# Parameterized Complexity of a Parallel Machine Scheduling Problem

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## Abstract

In this paper we consider the parameterized complexity of two versions of a parallel machine scheduling problem with precedence delays, unit processing times and time windows. In the first version – with exact delays – we assume that the delay between two jobs must be exactly respected, whereas in the second version – with minimum delays – the delay between two jobs is a lower bound on the time between them. Two parameters are considered for this analysis: the pathwidth of the interval graph induced by the time windows and the maximum precedence delay value. We prove that our problems are para-NP-complete with respect to any of the two parameters and fixed-parameter tractable parameterized by the pair of parameters.

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## 1 Introduction

While scheduling jobs with resources and precedence constraints has a wide range of industrial applications, these problems have been proven to be NP-hard in the strong sense even for very simple settings. For example minimizing the makespan of a schedule of jobs on two parallel processors assuming chain-like precedence constraints is already NP-hard [8]. Scheduling problems are usually denoted by the three field denotation $\alpha|\beta|\gamma$ of Graham [11]. Field $\alpha$ describes the machine setting, field $\beta$ describes the job relations and properties, and field $\gamma$ defines the optimization criterion - or is a star $\star$ for a decision problem. For example the NP-hard problem described above is denoted by $P2|chains|C_{\max}$.

In this paper we tackle two decision scheduling problems on $M$ parallel processors. We consider a set of $n$ jobs $\mathcal{T}$. Each job $i$ has a unit processing time and a time interval $[r_i, d_i)$ given for its computation. $r_i$ is the release date of job $i$, and $d_i$ its deadline. Jobs are linked by precedence constraints defined by an acyclic precedence graph $G$ with non-negative precedence delays $\ell_{i,j}$ on each arc $(i,j)$ of $G$. Two variants of the precedence delays are considered: exact precedence delays and minimum precedence delays, which will be denoted by $\ell^{ex}_{i,j}$ and $\ell^{min}_{i,j}$ in the standard notations.

A feasible schedule $\sigma$ defines for each job $i \in \mathcal{T}$ a starting time $\sigma(i) \in [r_i, d_i)$ so that no more than $M$ jobs are scheduled at the same time, and so that for each arc $(i,j)$ of $G$:

- in case of exact delays: $\sigma(i) + 1 + \ell^{ex}_{i,j} = \sigma(j)$,
- in case of minimum delays: $\sigma(i) + 1 + \ell^{min}_{i,j} \leq \sigma(j)$.

In particular we will explore the parameterized complexity of these problems with chain-like precedence constraints, denoted by $P|chains(\ell^{ex}_{i,j}), p_j = 1, r_j, d_j|\star$ and $P|chains(\ell^{min}_{i,j}), p_j = 1, r_j, d_j|\star$.

When restricting all processing times to be equal to one, NP-hardness results typically depend on the precedence relations used in the problem. Ullman [19] first showed that $P|prec, p_j = 1|C_{max}$ is strongly NP-hard. This was later improved by Lenstra et al [14] with a precedence graph of bounded height, then by Garey et al [10] in the case of an opposing forest. When we have release dates Brucker et al [5] showed that $P|intree, p_j = 1, r_j|C_{max}$ is strongly NP-hard with an intree as the precedence graph.

When further restricting the problem to chains with precedence delays, Yu et al [22] showed that $1|chains(\ell^{min}_{i,j}), p_j = 1|C_{\max}$ and $1|chains(\ell^{ex}_{i,j}), p_j = 1|C_{\max}$ are strongly NP-hard, even when all chains are of length two (i.e. if we only have coupled tasks). The problem $1|chains(\ell^{ex}_{i,j}), p_j = 1|\star$ was also proven NP-hard earlier by Orman in [17]. Note that apart from the delays themselves, there is nothing more to restrict in order to go around NP-hardness. Thus within the scope of classical complexity theory it becomes difficult to find the frontier between polynomial-time solvability and NP-hardness when delays are considered.

Parameterized complexity theory gives numerous tools for a refined analysis of such hard scheduling problems. Given a parameter $k$ and denoting $n$ the input size, a problem is called $fixed-parameter\ tractable$ (FPT) with respect to parameter $k$ if it can be solved in time $poly(n) \times f(k)$ with $f$ an arbitrary function [6]. Not only it tells us more about a problem than if we only showed NP-hardness, it allows us to further classify the NP-hard problems depending on whether they are FPT parameterized by $k$ or not.

When the studied problem is believed to not be FPT, the para-NP class is used as a parameterized version of NP: a problem is in para-NP with respect to parameter $k$ if there is a non-deterministic algorithm that solves it in time $poly(n) \times f(k)$ with $f$ an arbitrary function. In order to prove that a problem is para-NP-hard with respect to $k$, it is enough to prove that the un-parameterized problem is NP-hard for some fixed value of the parameter [9]. The W-hierarchy defined in [7] is an additional widely used tool in parameterized complexity. It is a sequence of intermediate complexity classes between FPT and para-NP, which allows us to further investigate the time complexity of a parameterized problem.

Until now quite few parameterized complexity results have been be proved for scheduling problems. Among the first ones related to our problems, Bodlaender proved in [3] that minimizing the makespan of unit processing times jobs on parallel machines is W[2]-Hard considering the number of machines as parameter. When precedence constraints are involved, several authors studied the parameter defined by the width $w$ of the precedence graph. Van Bevern et al. [20] proved that the problem $P2|prec, p_j \in \{1, 2\}|C_{\max}$ is W[2]-hard with respect to the width. In the same paper they also proved that if we add the maximum allowed lag of a job with respect to its earliest start time (ignoring resource constraints) to the width as parameter then the problem becomes FPT, even for more complex resource constraints (RCPSP). In [15] it is noted that the parameterized complexity of the problem with three processors, precedence constraints and unit execution times with respect to width $w$ is still open.

