Online Piercing of Geometric Objects

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Abstract

We consider the online version of the piercing set problem where geometric objects arrive one by one. The online algorithm must maintain a piercing set for the arrived objects by making irrevocable decisions. First, we show that any deterministic online algorithm that solves this problem has a competitive ratio of at least $\Omega(n)$, which even holds when the objects are one-dimensional intervals. On the other hand, piercing unit objects is equivalent to the unit covering problem which is well-studied in the online model. Due to this, all the results related to the online unit covering problem are preserved for the online unit piercing problem when the objects are translated from each other. Surprisingly, no upper bound was known for the unit covering problem when unit objects are anything other than balls and hypercubes. In this paper, we introduce the notion of α-aspect and α-aspect$_\infty$ objects. We give an upper bound of competitive ratio for α-aspect and α-aspect$_\infty$ objects in $\mathbb{R}^3$ and $\mathbb{R}^d$, respectively, with a scaling factor in the range $[1, k]$. We also propose a lower bound of the competitive ratio for bounded scaled objects like α-aspect objects in $\mathbb{R}^2$, axis-aligned hypercubes in $\mathbb{R}^d$, and balls in $\mathbb{R}^2$ and $\mathbb{R}^3$. For piercing α-aspect$_\infty$ objects in $\mathbb{R}^d$, we show that a simple deterministic algorithm achieves a competitive ratio of at most $\left(\frac{2}{\alpha}\right)^d \left((1+\alpha)^d - 1\right) \left(\lceil \log_{(1+\alpha)}(\frac{2k}{\alpha}) \rceil\right) + 1$. This result is very general in nature. One can obtain upper bounds for specific objects by specifying the value of α. By putting the value of $k = 1$ to the above result, we get an upper bound of the competitive ratio for the unit covering problem for various types of objects.

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1 Introduction

Piercing is one of the most important problems in computational geometry. A set $\mathcal{P} \subset \mathbb{R}^d$ of points is defined as a *piercing set* for a set $\mathcal{S}$ of geometric objects in $\mathbb{R}^d$, if each object in $\mathcal{S}$ contains at least one point from $\mathcal{P}$. Given the set $\mathcal{S}$ of objects, the piercing problem is to find a piercing set $\mathcal{P} \subset \mathbb{R}^d$ of minimum cardinality.

In this paper, we study the *online version* of the problem in which geometric objects arrive one by one, and the online algorithm needs to maintain a piercing set for the already arrived objects. Once a new geometric object σ arrives, the online algorithm needs to place a point $p \in \sigma$ if the existing piercing set does not already pierce it. An online algorithm may add points to the piercing set, but it cannot remove points from it, i.e., the online algorithm needs to take irrevocable decisions. The goal is to minimize the cardinality of the piercing set. We analyze the quality of our online algorithm by competitive analysis [3]. The *competitive ratio* of an online algorithm is $\sup_\beta \frac{ALG_\beta}{OPT_\beta}$, where β is an input sequence, and OPT_β and ALG_β are the cost of an optimal offline algorithm and the solution produced by the online algorithm, respectively, for the input sequence β. Here, the supremum is taken over all possible input sequences β.

The online version of this problem is inspired by an application in wireless networks [12]. Here the points model base stations, and the centers of objects model clients. The reception range of each client has some geometric shape (e.g., disks, hexagons, etc.). The algorithm needs to place a base station that serves a new uncovered client. Since installing the base station is expensive, so the overall goal is to place the minimum number of base stations.

1.1 Related Works

In the offline setting, the piercing set problem is well-studied problem [6, 11, 18]. For one-dimensional intervals, the minimum piercing set can be found in polynomial time, i.e., $O(n \log c^*)$, where c^* is the size of the minimum piercing set [19]. On the other hand, computing the minimum piercing set is NP-complete even for unit squares in the plane [15, 13]. Chan [6] proposed a polynomial-time approximation scheme for α-fat convex objects. Katz et al. [18] studied the problem in the dynamic setting for intervals.

A closely related well explored problem is the set-cover problem. Let X be a set of n elements, and let $\mathcal{S}$ be a family of subsets of X such that $|\mathcal{S}| = m$. A *set cover* is a collection of subsets $\mathcal{S}^* \subset \mathcal{S}$ such that their union covers the whole set X. The goal of the *set cover problem* is to find a set cover of minimum cardinality. Note that by interchanging the roles of subsets and points, the set-cover problem and the hitting-set problems are equivalent. Both problems are classical NP-hard problems [17]. In the offline setting, the minimum hitting set problem for intervals (points in $\mathbb{R}$) can be solved in polynomial time using a greedy algorithm. However, these problems remain NP-hard, even for unit disks or unit squares in $\mathbb{R}^2$ [13]. Alon et al. [1] initiated the study of the online set-cover problem. In their setting, the sets X and $\mathcal{S}$ are already known, but the order of arrivals of points is unknown. Upon the arrival of an uncovered point, the online algorithm must choose a subset that covers that point. The online algorithm presented by Alon et al. [1] achieves a competitive ratio of $O(\log n \log m)$. The results obtained in [1] also hold for the online hitting set problem. Even and Smorodinsky [12] studied the hitting set problem in an online setting, where both $\mathcal{S}$ and X are known in advance, but the order of arrivals of the input objects in $\mathcal{S}$ is unknown. In this setting, they proposed an algorithm with a competitive ratio of $O(\log n)$ for half-planes or unit disks. They also gave a matching lower bound of the competitive ratio for these cases. In the same paper, they proposed an online algorithm that achieves a tight bound of $\Theta(\log n)$ when objects are intervals and points are integers. Note that in our paper, the set $X = \mathbb{R}^d$, and the set $\mathcal{S}$ is not known in advance.

A related problem is the *unit covering* (a special variant of the set cover problem), where the objective is to cover a given set of n points in $\mathbb{R}^d$ with the minimum number of translated copies of an object. Charikar et al. [8] studied the problem in the online setting for unit balls in $\mathbb{R}^d$, where the points arrive one by one to the online algorithm, and the objective is to assign a newly arrived point to an existing unit ball or a new unit ball to cover it. They proposed a deterministic online algorithm with a competitive ratio of $O(2^d d \log d)$ and show that the lower bound of the competitive ratio for this problem is $\Omega(\log d/\log\log\log d)$. Dumitrescu et al. [9] improved both the upper and lower bound of the problem to $O(1.321^d)$ and $\Omega(d+1)$, respectively. They also proposed a lower bound of the competitive ratio for any centrally symmetric convex object as the illumination number(defined in Section 2) of the object. Recently, Dumitrescu and Tóth [10] proved that the competitive ratio of any deterministic online algorithm for the unit covering problem under L_∞ norm is at least 2^d. Surprisingly, we did not find any upper bound of the competitive ratio when objects are anything other than balls and hypercubes. In this paper, we raise this question and obtain a very general upper bound for the same.

