & \zeta_{11}^9 &\to \zeta_{11}^{63} = \zeta_{11}^8 \\
& \zeta_{11}^4 \to \zeta_{11}^{28} = \zeta_{11}^6 & \zeta_{11}^{10} &\to \zeta_{11}^{70} = \zeta_{11}^4 \\
& \zeta_{11}^5 \to \zeta_{11}^{35} = \zeta_{11}^2 & &
\end{aligned} \tag{5.3.28}$$

with $|\sigma| = 10$. Then

$$\begin{aligned} \sigma^2 : & 1 \mapsto 1 & \zeta_{11}^6 \to \zeta_{11}^8 \\ & \zeta_{11}^1 \to \zeta_{11}^5 & \zeta_{11}^7 \to \zeta_{11}^2 \\ & \zeta_{11}^2 \to \zeta_{11}^{10} & \zeta_{11}^8 \to \zeta_{11}^7 \\ & \zeta_{11}^3 \to \zeta_{11}^4 & \zeta_{11}^9 \to \zeta_{11}^1 \\ & \zeta_{11}^4 \to \zeta_{11}^9 & \zeta_{11}^{10} \to \zeta_{11}^6 \\ & \zeta_{11}^5 \to \zeta_{11}^3 & \end{aligned} \tag{5.3.29}$$

with $|\sigma^2| = (11-1)/2 = 5$. The $\mathbb{Z}$-linear combinations of ζ_p^j that belong to $\mathbb{Q}(\varepsilon)$ are those that are fixed by σ^2. The ones that are fixed are $a(\zeta_{11}^1 + \zeta_{11}^3 + \zeta_{11}^4 + \zeta_{11}^5 + \zeta_{11}^9) + b$ and $c(\zeta_{11}^2 + \zeta_{11}^6 + \zeta_{11}^7 + \zeta_{11}^8 + \zeta_{11}^{10}) + d$. For $p = 11$, $\varepsilon = \sqrt{-11}$ since $11 \equiv 3 \mod 4$. We have

$$\begin{aligned} \varepsilon &= \sqrt{-11} \\ &= \sum_{x \in \mathbb{Z}/11\mathbb{Z}} \zeta_p^{x^2} \\ &= \zeta_{11}^{0^2} + \zeta_{11}^{1^2} + \zeta_{11}^{2^2} + \zeta_{11}^{3^2} + \zeta_{11}^{4^2} + \zeta_{11}^{5^2} + \\ &\quad \zeta_{11}^{6^2} + \zeta_{11}^{7^2} + \zeta_{11}^{8^2} + \zeta_{11}^{9^2} + \zeta_{11}^{10^2} \\ &= 1 + 2(\zeta_{11}^1 + \zeta_{11}^3 + \zeta_{11}^4 + \zeta_{11}^5 + \zeta_{11}^9) \end{aligned} \tag{5.3.30}$$

This implies

$$\zeta_{11}^1 + \zeta_{11}^3 + \zeta_{11}^4 + \zeta_{11}^5 + \zeta_{11}^9 = \frac{-1 + \sqrt{-11}}{2}. \tag{5.3.31}$$

From our computation of $R(1,+)$, we see that the entries are of the form $a(\zeta_{11}^1 + \zeta_{11}^3 + \zeta_{11}^4 + \zeta_{11}^5 + \zeta_{11}^9) + b$. So, $R(1,+)$ and $R(n,+)$ are realized over $\mathbb{Z}[\frac{1}{2}(1+\sqrt{-11})]$.

Case $p = 19$

Since $\gcd(13, 18) = 1$, $i = 13$. We have $|\sigma| = 18$ and $|\sigma^2 = (19-1)/2 = 9$. The $\mathbb{Z}$-linear combinations of ζ_p^j that belong to $\mathbb{Q}(\varepsilon)$ are those that are fixed by σ^2. The ones that are fixed are $a(\zeta_{19}^1+\zeta_{19}^4+\zeta_{19}^5+\zeta_{19}^6+\zeta_{19}^7+\zeta_{19}^9+\zeta_{19}^{11}+\zeta_{19}^{16}+\zeta_{19}^{17})+b$ and $c(\zeta_{19}^2+\zeta_{19}^3+\zeta_{19}^8+\zeta_{19}^{10}+\zeta_{19}^{12}+\zeta_{19}^{13}+\zeta_{19}^{14}+\zeta_{19}^{15}+\zeta_{19}^{18})+d$. For $p=19$, $\varepsilon=\sqrt{-19}$ since $19\equiv 3 \mod 4$. We have

$$\begin{aligned}\varepsilon &= \sqrt{-19} \\ &= \sum_{x\in\mathbb{Z}/19\mathbb{Z}} \zeta_p^{x^2} \\ &= 1+2(\zeta_{19}^1+\zeta_{19}^4+\zeta_{19}^5+\zeta_{19}^6+\zeta_{19}^7+\zeta_{19}^9+\zeta_{19}^{11}+\zeta_{19}^{16}+\zeta_{19}^{17}).\end{aligned} \tag{5.3.32}$$

This implies

$$\zeta_{19}^1+\zeta_{19}^4+\zeta_{19}^5+\zeta_{19}^6+\zeta_{19}^7+\zeta_{19}^9+\zeta_{19}^{11}+\zeta_{19}^{16}+\zeta_{19}^{17}=\frac{-1+\sqrt{-19}}{2}. \tag{5.3.33}$$

From our computation of $R(1,+)$, we see that the entries are of the form $a(\zeta_{19}^1+\zeta_{19}^4+\zeta_{19}^5+\zeta_{19}^6+\zeta_{19}^7+\zeta_{19}^9+\zeta_{19}^{11}+\zeta_{19}^{16}+\zeta_{19}^{17})+b$. So $R(1,+)$ and $R(n,+)$ are realized in $\mathbb{Z}[\frac{1}{2}(1+\sqrt{-19})]$.

Case $p=23$

Since $\gcd(19,22)=1$, $i=19$. We have $|\sigma|=22$ and $|\sigma^2=(23-1)/2=11$. The $\mathbb{Z}$-linear combinations of ζ_p^j that belong to $\mathbb{Q}(\varepsilon)$ are those that are fixed by σ^2. The ones that are fixed are $a(\zeta_{23}^1+\zeta_{23}^4+\zeta_{23}^5+\zeta_{23}^6+\zeta_{23}^7+\zeta_{23}^9+\zeta_{23}^{11}+\zeta_{23}^{16}+\zeta_{23}^{17})+b$ and $c(\zeta_{23}^2+\zeta_{23}^3+\zeta_{23}^8+\zeta_{23}^{10}+\zeta_{23}^{12}+\zeta_{23}^{13}+\zeta_{23}^{14}+\zeta_{23}^{15}+\zeta_{23}^{18})+d$. For $p=23$, $\varepsilon=\sqrt{-23}$ since $23\equiv 3 \mod 4$. We have

$$\begin{aligned}\varepsilon &= \sqrt{-23} \\ &= \sum_{x\in\mathbb{Z}/23\mathbb{Z}} \zeta_p^{x^2} \\ &= 1+2(\zeta_{23}^1+\zeta_{23}^2+\zeta_{23}^3+\zeta_{23}^4+\zeta_{23}^6+\zeta_{23}^8+\zeta_{23}^9+\zeta_{23}^{12}+\zeta_{23}^{13}+\zeta_{23}^{16}+\zeta_{23}^{18}).\end{aligned} \tag{5.3.34}$$

This implies

$$\zeta_{23}^1+\zeta_{23}^2+\zeta_{23}^3+\zeta_{23}^4+\zeta_{23}^6+\zeta_{23}^8+\zeta_{23}^9+\zeta_{23}^{12}+\zeta_{23}^{13}+\zeta_{23}^{16}+\zeta_{23}^{18}=\frac{-1+\sqrt{-23}}{2}. \tag{5.3.35}$$

From our computation of $R(1,+)$, we see that the entries are of the form $a(\zeta_{23}^1 + \zeta_{23}^2 + \zeta_{23}^3 + \zeta_{23}^4 + \zeta_{23}^6 + \zeta_{23}^8 + \zeta_{23}^9 + \zeta_{23}^{12} + \zeta_{23}^{13} + \zeta_{23}^{16} + \zeta_{23}^{18}) + b$. So $R(1,+)$ and $R(n,+)$ are realized in $\mathbb{Z}[\frac{1}{2}(1+\sqrt{-23})]$.

5.4 Direct verification of the integrality of Wang's basis

Let $\mathfrak{s} = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$ and $\mathfrak{t} = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$. We want to prove that all of the entries of $R(1,+)(\mathfrak{s})$ and $R(1,+)(\mathfrak{t})$ lie in $\mathbb{Z}[(1+\varepsilon)/2]$ where ε is the Gauss quadratic sum. Let $p \geq 5$ be an odd prime. We will soon see that we encounter the very difficult problem of determine quadratic residues and non-residues for an arbitrary prime. In other words, this direct methods fails absolutely for the computation of $\mathbb{R}(1,+)(\mathfrak{s})$.