Now considering precedence delays, we can first mention the work of Bessy et al [2] on coupled tasks with due dates, a compatibility graph and a common deadline on a single machine. They consider the number of jobs that end before the due date as their parameter. They establish W[1]-hardness for this problem and propose a FPT algorithm when the maximum duration of a job (i.e. the processing time of the two tasks plus their delay) is bounded. Bodlaender et al in [4] studied the two problems we address in this paper but assuming that each chain $C$ has a time window $[r_C, d_C)$ instead of time windows for individual jobs like in our case. They considered two parameters: the first one is the number of chains and the second one is the thickness, i.e. the maximum number of overlapping chain time windows. On one hand they proved that for exact or minimum delays $1|chains(\ell_{i,j}), p_j = 1, r_C, d_C|\star$ is W[1]-hard with any of the two parameters, while the parallel machine variant $P|chains(\ell_{i,j}), p_j = 1, r_C, d_C|\star$ is W[2]-hard. On the other hand they proved that these problems are in XP (i.e. deterministic $n^{f(k)}$ time) if the delays are unary encoded.

Considering scheduling problems with job time windows, the pathwidth $\mu$ has been considered recently as a parameter by several authors [1, 16, 21]. Parameter $\mu$ is the pathwidth of the graph induced by jobs time intervals. It can be easily computed since $\mu + 1$ is the maximum number of overlapping job time windows at any given time. In particular Munier proved in [16] that scheduling unit processing times jobs with time windows and precedence constraints is FPT with parameter $\mu$. Note that van Bevern et al. [20] showed that $P2|prec, p_j \in \{1, 2\}|C_{max}$ is W[2]-hard parameterized by $w$, while this problem is likely to be FPT parameterized by $\mu$ according to Hanen and Munier [12]. With $\mu$ allowing such problems to be FPT when other parameters do not, it makes it all the more difficult to find hardness results relative to $\mu$.

The two studied parameters in our paper are pathwidth $\mu$ and the maximum precedence delay $\ell_{max} = \max_{(i,j)\ arc\ of\ G} \ell_{i,j}$. We first show in Section 2 that both decision problems $P|chains(\ell^{ex}_{i,j}), p_j = 1, r_j, d_j|\star$ and $P|chains(\ell^{min}_{i,j}), p_j = 1, r_j, d_j|\star$ are para-NP-complete parameterized by either one alone. In the case of pathwidth $\mu$ as the parameter we even prove that the single machine variant of both problems are para-NP-complete. Then in section 3 we prove that these problems are FPT parameterized by the couple $(\mu, \ell_{max})$.

## 2 Hardness reductions

In this section we prove para-NP-completeness of our two parallel machine scheduling problems parameterized by pathwidth $\mu$ or maximum delay $\ell_{max}$ alone. We even establish para-NP-completeness of the single-machine variant in the case of parameter $\mu$.

We use the following result from Flum and Grohe's book [9]:

▶ **Lemma 1** (Flum, Grohe 1998). *If a (nontrivial) problem $\mathcal{P}$ is* NP*-complete with a fixed value of some parameter $k$, then the parameterized problem $(\mathcal{P}, k)$ is para-*NP*-complete.*

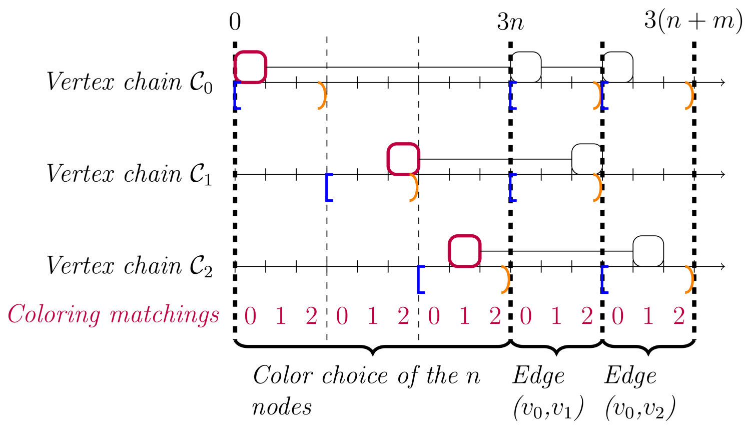
Thus the following scheduling problems will be shown to be NP-complete:

1. $1|chains(\ell^{ex}_{i,j}), p_j = 1, r_j, d_j|\star$ with $\mu = 1$,
2. $1|chains(\ell^{min}_{i,j}), p_j = 1, r_j, d_j|\star$ with $\mu = 2$,
3. $P|chains(\ell^{ex}_{i,j}), p_j = 1, r_j, d_j|\star$ with $\ell_{max} = 1$,
4. $P|chains(\ell^{min}_{i,j}), p_j = 1, r_j, d_j|\star$ with $\ell_{max} = 1$.

These problems are all trivially in NP by guessing the starting time of each job then checking if this leads to a feasible schedule. For the hardness proofs all reductions will start from the (strongly) NP-hard 3-COLORING graph problem [13]. Let $G = (V, E)$ be the input graph. Let $v_0, \ldots, v_{n-1}$ be the vertices in $V$ and $e_0, \ldots, e_{m-1}$ be the edges in $E$. Let $n = |V|$ and $m = |E|$. The colors will be named 0, 1 and 2.