Another relevant problem is the *chasing convex objects*, initiated by Friedman and Linial [14]. Here, convex objects $\sigma_i \subseteq \mathbb{R}^d$ arrives to the online algorithm, and the algorithm needs to place a point $p_i \in \sigma_i$ such that the total distances between successive points $\Sigma_{i=1}^{n-1}||p_i - p_{i+1}||$ is minimized, where n is length of input sequence. They proved that no online algorithm could achieve a competitive ratio lower than $\sqrt{d}$, and gave an online algorithm for $d = 2$. Bubeck et al. [5] presented an online algorithm with a competitive ratio $2^{O(d)}$ for this problem. Parallel to this, Bubeck et al. [4] presented an online algorithm having competitive ratio $O(\min(d, \sqrt{d \log n}))$ using the notion of Steiner point when objects are nested, i.e., $\sigma_1 \supseteq \sigma_2 \supseteq \ldots \sigma_n$. Later, Sellke [21] and Argue et al. [2] independently improved the result by showing that there exists an online algorithm having a competitive ratio of $O(\sqrt{d \log n})$ and $O(\min(d, \sqrt{d \log n}))$, respectively, for the chasing convex object problem.

1.2 Our Contributions

First, we prove that the competitive ratio of every deterministic online algorithm for piercing objects is at least $\Omega(n)$, which holds even when the input objects are one-dimensional intervals (Theorem 1, Section 2.1). As a result, next, we concentrate on when input objects are translated copies of an object. We show that for the translates of a centrally symmetric convex object C, the competitive ratio of any deterministic online algorithm is at least the illumination number of the object C (Theorem 2, Section 2.1). Next, we show that piercing translated copies of an object is equivalent to the unit covering problem (Theorem 6, Section 2.2). As an implication, all the results for the online unit covering problem obtained in [9] and [10] would be applicable to the unit piercing problem. Motivated by these, we ask what would happen when objects are neither of arbitrary size nor of unit size, but something in between: bounded scaled, and the shape of the object could vary. For this, we introduce the concept of α-aspect and α-aspect$_\infty$ objects in $\mathbb{R}^d$ (Section 2.3). The α-aspect objects are invariant under translation, rotation and scaling. Whereas, α-aspect$_\infty$ objects are invariant under translation and scaling. For any object in $\mathbb{R}^d$, the value of α is in the interval $(0, 1]$. We consider objects with scaling factors in the range $[1, k]$ for any fixed $k \in \mathbb{R}$.

1. First, we prove that for piercing α-aspect$_\infty$ objects in $\mathbb{R}^d$, there exists a deterministic online algorithm whose competitive ratio is at most $\left(\frac{2}{\alpha}\right)^d \left((1+\alpha)^d - 1\right) \left(\lceil \log_{(1+\alpha)}(\frac{2k}{\alpha}) \rceil\right) + 1$ (Theorem 7, Section 3).
2. Next, we show that for piercing α-aspect objects in $\mathbb{R}^3$, there exists an online algorithm that achieves a competitive ratio of at most $\frac{2}{1-\cos(\theta)} \lceil \log_{(1+x)}(\frac{2k}{\alpha}) \rceil + 1$, where $\theta = \frac{1}{2}\cos^{-1}\left(\frac{1}{2} + \frac{1}{1+\sqrt{1+4\alpha^2}}\right)$ and $x = \frac{\sqrt{1+4\alpha^2}-1}{2}$ (Theorem 12, Section 3). We achieve a similar result for piercing α-aspect objects in $\mathbb{R}^2$ (Theorem 16, Section 3).

3. Then, we prove that the competitive ratio of every deterministic online algorithm for piercing α-aspect objects in $\mathbb{R}^2$ is at least $\lfloor \log_{2/\alpha}(k) \rfloor + 1$ (Theorem 19, Section 4).
4. Later, we consider the case of axis-aligned d-dimensional hypercube. We show the competitive ratio of every deterministic online algorithm for piercing axis-aligned hypercubes in $\mathbb{R}^d$ is at least $d(\lfloor \log_2(k) \rfloor) + 2^d$ (Theorem 21, Section 4).
5. At last, we consider balls in $\mathbb{R}^3$ (resp., $\mathbb{R}^2$), and we propose that the competitive ratio of every deterministic online algorithm for piercing balls in $\mathbb{R}^3$ (respectively, $\mathbb{R}^2$) with radius in the range $[1, k]$ is at least $3\lfloor \log_{(4+\epsilon)}(k) \rfloor + 4$ (respectively $2\lfloor \log_{(\frac{8}{3}+\epsilon)}(k) \rfloor + 3$), where ϵ is a very small constant close to zero (Theorem 22, Section 4).

To obtain the upper bound, we consider a very natural algorithm, Algorithm-Center, that works as follows. On receiving a new input object σ, if it is not pierced by the existing piercing set, our online algorithm adds the center of σ to the piercing set. For a hypercube σ, *center* is a point in σ from which maximum distance from any point of σ is minimized. For α-aspect and α-aspect$_\infty$ objects, a generalized definition of the center is given in Section 2.3.

1.3 Organization

In Section 2, initially, we propose a lower bound for piercing arbitrarily sized intervals followed by a lower bound for piercing a set of translated copies of a centrally symmetric convex object. Then, we discuss the equivalence of online unit piercing and online unit covering problems for translated copies of the unit object. At last, in this section, we introduce α-aspect objects and α-aspect$_\infty$ objects. In Section 3, we present an upper bound for α-aspect and α-aspect$_\infty$ objects having width in the range $[1, k]$. Later, in Section 4, we propose lower bounds for various bounded scaled objects like two-dimensional α-aspect objects, d-dimensional axis-aligned hypercubes in $\mathbb{R}^d$, and two and three-dimensional balls. Finally, in Section 5, we conclude.

2 Notation and Preliminaries

We use $[n]$ to denote the set $\{1, 2, \ldots, n\}$. If not explicitly mentioned, we use the term *object* to denote a simply connected compact set in $\mathbb{R}^d$ with a nonempty interior. The interior of an object C is represented by $\text{int}(C)$. A convex object is said to be a *centrally symmetric* if it is invariant under point reflection through its center. The *illumination number* of an object C, denoted by $I(C)$, is the minimum number of smaller homothetic copies of C whose union contains C [20].

2.1 Lower Bounds

First, we observe that the competitive ratio of the piercing problem has a pessimistic lower bound of $\Omega(n)$, which even holds for one-dimensional intervals.

▶ **Theorem 1.** *The competitive ratio of every deterministic online algorithm for piercing intervals is at least $\Omega(n)$, where n is the length of the input sequence.*

Proof. Let σ_1 be the first interval presented by the adversary to the online algorithm. Let p_1 be a point placed by the online algorithm to pierce the interval σ_1. The point p_1 partitions the interval σ_1 into two parts, of which, let σ_1^L be a larger part that does not contain the point p_1. Now, the adversary can place an interval σ_2 completely contained in σ_1^L (see Figure 1). For the new interval σ_2, any online algorithm needs a new piercing point p_2. Now

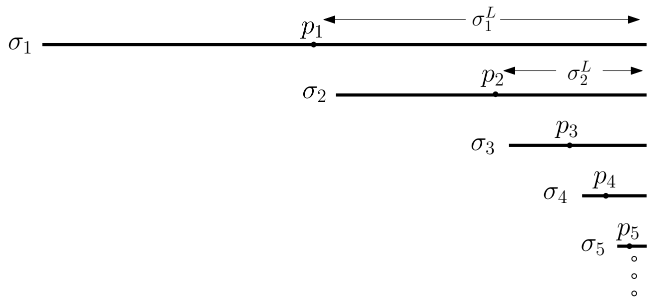

Figure 1 Input instance of intervals.

again, one can define a partition σ_2^L of σ_2 depending on the position of the point p_2 such that σ_2^L does not contain p_2 and the adversary will place an interval σ_3 completely contained in σ_2^L. In this way, the adversary can adaptively construct n intervals for which any online algorithm needs n distinct points to pierce, while the offline optimum needs only one point. Hence, the lower bound of the competitive ratio is $\Omega(n)$. ◀

Now, we consider when objects are translated copies of a centrally symmetric convex object.