Let v be vector of dimension p whose entries are 1. That is,

$$v := (1, 1, \cdots, 1)'. \tag{5.4.1}$$

Given the quadratic form $Q(x) = x^2/p$, we know that $\rho(\mathfrak{t})$, the Weil representation of $\mathfrak{t}$, is a diagonal matrix whose (j,j)-th entry is $\zeta_p^{j^2}$. We also see that $\rho(\mathfrak{t})^k$ where $0 \leq k \leq p-1$ is also a diagonal matrix whose (j,j)-th entry is $\zeta_p^{k \cdot j^2}$. Then

$$\rho(\mathfrak{t})^k \cdot v = (1, \zeta_p^k, \zeta_p^{2^2 k}, \zeta_p^{3^2 k}, \cdots, \zeta_p^{(p-1)^2 \cdot k})'. \tag{5.4.2}$$

The invariant subspace W's basis is given by $\{v, \rho(\mathfrak{t})v, \cdots, \rho(\mathfrak{t})^{(p-1)/2} v\}$. Index the basis vectors as b_j where j is the exponent of $\rho(\mathfrak{t})$. So $b_0 = v, b_1 = \rho(\mathfrak{t})v, \cdots, b_{\frac{p-1}{2}} = \rho(\mathfrak{t})^{(p-1)/2} v$. That is,

$$b_0 = (1, 1, \cdots, 1), \tag{5.4.3}$$

$$b_1 = (1, \zeta_p^1, \zeta_p^4, \zeta_p^9, \cdots, \zeta_p^{(p-1)^2}), \tag{5.4.4}$$

$$b_2 = (1, \zeta_p^2, \zeta_p^8, \zeta_p^{18}, \cdots, \zeta_p^{2(p-1)^2}), \tag{5.4.5}$$

$$b_3 = (1, \zeta_p^3, \zeta_p^{12}, \zeta_p^{27}, \cdots, \zeta_p^{3(p-1)^2}), \tag{5.4.6}$$

and so on until

$$b_{\frac{p-1}{2}} = (1, \zeta_p^{(p-1)/2}, \zeta_p^{2^2\cdot(p-1)/2}, \zeta_p^{3^2\cdot(p-1)/2}, \cdots, \zeta_p^{((p-1)^2\cdot(p-1)/2)}. \qquad (5.4.7)$$

We wish to find the coordinates of $R(1,+)$ with respect to this basis. So we multiply $\rho(\mathfrak{t})$ by each of these basis vectors. Then the image of $\rho(\mathfrak{t})\cdot b_0$ is the basis vector $b_1 = \rho(\mathfrak{t})v$. The image of $\rho(\mathfrak{t})\cdot b_1$ is the basis vector $b_2 = \rho(\mathfrak{t})^2 v$. So $\rho(\mathfrak{t})\cdot b_i = b_{i+1}$ for $0 \leq i < (p-1)/2$. For the basis vector $b_{\frac{p-1}{2}}$, we have

$$\begin{aligned} \ell &:= \rho(\mathfrak{t}) \cdot b_{\frac{p-1}{2}} \\ &= \rho(\mathfrak{t}) \cdot \rho(\mathfrak{t})^{(p-1)/2} \cdot v \\ &= \rho(\mathfrak{t})^{(p+1)/2} \cdot v \\ &= (1, \zeta_p^{(p+1)/2}, \zeta_p^{(4\cdot(p+1)/2)}, \cdots, \zeta_p^{((p-1))^2(p+1)/2}). \end{aligned} \qquad (5.4.8)$$

So the (j)-th coordinate of ℓ is $\zeta_p^{j^2\cdot(p+1)/2}$. For the first coordinate, $j=0$, it is 1. The second coordinate is of ℓ is $\zeta_p^{(p+1)/2}$. Denote the i-the coordinate of the a vector v as $v(i)$ and denote the coordinate vector of ℓ with respect to Wang's basis as w. Then

$$\ell = w(0)b_0 + w(1)b_1 + \cdots + w((p-1)/2))b_{(p-1)/2}. \qquad (5.4.9)$$

In coordinates, the vector ℓ is given by

$$\begin{bmatrix} \ell(0) \\ \ell(1) \\ \ell(2) \\ \ell(3) \\ \vdots \\ \ell(p-1) \end{bmatrix} = \begin{bmatrix} w(0)b_0(0) \\ w(0)b_0(1) \\ w(0)b_0(2) \\ w(0)b_0(3) \\ \vdots \\ w(0)b_0\left(\frac{p-1}{2}\right) \end{bmatrix} + \cdots + \begin{bmatrix} w\left(\frac{p-1}{2}\right) b_{\frac{p-1}{2}}(0) \\ w\left(\frac{p-1}{2}\right) b_{\frac{p-1}{2}}(1) \\ w\left(\frac{p-1}{2}\right) b_{\frac{p-1}{2}}(2) \\ w\left(\frac{p-1}{2}\right) b_{\frac{p-1}{2}}(3) \\ \vdots \\ w\left(\frac{p-1}{2}\right) b_{\frac{p-1}{2}}\left(\frac{p-1}{2}\right) \end{bmatrix} \qquad (5.4.10)$$

$$\begin{bmatrix} 1 \\ \zeta_p^{\frac{p+1}{2}} \\ \zeta_p^{\frac{4(p+1)}{2}} \\ \zeta_p^{\frac{9(p+1)}{2}} \\ \vdots \\ \zeta_p^{\frac{(p-1)^2(p+1)}{2}} \end{bmatrix} = \begin{bmatrix} w(0) \\ w(0) \\ w(0) \\ w(0) \\ \vdots \\ w(0) \end{bmatrix} + \begin{bmatrix} w(1) \\ w(1)\zeta_p^1 \\ w(1)\zeta_p^4 \\ w(1)\zeta_p^9 \\ \vdots \\ w(1)\zeta_p^{(p-1)^2} \end{bmatrix} + \cdots + \begin{bmatrix} w\left(\frac{p-1}{2}\right) 1 \\ w\left(\frac{p-1}{2}\right) \zeta_p^{\frac{p-1}{2}} \\ w\left(\frac{p-1}{2}\right) \zeta_p^{2^2 \cdot \frac{p-1}{2}} \\ w\left(\frac{p-1}{2}\right) \zeta_p^{3^2 \cdot \frac{p-1}{2}} \\ \vdots \\ w\left(\frac{p-1}{2}\right) \zeta_p^{(p-1)^2 \cdot \frac{p-1}{2}} \end{bmatrix}. \tag{5.4.11}$$

That is,

$$\begin{aligned} \ell(0) &= 1 \\ &= w(0) \cdot b_0(0) + w(1) \cdot b_1(0) + \cdots + w((p-1)/2) \cdot b_{(p-1)/2}(0) \\ &= w(0) \cdot 1 + w(1) \cdot 1 + \cdots + w((p-1)/2) \cdot 1 \\ &= w(0) + w(1) \cdot 1 + \cdots + w((p-1)/2) \cdot 1, \end{aligned} \tag{5.4.12}$$

$$\begin{aligned} \ell(1) &= \zeta_p^{(p+1)/2} \\ &= w(0) \cdot b_0(1) + w(1) \cdot b_1(1) + \cdots + w((p-1)/2) \cdot b_{(p-1)/2}(1) \\ &= w(0) + w(1) \cdot \zeta_p^1 + \cdots + w((p-1)/2) \cdot \zeta_p^{(p-1)/2}, \end{aligned} \tag{5.4.13}$$

$$\begin{aligned} \ell(2) &= \zeta_p^{4(p+1)/2} \\ &= w(0) \cdot b_0(2) + w(1) \cdot b_1(2) + \cdots + w((p-1)/2) \cdot b_{(p-1)/2}(2) \\ &= w(0) + w(1) \cdot \zeta_p^2 + \cdots + w((p-1)/2) \cdot \zeta_p^{2^2 \cdot (p-1)/2}, \end{aligned} \tag{5.4.14}$$

and so on until

$$\begin{aligned}\ell(p-1) &= \zeta_p^{(p-1)^2\cdot(p+1)/2} \\ &= w(0)\cdot b_0((p-1)) + w(1)\cdot b_1((p-1)) + \cdots \\ &\qquad + w((p-1)/2)\cdot b_{(p-1)/2}(p-1) \\ &= w(0) + w(1)\cdot \zeta_p^{(p-1)/2} + \cdots \\ &\qquad + w((p-1)/2)\cdot \zeta_p^{(p-1)/2((p-1)^2)}.\end{aligned} \tag{5.4.15}$$

It is clear that $b_i(0) = 1$ for $0 \leq i \leq (p-1)/2$ and $b_0(k) = 1$ for $0 \leq k \leq (p-1)$. These relations show that $\zeta_p^{j^2\cdot(p+1)/2}$ is an $\mathcal{O}$-linear combination of the sums of squares of ζ_p^k. We want to show that $\mathcal{O}$ is $\mathbb{Z}[\frac{1+\varepsilon}{2}]$.

Let B denote the $p\times\frac{p+1}{2}$ dimensional matrix consisting of the basis vectors $b_0, \cdots b_{\frac{p-1}{2}}$. The matrix B has full column rank so it has a unique left inverse $(B^\intercal B)^{-1}B^\intercal$. Let L be the $p\times\frac{p+1}{2}$-dimensional matrix whose column vectors are $\rho(\mathsf{t})\cdot b_0,\ \rho(\mathsf{t})^2\cdot b_0,\ \cdots, \rho(\mathsf{t})^{\frac{p+1}{2}}\cdot b_0$. Denote W as the coordinate matrix of L with respect to Wang's basis. Then $L = BW$ and since B has a left inverse,

$$W = B^{-1}L = (B^\intercal B)^{-1}B^\intercal \cdot L.$$

So the coordinates of ℓ with respect to Wang's basis are given by $(B^\intercal B)^{-1}B^\intercal\cdot\ell$. Next, we see that

$$B^\intercal B = \begin{bmatrix} b_0^\intercal b_0 & b_0^\intercal b_1 & b_0^\intercal b_2 & \cdots & b_0^\intercal b_{\frac{p-1}{2}} \\ b_1^\intercal b_0 & b_1^\intercal b_1 & b_1^\intercal b_2 & \cdots & b_1^\intercal b_{\frac{p-1}{2}} \\ b_2^\intercal b_0 & b_2^\intercal b_1 & b_2^\intercal b_2 & \cdots & b_2^\intercal b_{\frac{p-1}{2}} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ b_{\frac{p-1}{2}}^\intercal b_0 & b_{\frac{p-1}{2}}^\intercal b_1 & b_{\frac{p-1}{2}}^\intercal b_2 & \cdots & b_{\frac{p-1}{2}}^\intercal b_{\frac{p-1}{2}} \end{bmatrix} \tag{5.4.16}$$

It is clear that $b_0^\intercal b_0 = p,\ b_0^\intercal b_1 = \varepsilon$. Equations 5.3.5 and 5.3.3 showed that

$$\begin{aligned} \varepsilon &= 1 + 2 \sum_{k \in (\mathbb{Z}/p\mathbb{Z})^\times \ k \ a \ square} \zeta_p^k \\ &= -1 - 2 \left(\sum_{j \in (\mathbb{Z}/p\mathbb{Z})^\times \ j \ not \ a \ square} \zeta_p^j \right). \end{aligned} \tag{5.4.17}$$