### 2.1 NP-hardness of $1|chains(\ell^{ex}_{i,j}), p_j = 1, r_j, d_j|\star$ with $\mu = 1$

We build an instance of $1|chains(\ell^{ex}_{i,j}), p_j = 1, r_j, d_j|\star$ with $\mu = 1$. An example is given in Figure 1. We have $n$ *vertex chains* $\mathcal{C}_i$ with $deg(v_i) + 1$ jobs in chain $\mathcal{C}_i$, $0 \leq i \leq n - 1$ with $deg(v_i)$ the degree of node $v_i$ in $G$. We define vertex chain $\mathcal{C}_i$ the following way:



0
3n
3(n + m)
Vertex chain $\mathcal{C}_0$
Vertex chain $\mathcal{C}_1$
Vertex chain $\mathcal{C}_2$
Coloring matchings
0 1 2 0 1 2 0 1 2 0 1 2 0 1 2
Color choice of the n nodes
Edge ($v_0$,$v_1$)
Edge ($v_0$,$v_2$)

**Figure 1** An instance of $1|chains(\ell^{ex}_{i,j}), p_j = 1, r_j, d_j|\star$ representing a graph coloring. We have $G = (V, E)$ with $V = \{v_0, v_1, v_2\}$ and $E = (\{v_0, v_1\}, \{v_0, v_2\})$. This schedule corresponds to the coloring $(0, 2, 1)$.

▶ **Definition 2** (Vertex chain $\mathcal{C}_i$). *We segment time into $m + 1$ segments: a color choice segment $[0, 3n)$ and $m$ edge check selection segments of length 3 along $[3n, 3(n + m))$. We describe the chain from left to right:*

**(1)** *Color choice segment $[0, 3n)$*
- *The first job of chain $\mathcal{C}_i$ has time window $[3i, 3(i + 1))$.*
  - **a.** *If $v_i$ appears in no edge of $G$: end the chain.*
  - **b.** *Else: set $3(n - i) - 1$ as the current exact delay after this job.*

**(2)** *Edge check segment $[3(n + j), 3(n + j + 1))$, $0 \le j \le m - 1$*
*For $j$ in $[0, m - 1]$:*
*Let edge $e_j = \{v_{i_1}, v_{i_2}\}, i_1 < i_2$.*
- **a.** *Vertex chain $\mathcal{C}_i$ with $i \notin \{i_1, i_2\}$*
  - *Add 3 to the current exact delay after the currently latest job of chain $\mathcal{C}_i$.*
- **b.** *Vertex chain $\mathcal{C}_i$ with $i = i_1$ or $i = i_2$*
  - *Set a job with time window $[3(n + j), 3(n + j + 1))$*
    - **i.** *If $e_j$ is the last edge where $v_i$ appears: end the chain.*
    - **ii.** *Else: set 2 as the current exact delay after this job.*

▶ Remark 3. The created instance has pathwidth 1. In the color choice segment: for $i \in [0, n - 1]$ there is exactly one job to be scheduled in time window $[3i, 3(i + 1))$: the first job of vertex chain $\mathcal{C}_i$. In the edge check segments: for $j \in [0, m - 1]$ if edge $e_j = \{v_{i_1}, v_{i_2}\}$ then there are exactly two jobs to be scheduled in time window $[3(n + j), 3(n + j + 1))$: one from vertex chain $\mathcal{C}_{i_1}$ and one from vertex chain $\mathcal{C}_{i_2}$. Thus there are indeed at most two overlapping time windows at any given time.

Let $i \in [0, n - 1]$. Vertex chain $\mathcal{C}_i$ has three possible starting times in $[3i, 3(i + 1))$ which corresponds to the three color choices of node $v_i$. Then this color choice is propagated to every edge check segment $[3(n + j), 3(n + j + 1))$ where node $v_i$ is a part of edge $e_j$, $j \in [0, m - 1]$. The following lemma ensures that the color choices are faithfully propagated.

▶ **Lemma 4.** *Let $0 \le i \le n - 1$. In any feasible schedule, if vertex chain $\mathcal{C}_i$ starts at time $3i + k$ with $k \in \{0, 1, 2\}$, then all jobs $J$ in this chain are scheduled at time $r(J) + k$, where $r(J)$ is the release date of job $J$.*

**Proof.** Suppose we have a feasible schedule where vertex chain $\mathcal{C}_i$ starts at time $3i + k$ with $k \in \{0, 1, 2\}$.

- If vertex $v_i$ is part of no edge in the graph:
  Then by Definition 2 vertex chain $\mathcal{C}_i$ only has one job. It is scheduled at time $3i + k$, which is indeed the release date of this job plus $k$.
- If vertex $v_i$ is part of at least one edge in the graph:
  Let $j_0 < j_1 < \ldots < j_{deg(v_i)-1}$ be the indices of the edges $e_j$ such that $v_i \in e_j$. We prove by induction on $l \in [0, deg(v_i) - 1]$ that the job of vertex chain $\mathcal{C}_i$ which has time window $[3(n + j_l), 3(n + j_l + 1))$ is scheduled at time $3(n + j_l) + k$.
  - Consider the job of vertex chain $\mathcal{C}_i$ which has time window $[3(n + j_0), 3(n + j_0 + 1))$. By Definition 2 this job is the successor of the first job in the chain and the exact delay between them is $3(n-i)-1+3j_0$. Thus if the first job of the chain is scheduled at time $3i + k$, then this following job is scheduled at time $(3i + k) + 1 + (3(n - i) - 1 + 3j_0) = 3(n + j_0) + k$, which is indeed the release date of this job plus $k$.
  - Let $l \in [1, deg(v_i) - 1]$. Suppose the job of vertex chain $\mathcal{C}_i$ which has time window $[3(n+j_{l-1}), 3(n+j_{l-1}+1))$ is scheduled at time $3(n+j_{l-1})+k$. Then by Definition 2 there is an exact delay $2 + 3(j_l - j_{l-1} - 1)$ before the next job of the chain. Thus the next job of the chain is scheduled at time $(3(n+j_{l-1})+k)+1+(2+3(j_l-j_{l-1}-1)) = 3(n + j_l) + k$, which is indeed the release date of this job plus $k$.