▶ **Theorem 2.** *The competitive ratio of every deterministic online algorithm for piercing a set of translated copies of a centrally symmetric convex object C is at least $I(C)$, where $I(C)$ denotes the illumination number of C.*

The proof can be adapted from [9, Thm 4]. Also, note that Theorem 2 can be obtained as a result of Theorem 6 and [9, Thm 4]. It is known that $I(C) = 2^d$ for any full-dimensional parallelepiped in $\mathbb{R}^d$ [9, 20]. Consequently, the value of $I(C) = 2^d$, for d-dimensional hypercube. Therefore, we have the following.

▶ **Corollary 3.** *The competitive ratio of every deterministic online algorithm for piercing a set of axis-parallel unit hypercubes in $\mathbb{R}^d$ is at least 2^d.*

2.2 Unit Covering vs Unit Piercing

Let C be an object, and let $-C$ be the reflection of C through a point $c \in C$ (see Figure 2).

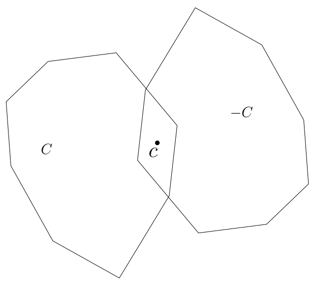

Figure 2 Reflection of an object C through a point c.

▶ **Definition 4** (Unit Piercing Problem). *Given a family $\mathcal{S}$ of translated copies of an object C, in the unit piercing problem, we need to pierce each object by placing the minimum number of points in $\mathbb{R}^d$.*

▶ **Definition 5** (Unit Covering Problem). *Given a set of points $\mathcal{P} \subseteq \mathbb{R}^d$, in the unit covering problem, we need to place the minimum number of translated copies of an object $-C$ to cover all the points in $\mathcal{P}$.*

The following theorem connects the above two problems.

▶ **Theorem 6.** *The unit piercing problem is equivalent to the unit covering problem.*

As a consequence of Theorem 6, all the results related to the online unit covering problem (summarized in Table 1) studied in [9] and [10] are carried for the online piercing problem when objects are translates of each other.

Table 1 Summary of known results for unit piercing problem.

Geometric Objects	Lower Bound	Upper Bound
Unit Intervals	2 [8]	2 [7]
Fixed oriented unit squares	4 [9, 10]	4 [7]
Fixed oriented unit hypercubes in $\mathbb{R}^d$	2^d [9, 10]	2^d [7, 10]
Unit disks in $\mathbb{R}^2$	4 [9]	5 [9]
Unit balls in $\mathbb{R}^3$	5 [9]	12 [9]
Unit balls in $\mathbb{R}^d$, $d > 3$	$d+1$ [9]	$O(1.321^d)$ [9]
Translated copies of a centrally symmetric convex object C	$I(C)$ [9]	★ [Corollary 11]

★ Result obtained in this paper.

2.3 Objects with Fixed Aspect Ratio

Let σ be an object and x be any point in σ. Let $\alpha(x)$ be the ratio between minimum and maximum distance from x to the boundary $\delta(\sigma)$ of the object σ under L_2 norm. In other words, $\alpha(x) = \frac{\min_{y\in\delta(\sigma)} d(x,y)}{\max_{y\in\delta(\sigma)} d(x,y)}$, where $d(x,y)$ is the distance (under L_2 norm) between x and y (refer to Figure 3a). The *aspect ratio* $\alpha(\sigma)$ of an object σ is defined as the maximum value of $\alpha(x)$ for any point $x \in \sigma$, i.e., $\alpha(\sigma) = \max\{\alpha(x) : x \in \sigma\}$. An object is said to be α*-aspect object* if its aspect ratio is α. A point $c \in \sigma$ with $\alpha(c) = \alpha(\sigma)$ is defined as the *center* of the object σ (see Figure 3a). The minimum (resp., maximum) distance (under L_2 norm) from the center to the boundary of the object is referred to as the *width* (resp., *height*) of the object. Note that for any object σ, we have $0 < \alpha(\sigma) \leq 1$. The maximum possible value of α is attained when the object is ball.

Observe following properties of an α-aspect object.

▶ **Property 1.** *Let $\sigma \subseteq \mathbb{R}^d$ be an α-aspect object, then*
- *any arbitrary rotated object σ' of σ is an α-aspect object;*
- *any translated object σ' of σ is an α-aspect object;*
- *the scaled object $\lambda\sigma$ is an α-aspect object, where $\lambda \in \mathbb{R}^+$.*

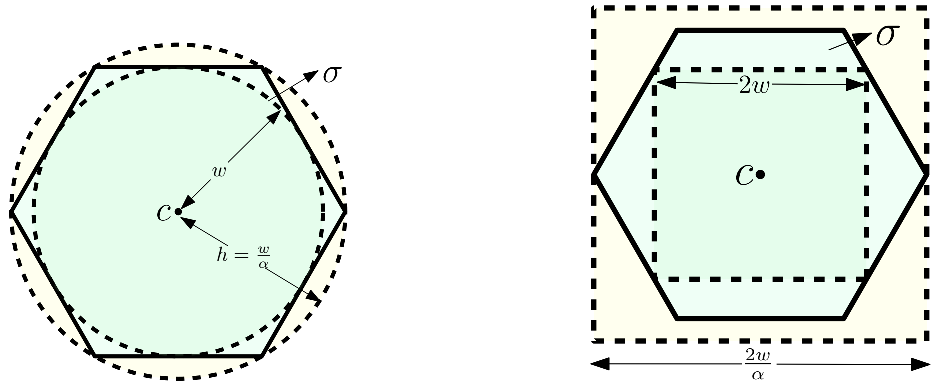

Figure 3 Geometric interpretation of (a) aspect ratio and (b) aspect$_\infty$ ratio.

▶ **Property 2.** *Let $\sigma \subseteq \mathbb{R}^d$ be an α-aspect object centered at a point c with width w, then*
- *a ball of radius w centered at c is completely contained in σ, and*
- *a ball of radius $\frac{w}{\alpha}$ centered at c contains the object σ.*

Considering L_∞ norm instead of L_2 norm, similar to the above, one can define the aspect$_\infty$ ratio, center, width and height of an object. An object with aspect$_\infty$ ratio α is said to be an α-aspect$_\infty$ object. Note that for any object σ, the value of $\alpha_\infty(\sigma)$ is also greater than zero and the maximum possible value of $\alpha_\infty(\sigma)$ is one which is attained for the hypercube. Analogous to Property 1 and 2, we have the following.