If k is a square modulo p, then

$$b_0^\intercal b_k = \sum_{i=0}^{p-1} b_k(i) = \zeta_p^{k \cdot i^2} = \varepsilon. \tag{5.4.18}$$

and if k is not square modulo p, then

$$b_0^\intercal b_k = \sum_{i=0}^{p-1} b_k(i) = -\varepsilon. \tag{5.4.19}$$

By a result of Gauss[17], we know that

$$\sum_{x=0}^{p-1} \zeta_p^{ax^2} = \left(\frac{a}{p}\right) \varepsilon \tag{5.4.20}$$

where $\left(\frac{a}{p}\right)$ denotes the Legendre symbol. For $j, k \in \mathbb{Z}/p\mathbb{Z}$,

$$\begin{aligned} b_j^\intercal b_k &= \sum_{i=0}^{p-1} b_j(i) b_k(i) \\ &= \sum_{i=0}^{p-1} \zeta_p^{j \cdot i^2} \cdot \zeta_p^{k \cdot i^2} \\ &= \sum_{i=0}^{p-1} \zeta_p^{(j+k) \cdot i^2} \\ &= \left(\frac{j+k}{p}\right) \varepsilon. \end{aligned} \tag{5.4.21}$$

So we have that

$$b_j^\intercal b_k = \left(\frac{j+k}{p}\right) \varepsilon. \tag{5.4.22}$$

That gives us

$$B^{\intercal}B = \begin{bmatrix} p & \varepsilon & \left(\frac{2}{p}\right)\varepsilon & \cdots & \left(\frac{(p-1)/2}{p}\right)\varepsilon \\ \varepsilon & \left(\frac{2}{p}\right)\varepsilon & \left(\frac{3}{p}\right)\varepsilon & \cdots & \left(\frac{(p+1)/2}{p}\right)\varepsilon \\ \left(\frac{2}{p}\right)\varepsilon & \left(\frac{3}{p}\right)\varepsilon & \left(\frac{4}{p}\right)\varepsilon & \cdots & \left(\frac{(p+3)/2}{p}\right)\varepsilon \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \left(\frac{(p-1)/2}{p}\right)\varepsilon & \left(\frac{(p+1)/2}{p}\right)\varepsilon & \left(\frac{(p+3)/2}{p}\right)\varepsilon & \cdots & \left(\frac{(p-1)}{p}\right)\varepsilon \end{bmatrix} \tag{5.4.23}$$

$$= \varepsilon \cdot \begin{bmatrix} \varepsilon & 1 & \left(\frac{2}{p}\right) & \cdots & \left(\frac{(p-1)/2}{p}\right) \\ 1 & \left(\frac{2}{p}\right) & \left(\frac{3}{p}\right) & \cdots & \left(\frac{(p+1)/2}{p}\right) \\ \left(\frac{2}{p}\right) & \left(\frac{3}{p}\right) & \left(\frac{4}{p}\right) & \cdots & \left(\frac{(p+3)/2}{p}\right) \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \left(\frac{(p-1)/2}{p}\right) & \left(\frac{(p+1)/2}{p}\right) & \left(\frac{(p+3)/2}{p}\right) & \cdots & \left(\frac{(p-1)}{p}\right) \end{bmatrix} \tag{5.4.24}$$

So we proved the following

Lemma 5.4.1. *All but one of the symmetric $(p+1)/2$-dimensional matrix $B^{\intercal}B$'s entries are of the the form $\pm\varepsilon$. The $(0,0)$ entry is p.*

Also we have the following lemma.

Lemma 5.4.2. *The determinant and cofactors of $\varepsilon^{-1}B^{\intercal}B$ lie in $\mathbb{Z}[\varepsilon]$.*

Proof. The first row of the adjugate matrix of $1\varepsilon B^{\intercal}B$ (the cofactors) are all integers. So computing the determinant using the first row yields $C_{00}\varepsilon + d$ where d is an integer. □

The characteristic polynomial of $\rho(\mathfrak{t})$ is $x^p - 1 = (x-1)(x^{p-1} + x^{p-2} + \cdots + x + 1) = (x-1)\Phi_p(x)$, where $\Phi_p(x)$ is the p-th cyclotomic polynomial. Let $R(n,+)(T)$ denote the $(p+1)/2$ representation of T. Let $m(x)$ denote the minimal polynomial of $R(n,+)(T)$. Then $m(x)$ divides the characteristic polynomial of $\rho(\mathfrak{t})$. $R(1,+)(T)$ has the following form (after making the necessary substitutions):

$$R(n,+)(T) = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & -a_{(p+1)/2} \\ 1 & 0 & 0 & \cdots & 0 & -a_{(p-1)/2} \\ 0 & 1 & 0 & \cdots & 0 & -a_{(p-3)/2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & \cdots & 1 & -a_1 \end{bmatrix}. \tag{5.4.25}$$

Let $r = (p-1)/2$. Using the fact that a matrix and its transpose have the same minimal polynomial and Wang's Lemma 4[36], the minimal polynomial $m(x)$ is given by

$$m(x) = x^{r+1} + a_1 x^r + a_2 x^{r-1} + \cdots + a_r x + a_{r+1}. \tag{5.4.26}$$

Since both polynomials are completely reducible over $\mathbb{Q}[\zeta_p]$, $m(x)$ factors as $(x-1)(x-\theta_1)(x-\theta_2)\cdots(x-\theta_r)$. The θ_i are roots of $x^p - 1$ as well. ζ_p^i where $0 \le i \le p-1$ are roots of $x^p - 1$. There are $(p+1)/2$ roots of $m(x)$. By a theorem of Viète (or Vieta)[35] we can write the coefficients of the polynomial in terms of its roots:

$$\begin{aligned} a_1 &= -(1 + \theta_1 + \theta_2 + \theta_3 + \cdots + \theta_{(p-1)/2}), \\ a_2 &= (1 \cdot \theta_1 + 1 \cdot \theta_2 + \cdots + 1 \cdot \theta_{(p-1)/2}) \\ &\quad + (\theta_1 \cdot \theta_2 + \theta_1 \cdot \theta_3 + \cdots \theta_1 \cdot \theta_{(p-1)/2}) \\ &\quad + (\theta_2 \cdot \theta_3 + \theta_2 \cdot \theta_4 + \cdots \theta_2 \cdot \theta_{(p-1)/2}) \\ &\qquad\qquad \vdots \\ &\quad + \theta_{(p-3)/2} \cdot \theta_{(p-1)/2}, \\ \vdots &\qquad\qquad \vdots \\ a_{(p+1)/2} &= (-1)^{(p+1)/2} \cdot 1 \cdot \theta_1 \cdot \theta_2 \cdot \cdots \cdot \theta_{(p-1)/2}. \end{aligned} \tag{5.4.27}$$

The number of summands in a_i is given by

$$\binom{(p+1)/2}{i}. \tag{5.4.28}$$

First we consider the $a_{(p+1)/2}$ term. Since $\theta_j = \zeta_p^{j(p-j)/2}$,

$$\begin{aligned}(-1)^{(p+1)/2} \cdot \prod_{j=0}^{(p-1)/2} \theta_j &= (-1)^{(p+1)/2} \cdot \prod_{j=0}^{(p-1)/2} \zeta_p^{j(p-j)/2} \\ &= (-1)^{(p+1)/2} \zeta_p^x\end{aligned} \tag{5.4.29}$$

where

$$\begin{aligned}x &= \sum_{j=0}^{(p-1)/2} j(p-j)/2 \\ &= \frac{1}{2} \sum_{j=0}^{(p-1)/2} j(p-j) \\ &= \frac{1}{2} \sum_{j=0}^{(p-1)/2} (jp - j^2) \\ &= \frac{1}{2} \left[\frac{p}{2} \left(\frac{p-1}{2} \cdot \frac{p+1}{2} \right) - \frac{1}{6} \left(\frac{p-1}{2} \cdot \frac{p+1}{2} \cdot \left(2 \cdot \frac{p-1}{2} + 1 \right) \right) \right] \\ &= \frac{1}{2} \left[\frac{p}{2} \left(\frac{p-1}{2} \cdot \frac{p+1}{2} \right) - \frac{1}{6} \left(\frac{p-1}{2} \cdot \frac{p+1}{2} \cdot (p) \right) \right] \\ &= \frac{1}{2} \left[\frac{2p}{6} \left(\frac{p-1}{2} \cdot \frac{p+1}{2} \right) \right] \\ &= \frac{p(p-1)(p+1))}{24}.\end{aligned} \tag{5.4.30}$$

We claim that $(p-1)p(p+1)$ is divisible by 24. One of the three terms $p-1, p$, or $p+1$ is divisible by three. Since $p-1$ and $p+1$ are consecutive even numbers, both are divisible by two and one of them is divisible by four. Let k be an even number. If $p-1 = 2k$ then $p-1$ is divisible by 4 and $p-1+2 = p+1 = 2k+2 = 2(k+1)$ is divisible by 2. If k is odd, then $p+1$ is divisible by four while $p-1$ is divisible by 2. So $(p-1)p(p+1)$ is divisible by $3 \cdot 2 \cdot 4 = 24$. If we require $p > 3$, then $(p-1)(p+1)$ is divisible by 24.

Denote this quotient by m. Substituting for x yields

$$\begin{aligned}(-1)^{(p+1)/2} \cdot \prod_{j=0}^{(p-1)/2} \theta_j &= (-1)^{(p+1)/2}\zeta_p^x \\ &= (-1)^{(p+1)/2} \exp\left(\frac{2\pi i}{p} \cdot \frac{p(p^2-1)}{24}\right) \\ &= (-1)^{(p+1)/2} \exp\left(\frac{2\pi i \cdot (p^2-1)}{24}\right) \\ &= (-1)^{(p+1)/2} \exp\left(2\pi i \cdot m\right) \\ &= (-1)^{(p+1)/2}.\end{aligned} \tag{5.4.31}$$

So $a_{(p+1)/2} = 1$ or $a_{(p+1)/2} = -1$ for $p \geq 5$.

For the a_1 term, which consists of $\binom{(p+1)/2}{1} = (p+1)/2$ summands, we note that is the negative of the trace of the representation $R(n,+)(T)$. That is,

$$\begin{aligned} a_1 &= -(1+\theta_1+\cdots\theta_{(p-1)/2}) \\ -a_1 &= (1+\theta_1+\cdots\theta_{(p-1)/2}) \\ &= 1+\sum_{\text{x a square}} \zeta_p^x \\ &= -\sum_{\text{y not a square}} \zeta_p^y \\ &= \frac{1-\varepsilon}{2} \\ &= \mathrm{Tr}(R(n,+)).\end{aligned} \tag{5.4.32}$$

The trace of a matrix is an invariant. The character table of $\mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})$ tells us that the trace of $R(n,+)(T)$ lies in $\mathcal{O} = \mathbb{Z}[\frac{1}{2}(1+\varepsilon)]$. So we have a candidate ring as $a_{(p+1)/2}$ is also contained $\mathcal{O}$. So a_1 can be expressed either as the sum of ζ_p^x where x are all squares (including zero) or non-squares.