  This proves the lemma for all the jobs of vertex chain $\mathcal{C}_i$ in an edge check segment. ◀

Then for each edge $e_j = \{v_{i_1}, v_{i_2}\}, i_1 < i_2$, the color choices of $v_{i_1}$ and $v_{i_2}$ are confronted in edge check segment $[3(n + j), 3(n + j + 1))$. If both nodes chose the same color then both jobs in this edge check segment would be scheduled at the same time, which would invalidate our schedule in this single-machine instance. Conversely if we start from a valid coloring, then there will never be two jobs scheduled at the same time in an edge check segment. This is the key ingredient behind the reduction.

▶ **Proposition 5.** *G is 3-colorable if and only if there exists a feasible schedule for this instance of EXACT DELAYS.*

**Proof.** ($\implies$) Suppose we have $(c_0, \ldots, c_{n-1}) \in \{0, 1, 2\}^n$ a 3-coloring of $G$ where vertex $v_i$ has color $c_i$. We propose the schedule where for all $0 \le i \le n - 1$, chain $\mathcal{C}_i$ starts at time $3i + c_i$. Then in every edge $e_j \in E$ where vertex $v_i$ appears, we schedule the job of chain $\mathcal{C}_i$ which is in edge check segment $[3(n + j), 3(n + j + 1))$ at time $3(n + j) + c_i$.

We show that the jobs in different chains do not interfere with each other. Since the time windows do not overlap in the color choice segment, only the edge check segments remain to be checked. Let $e_j = \{v_{i_1}, v_{i_2}\}$ be an edge in $E$. By definition of the vertex chains, only chains $\mathcal{C}_{i_1}$ and $\mathcal{C}_{i_2}$ have a job to be scheduled in time window $[3(n + j), 3(n + j + 1))$. In our schedule the job of chain $\mathcal{C}_{i_1}$ is scheduled at time $3(n + j) + c_{i_1}$ and the job of chain $\mathcal{C}_{i_2}$ at time $3(n + j) + c_{i_2}$. Since $(c_0, \ldots, c_{n-1})$ is a 3-coloring and $\{v_{i_1}, v_{i_2}\} \in E$, we have $c_{i_1} \neq c_{i_2}$. Thus both jobs are scheduled at different times and the jobs in edge check segment $[3(n + j), 3(n + j + 1))$ do not interfere with each other. Thus the proposed schedule is feasible.

($\impliedby$) Suppose we have a feasible schedule. For all $0 \le i \le n - 1$, let $s_i \in \{0, 1, 2\}$ be such that $3i + s_i$ is the starting time of chain $\mathcal{C}_i$. We show that $(s_0, \ldots, s_{n-1})$ is a 3-coloring of $G$. By contradiction suppose there is an edge $e_j = \{v_{i_1}, v_{i_2}\} \in E$ such that $s_{i_1} = s_{i_2}$. Then by Lemma 4 the jobs of chains $i_1$ and $i_2$ that must be scheduled in edge check segment $[3(n+j), 3(n+j+1))$ are scheduled at the same time $3(n+j)+s_{i_1}$. Thus the schedule is not feasible, which leads to a contradiction. Thus $(s_0, \ldots, s_{n-1})$ is indeed a 3-coloring of $G$. ◀

This proves that $1|chains(\ell^{ex}_{i,j}), p_j = 1, r_j, d_j|\star$ with $\mu = 1$ is NP-hard, which concludes the para-NP-completeness proof of the corresponding parameterized problem.

▶ **Theorem 6.** *$1|chains(\ell^{ex}_{i,j}), p_j = 1, r_j, d_j|\star$ is para-NP-complete parameterized by pathwidth $\mu$.*

## 2.2 NP-hardness of $1|chains(\ell^{min}_{i,j}), p_j = 1, r_j, d_j|\star$ with $\mu = 2$

We build an instance of $1|chains(\ell^{min}_{i,j}), p_j = 1, r_j, d_j|\star$ with $\mu = 1$. We begin in a similar way: for each node we have a vertex chain $\mathcal{C}'_i$ with three possible starting times, each corresponding to a color choice, then we want to propagate this color choice. However now that the delays are not exact anymore, the color choice cannot be propagated properly as previously. More constraints are needed in order to deal with the extra flexibility coming from the minimum delays.

One way is to add a closing segment $[3(n+m), 3(2n+m))$ at the end and two gadget chains $\mathcal{C}'_{i,1}, \mathcal{C}'_{i,2}$ per node, each composed of two jobs. As shown in Figure 2 the gadget chains will fill the two gaps at the start and at the end of each vertex chain.

▶ **Definition 7** (Vertex chain $\mathcal{C}'_i$). *We segment time into $m+2$ segments: a color selection segment $[0, 3n)$, $m$ edge check selection segments along $[3n, 3(n+m))$ and a closing segment $[3(n+m), 3(2n+m))$. We describe the chain from left to right:*

**(1)** *Color selection segment $[0, 3n)$*
- *The first job of chain $\mathcal{C}'_i$ has time window $[3i, 3(i+1))$.*
- *Set $3(n-i)-1$ as the current minimum delay after this job.*

**(2)** *Edge check segment $[3(n+j), 3(n+j+1))$, $0 \le j \le m-1$*
*For $j$ in $[0, m-1]$:*
*Let edge $e_j = \{v_{i_1}, v_{i_2}\}, i_1 < i_2$.*
**a.** *Vertex chain $\mathcal{C}'_i$ with $i \notin \{i_1, i_2\}$*
- *Add 3 to the current minimum delay after the currently latest job of chain $\mathcal{C}'_i$*

**b.** *Vertex chain $\mathcal{C}'_i$ with $i = i_1$ or $i = i_2$*
- *Set a job with time window $[3(n+j), 3(n+j+1))$*
- *Set 2 as the current minimum delay after this job*

**(3)** *Closing segment $[3(n+m), 3(2n+m))$*
- *Add $3i$ to the current minimum delay of the currently latest job of chain $\mathcal{C}'_i$.*
- *Set a job with time window $[3(n+i+m), 3(n+i+1+m))$ as the last job of vertex chain $\mathcal{C}'_i$.*

▶ **Definition 8** (Gadget chains $\mathcal{C}'_{i,1}, \mathcal{C}'_{i,2}$). *For both gadget chains $\mathcal{C}'_{i,1}, \mathcal{C}'_{i,2}$ relative to vertex $v_i$, the first job must be scheduled in time window $[3i, 3(i+1))$, the second one in time window $[3(n+m+i), 3(n+m+i+1))$, and there is a minimum delay $3(n+m)-1$ between them.*

▶ Remark 9. The created instance has indeed pathwidth 2: gadget chains add two more time windows at the beginning and the end of each vertex chain, so there are at most three time windows overlapping at any time.