▶ **Property 3.** *Let $\sigma \subseteq \mathbb{R}^d$ be an α-aspect$_\infty$ object, then*
- *any translated object σ' of σ is an α-aspect$_\infty$ object;*
- *the scaled object $\lambda\sigma$ is an α-aspect$_\infty$ object, where $\lambda \in \mathbb{R}^+$.*

▶ **Property 4.** *Let $\sigma \subseteq \mathbb{R}^d$ be an α-aspect$_\infty$ object object centered at a point c with width w, then*
- *a hypercube of side length $2w$ centered at c is completely contained in σ, and*
- *a hypercube of side length $\frac{2w}{\alpha}$ centered at c contains the object σ.*

3 Upper Bound for Bounded Scaled Objects

3.1 For Objects with Fixed Aspect$_\infty$ Ratio

In this section, we present an upper bound of the competitive ratio for piercing objects with a fixed α-aspect$_\infty$ ratio. In the proof of the following theorem, all the distances, if explicitly not mentioned, are under L_∞ norm, and all the hypercubes are axis-parallel.

▶ **Theorem 7.** *For piercing α-aspect$_\infty$ objects in $\mathbb{R}^d$ having width in the range $[1, k]$,* Algorithm-Center *achieves a competitive ratio of at most $\left(\frac{2}{\alpha}\right)^d \left((1+\alpha)^d - 1\right) \log_{(1+\alpha)}(\frac{2k}{\alpha}) + 1$.*

Proof. Let $\mathcal{I}$ be the set of input α-aspect$_\infty$ objects presented to the algorithm. Let $\mathcal{A}$ and $\mathcal{O}$ be the piercing set returned by Algorithm-Center and the offline optimal, respectively, for $\mathcal{I}$. Let $p \in \mathcal{O}$ be a piercing point and let $\mathcal{I}_p \subseteq \mathcal{I}$ be the set of input α-aspect$_\infty$ objects that

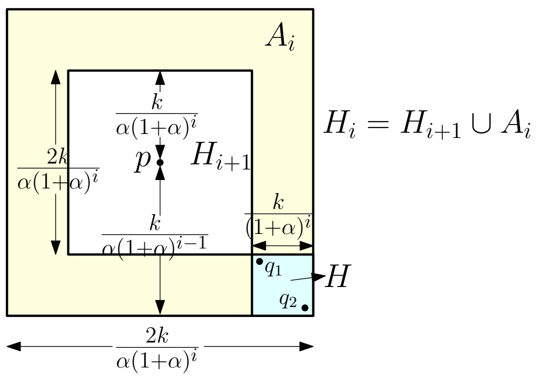

Figure 4 Illustration.

are pierced by the point p. Let $\mathcal{A}_p \subseteq \mathcal{A}$ be the set of piercing points placed by ALGORITHM-CENTER to pierce all the input objects in $\mathcal{I}_p$. It is easy to see that $\mathcal{A} = \cup_{p \in \mathcal{O}} \mathcal{A}_p$. Therefore, the competitive ratio of our algorithm is upper bounded by $\max_{p \in \mathcal{O}} |\mathcal{A}_p|$.

Let us consider any point $a \in \mathcal{A}_p$. Since a is the center of an α-aspect$_\infty$ object $\sigma \in \mathcal{I}_p$ containing the point p having width at most k, the distance (under L_∞ norm) between a and p is at most $\frac{k}{\alpha}$. Therefore, a hypercube H_1 of side length $\frac{2k}{\alpha}$, centered at p, contains all the points in $\mathcal{A}_p$. Let H_i be a hypercube centered at p having side length $\frac{2k}{\alpha(1+\alpha)^{i-1}}$, where $i \in [m]$ and m is the smallest integer such that $\frac{2k}{\alpha(1+\alpha)^{m-1}} \leq 1$. Note that $H_1, H_2, \ldots, H_m$ are concentric hypercubes. Let us define the annular region $A_i = H_i \setminus H_{i+1}$, where $i \in [m-1]$. Let $\mathcal{A}_{p,m} = \mathcal{A}_p \cap H_m$, and $\mathcal{A}_{p,i} = \mathcal{A}_p \cap A_i$ be the subset of $\mathcal{A}_p$ that is contained in the region A_i, for $i \in [m-1]$. Since the distance (under L_∞ norm) between any two points in H_m is at most one, and the width of any object in $\mathcal{I}_p$ is at least one; therefore, any object belonging to $\mathcal{I}_p$ having center in H_m contains the entire hypercube H_m. As a result, our online algorithm places at most one piercing point in H_m. Thus, $|\mathcal{A}_{p,m}| \leq 1$.

▶ **Lemma 8.** $|\mathcal{A}_{p,i}| \leq 2^d \left((1+\frac{1}{\alpha})^d - (\frac{1}{\alpha})^d\right)$, *where* $i \in [m-1]$.

Proof. Since $A_i = H_i \setminus H_{i+1}$, the distance (under L_∞ norm) from the center p to the boundary of H_i and H_{i+1} is $\frac{k}{\alpha(1+\alpha)^{i-1}}$ and $\frac{k}{\alpha(1+\alpha)^i}$, respectively. So the annular region A_i can contain hypercubes of side length $\frac{k}{(1+\alpha)^i}$. It is easy to observe that, the annular region A_i is the union of at most $2^d \left((1+\frac{1}{\alpha})^d - (\frac{1}{\alpha})^d\right)$ disjoint hypercubes, each having side length $\frac{k}{(1+\alpha)^i}$. To complete the proof, next, we argue that our online algorithm places at the most one piercing point in each of these hypercubes to pierce the objects in $\mathcal{I}_p$. Let H be any such hypercube of side length $\frac{k}{(1+\alpha)^i}$, and let $q_1 \in H$ be a piercing point placed by our online algorithm. For a contradiction, let us assume that our online algorithm places another piercing point $q_2 \in H$, where q_2 is the center of an object $\sigma \in \mathcal{I}_p$. Since σ contains both the points p and q_2, and the distance (under L_∞ norm) between them is at least $\frac{k}{\alpha(1+\alpha)^i}$, therefore, the height of the object σ is at least $\frac{k}{\alpha(1+\alpha)^i}$ and the width is at least $\frac{k}{(1+\alpha)^i}$. Note that the distance (under L_∞ norm) between any two points in H is at most $\frac{k}{(1+\alpha)^i}$, as a result, σ is already pierced by q_1. This contradicts our algorithm. Thus, the region H contains at most one piercing point of $\mathcal{A}_{p,i}$. Hence, the lemma follows. ◁

Since $\cup \mathcal{A}_{p,i} = \mathcal{A}_p$, we have $|\mathcal{A}_p| \leq 2^d \left((1+\frac{1}{\alpha})^d - (\frac{1}{\alpha})^d\right)(m-1) + 1$. As the value of $m \leq \log_{(1+\alpha)} \left(\frac{2k}{\alpha}\right) + 1$, the theorem follows. ◀

When objects are fixed oriented hypercubes and congruent balls in $\mathbb{R}^d$, then the value of α is 1 and $\frac{1}{\sqrt{d}}$, respectively. As a result, we have the following.