We did try to use two facts from number theory. First states that a positive square-free integer can be written as a sum of three squares if the squares are not congruent to 7 modulo 8. However, the set of quadratic residues does not form a group under modulo addition so that approach will not be practical. This issue is evident when we attempt to determine a_2. The a_2 term consists of $\binom{(p+1)/2}{2} = (p+1)/2 \cdot (p-1)/2 \cdot 1/2 = (p^2-1)/8$ elements. By the formula for a_2, we see that it contains $-a_1 - 1$. $-a_1 - 1$ is in $\mathcal{O}$. Let's investigate this pattern and see if we can generalize it to all primes. Let $x = -a_1 - 1$ and let $p = 13$. Then $(p+1)/2 = 7$ and $(p-1)/2 = 6$. Then

$$\begin{aligned}
a_2 &= -x + \theta_1\theta_2 + \theta_1\theta_3 + \theta_1\theta_4 + \theta_1\theta_5 + \theta_1\theta_6 \\
&\quad + \theta_2\theta_3 + \theta_2\theta_4 + \theta_2\theta_5 + \theta_2\theta_6 \\
&\quad + \theta_3\theta_4 + \theta_3\theta_5 + \theta_3\theta_6 \\
&\quad + \theta_4\theta_5 + \theta_4\theta_6 \\
&\quad + \theta_5\theta_6 \\
&= -x + \theta_1(x - \theta_1) \\
&\quad + \theta_2(x - \theta_1 - \theta_2) \\
&\quad + \theta_3(x - \theta_1 - \theta_2 - \theta_3) \\
&\quad + \theta_4(x - \theta_1 - \theta_2 - \theta_3 - \theta_4) \\
&\quad + \theta_5(x - \theta_1 - \theta_2 - \theta_3 - \theta_4 - \theta_5).
\end{aligned} \tag{5.4.33}$$

Expanding the terms yields

$$\begin{aligned}
a_2 &= -x + \theta_1 x - \theta_1^2 \\
&\quad + \theta_2 x - \theta_1\theta_2 - \theta_2^2 \\
&\quad + \theta_3 x - \theta_3\theta_1 - \theta_3\theta_2 - \theta_3^2 \\
&\quad + \theta_4 x - \theta_4\theta_1 - \theta_4\theta_2 - \theta_4\theta_3 - \theta_4^2 \\
&\quad + \theta_5 x - \theta_5\theta_1 - \theta_5\theta_2 - \theta_5\theta_3 - \theta_5\theta_4 - \theta_5^2.
\end{aligned} \tag{5.4.34}$$

Rearranging gives us

$$
\begin{aligned}
a_2 = -x + x(\theta_1 + \theta_2 + \theta_3 + \theta_4 + \theta_5) \\
- \theta_1(\theta_1 + \theta_2 + \theta_3 + \theta_4 + \theta_5) \\
- \theta_2(\theta_2 + \theta_3 + \theta_4 + \theta_5) \\
- \theta_3(\theta_3 + \theta_4 + \theta_5) \\
- \theta_4(\theta_4 + \theta_5) \\
- \theta_5(\theta_6),
\end{aligned}
\tag{5.4.35}
$$

$$
\begin{aligned}
a_2 = -x + x(x - \theta_6) \\
- \theta_1(x - \theta_6) \\
- \theta_2(x - \theta_1 - \theta_6) \\
- \theta_3(x - \theta_1 - \theta_2 - \theta_6) \\
- \theta_4(x - \theta_1 - \theta_2 - \theta_3 - \theta_6) \\
- \theta_5(x - \theta_1 - \theta_2 - \theta_3 - \theta_4 - \theta_5)
\end{aligned}
\tag{5.4.36}
$$

which does not simplify enough. This direct approach fails. In the next chapter, we use Galois-theoretic methods to prove our result.

CHAPTER 6 SL(2,p)'s Minimal Integral Models Of ξ_i

6.1 Introduction

Let $\rho : G \to \mathrm{GL}(V)$ be an irreducible complex representation of a finite group G of exponent e. A famous theorem of Brauer states that there is a choice of basis for V (i.e. a *model* for ρ) so that $\rho(g)$ is a matrix with entries in the cyclotomic field $\mathbb{Q}(\zeta_e)$, for all $g \in G$. Since the entries of the character table for G are always algebraic integers, it is natural to formulate a much stronger conjecture stating that, in fact, the basis can be chosen so that the matrix entries lie in the ring of integers $\mathbb{Z}[\zeta_e] \subseteq \mathbb{Q}(\zeta_e)$ [5],[19]. In this case, we say that ρ has an *integral* model over the ring $R = \mathbb{Z}[\zeta_e]$. Proving the existence of such integral models is a notoriously difficult problem in integral representation theory. Even when existence can be ascertained, the arguments often cannot be directly adapted to explicitly construct such integral models.

In the case $G = \mathrm{SL}_2(\mathbb{F}_p)$, for $p \geq 3$ a prime, Udo Riese [26] proved the existence of integral models over $\mathbb{Z}[\zeta_e]$. The question remains however whether the result is best possible, or whether for this particular family of groups there are integral models defined over proper subrings $R \subset \mathbb{Z}[\zeta_e]$. For each irreducible character χ, the entries of the character table provide a minimal ring of definition $R_{min}(\chi)$, but it is not clear a priori whether such a *minimal* integral model over $R_{min}(\chi)$ actually exists. For example, the character of the Steinberg representation $\chi = St$ (the unique irreducible character of dimension p) has entries in $\mathbb{Z}$, thus $R_{min}(St) = \mathbb{Z}$. It can be shown that an integral model for St exists over $\mathbb{Z}$ [26, Prop. 1], [14], thus providing a minimal integral model for St. Similarly, minimal integral models can easily be found for the characters belonging to the irreducible principal series of G.

In this chapter we explicitly provide minimal integral models for the Weil characters arising from the reducible principal series of $\mathrm{SL}_2(\mathbb{F}_p)$. For each $p \geq 3$, there are precisely two non-isomorphic such irreducible characters (representations) ξ_1 and ξ_2, each of dimension $(p+1)/2$. The entries in the character table give

$$R_{min}(\xi_i) = \mathbb{Z}\left[\frac{1+\sqrt{\varepsilon p}}{2}\right], \quad \varepsilon = \begin{cases} 1 & \text{if } p \equiv 1 \mod 4 \\ -1 & \text{if } p \equiv 3 \mod 4 \end{cases}$$

for both $i = 1, 2$. We therefore seek explicit integral models defined over this quadratic ring $R = R_{min}(\xi_i)$, which in all cases coincides with the ring of integers of $\mathbb{Q}(\sqrt{\varepsilon p})$. In [26], it is conjectured that such minimal integral models should always exist, and existence is proved under the restriction $p \equiv 5 \mod 8$ [26, Prop. 4]. The methods of [26] are based on class-field theory and do not provide *explicit* integral models even under the more restrictive assumptions. In this chapter (Theorems 6.3.1 and 6.3.3) we prove the existence of minimal integral models for any prime p, with no restrictions, and we provide them explicitly. Our methods are based on recent work of Yilong Wang [36], who provided explicit integral models for ξ_i over $\mathbb{Z}[\zeta_p]$. By extensive explicit calculations, we observed that in fact Wang's models are defined over the minimal rings $R_{min}(\xi_i)$. This is what we prove in this chapter, by studying the action of the Galois group $\mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$ on Wang's integral models.

We now summarize the contents of this chapter. In Section 6.2, we collect all the necessary formulas in the section regarding the Weil representation and define a more modern set of notation. The notation will differ from the previous chapters. We explain how to obtain explicit models for the Weil characters of $\mathrm{SL}_2(\mathbb{F}_p)$ from the Weil representation, and we recall Wang's construction of integral models for ξ_1 and ξ_2 [36]. In Section 6.3 we prove our main results Thms. 6.3.1, 6.3.2 and 6.3.3, which show that Wang's integral models are in fact *minimal*. Finally, in Section 6.4 we provide explicit examples of the minimal integral models for $p = 7$ and $p = 13$ and for both ξ_1 and ξ_2.

6.2 Integral models for Weil characters

We write the Weil representations of the generators of $\mathrm{SL}_2(\mathbb{F}_p)$,$\mathfrak{s} = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$ and $\mathfrak{t} = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$, in terms of a quadratic form Q and its associated bilinear form B:

$$\rho_Q(\mathfrak{s}) = M(\psi_{\mathfrak{s}}) = \frac{1}{\sqrt{\varepsilon p}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & e^{-2\pi i B(1,1)} & e^{-2\pi i B(1,2)} & \cdots & e^{-2\pi i B(1,p-1)} \\ 1 & e^{-2\pi i B(2,1)} & e^{-2\pi i B(2,2)} & \cdots & e^{-2\pi i B(2,p-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & e^{-2\pi i B(p-1,1)} & e^{-2\pi i B(p-1,2)} & \cdots & e^{-2\pi i B(p-1,p-1)} \end{bmatrix} \quad (6.2.1)$$

and

$$\rho_Q(\mathfrak{t}) = M(\psi_{\mathfrak{t}}) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & e^{2\pi i Q(1)} & 0 & \cdots & 0 & 0 \\ 0 & 0 & e^{2\pi i Q(2)} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \\ 0 & 0 & 0 & \cdots & 0 & e^{2\pi i Q(p-1)} \end{bmatrix}. \quad (6.2.2)$$

The isomorphism class of each Weil representation only depends on the equivalence class of the quadratic form Q. Note that there are two nonequivalent quadratic forms on $C_p = (\mathbb{Z}/p\mathbb{Z}, +)$:

$$Q_1(x) = x^2/p \quad \text{and} \quad Q_2(x) = cx^2/p \quad (6.2.3)$$

where c is a quadratic non-residue modulo p. Therefore we obtain two non-isomorphic Weil representations

$$\rho_1, \rho_2 : \mathrm{SL}_2(\mathbb{F}_p) \longrightarrow \mathrm{GL}(V), \quad (6.2.4)$$

along with their explicit (non-integral) models defined over the ring $\mathbb{Z}[1/p, \zeta_p]$.