A full example is given in Figure 2. For our proof the goal is to show that adding these gadget chains is enough to get an analogue result to Lemma 4. Note that if we only use the definition of the chains like in the reduction of Section 2.1, we only get this weaker result:

▶ **Lemma 10.** *Let $0 \le i \le n-1$. In any feasible schedule, if a chain starts at time $3i+k$ with $k \in \{0, 1, 2\}$, then all jobs $J$ in this chain are scheduled at time $r(J)+k$ or later, where $r(J)$ is the release date of job $J$.*

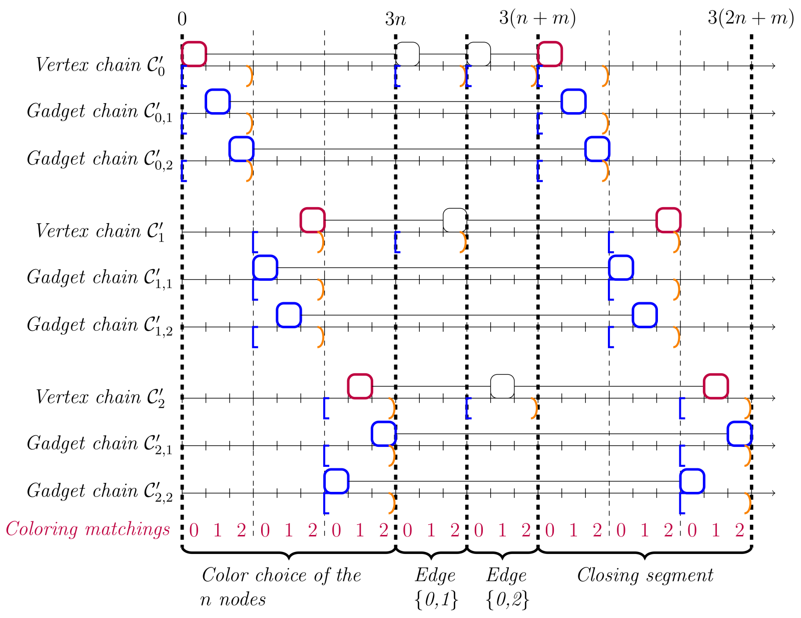

0
3n
3(n + m)
3(2n + m)
Vertex chain $\mathcal{C}'_0$
Gadget chain $\mathcal{C}'_{0,1}$
Gadget chain $\mathcal{C}'_{0,2}$
Vertex chain $\mathcal{C}'_1$
Gadget chain $\mathcal{C}'_{1,1}$
Gadget chain $\mathcal{C}'_{1,2}$
Vertex chain $\mathcal{C}'_2$
Gadget chain $\mathcal{C}'_{2,1}$
Gadget chain $\mathcal{C}'_{2,2}$
Coloring matchings
0 1 2 0 1 2 0 1 2 0 1 2 0 1 2 0 1 2 0 1 2 0 1 2
Color choice of the n nodes
Edge {0,1}
Edge {0,2}
Closing segment

**Figure 2** An instance of $1|chains(\ell_{i,j}^{min}), p_j = 1, r_j, d_j|\star$ representing a graph coloring. We have $G = (V, E)$ with $V = \{v_0, v_1, v_2\}$ and $E = (\{v_0, v_1\}, \{v_0, v_2\})$. This schedule corresponds to the coloring $(0, 2, 1)$.

**Proof.** For gadget chains $\mathcal{C}'_{i,1}, \mathcal{C}'_{i,2}$ this comes from the minimum delay $3(n+m)-1$ between their two jobs. For vertex chain $\mathcal{C}'_i$: suppose we have a feasible schedule where vertex chain $\mathcal{C}'_i$ starts at time $3i + k$ with $k \in \{0, 1, 2\}$.

- If vertex $v_i$ is part of no edge in the graph:
  Then by Definition 7 vertex chain $\mathcal{C}_i$ only has two jobs and there is a minimum delay $3(n-i)-1+3m+3i = 3(n+m)-1$ between them. Thus if the first job is scheduled at time $3i+k$, then the job in the closing segment is scheduled at time $(3i+k)+1+(3(n+m)-1) = 3(n+m+i)+k$, which is indeed the release date of this job plus $k$.
- If vertex $v_i$ is part of at least one edge in the graph:
  Let $j_0 < j_1 < \ldots < j_{deg(v_i)-1}$ be the indices of the edges $e_j$ such that $v_i \in e_j$. We prove the lemma for all the jobs of vertex chain $\mathcal{C}_i$ in an edge check segment by induction on $l \in [0, deg(v_i)-1]$ the same way as in Lemma 4. Then only the job of the chain in the closing segment is left. By Definition 7 there is a minimum delay $2+3(m-1-j_{deg(v_i)-1})+3i$ between this job and the one in edge check segment $[3(n+j_{deg(v_i)-1}), 3(n+j_{deg(v_i)-1}+1))$. Since we know from the induction that the latter job is scheduled at time $3(n+j_{deg(v_i)-1})+k$ or later, this means that the job of vertex chain $\mathcal{C}_i$ in the closing segment is scheduled at time $(3(n+j_{deg(v_i)-1})+k)+1+(2+3(m-1-j_{deg(v_i)-1})+3i) = 3(n+m+i)+k$ or later, which is indeed the release date of this job plus $k$. ◀

By taking into account the constraints added by the gadget chains at the beginning and at the end of each vertex chain, we are able to prove the needed key property.