▶ **Corollary 9.** *For piercing axis-aligned d-dimensional hypercubes with side length in the range* $[1, k]$, Algorithm-Center *achieves a competitive ratio of at most* $2^d\left(2^d - 1\right)\log_2(2k) + 1$.

▶ **Corollary 10.** *For piercing d-dimensional balls with radius in the range* $[1, k]$, Algorithm-Center *achieves a competitive ratio of at most*
$(2\sqrt{d})^d\left((1 + \frac{1}{\sqrt{d}})^d - 1\right)\log_{(1+\frac{1}{\sqrt{d}})}(2k\sqrt{d}) + 1$.

Unit Covering Problem in $\mathbb{R}^d$

In Theorem 7, if we place the value of $k = 1$, then due to Theorem 6, we have following result for the unit covering problem.

▶ **Corollary 11.** *For the unit covering problem in $\mathbb{R}^d$, there exists a deterministic online algorithm whose competitive ratio is at most* $\left(\frac{2}{\alpha}\right)^d\left((1 + \alpha)^d - 1\right)\log_{(1+\alpha)}(\frac{2}{\alpha}) + 1$, *where α is the aspect$_\infty$ ratio of the unit object.*

3.2 For α-Aspect Objects in $\mathbb{R}^3$

▶ **Theorem 12.** *For piercing α-aspect object in $\mathbb{R}^3$ having width in the range $[1, k]$,* Algorithm-Center *achieves a competitive ratio of at most* $\frac{2}{1-\cos(\theta)}\lceil\log_{(1+x)}(\frac{2k}{\alpha})\rceil + 1$, *where* $\theta = \frac{1}{2}\cos^{-1}\left(\frac{1}{2} + \frac{1}{1+\sqrt{1+4\alpha^2}}\right)$ *and* $x = \frac{\sqrt{1+4\alpha^2}-1}{2}$.

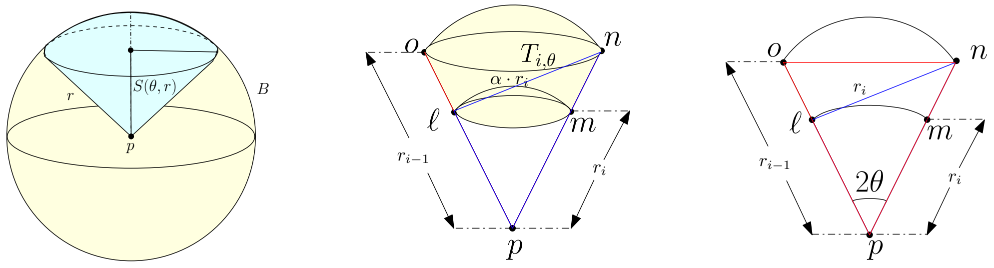

Figure 5 (a) Partitioning the ball B of radius r using spherical sectors $S(\theta, r)$; (b) Description of spherical sector $S(\theta, r_{i-1})$ and spherical block $T_{i,\theta}$; (c) Projection of spherical sector $S(\theta, r_{i-1})$.

Proof. Let $\mathcal{I}$ be the set of input α-aspect objects in $\mathbb{R}^3$ presented to the algorithm. Let $\mathcal{A}$ and $\mathcal{O}$ be two piercing sets for $\mathcal{I}$ returned by Algorithm-Center and the offline optimal, respectively. Let p be any piercing point of the offline optima $\mathcal{O}$. Let $\mathcal{I}_p \subseteq \mathcal{I}$ be the set of input objects pierced by the point p. Let $\mathcal{A}_p$ be the set of piercing points placed by our algorithm to pierce all the objects in $\mathcal{I}_p$. To prove the theorem, we will give an upper bound of $|\mathcal{A}_p|$.

Let us consider any point $a \in \mathcal{A}_p$. Since a is the center of an α-aspect object $\sigma \in \mathcal{I}_p$ containing the point p, with width at most k (height at most $\frac{k}{\alpha}$), the distance between a and p is at most $\frac{k}{\alpha}$. Therefore, a ball B_1 of radius $\frac{k}{\alpha}$, centered at p, contains all the points in $\mathcal{A}_p$. Let $x = \frac{\sqrt{1+4\alpha^2}-1}{2}$ be a positive constant. Let B_i be a ball centered at p having radius $r_i = \frac{k}{\alpha(1+x)^{i-1}}$, where $i \in [m]$ and m is the smallest integer such that $\frac{k}{\alpha(1+x)^{m-1}} \leq \frac{1}{2}$. Note that $B_1, B_2, \ldots, B_m$ are concentric balls, centered at p. Let $\theta = \frac{1}{2}\cos^{-1}\left(\frac{1}{2} + \frac{1}{1+\sqrt{1+4\alpha^2}}\right)$ be a constant angle in $(0, \frac{\pi}{10}]$. Let $S(\theta, i)$ be a *spherical sector* obtained by taking the portion of the ball B_i by a conical boundary with apex at the center p of the ball and θ as the half of the cone angle (for an illustration see figure 5a). For any $i \in [m-1]$, let us define the ith spherical block $T_{i,\theta} = S(\theta, i) \setminus S(\theta, i+1)$.

▷ Claim 13. The distance between any two points in $T_{i,\theta}$ is at most αr_i.

Proof. Observe Figure 5b, where a detail of a spherical block is depicted. Note that the maximum distance between any two points in $T_{i,\theta}$ is at most $\max\{\overline{ln}, \overline{on}\}$. First, consider the triangle $\triangle \ell pn$ (see Figure 5b and 5c). By cosine rule of triangle, we have:

$$\begin{aligned}
\overline{\ell n}^2 &= \overline{p\ell}^2 + \overline{pn}^2 - 2\overline{p\ell}\;\overline{pn}\cos(2\theta) \\
&= \left(\frac{k}{\alpha(1+x)^{i-1}}\right)^2 + \left(\frac{k}{\alpha(1+x)^{i-2}}\right)^2 - 2\left(\frac{k}{\alpha(1+x))^{i-1}}\right)\left(\frac{k}{\alpha(1+x))^{i-2}}\right)\cos(2\theta) \\
&= \left(\frac{k}{\alpha(1+x)^{i-2}}\right)^2 \left(\left(\frac{1}{(1+x)}\right)^2 + 1 - 2\left(\frac{1}{(1+x)}\cos(2\theta)\right)\right) \\
&= \left(\frac{k}{\alpha(1+x)^{i-1}}\right)^2 \left(1 + (1+x)^2 - 2(1+x)\cos(2\theta)\right)
\end{aligned}$$

Since $\theta = \frac{1}{2}\cos^{-1}\left(\frac{1}{2} + \frac{1}{1+\sqrt{1+4\alpha^2}}\right)$, and $x = \frac{\sqrt{1+4\alpha^2}-1}{2}$, therefore, $cos(2\theta) = \frac{(x+2)}{2(x+1)}$ and $x^2 + x = \alpha^2$. Now substituting these values in the above equation, we get

$$\overline{\ell n}^2 = r_i^2\left(1 + (1+x)^2 - (2+x)\right) = r_i^2\left(1 + 1 + x^2 + 2x - 2 - 2x\right) = r_i^2\left(x^2 + x\right) = (\alpha r_i)^2.$$