The Weil representations ρ_1, ρ_2 defined by (6.2.4) decompose into irreducible representations $\rho_i = \xi_i \oplus \pi_i$, $i = 1, 2$. The characters ξ_1, ξ_2 are the two *Weil characters* of

$\mathrm{SL}_2(\mathbb{F}_p)$ of dimension $(p+1)/2$; they belong to the *principal series* of $\mathrm{SL}_2(\mathbb{F}_p)$. The other two Weil characters π_1, π_2 are of dimension $(p-1)/2$ and they belong to the *cuspidal series.*

We now construct an explicit integral model over $\mathbb{Z}[\zeta_p]$ for the principal series Weil characters ξ_1, ξ_2, following [36]. Let $V^+ \subseteq V = \{\text{functions: } C_p \to \mathbb{C}\}$ be the subspace of *even* functions, satisfying $f(-x) = f(x)$ for all $x \in C_p$. This subspace is an invariant subspace for both ρ_1 and ρ_2, and the principal series Weil characters ξ_1, ξ_2 are the restrictions of ρ_1, ρ_2 to V^+. To write down an explicit model for these characters, consider the basis for V^+ given by the even delta functions:

$$b_0^+ := \delta_0,\ b_1^+ := \delta_1 + \delta_{p-1},\ \cdots,\ b_{(p-1)/2}^+ := \delta_{(p-1)/2} + \delta_{(p+1)/2}.$$

Then the explicit model for the Weil representation given by (6.2.1) and (6.2.2) can be used to give an explicit model for ξ_1 and ξ_2 over the ring $\mathbb{Z}[1/p, \zeta_p]$. Let S and T denote the matrices of the Weil representations $\rho(\mathfrak{s})$ and $\rho(\mathfrak{t})$ respectively, restricted to the basis of even functions. Then

$$S_{jk} = \begin{cases} \dfrac{\sqrt{\varepsilon p}}{\varepsilon p} & 0 \le j \le \frac{p-1}{2} \text{ and } k = 0 \\ \dfrac{2\sqrt{\varepsilon p}}{\varepsilon p} & j = 0 \text{ and } 1 \le k \le \frac{p-1}{2} \\ \dfrac{1}{\sqrt{\varepsilon p}}\left(e^{-2\pi i B(j,k)} + e^{2\pi i B(j,k)}\right) & \text{otherwise} \end{cases} \tag{6.2.5}$$

and

$$T_{jk} = \begin{cases} e^{2\pi i Q(j)} & \text{if } j = k \\ 0 & \text{otherwise.} \end{cases} \tag{6.2.6}$$

Note that these models are not integral, since they can only be defined over the ring $\mathbb{Z}[1/p, \zeta_p]$.

Remark 6.2.1. *The $(p-1)/2$-dimensional cuspidal series representations π_1 and π_2 can be constructed similarly by restricting ρ to the subspace $V^- \subseteq V$ of* odd *functions, those satisfying $f(-x) = -f(x)$. This subspace has basis*

$$b_0^- := \delta_1 - \delta_{(p-1)}, \quad \cdots, \quad b_{(p-3)/2}^- := \delta_{(p-1)/2} - \delta_{(p+1)/2}.$$

Let S and T denote the matrices of the Weil representations $\rho(\mathfrak{s})$ and $\rho(\mathfrak{t})$ respectively, restricted to the basis of odd delta functions. Then S and T are given by

$$S_{jk} = \frac{1}{\sqrt{\varepsilon p}} \left(e^{-2\pi i B(j+1,k+1)} - e^{2\pi i B(j+1,k+1)} \right),$$

and

$$T_{jk} = \begin{cases} e^{2\pi i Q(j+1)} & \text{if } \; j = k \\ 0 & \text{otherwise.} \end{cases}$$

These formulas provide explicit models for π_1 and π_2 over $\mathbb{Z}[1/p, \zeta_p]$.

Wang [36] provides an *integral* model for ξ_2 over the ring $\mathbb{Z}[\zeta_p]$, by using a basis consisting of circulant vectors. His construction can also be employed to provide an integral model for ξ_1, also defined over the ring $\mathbb{Z}[\zeta_p]$. Note that the existence of such integral models for ξ_1, ξ_2 was proved by Riese in [26], but no explicit integral models were given. As far as the authors know, the first integral models for ξ_1 were later given by Gilmore, Massbaum, and van Wamelen[13] and Wang [36] was the first to provide integral models for ξ_2.

To recall Wang's construction, let $Q = Q_1, Q_2$ be one of the quadratic forms (6.2.3), corresponding to each Weil character ξ_1, ξ_2. Let $r = (p-1)/2$ and let $\theta_j = e^{2\pi i Q(j)}$ for $0 \leq j \leq r$, be the eigenvalues of the T-matrix (6.2.2). Note that $\theta_j \neq \theta_k$ for all $j \neq k$.

Consequently, the Vandermonde matrix

$$V_Q = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \theta_1 & \theta_1^2 & \theta_1^3 & \cdots & \theta_1^r \\ 1 & \theta_2 & \theta_2^2 & \theta_2^3 & \cdots & \theta_2^r \\ 1 & \theta_3 & \theta_3^2 & \theta_3^3 & \cdots & \theta_3^r \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & \theta_r & \theta_r^2 & \theta_r^3 & \cdots & \theta_r^r \end{bmatrix} \tag{6.2.7}$$

is invertible. Wang proves the following:

Theorem 6.2.2 ([36], Thm. 1)**.** *Suppose $p \geq 5$ is an odd prime. Let S, T be the matrices* (6.2.5) *and* (6.2.6)*. Then the matrices $V_Q^{-1} S V_Q$ and $V_Q^{-1} T V_Q$ have entries in $\mathbb{Z}[\zeta_p]$.*

By Wang's theorem, setting

$$\xi_i(\mathfrak{s}) = V_{Q_i}^{-1} S V_{Q_i}, \quad \xi_i(\mathfrak{t}) = V_{Q_i}^{-1} T V_{Q_i}$$

yields explicit integral models for ξ_i, $i = 1, 2$ over the ring $\mathbb{Z}[\zeta_p]$.

Remark 6.2.3. *In his proof, Wang shows that the representation ξ_2 is defined over $\mathbb{Z}[\zeta_p]$ for $p \equiv 1 \mod 4$ and $\mathbb{Z}[\zeta_p, i]$ for $p \equiv 3 \mod 4$. However, upon further inspection of Proposition 2.4, the proof readily generalizes to show integrality over $\mathbb{Z}[\zeta_p]$ for all odd primes and for both ξ_1 and ξ_2.*

Remark 6.2.4. *As mentioned in the previous chapter, Zemel [44] has recently constructed integral models over $\mathbb{Z}[\zeta_p]$ for the cuspidal series Weil characters π_1 and π_2.*

6.3 Minimal integral models and proof of the Main Theorem

In this section we prove that the integral models given by Theorem 6.2.2 are in fact defined over the ring of integers of $\mathbb{Q}(\sqrt{\varepsilon p})$, thus providing *minimal* integral models for ξ_1 and ξ_2. We do so by analyzing the action of the Galois group $\mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$ on the integral models. The key observation in this analysis is a surprising compatibility between the Galois actions on the Vandermonde matrix V_Q and on the entries of the matrices of the Weil representations ρ_Q, for $Q = Q_1, Q_2$. Investigating this compatibility for more general quadratic forms Q might be of independent interest.

Recall that for an odd prime p, the quadratic Gauss sums can be evaluated as follows:

$$\sum_{x \in \mathbb{Z}/p\mathbb{Z}} \zeta_p^{x^2} = \begin{cases} \sqrt{p} & \text{for } p \equiv 1 \mod 4 \\ \\ \sqrt{-p} & \text{for } p \equiv 3 \mod 4, \end{cases}$$

and therefore the quadratic field $\mathbb{Q}(\sqrt{\varepsilon p})$ is always contained in $\mathbb{Q}(\zeta_p)$. By the fundamental theorem of Galois theory, this subfield must correspond to the subgroup $H \subseteq \mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$ of index 2,

$$\begin{array}{ccc} \mathbb{Q}(\zeta_p) & & \{e\} \\ \Big| {\scriptstyle r} & & \Big| {\scriptstyle r} \\ \mathbb{Q}(\sqrt{\varepsilon p}) & \Longleftrightarrow & H = \langle \gamma^2 \rangle \\ \Big| {\scriptstyle 2} & & \Big| {\scriptstyle 2} \\ \mathbb{Q} & & \langle \gamma \rangle \end{array} \tag{6.3.1}$$

where in the diagram we chose a generator $\langle \gamma \rangle = \mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q}) \simeq \mathbb{Z}/(2r)\mathbb{Z}$ and $r = (p-1)/2$. The generator γ can be written down as an automorphism $\gamma : \zeta_p \to \zeta_p^j$ where $\gcd(j, 2r) = 1$. Then $\gamma^2 : \zeta_p \to \zeta_p^{j^2}$ acts on the exponents of ζ_p by sending squares to squares and non-squares to non-squares. We can write the sums of square exponents and sums of nonsquares exponents in terms of the quadratic Gauss sum:

$$\sum_{k \in (\mathbb{F}_p)^\times,\ k \ a \ square} \zeta_p^k = \frac{-1 + \sqrt{\varepsilon p}}{2} \quad \text{and} \quad \sum_{j \in (\mathbb{F}_p)^\times,\ j \ not \ a \ square} \zeta_p^j = \frac{-1 - \sqrt{\varepsilon p}}{2}$$

and from Remark 2.1.10, it is clear that they are always contained in the ring of integers $\mathbb{Z}[\frac{1}{2}(1 + \sqrt{\varepsilon p})] \subset \mathbb{Q}(\sqrt{\varepsilon p})$.