▶ **Lemma 11.** *In any feasible schedule, if a vertex chain $\mathcal{C}_i$ starts at time $3i + k$ with $k \in \{0, 1, 2\}$, then all jobs $J$ in vertex chain $\mathcal{C}_i$ which are in an edge check segment have to be scheduled at time $r(J) + k$, where $r(J)$ is the release date of job $J$.*

**Proof.** Consider a feasible schedule. Let $0 \leq i \leq n-1$. Three jobs are scheduled in time window $[3i, 3(i+1))$: the first job of vertex chain $\mathcal{C}_i$ and the first job of the two gadget chains relative to it. Let $3i+k$ (resp. $3i+k'_1$, $3i+k'_2$) be the starting time of the first job of vertex chain $\mathcal{C}_i$ (resp. gadget chains $\mathcal{C}'_{i,1}, \mathcal{C}'_{i,2}$). We have $k, k'_1$ and $k'_2$ in $\{0,1,2\}$ and since we have a feasible schedule the three values are different from each other. Thus one chain starts at time $3i+2$. By Lemma 10 and the time window $[3(n+m+i), 3(n+m+i+1))$ of the last job, this means that this last job must be exactly scheduled at time $3(n+m+i)+2$. Now consider the chain which starts at time $3i+1$. By Lemma 10 its last job must be scheduled at time $3(n+m+i)+1$ or $3(n+m+i)+2$. However the later time position is already taken by the chain starting at time $3i+2$, which means that this last job must be exactly scheduled at time $3(n+m+i)+1$. Finally by the same reasoning we get that the chain starting at time $3i$ must have its last job scheduled at time $3(n+m+i)$.

This means that whatever the starting time $3i+k$ is for vertex chain $\mathcal{C}_i$, its last job must be scheduled at time $3(n+m+i)+k$. So all the delays in the chain must be equal to their minimum and thus by Lemma 10 all the jobs $J$ in the vertex chain must be scheduled at time $r(J)+k$. ◀

Now in any feasible schedule we proved that we have the same guarantee on the position of the edge check jobs as we had with Lemma 4 in the exact delay case. Thus we are able to propagate the color choices accurately and complete the reduction the same way.

▶ **Proposition 12.** *$G$ is 3-colorable if and only if there exists a feasible schedule for this instance of MIN DELAYS.*

**Proof.** ($\Longrightarrow$) Suppose we have $(c_0, \dots, c_{n-1}) \in \{0,1,2\}^n$ a 3-coloring of $G$ where vertex $v_i$ has color $c_i$. We propose a schedule where for all $0 \leq i \leq n-1$, vertex chain $\mathcal{C}'_i$ starts at time $3i + c_i$ and gadget chains $\mathcal{C}'_{i,1}, \mathcal{C}'_{i,2}$ start in the two remaining time positions $3i+k_1, 3i+k_2$ in $[3i, 3(i+1))$ (with $k_1 \neq k_2$). Plus we require all delays to match their minimum value.

Then, according to Definition 7 and going from left to right as we did in the proof of Lemma 10, we know that in every edge $e_j \in E$ where node $v_i$ appears, the job of vertex chain $\mathcal{C}'_i$ which is in edge check segment $[3(n+j), 3(n+j+1))$ is scheduled at time $3(n+j)+c_i$, and the last job of $\mathcal{C}'_i$ is scheduled at time $3(n+m+i)+c_i$. Plus from Definition 8 we know that the last job of gadget chain $\mathcal{C}'_{i,1}$ (resp. $\mathcal{C}'_{i,2}$) is scheduled $3(n+mi)+k_1$ (resp. $3(n+m+i)+k_2$).

We show that the jobs in different chains do not interfere with each other. For the color choice segment we know that vertex chain $\mathcal{C}'_i$ and gadget chains $\mathcal{C}'_{i,1}, \mathcal{C}'_{i,2}$ start respectively at times $3i+c_i, 3i+k_1$, and $3i+k_2$ with $c_i, k_1$ and $k_2$ in $\{0,1,2\}$ and different from each other. For the closing segment we determined that vertex chain $\mathcal{C}'_i$ and gadget chains $\mathcal{C}'_{i,1}, \mathcal{C}'_{i,2}$ end respectively at times $3(n+m+i)+c_i, 3(n+m+i)+k_1$, and $3(n+m+i)+k_2$, again with $c_i, k_1$ and $k_2$ in $\{0,1,2\}$ and different from each other. Thus only the edge check segments remain to be checked. Let $e_j = \{v_{i_1}, v_{i_2}\}$ be an edge in $E$. By definition of the vertex chains, only vertex chains $\mathcal{C}'_{i_1}$ and $\mathcal{C}'_{i_2}$ have a job to be scheduled in time window $[3(n+j), 3(n+j+1))$. In our schedule the job of chain $\mathcal{C}_{i_1}$ is scheduled at time $3(n+j)+c_{i_1}$ and the job of chain $\mathcal{C}_{i_2}$ at time $3(n+j)+c_{i_2}$. Since $(c_0, \dots, c_{n-1})$ is a 3-coloring and $\{v_{i_1}, v_{i_2}\} \in E$, we have $c_{i_1} \neq c_{i_2}$. Thus both jobs are scheduled at different times and the jobs in edge check segment $[3(n+j), 3(n+j+1))$ do not interfere with each other. Thus the proposed schedule is feasible.