Now, consider the triangle $\triangle opn$ (see Figure 5b and 5c). Here we have:

$$\begin{aligned}
\overline{on}^2 &= 2\left(\frac{k}{\alpha(1+x)^{i-2}}\right)^2 - 2\left(\frac{k}{\alpha(1+x)^{i-2}}\right)^2\cos(2\theta) = 2\left(\frac{k}{\alpha(1+x)^{i-2}}\right)^2(1-\cos(2\theta)) \\
&= 2\left(\frac{k}{\alpha(1+x)^{i-1}}\right)^2(1+x)^2(1-\cos(2\theta)) = 2r_i^2(1+x)^2(1-\cos(2\theta)).
\end{aligned}$$

Now substituting the values of $cos(2\theta) = \frac{(x+2)}{2(x+1)}$ and $x^2 + x = \alpha^2$ in the above equation, we get

$$\begin{aligned}
\overline{on}^2 &= 2r_i^2(1+x)^2\left(1 - \frac{(x+2)}{2(x+1)}\right) = 2r_i^2(1+x)^2\left(\frac{2(x+1)-(x+2)}{2(x+1)}\right) \\
&= r_i^2(1+x)x = (\alpha r_i)^2.
\end{aligned}$$

Note that $\overline{ln} = \overline{on} = \alpha r_i$, thus, αr_i is the maximum distance between any two points in the region $T_{i,\theta}$. ◁

▶ **Lemma 14.** *For each $i \in [m]$, our algorithm places at most one piercing point in the spherical block $T_{i,\theta}$ to pierce any object in $\mathcal{I}_p$.*

Proof. Let q_1 be the first piercing point placed by Algorithm-Center in $T_{i,\theta}$. For a contradiction, let us assume that Algorithm-Center places another piercing point $q_2 \in T_{i,\theta}$, where q_2 is center of some object $\sigma \in \mathcal{I}_p$. Since σ contains both points p and q_2, and the distance between them is at least r_i, therefore, the height and width of σ are at least r_i and αr_i, respectively. Due to Claim 13, the distance between any two points in the spherical block $T_{i,\theta}$ is at most αr_i, as a result, σ is already pierced by q_1. This contradicts that our algorithm places two piercing points in $T_{i,\theta}$. Hence, Algorithm-Center places at the most one piercing point in the spherical block $T_{i,\theta}$ to pierce objects in $\mathcal{I}_p$. ◁

Since the solid angle of the spherical sector $S(\theta, 1)$ is $2\pi(1-\cos(\theta))$ steradians, and the solid angle of a ball, measured from any point in its interior, is 4π steradians, therefore, we need at most $\frac{2}{1-\cos\theta}$ spherical sectors to cover the ball B_1 [16, §4.8.3]. Combining this with Lemma 14, we have $|\mathcal{A}_p| \leq \frac{2m}{(1-\cos\theta)}$. We can give a slightly better estimation, as follow. Since the distance between any two points in the innermost ball B_m is at most one, therefore any object in $\mathcal{I}_p$ having center $q \in B_m$ contains the entire ball B_m. As a result, our online algorithm places at most one piercing point in B_m. Thus, $|\mathcal{A}_p| \leq \frac{2(m-1)}{1-\cos(\theta)} + 1 \leq \frac{2(\log_{(1+x)}(\frac{2k}{\alpha}))}{(1-\cos(\theta))} + 1$. This completes the proof of the theorem. ◀

As a consequence of Property 1 and Theorem 12, we have the following.

▶ **Corollary 15.** *For piercing arbitrary oriented scaled copies of an object in $\mathbb{R}^3$ with aspect ratio α and having width in the range $[1, k]$,* Algorithm-Center *achieves a competitive ratio of at most $\frac{2}{1-\cos(\theta)}\lceil\log_{(1+x)}(\frac{2k}{\alpha})\rceil + 1$, where $\theta = \frac{1}{2}\cos^{-1}\left(\frac{1}{2} + \frac{1}{1+\sqrt{1+4\alpha^2}}\right)$ and $x = \frac{\sqrt{1+4\alpha^2}-1}{2}$.*

Analogous to Theorem 12, we have similar result for α-aspect objects in $\mathbb{R}^2$. Here, we consider circular sector $C(\theta, r)$ with central angle θ instead of spherical sector $S(\theta, r)$ with central angle 2θ.

▶ **Theorem 16.** *For piercing α-aspect objects in $\mathbb{R}^2$ having width in the range $[1, k]$,* Algorithm-Center *achieves a competitive ratio of at most $\frac{2\pi}{\theta}(\log_{(1+x)}(2k/\alpha)) + 1$, where $\theta = \cos^{-1}\left(\frac{1}{2} + \frac{1}{1+\sqrt{1+4\alpha^2}}\right)$ and $x = \frac{\sqrt{1+4\alpha^2}-1}{2}$.*

Observe that for hypercubes in $\mathbb{R}^d$, the value of α is $\frac{1}{\sqrt{d}}$. As a result, we have the upper bound as follows.

▶ **Corollary 17.** *For piercing arbitrary oriented hypercubes in $\mathbb{R}^3$ (respectively, $\mathbb{R}^2$) with a side-length in the range $[1, k]$,* Algorithm-Center *achieves a competitive ratio of at most $223.98\lceil\log_2(2\sqrt{3}k)\rceil + 1$ (respectively, $26.67\lceil\log_2(2\sqrt{2}k)\rceil + 1$).*

Observe that for balls in $\mathbb{R}^2$ and $\mathbb{R}^3$, the value of α is one. As a result, for $d = 2$ and 3, we have a better estimation of the upper bound than Corollary 10. The result is as follows

▶ **Corollary 18.** *For piercing balls in $\mathbb{R}^3$ (respectively, $\mathbb{R}^2$) with a radius in the range $[1, k]$,* Algorithm-Center *achieves a competitive ratio of at most $58.861\lceil\log_2(2k)\rceil + 1$ (respectively, $14.4\lceil\log_2(2k)\rceil + 1$).*

4 Lower Bound for Bounded Scaled Objects

To obtain a lower bound, we think of a game between two players: Alice and Bob. Here, Alice plays the role of an adversary, and Bob plays the role of an online algorithm. In each round of the game, Alice presents an object such that Bob needs to place a new piercing point, i.e., the object does not contain any of the previously placed piercing points. To obtain a lower bound of the competitive ratio of $\Omega(z)$, it is enough to show that Alice can present a sequence of z nested objects in a sequence of z consecutive rounds of the game such that an offline optimum algorithm uses only one point to pierce all of these objects. First, we consider when objects are α-aspect objects in $\mathbb{R}^2$, followed by axis parallel hypercubes in $\mathbb{R}^d$.