Let S and T be the matrices given in (6.2.5) and (6.2.6), let V_Q be the Vandermonde matrix (6.2.7), and let $T' = V_Q^{-1} T V_Q$. We first prove the integrality of T' over the quadratic ring:

Theorem 6.3.1. *The matrix entries of T' lie in $\mathbb{Z}[\frac{1}{2}(1+\sqrt{\varepsilon p})]$, the ring of integers of $\mathbb{Q}(\sqrt{\varepsilon p})$.*

Proof. Note that T' is the following circulant matrix

$$T' = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & -a_{r+1} \\ 1 & 0 & 0 & \cdots & 0 & -a_r \\ 0 & 1 & 0 & \cdots & 0 & -a_{r-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & \cdots & 1 & -a_1 \end{bmatrix},$$

whose characteristic polynomial $m(x) \in \mathbb{Q}(\zeta_p)[x]$ is given by

$$m(x) = x^{r+1} + a_1 x^r + a_2 x^{r-1} + \cdots + a_r x + a_{r+1}.$$

Thus to prove the Theorem it suffices to show that the coefficients of $m(x)$ are contained in $\mathbb{Z}[\frac{1}{2}(1+\sqrt{\varepsilon p})]$. Since T and T' are conjugate matrices, we know that $m(x)$ splits as $(x-1)(x-\theta_1)(x-\theta_2)\cdots(x-\theta_r)$ in $\mathbb{Q}(\zeta_p)$, where the θ_i's are the eigenvalues/diagonal entries of T. In the case $Q = Q_1$, each θ_j is of the form ζ_p^s, where s is a square mod p, while in the case $Q = Q_2$, each θ_i is of the form ζ_p^n, where n is not a square mod p. In each case, the set of roots of the polynomial $m(x)$ is permuted by the index 2 subgroup $H \subseteq \mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$ defined in (6.3.1). So H fixes the coefficients of $m(x)$, since those are expressible in terms of elementary symmetric polynomials in the θ_j's. Therefore the coefficients a_i of $m(x)$ lie in the quadratic extension $\mathbb{Q}(\sqrt{\varepsilon p})$. In addition, since all the roots of unity ζ_p^j belong to the ring of integers $\mathbb{Z}[\zeta_p]$ and since the symmetric polynomials have integral coefficients, it must follow that the coefficients a_j actually belong to the ring of integers of $\mathbb{Q}(\sqrt{\varepsilon p})$. □

For any element $\alpha \in \mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$ and any matrix $M = (m_{jk})$ with entries $m_{jk} \in \mathbb{Q}(\zeta_p)$, we define $\alpha(M) = (\alpha(m_{jk}))$ to be the matrix obtained from M by applying the field automorphism α to each entry. More generally, if $\rho : G \to \mathrm{GL}_n(\mathbb{Q}(\zeta_p))$ is a model for a group representation, we denote by ρ^α the model for the group representation $\rho^\alpha(g) = \alpha(\rho(g))$, obtained by applying the field automorphism α to each matrix entry $\rho(g)$, $g \in G$.

In particular, let $\tau = \gamma^2$ be the generator of the subgroup H of the Galois group of $\mathbb{Q}(\zeta_p)$ defined in (6.3.1). Clearly we have $|\tau| = |H| = r$. Consider the action of τ on the Vandermonde matrix V_Q. Since $\tau(\zeta_p^x) = \zeta_p^y$ where x, y are either both squares or both non-squares mod p, and since τ fixes the number 1, it follows that $\tau(V_Q)$ is a matrix obtained by permuting the rows of V_Q. Let P denote the permutation matrix corresponding to this permutation of the rows of V_Q:

$$\tau(V_Q) = P \cdot V_Q. \tag{6.3.2}$$

Since $|\tau| = r$, the order of P is also r. P fixes the first row and therefore P^{-1} also fixes the first row. We can consider P as a $r-1$ cycle. We also note that $P^{-1} = P^\intercal$, since the inverse of a permutation matrix is its transpose.

Interestingly, it turns out that the permutation matrix P also gives the Galois action of τ on the Weil representation models for ξ_1, ξ_2 given by the matrices S, T defined in (6.2.5) and (6.2.6). This non-trivial compatibility between the Galois action on the Vandermonde matrix V_Q and the Weil representation models is the key observation of this article:

Theorem 6.3.2. *Let $\xi_1, \xi_2 : \mathrm{SL}_2(\mathbb{F}_p) \to \mathrm{GL}_{r+1}(\mathbb{Q}(\zeta_p))$ be the explicit models for the principal series Weil characters determined by $\xi_i(\mathfrak{s}) = S, \xi_i(\mathfrak{t}) = T$, where S, T are given in* (6.2.5) *and* (6.2.6)*. Then for each $i = 1, 2$ and all primes $p > 2$, the permutation matrix P given in* (6.3.2) *satisfies*

$$\xi_i^\tau(g) = P\,\xi_i(g)\,P^{-1}$$

for all $g \in \mathrm{SL}_2(\mathbb{F}_p)$.

Proof. As is well-known, the character table entries for ξ_i are defined over the quadratic field $\mathbb{Q}(\sqrt{\varepsilon p})$. Since this the fixed field of the subgroup $H = \langle \tau \rangle \subseteq \mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$, it follows that the character table entries of ξ_i and ξ_i^τ are the same, so $\xi_i^\tau \cong \xi_i$. This implies that there exists a matrix $M \in \mathrm{GL}(V)$ (uniquely defined up to multiplication by a scalar) such that

$$\xi_i^\tau(g) = M^{-1} \cdot \xi_i(g) \cdot M \tag{6.3.3}$$

for all $g \in \mathrm{SL}_2(\mathbb{F}_p)$. We want to show that $M = \lambda P^{-1}$ where $\lambda \in \mathbb{C}$ and P is the permutation matrix (6.3.2), determined by the relation $\tau(V_Q) = PV_Q$. First, recall that the matrix $T' = V_Q^{-1} T V_Q$ has entries in $\mathbb{Q}(\sqrt{\varepsilon p})$, by Theorem 6.3.1, and therefore $\tau(T') = T'$. It follows that

$$\tau(T) = \tau(V_Q) \cdot T' \cdot \tau(V_Q)^{-1} = PV_Q \cdot T' \cdot V_Q^{-1} P^{-1} = PTP^{-1}.$$

On the other hand, we know from (6.3.3) that

$$\tau(T) = M^{-1} T M$$

so that the matrix T must commute with the matrix PM. Since T is a diagonal matrix whose entries θ_i are all *distinct*, it follows that $PM = D$ is also a diagonal matrix, and $M = P^{-1} D$. We want to show that $D = \lambda \cdot \mathbb{1}$ in fact scalar. To show this, we also need to employ the S-matrix. Let

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_r \end{bmatrix}.$$

Again recall from (6.3.3) that $\tau(S) = M^{-1} \cdot S \cdot M$, so that

$$\tau(S) = M \cdot S \cdot M^{-1} = P^{-1}D \cdot S \cdot D^{-1}P$$

Recall that S has the form

$$S = \frac{\sqrt{\varepsilon p}}{\varepsilon p} \begin{bmatrix} 1 & 2 & 2 & \cdots & 2 \\ 1 & s_{11} & s_{12} & \cdots & s_{1r} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & s_{r1} & s_{r2} & \cdots & s_{rr} \end{bmatrix},$$

for some entries s_{jk}. In particular, since the first row and first column of S consists of elements of $\mathbb{Q}(\sqrt{\varepsilon p})$, the action of τ leaves them fixed. In addition, by definition the permutation matrix P also fixes the first row and the first column, and so does $P^{-1} = P^{\intercal}$. Therefore we can write

$$P \cdot \tau(S) \cdot P^{-1} = \frac{\sqrt{\varepsilon p}}{\varepsilon p} \begin{bmatrix} 1 & 2 & 2 & \cdots & 2 \\ 1 & x_{11} & x_{12} & \cdots & x_{1r} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & x_{r1} & x_{r2} & \cdots & x_{rr} \end{bmatrix},$$

for some entries x_{jk}'s. So

$$D^{-1} \cdot P \cdot \tau(S) \cdot P^{-1} \cdot D = \frac{\sqrt{\varepsilon p}}{\varepsilon p} \begin{bmatrix} 1 & 2\lambda_1\lambda_2^{-1} & 2\lambda_1\lambda_3^{-1} & \cdots & 2\lambda_1\lambda_r^{-1} \\ \lambda_2\lambda_1^{-1} & * & * & \cdots & * \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \lambda_r\lambda_1^{-1} & * & * & \cdots & * \end{bmatrix}$$

$$= \frac{\sqrt{\varepsilon p}}{\varepsilon p} \begin{bmatrix} 1 & 2 & 2 & \cdots & 2 \\ 1 & s_{11} & s_{12} & \cdots & s_{1r} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & s_{r1} & s_{r2} & \cdots & s_{rr} \end{bmatrix} = S.$$

Equating the first column and row of these matrices we conclude that $\lambda_j = \lambda_k = \lambda$ are all equal, and thus $M = \lambda P^{-1}$. $\square$

Let now $S' := V_Q^{-1} \cdot S \cdot V_Q$. Theorem 6.3.2 enables us to prove the following theorem.

Theorem 6.3.3. *S' has matrix entries in $\mathbb{Z}[\frac{1}{2}(1+\sqrt{\varepsilon p})]$, the ring of integers of $\mathbb{Q}(\sqrt{\varepsilon p})$.*

Proof. Again let τ be the generator of the subgroup $H \subseteq \mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$ defined in (6.3.1). We want to show that $\tau(S') = S'$. By Theorem 6.3.3, we have:

$$
\begin{aligned}
\tau(S') &= \tau(V_Q^{-1} \cdot S \cdot V_Q) \\
&= \tau(V_Q^{-1}) \cdot \tau(S) \cdot \tau(V_Q) \\
&= V_Q^{-1} \cdot P^{-1} \cdot P \cdot S \cdot P^{-1} \cdot P \cdot V_Q \\
&= V_Q^{-1} \cdot S \cdot V_Q \\
&= S'.
\end{aligned}
$$

Since τ fixes S', the entries of S' lie in the quadratic extension $\mathbb{Q}(\sqrt{\varepsilon p})$. But we also know from Thm. 6.2.2 that the entries of S' belong to the ring of algebraic integers $\mathbb{Z}[\zeta_p]$, therefore they must lie in the ring of integers of $\mathbb{Q}(\sqrt{\varepsilon p})$. $\square$

By Theorems 6.3.1 and 6.3.3, setting

$$\xi_i(\mathfrak{s}) = S' \text{ and } \xi_i(\mathfrak{t}) = T'$$

gives integral models for the principal series Weil characters over the ring of integers of $\mathbb{Q}(\sqrt{\varepsilon p})$, therefore giving minimal integral models for these characters.