($\Longleftarrow$) Suppose we have a feasible schedule. We reuse the same coloring as in the proof of Proposition 5: for all $0 \leq i \leq n-1$, let $s_i \in \{0,1,2\}$ be such that $3i+s_i$ is the starting time of chain $\mathcal{C}'_i$. We show that $(s_0, \dots, s_{n-1})$ is a 3-coloring of $G$. Considering any edge

$e_j = \{v_{i_1}, v_{i_2}\} \in E$, Lemma 11 ensures that the job of vertex chain $\mathcal{C}_{i_1}$ (resp. $\mathcal{C}_{i_2}$) in edge check segment $[3(n+j), 3(n+j+1))$ is scheduled at time $3(n+j)+s_{i_1}$ (resp. $3(n+j)+s_{i_2}$), with $s_{i_1}$ (resp. $s_{i_2}$) the starting time of vertex chain $\mathcal{C}_{i_1}$ (resp. $\mathcal{C}_{i_2}$). Since this is a feasible schedule we have: $3(n+j)+s_{i_1} \neq 3(n+j)+s_{i_2}$, which means: $s_{i_1} \neq s_{i_2}$. Thus the two nodes of edge $e_j$ have indeed different colors. ◀

This proves that $1|chains(\ell_{i,j}^{min}), p_j = 1, r_j, d_j|\star$ with $\mu = 2$ is NP-hard, which concludes the para-NP-completeness proof of the corresponding parameterized problem.

▶ **Theorem 13.** $1|chains(\ell_{i,j}^{min}), p_j = 1, r_j, d_j|\star$ *is para-*NP*-complete parameterized by path-width* $\mu$*.*

## 2.3 NP-hardness of $P|chains(\ell_{i,j}^{ex}), p_j = 1, r_j, d_j|\star$ with $\ell_{max} = 1$

We build an instance of $P|chains(\ell_{i,j}^{ex}), p_j = 1, r_j, d_j|\star$ with $\ell_{max} = 1$. We still have $n$ *vertex chains*, one per node $v_i$ in graph $G$, and we want to represent and check a coloring of $G$ in a similar way to Section 2.1: choose the color of the nodes with the starting time of the vertex chains, then propagate these choices and check that two nodes of an edge do not have the same color. However with the change of parameter we must worry about the maximum delay value instead of the overlapping of time windows. Here we want to propagate the color choices while keeping the delays small.

We propose to add extra intermediate jobs that we call *propagators* at every other time position. This way the color choices can be propagated along the odd time positions with exact delays of length 1 while the even time positions are kept for edge checking. We set $M = n$ as the number of machines in order to make room for these propagators. We define vertex chain $\mathcal{C}_i$ the following way:

▶ **Definition 14** (Vertex chain $\mathcal{C}_i$ with $\ell_{max} = 1$)**.** *We define $\mathcal{C}_i$ as a chain of $3(n+m-i)+deg(v_i)$ jobs. These jobs will fulfill two roles:*

- *Propagators $O_{j,k}^i$: $O_{i,0}^i$ will give the color choice of node $v_i$. The other $3(n+m-i)-1$ jobs $O_{j,k}^i$ ($i \leq j \leq n+m-1$, $0 \leq k \leq 2$) will propagate this color choice along the whole chain while keeping the maximum delay value at 1. Job $O_{j,k}^i$ will have time window $[6j+2k, 6(j+1)+2k)$.*
- *Edge jobs $J_j^i$: the $deg(v_i)$ jobs $J_{n+j}^i$ will represent the color choice of node $v_i$ in every edge $e_j$ where node $v_i$ is in ($0 \leq j \leq m-1$). Job $J_{n+j}^i$ will have time window $[6(n+j), 6(n+j+1))$.*

*We segment time into $m+1$ segments: a color choice segment $[0, 6n)$ and $m$ edge check segments along $[6n, 6(n+m))$. We describe the chain from left to right:*

**(1)** *Color choice segment $[0, 6n)$*
- *Set the first job $O_{i,0}^i$ of $\mathcal{C}_i$ in time window $[6i, 6(i+1))$.*
- *Add a unit-time exact delay then a job, and do this $3(n-i)-1$ times in a row. These jobs are named $O_{i,1}^i$, $O_{i,2}^i$, $O_{i+1,0}^i$, …, $O_{n,2}^i$.*

**(2)** *Edge check segment $[6(n+j), 6(n+j+1))$, $0 \leq j \leq m-1$*
*Let edge $e_j = \{v_{i_1}, v_{i_2}\}, i_1 < i_2$. This segment will check if the vertices $v_{i_1}$ and $v_{i_2}$ have different colors.*
 **a.** *Vertex chain $\mathcal{C}_i$ with $i \notin \{i_1, i_2\}$: add a unit-time exact delay then job $O_{n+j,0}^i$ then a unit-time exact delay then job $O_{n+j,1}^i$ then a unit-time exact delay then job $O_{n+j,2}^i$.*
 **b.** *Vertex chain $\mathcal{C}_i$ with $i = i_1$ or $i = i_2$: add job $J_{n+j}^i$ then an exact delay of length zero then job $O_{n+j,0}^i$ then a unit-time exact delay then job $O_{n+j,1}^i$ then a unit-time exact delay then job $O_{n+j,2}^i$.*

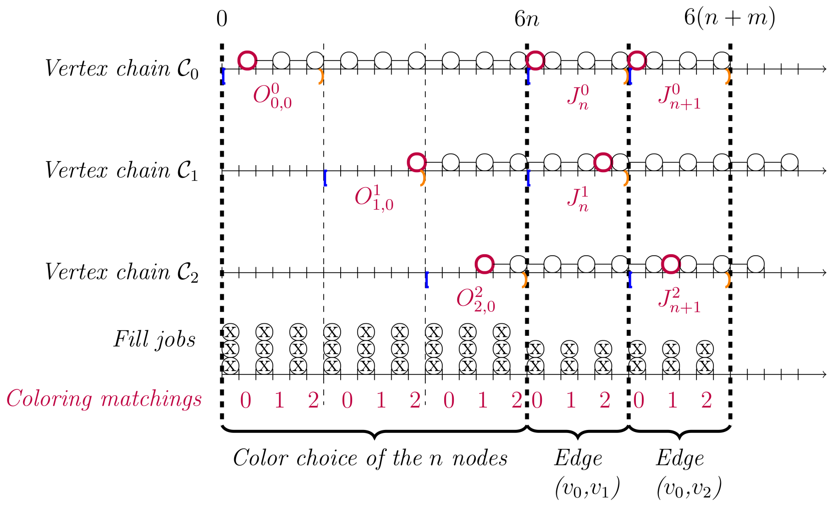


**Figure 3** An instance of $P|chains(\ell^{ex}_{i,j}), p_j = 1, r_j, d_j|\star$ representing a graph coloring. We have $G = (V, E)$ with $V = \{v_0, v_1, v_2\}$ and $E = (\{v_0, v_1\}, \{v_0, v_2\})$. There are $M = n = 3$ machines and this schedule corresponds to the coloring $(0, 2, 1)$.