4.1 α-Aspect Objects in $\mathbb{R}^2$

▶ **Theorem 19.** *The competitive ratio of every deterministic online algorithm for piercing α-aspect objects in $\mathbb{R}^2$ having width in the range $[1,k]$ is at least $\log_{(\frac{2}{\alpha})} k$ for $k > \frac{2}{\alpha}$.*

Proof. To prove the lower bound, we adaptively construct a sequence of $m = \log_{(\frac{2}{\alpha})} k$ objects each with aspect ratio α and width in the range $[1,k]$ such that any online algorithm needs at least m piercing points to pierce them; while the offline optimal needs just one point. In each round of the game, the adversary presents an object of width in the range $[1,k]$, and the online algorithm needs to place a piercing point to pierce the presented object if it is already not pierced by any of the previously placed piercing points. Let $w(\sigma)$ and $h(\sigma)$ be width and height of an object σ, respectively. Let σ_1, having width $w(\sigma_1) = k$, be the first input object presented to the algorithm. For the sake of simplicity, let us assume that the center c_1 of σ_1 coincides with the origin. All remaining objects $\sigma_2, \sigma_3, \ldots, \sigma_m$ are presented adaptively depending on the position of the piercing points $p_1, p_2, \ldots, p_{m-1}$ placed by the algorithm. For $i = 1, 2, \ldots, m = \log_{(\frac{2}{\alpha})} k$, we maintain the following two invariants.

(1) The object σ_i having width $k\left(\frac{\alpha}{2+\epsilon}\right)^{i-1}$ is not pierced by any of the previously placed piercing point p_j, where $j < i$.

(2) The object σ_i is totally contained in the object σ_{i-1}.

Invariant (1) ensures that any online algorithm needs $m+1$ piercing points; while invariant (2) ensures that all the objects $\sigma_1, \sigma_2, \ldots, \sigma_m$ can be pierced by a single point. For $i = 1$, both invariants hold trivially. At the beginning of round $i = 1, 2, \ldots, m-1$, assume that both invariants hold. Depending on the position of the previously placed piercing point p_i, in the $(i+1)$th round of the game, the object σ_{i+1}, having width $w(\sigma_{i+1}) = k\left(\frac{\alpha}{2+\epsilon}\right)^{i}$ $\left(h(\sigma_{i+1}) = \frac{k}{2+\epsilon}\left(\frac{\alpha}{2+\epsilon}\right)^{i-1}\right)$, is presented to the algorithm. Here $\epsilon > 0$ is an arbitrary constant very close to zero. The center c_{i+1} of σ_{i+1} is defined as the following.

$$c_{i+1} = \begin{cases} c_i + \frac{w(\sigma_i)}{2}, & \text{if } p_i(x_1) \le c_i(x_1), \\ c_i - \frac{w(\sigma_i)}{2}, & \text{otherwise } (i.e., p_i(x_1) > c_i(x_1)), \end{cases}$$

where $p_i(x_1)$ and $c_i(x_1)$ denotes the first coordinate of p_i and c_i, respectively.

First, we show that σ_{i+1} is totally contained in σ_i. Observe that, depending on the position of p_i, the center of σ_{i+1} is either $c_i + \frac{w(\sigma_i)}{2}$ or $c_i - \frac{w(\sigma_i)}{2}$. In both cases, we have $dist(c_i, c_{i+1}) = \frac{w(\sigma_i)}{2}$. On the other hand, $h(\sigma_{i+1}) = \frac{k}{2+\epsilon}\left(\frac{\alpha}{2+\epsilon}\right)^{i-1} < \frac{k}{2}\left(\frac{\alpha}{2+\epsilon}\right)^{i-1} = \frac{w(\sigma_i)}{2}$. Hence, σ_{i+1} is totally contained in σ_i. Thus, invariant (2) is maintained.

Note that $dist(p_i, c_{i+1})$ is greater than the height of σ_{i+1} since

$$dist(p_i, c_{i+1}) > \frac{w(\sigma_i)}{2} = \frac{k}{2}\left(\frac{\alpha}{2+\epsilon}\right)^{i-1} > \frac{k}{2+\epsilon}\left(\frac{\alpha}{2+\epsilon}\right)^{i-1} = h(\sigma_{i+1}).$$

The first inequality follows from the definition of c_{i+1}. Thus, σ_{i+1} does not contain the point p_i. Due to induction hypothesis, σ_i does not contain any of the previously placed piercing point p_j for $j < i$ and from invariant (2) we know that σ_{i+1} is contained in σ_i. Hence, σ_{i+1} does not contain any of the previously placed piercing point p_j, for $j < i+1$. Thus, invariant (1) is maintained.

Note that when $m > \log_{\frac{2+\epsilon}{\alpha}} k + 1$, the width of the object σ_m, i.e., $k\left(\frac{\alpha}{2+\epsilon}\right)^{m-1}$ is less than one. In other words, for any $m \le \log_{\frac{2+\epsilon}{\alpha}} k + 1$, we can construct the input sequence satisfying both invariants. Since $\epsilon > 0$ is arbitrarily constant very close to 0, we choose the value of $m = \log_{\frac{2}{\alpha}} k$. Hence, the theorem follows. ◀

4.2 Fixed Oriented Hypercubes in $\mathbb{R}^d$

In this subsection, all the hypercubes are axis parallel. First, we establish the following essential ingredient to prove our main result.

▶ **Lemma 20.** GAME-OF-SAME-SIDE(r) ($GSS(r)$):

For any $r \in \mathbb{R}^+$, in a consecutive d rounds of the game, Alice can adaptively present d hypercubes $\sigma_1, \sigma_2, \ldots, \sigma_d$, each having side length r, such that

(i) *Bob needs to place d points to pierce them;*

(ii) *the common intersection region $Q = \cap_{i=1}^{d} \sigma_i$ is nonempty;*

(iii) *moreover, Q contains an* empty *hypercube E, of side length at least $\frac{r}{2} - \epsilon$, not containing any of the d piercing points placed by Bob. Here $\epsilon > 0$ is a very small constant close to 0.*

We refer a consecutive d rounds of the game satisfying the above lemma as a $GSS(r)$. The first hypercube σ_1 is denoted as the *starter* and the hypercube E as the *empty hypercube* since it does not contain any of the piercing points placed by Bob in the $GSS(r)$. Note that any hypercube of side length r could be the starter of this d round game.

Proof. Let σ_1 be a hypercube of side length r presented by Alice in the first round of the game. Throughout the proof of this claim, for the sake of simplicity, we assume that the center of σ_1 is the origin (if not, we can always translate the coordinate system to make it). Alice presents the remaining hypercubes $\sigma_2, \ldots, \sigma_d$ adaptively depending on Bob's moves. We maintain the following two invariants: For $i = 1, \ldots, d$, when Alice presents hypercubes $\sigma_1, \ldots, \sigma_i$, each of side length r, and Bob presents piercing points $p_1, \ldots, p_i$,

(I) the hypercube σ_i is not pierced by any of the previously placed piercing point p_j, where $j < i$;

(II) the common intersection region $Q_i = \cap_{j=1}^{i} \sigma_j$ is a hyperrectangle whose first $(i-1)$ sides are of length $\frac{r}{2} - \epsilon$ each, and each of the remaining sides are of length r. Moreover, Q_i does not contain any of the points p_j, where $j < i$.