6.4 Examples

We now show the computations giving the explicit minimal integral models for ξ_1 and ξ_2 for the primes $p = 7$ and $p = 13$. We used SageMath[30] and MATLAB[21] to make and verify the computations.

Example 6.4.1. *Let $p = 7$. For the equivalent representation of ξ_1, let $c = 1$. So, $Q_1(x) = x^2/7$, $B_1(x, y) = 2xy/7$ and the Gauss sum is given by*

$$\sum_{x \in \mathbb{Z}/7\mathbb{Z}} \zeta_7^{x^2} = 2\zeta_7^4 + 2\zeta_7^2 + 2\zeta_7 + 1 = \sqrt{-7}.$$

Then we have

$$S = \frac{-\sqrt{7}}{7} \begin{bmatrix} 1 & 2 & 2 & 2 \\ 1 & \zeta_7^5 + \zeta_7^2 & \zeta_7^4 + \zeta_7^3 & \zeta_7^6 + \zeta_7 \\ 1 & \zeta_7^4 + \zeta_7^3 & \zeta_7^6 + \zeta_7 & \zeta_7^5 + \zeta_7^2 \\ 1 & \zeta_7^6 + \zeta_7 & \zeta_7^5 + \zeta_7^2 & \zeta_7^4 + \zeta_7^3 \end{bmatrix},$$

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \zeta_7 & 0 & 0 \\ 0 & 0 & \zeta_7^4 & 0 \\ 0 & 0 & 0 & \zeta_7^2 \end{bmatrix},$$

$$V_Q = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \zeta_7 & \zeta_7^2 & \zeta_7^3 \\ 1 & \zeta_7^4 & \zeta_7 & \zeta_7^5 \\ 1 & \zeta_7^2 & \zeta_7^4 & \zeta_7 \end{bmatrix},$$

$$S' = V_Q^{-1} S V_Q = \begin{bmatrix} -1 & \frac{1}{2}(1 - \sqrt{-7}) & 0 & 0 \\ \frac{1}{2}(-1 - \sqrt{-7}) & 1 & 0 & 0 \\ \frac{1}{2}(1 - \sqrt{-7}) & \frac{1}{2}(1 + \sqrt{-7}) & 0 & -1 \\ 1 & -1 & 1 & 0 \end{bmatrix},$$

and

$$T' = V_Q^{-1} T V_Q = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & \frac{1}{2}(1-\sqrt{-7}) \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & \frac{1}{2}(1+\sqrt{-7}) \end{bmatrix}.$$

For the equivalent representation of ξ_2*, we choose* $c = 3$*. So,* $Q_2(x) = 3x^2/7$*,* $B_2(x,y) = 6xy/7$ *and the Gauss sum is given by*

$$\sum_{x \in \mathbb{Z}/7\mathbb{Z}} \zeta_7^{3x^2} = -2\zeta_7^4 - 2\zeta_7^2 - 2\zeta_7 - 1 = -\sqrt{-7}.$$

Then we have

$$S = \frac{\sqrt{-7}}{7} \begin{bmatrix} 1 & 2 & 2 & 2 \\ 1 & \zeta_7^6 + \zeta_7 & \zeta_7^5 + \zeta_7^2 & \zeta_7^4 + \zeta_7^3 \\ 1 & \zeta_7^5 + \zeta_7^2 & \zeta_7^4 + \zeta_7^3 & \zeta_7^6 + \zeta_7 \\ 1 & \zeta_7^4 + \zeta_7^3 & \zeta_7^6 + \zeta_7^1 & \zeta_7^5 + \zeta_7^2 \end{bmatrix},$$

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \zeta_7^3 & 0 & 0 \\ 0 & 0 & \zeta_7^5 & 0 \\ 0 & 0 & 0 & \zeta_7^6 \end{bmatrix},$$

$$V_Q = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \zeta_7^3 & \zeta_7^6 & \zeta_7^2 \\ 1 & \zeta_7^5 & \zeta_7^3 & \zeta_7 \\ 1 & \zeta_7^6 & \zeta_7^5 & \zeta_7^4 \end{bmatrix},$$

$$S' = V_Q^{-1} S V_Q = \begin{bmatrix} -1 & \frac{1}{2}(1+\sqrt{-7}) & 0 & 0 \\ \frac{1}{2}(-1+\sqrt{-7}) & 1 & 0 & 0 \\ \frac{1}{2}(1+\sqrt{-7}) & \frac{1}{2}(1-\sqrt{-7}) & 0 & -1 \\ 1 & -1 & 1 & 0 \end{bmatrix},$$

and

$$T' = V_Q^{-1} T V_Q = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & \frac{1}{2}(1+\sqrt{7}) \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & \frac{1}{2}(1-\sqrt{7}) \end{bmatrix}.$$

Example 6.4.2. *Let $p = 13$. For the equivalent representation of ξ_1, let $c = 1$. So, $Q_1(x) = x^2/13$, $B_1(x,y) = 2xy/13$ and the Gauss sum is given by*

$$\sum_{x \in \mathbb{Z}/13\mathbb{Z}} \zeta_{13}^{x^2} = -2\zeta_{13}^{11} - 2\zeta_{13}^{8} - 2\zeta_{13}^{7} - 2\zeta_{13}^{6} - 2\zeta_{13}^{5} - 2\zeta_{13}^{2} - 1 = \sqrt{13}.$$

Then we have

$$S = \frac{\sqrt{13}}{13} \begin{bmatrix} 1 & 2 & 2 & 2 & 2 & 2 & 2 \\ 1 & \zeta_{13}^{11}+\zeta_{13}^{2} & \zeta_{13}^{9}+\zeta_{13}^{4} & \zeta_{13}^{7}+\zeta_{13}^{6} & \zeta_{13}^{8}+\zeta_{13}^{5} & \zeta_{13}^{10}+\zeta_{13}^{3} & \zeta_{13}^{12}+\zeta_{13} \\ 1 & \zeta_{13}^{9}+\zeta_{13}^{4} & \zeta_{13}^{8}+\zeta_{13}^{5} & \zeta_{13}^{12}+\zeta_{13} & \zeta_{13}^{10}+\zeta_{13}^{3} & \zeta_{13}^{7}+\zeta_{13}^{6} & \zeta_{13}^{11}+\zeta_{13}^{2} \\ 1 & \zeta_{13}^{7}+\zeta_{13}^{6} & \zeta_{13}^{12}+\zeta_{13} & \zeta_{13}^{8}+\zeta_{13}^{5} & \zeta_{13}^{11}+\zeta_{13}^{2} & \zeta_{13}^{9}+\zeta_{13}^{4} & \zeta_{13}^{10}+\zeta_{13}^{3} \\ 1 & \zeta_{13}^{8}+\zeta_{13}^{5} & \zeta_{13}^{10}+\zeta_{13}^{3} & \zeta_{13}^{11}+\zeta_{13}^{2} & \zeta_{13}^{7}+\zeta_{13}^{6} & \zeta_{13}^{12}+\zeta_{13} & \zeta_{13}^{9}+\zeta_{13}^{4} \\ 1 & \zeta_{13}^{10}+\zeta_{13}^{3} & \zeta_{13}^{7}+\zeta_{13}^{6} & \zeta_{13}^{9}+\zeta_{13}^{4} & \zeta_{13}^{12}+\zeta_{13} & \zeta_{13}^{11}+\zeta_{13}^{2} & \zeta_{13}^{8}+\zeta_{13}^{5} \\ 1 & \zeta_{13}^{12}+\zeta_{13} & \zeta_{13}^{11}+\zeta_{13}^{2} & \zeta_{13}^{10}+\zeta_{13}^{3} & \zeta_{13}^{9}+\zeta_{13}^{4} & \zeta_{13}^{8}+\zeta_{13}^{5} & \zeta_{13}^{7}+\zeta_{13}^{6} \end{bmatrix},$$

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \zeta_{13} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \zeta_{13}^{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \zeta_{13}^{9} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \zeta_{13}^{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \zeta_{13}^{12} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \zeta_{13}^{10} \end{bmatrix},$$

$$V_Q = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \zeta_{13} & \zeta_{13}^{2} & \zeta_{13}^{3} & \zeta_{13}^{4} & \zeta_{13}^{5} & \zeta_{13}^{6} \\ 1 & \zeta_{13}^{4} & \zeta_{13}^{8} & \zeta_{13}^{12} & \zeta_{13}^{3} & \zeta_{13}^{7} & \zeta_{13}^{11} \\ 1 & \zeta_{13}^{9} & \zeta_{13}^{5} & \zeta_{13} & \zeta_{13}^{10} & \zeta_{13}^{6} & \zeta_{13}^{2} \\ 1 & \zeta_{13}^{3} & \zeta_{13}^{6} & \zeta_{13}^{9} & \zeta_{13}^{12} & \zeta_{13}^{2} & \zeta_{13}^{5} \\ 1 & \zeta_{13}^{12} & \zeta_{13}^{11} & \zeta_{13}^{10} & \zeta_{13}^{9} & \zeta_{13}^{8} & \zeta_{13}^{7} \\ 1 & \zeta_{13}^{10} & \zeta_{13}^{7} & \zeta_{13}^{4} & \zeta_{13} & \zeta_{13}^{11} & \zeta_{13}^{8} \end{bmatrix},$$

$$S' = V_Q^{-1} S V_Q = \begin{bmatrix} \frac{1}{2}(3+\sqrt{13}) & \frac{1}{2}(1+\sqrt{13}) & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2}(5+\sqrt{13}) & -\frac{1}{2}(3+\sqrt{13}) & 0 & 0 & 0 & 0 & 0 \\ 3+\sqrt{13} & \frac{1}{2}(5+\sqrt{13}) & 0 & 0 & 0 & 0 & -1 \\ -4-\sqrt{13} & -\frac{1}{2}(5+\sqrt{13}) & 0 & 0 & 1 & 0 & 0 \\ 3+\sqrt{13} & \frac{1}{2}(3+\sqrt{13}) & 0 & 1 & 0 & 0 & 0 \\ -\frac{1}{2}(5+\sqrt{13}) & -\frac{1}{2}(1+\sqrt{13}) & 0 & 0 & 0 & -1 & 0 \\ \frac{1}{2}(3+\sqrt{13}) & 1 & -1 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and

$$T' = V_Q^{-1} T V_Q = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}(1+\sqrt{13}) \\ 0 & 1 & 0 & 0 & 0 & 0 & \frac{1}{2}(3+\sqrt{13}) \\ 0 & 0 & 1 & 0 & 0 & 0 & -\frac{1}{2}(5+\sqrt{13}) \\ 0 & 0 & 0 & 1 & 0 & 0 & \frac{1}{2}(5+\sqrt{13}) \\ 0 & 0 & 0 & 0 & 1 & 0 & -\frac{1}{2}(3+\sqrt{13}) \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{2}(1+\sqrt{13}) \end{bmatrix}.$$