An illustration of the planar version of the game appears in Figure 6. For $i = 1$, both the invariants trivially hold. At the beginning of round i (for $i = 2, \ldots, d$), assume that both invariants hold. Let us define a translation vector $\mathbf{v}_i \in \mathbb{R}^d$ as $\mathbf{v}_i = (s(1)\left(\frac{r}{2} + \epsilon\right), s(2)\left(\frac{r}{2} + \epsilon\right), \ldots, s(i-1)\left(\frac{r}{2} + \epsilon\right), 0, \ldots, 0)$, where for any $j < i$, we have

$$s(j) = \begin{cases} +1, & \text{if } p_j(x_j) < 0, \text{ where } p_j(x_j) \text{ is } jth \text{ coordinate of } p_j, \\ -1, & \text{otherwise.} \end{cases}$$

We define $\sigma_i = \sigma_1 + \mathbf{v}_i$. For any $j < i$, due to the definition of the jth component of the translation vector $\mathbf{v}_i$, the hypercube σ_i does not contain the point p_j. Hence, invariant (I) is maintained.

Note that $\sigma_1 \cap \sigma_i$ is a hyperrectangle whose first $(i-1)$ sides are of length $\frac{r}{2} - \epsilon$ each, and each of the remaining sides are of length r. On the other hand, from the assumption, we know that Q_{i-1} is a hyperrectangle whose first $(i-2)$ sides are of length $\frac{r}{2} - \epsilon$ each, and each of the remaining sides are of length r. Therefore, $Q_i = Q_{i-1} \cap \sigma_i$ is a hyperrectangle whose first $(i-1)$ sides are of length $\frac{r}{2} - \epsilon$ each, and each of the remaining sides are of length r. Since σ_i does not contain any of the points p_j where $j < i$, as a result the hyperrectangle $Q_i \subseteq \sigma_i$ also does not contain any of them. Therefore, invariant (II) is also maintained.

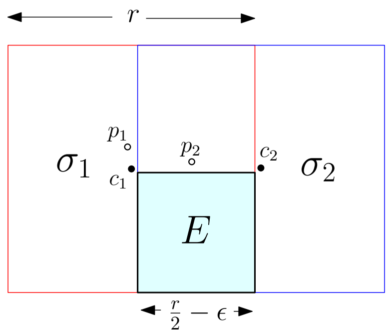

Figure 6 Illustration of the GAME-OF-SAME-SIDE(r) for $d = 2$. Initially Alice presented hypercube σ_1 of side length r in the first round of the game. Then, Alice presents the second hypercube σ_2 adaptively, depending on Bob's moves, such that $p_1 \notin \sigma_2$. Also, the common intersection region $\cap_{j=1}^{2}\sigma_j$ is a rectangle whose sides are of length $\frac{r}{2} - \epsilon$ and r. Moreover, the intersection region contains an *empty* hypercube E, of side length at least $\frac{r}{2} - \epsilon$, not containing p_1 and p_2. Here $\epsilon > 0$ is a very small constant close to 0.

At the end of the dth round of the game, we have the hyperrectangle $Q = Q_d$ (whose dth side is only of length r, other sides are of length $\frac{r}{2} - \epsilon$ each) that does not contain any of the points p_j where $j < d$, but it may contain the point p_d. Depending on the dth coordinate of the point p_d, we can construct a hypercube $E \subset Q$ of length $\frac{r}{2} - \epsilon$ that does not contain the point p_d. Hence, the lemma follows. ◀

Now, we prove the following main theorem.

▶ **Theorem 21.** *The competitive ratio of every deterministic online algorithm for piercing fixed oriented hypercubes in $\mathbb{R}^d$ with side length in the range $[1, k]$ is at least $d\left(\log_2\left(\frac{k}{1+2\epsilon}\right)\right) + 2^d$.*

Proof. To prove the main theorem, we maintain the following two invariants: For $i = 1, \ldots, z = d\left(\left(\frac{k}{1+2\epsilon}\right) - 1\right) + 2^d$, when Alice presents hypercubes $\sigma_1, \ldots, \sigma_i$ and Bob presents piercing points $p_1, \ldots, p_i$,

(III) The hypercube σ_i is not pierced by any of the previously placed piercing point p_j, where $j < i$.

(IV) The intersection region $\cap_{j=1}^{i}\sigma_j$ is nonempty.

Invariant (III) implies that Bob is forced to place z piercing points, while Invariant (IV) ensures that all the hypercubes $\sigma_1, \sigma_2, \ldots, \sigma_z$ can be pierced by a single piercing point.

We evoke the $GSS(r)$ for $t > \left(\log_2\left(\frac{k}{1+2\epsilon}\right) - 1\right)$ times with different values of $r = k, \frac{k}{2} - \epsilon, \ldots, \frac{k}{2^t} - \frac{2^t-1}{2^{t-1}}\epsilon$. More specifically, the empty hypercube $E_{t'}$ generated by $GSS(\frac{k}{2^{t'-1}} - \frac{2^{t'}-1}{2^{t'-1}}\epsilon)$ plays the role of the starter hypercube for the next evokation of $GSS(\frac{k}{2^{t'}} - \frac{2^{t'}-1}{2^{t'-1}}\epsilon)$, where $1 \leq t' < t$. Since the empty hypercube of t'th game plays the role of the starter of $(t'+1)$th game, it is straightforward to see that the set of td hypercubes $\sigma_1, \sigma_2, \ldots, \sigma_{td}$ and set of points $p_1, p_2, \ldots, p_{td}$ satisfies both invariants (III) and (IV). At the end of the tdth GSS, the empty hypercube E_{td} has side length at least 1. Due to Corollary 3, starting with the hypercube E_{td}, Alice can adaptively present 2^d (including E_{td}) hypercubes each of side length 1 such that both invariants (III) and (IV) are maintained. This completes the proof of the theorem. ◀

4.3 Balls in $\mathbb{R}^2$ and $\mathbb{R}^3$

Similar to the proof of Theorem 21, one can prove the following lower bound result for balls in $\mathbb{R}^3$ (respectively, $\mathbb{R}^2$). Also, note that we could not generalize this to higher dimensional balls.

▶ **Theorem 22.** *The competitive ratio of every deterministic online algorithm for piercing balls in $\mathbb{R}^3$ (respectively, $\mathbb{R}^2$) with radius in the range $[1, k]$ is at least $3\log_4 k+1$ (respectively, $2\log_{(\frac{8}{3})} k+1$).*

5 Conclusion

We show that no online algorithm can obtain a competitive ratio lower than $\Omega(n)$ for piercing n intervals. Due to this pessimistic result, we restricted our attention to special kinds of objects, i.e., bounded scaled objects. We propose upper bounds for piercing bounded scaled α-aspect$_\infty$ objects in higher dimension. By placing specific values of α, the tight asymptotic bounds can be obtained for bounded scaled objects (like intervals, squares, disks, and α-aspect polygons) in $\mathbb{R}$ and $\mathbb{R}^2$. For bounded scaled α-aspect$_\infty$ objects in $\mathbb{R}^d$, it is possible to generalize our lower bound result obtained for hypercubes. For higher dimensions, for example, for d-dimensional hypercubes, there is a huge gap between the lower and the upper bound. We propose bridging these gaps as a future direction of research. We only consider the deterministic model. This raises the question of whether randomization helps.

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