For the equivalent representation of ξ_2*, let* $c = 2$*. So,* $Q_2(x) = 2x^2/13$*,* $B_2(x,y) = 4xy/13$ *and the Gauss sum is given by*

$$\sum_{x \in \mathbb{Z}/13\mathbb{Z}} \zeta_{13}^{x^2} = 2\zeta_{13}^{11} + 2\zeta_{13}^{8} + 2\zeta_{13}^{7} + 2\zeta_{13}^{6} + 2\zeta_{13}^{5} + 2\zeta_{13}^{2} + 1 = -\sqrt{13}.$$

Then we have

$$S = \frac{-\sqrt{13}}{13} \begin{bmatrix} 1 & 2 & 2 & 2 & 2 & 2 & 2 \\ 1 & \zeta_{13}^{9}+\zeta_{13}^{4} & \zeta_{13}^{8}+\zeta_{13}^{5} & \zeta_{13}^{12}+\zeta_{13} & \zeta_{13}^{10}+\zeta_{13}^{3} & \zeta_{13}^{7}+\zeta_{13}^{6} & \zeta_{13}^{11}+\zeta_{13}^{2} \\ 1 & \zeta_{13}^{8}+\zeta_{13}^{5} & \zeta_{13}^{10}+\zeta_{13}^{3} & \zeta_{13}^{11}+\zeta_{13}^{2} & \zeta_{13}^{7}+\zeta_{13}^{6} & \zeta_{13}^{12}+\zeta_{13} & \zeta_{13}^{9}+\zeta_{13}^{4} \\ 1 & \zeta_{13}^{12}+\zeta_{13} & \zeta_{13}^{11}+\zeta_{13}^{2} & \zeta_{13}^{10}+\zeta_{13}^{3} & \zeta_{13}^{9}+\zeta_{13}^{4} & \zeta_{13}^{8}+\zeta_{13}^{5} & \zeta_{13}^{7}+\zeta_{13}^{6} \\ 1 & \zeta_{13}^{10}+\zeta_{13}^{3} & \zeta_{13}^{7}+\zeta_{13}^{6} & \zeta_{13}^{9}+\zeta_{13}^{4} & \zeta_{13}^{12}+\zeta_{13} & \zeta_{13}^{11}+\zeta_{13}^{2} & \zeta_{13}^{8}+\zeta_{13}^{5} \\ 1 & \zeta_{13}^{7}+\zeta_{13}^{6} & \zeta_{13}^{12}+\zeta_{13} & \zeta_{13}^{8}+\zeta_{13}^{5} & \zeta_{13}^{11}+\zeta_{13}^{2} & \zeta_{13}^{9}+\zeta_{13}^{4} & \zeta_{13}^{10}+\zeta_{13}^{3} \\ 1 & \zeta_{13}^{11}+\zeta_{13}^{2} & \zeta_{13}^{9}+\zeta_{13}^{4} & \zeta_{13}^{7}+\zeta_{13}^{6} & \zeta_{13}^{8}+\zeta_{13}^{5} & \zeta_{13}^{10}+\zeta_{13}^{3} & \zeta_{13}^{12}+\zeta_{13} \end{bmatrix},$$

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \zeta_{13}^{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \zeta_{13}^{8} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \zeta_{13}^{5} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \zeta_{13}^{6} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \zeta_{13}^{11} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \zeta_{13}^{7} \end{bmatrix},$$

$$V_Q = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \zeta_{13}^{2} & \zeta_{13}^{4} & \zeta_{13}^{6} & \zeta_{13}^{8} & \zeta_{13}^{10} & \zeta_{13}^{12} \\ 1 & \zeta_{13}^{8} & \zeta_{13}^{3} & \zeta_{13}^{11} & \zeta_{13}^{6} & \zeta_{13} & \zeta_{13}^{9} \\ 1 & \zeta_{13}^{5} & \zeta_{13}^{10} & \zeta_{13}^{2} & \zeta_{13}^{7} & \zeta_{13}^{12} & \zeta_{13}^{4} \\ 1 & \zeta_{13}^{6} & \zeta_{13}^{12} & \zeta_{13}^{5} & \zeta_{13}^{11} & \zeta_{13}^{4} & \zeta_{13}^{10} \\ 1 & \zeta_{13}^{11} & \zeta_{13}^{9} & \zeta_{13}^{7} & \zeta_{13}^{5} & \zeta_{13}^{3} & \zeta_{13} \\ 1 & \zeta_{13}^{7} & \zeta_{13} & \zeta_{13}^{8} & \zeta_{13}^{2} & \zeta_{13}^{9} & \zeta_{13}^{3} \end{bmatrix},$$

$$S' = V_Q^{-1} S V_Q = \begin{bmatrix} \frac{1}{2}(3-\sqrt{13}) & \frac{1}{2}(1-\sqrt{13}) & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2}(-5+\sqrt{13}) & \frac{1}{2}(-3+\sqrt{13}) & 0 & 0 & 0 & 0 & 0 \\ 3-\sqrt{13} & \frac{1}{2}(5-\sqrt{13}) & 0 & 0 & 0 & 0 & -1 \\ -4+\sqrt{13}) & \frac{1}{2}(-5+\sqrt{13}) & 0 & 0 & 1 & 0 & 0 \\ 3-\sqrt{13} & \frac{1}{2}(3-\sqrt{13}) & 0 & 1 & 0 & 0 & 0 \\ \frac{1}{2}(-5+\sqrt{13}) & \frac{1}{2}(-1+\sqrt{13}) & 0 & 0 & 0 & -1 & 0 \\ \frac{1}{2}(3-\sqrt{13}) & 1 & -1 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and

$$T' = V_Q^{-1} T V_Q = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(-1+\sqrt{13}) \\ 0 & 1 & 0 & 0 & 0 & 0 & \frac{1}{2}(3-\sqrt{13}) \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{2}(-5+\sqrt{13}) \\ 0 & 0 & 0 & 1 & 0 & 0 & \frac{1}{2}(5-\sqrt{13}) \\ 0 & 0 & 0 & 0 & 1 & 0 & \frac{1}{2}(-3+\sqrt{13}) \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{2}(1-\sqrt{13}) \end{bmatrix}.$$

SUMMARY

We gave a construction of Weil representation from the Heisenberg group. We used the results from Nobs[23] and Wolfart[24] explicitly compute all of the irreducible representations of $\mathrm{SL}_2(\mathbb{F}_p)$ for the primes $p = 3$ and $p = 5$. The explicitly calculations provide concrete examples which are not found in existing literature. In addition, we noted that Yilong Wang and Samuel Wilson wrote a GAP package to compute all of the irreducible representations of $\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$. For the examples given, we readily see that the non-trivial representations are not integral using the methods/bases given in Nobs[23] and Wolfart[24]. We discussed the existence of integral representations using Riese's paper and provided an alternative proof regarding existence. We explored the integrality results of Wang, Candelori, and Zemel. Then we defined the notion of a minimal integral model was defined.

Using Wang's basis, we proved that we achieve the minimal integral model for the principal series Weil representations. So we answered Riese's question "Can ξ be realized over the ring of integers of $\mathbb{Q}(\sqrt{\varepsilon q})$?" when q is a prime. Our result gives a stronger statement than Riese's Proposition 4 for odd primes q:

Proposition 4. *Suppose q is a rational square or that $q \equiv 5 \mod 8$. Then the Weil character can be realized over $R = \mathbb{Z}[(1+\sqrt{q})/2]$.*

since our proof shows the minimal integral model is the ring of integers of $\mathbb{Q}(\varepsilon\sqrt{q})$. This is our central result.

The following questions remain:

1. Is the minimal integral model for the principal series Weil representation was unique?
2. What is a basis that yields the minimal integral models for the cuspidal series Weil representations of $\mathrm{SL}_2(\mathbb{F}_p)$?
3. What is a basis that yields the minimal integral models for the discrete cuspidal

series of $\mathrm{SL}_2(\mathbb{F}_p)$?

4. How about all of the above for $\mathrm{SL}_2(\mathbb{F}_q)$?
5. We know that Nobs and Wolfart's[23][24] methods do not yield integrality for $\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$. So what bases will? What are the minimal integral models?

These questions will require different techniques and we encourage the reader to pursue these questions either independently or with us.

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ABSTRACT

INTEGRAL REPRESENTATIONS OF $\mathbf{SL}_2(\mathbb{Z}/n\mathbb{Z})$

by

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Advisors: Dr. Luca Candelori and Dr. Andrew Salch

Major: Mathematics

Degree: Doctor of Philosophy

The aim of this work is to determine for which commutative rings integral representations of $\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$ exist and to explicitly compute them. We start with $R = \mathbb{Z}/p\mathbb{Z}$ and then consider $\mathbb{Z}/p^\lambda\mathbb{Z}$. A new approach will be used to do this based on the Weil representation. We then consider general finite rings $\mathbb{Z}/n\mathbb{Z}$ by extending methods described in [26]. We make extensive use of group theory, linear representations of finite groups, ring theory, algebraic geometry, and number theory. From number theory we will employ results regarding modular forms, Legendre symbols, Hilbert symbols, and quadratic forms. We consider the works of André Weil[38], Alexandre Nobs[23][24] and Udo Riese[26]. We explicitly compute the irreducible representations for several odd primes using Nobs and Wolfart's methods. Then we will explore Riese's[26] construction of the integral representations of $\mathrm{SL}_2(A_\lambda)$ and explicitly compute them as the paper only proves the existence. We will use integrality results of the Weil representations by Luca Candelori, Shaul Zemel, and Yilong Wang (to appear). Then we will extend Reise's results to construct integral representations for rings that are not of the form A_λ.

AUTOBIOGRAPHICAL STATEMENT

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