Note that time intervals of length 6 are used instead of length 3. This way three odd starting times are available for each vertex chain. However *fill jobs* must be added at every even time position of the color choice segment: $M$ of them so that only these odd starting times are actually allowed. Plus now that we are in a parallel-machine framework instead of a single-machine one, more *fill jobs* are needed at the even time positions of the edge check segments: $M - 1$ of them so that one edge job at any of these positions is allowed but two would invalidate the schedule.

▶ **Definition 15** (Fill jobs). *Fill jobs are chains of one job with a time window of length 1. $M$ fill jobs are set at every even time position in color choice segment $[0, 6n)$ and $M - 1$ fill jobs are set at every even time position in time segment $[6n, 6(n + m))$.*

An example is given in Figure 3. With the addition of fill jobs the reduction can now be proved correct in a similar way to Section 2.1: determine the exact positions of all jobs in a vertex chain given its starting time, then show that a schedule is feasible if and only if whenever two vertex chains have an edge job in the same edge check segment they must start at different times.

▶ **Proposition 16.** *$G$ is 3-colorable if and only if there exists a feasible schedule for this instance of $P|chains(\ell^{ex}_{i,j}), p_j = 1, r_j, d_j|\star$.*

First we determine the positions of all jobs in a vertex chain given its starting time. In particular we confirm that in any feasible schedule propagators are always scheduled at odd time positions and edge jobs at even time positions:

▶ **Lemma 17.** *Let $0 \le i \le n - 1$. In any feasible schedule, if vertex chain $\mathcal{C}_i$ starts at time $6i + 2l + 1$ with $l \in \{0, 1, 2\}$, then all the jobs $O^i_{j,k}$ (resp. $J^i_j$) in this chain are scheduled at time $r(O^i_{j,k}) + 2l + 1$ (resp. $r(J^i_j) + 2l$), where $r(J)$ is the release date of job $J$.*

**Proof.** Suppose we have a feasible schedule where vertex chain $\mathcal{C}_i$ starts at time $6i + 2l + 1$ with $l \in \{0, 1, 2\}$.

- Propagators $O^i_{j,k}$: by Definition 14 there is always either a unit-time exact delay or a job $J^i_j$ between two consecutive jobs $O^i_{j,k}$ in vertex chain $\mathcal{C}_i$. Thus if the first job $O^i_{i,0}$ is scheduled at time $6i+2l+1 = r(O^i_{i,0})+2l+1$, then we know that the next propagator $O^i_{i,1}$ is scheduled at time $(6i+2l+1)+2 = r(O^i_{i,1})+2l+1$, and so on. By induction on the couple $(j, k)$ with $i \leq j \leq n + m - 1$ and $0 \leq k \leq 2$, we get that all the jobs $O^i_{j,k}$ in vertex chain $\mathcal{C}_i$ are scheduled at time $[6i+2l+1]+2\times(3(j-i)+k) = 6j+2k+2l+1 = r(O^i_{j,k})+2l+1$.
- Edge jobs $J^i_j$: let $j_0 < j_1 < \ldots < j_{deg(v_i)-1}$ be the indices of the edges $e_j$ such that $v_i \in e_j$ (if there are any). Let $p \in [0..deg(v_i) - 1]$. According to Definition 14 job $O^i_{j_p,0}$ has the same time window $[6(n + j_p), 6(n + j_p + 1))$ as job $J^i_{j_p}$ and it is scheduled right before it. Therefore according to our previous point about propagators $O^i_{j,k}$, job $J^i_{j_p}$ is scheduled at time $[r(O^i_{j_p,0}) + 2l + 1] - 1 = r(J^i_{j_p}) + 2l$. ◀

Now we are able to prove Proposition 16:

**Proof.** ( $\implies$ ) Suppose we have $(c_0, \ldots, c_{n-1}) \in \{0, 1, 2\}^n$ a 3-coloring of $G$ where vertex $v_i$ has color $c_i$. We propose the schedule $\sigma$ where for all $0 \leq i \leq n - 1$, chain $\mathcal{C}_i$ starts at time $6i + 2c_i + 1$. Then by Definition 14 and abiding by exact delays, in every edge $e_j \in E$ where vertex $v_i$ is in, job $J^i_j$ is scheduled at time $r(J^i_j) + 2c_i = 6(n + j) + 2c_i$.

We show that there are never more than $M = n$ jobs scheduled at any time position. Since there is no fill job at an odd time position and two jobs of the same chain cannot be scheduled at the time, there are at most $n$ jobs scheduled at every odd time. Thus only the even time positions remain to be checked. All the chains $\mathcal{C}_i$ start at an odd time $6i + 2c_i + 1$, so by Definition 14 and abiding by exact delays every job $O^i_{j,k}$ is scheduled at time $r(O^i_{j,k}) + 2c_i + 1$, which is odd since all the release dates are even according to Definition 14. Plus it means that only fill jobs are scheduled at the even time positions of color choice segment $[0, 6n)$. Thus only the even time positions of the edge check segments in $[6n, 6(n + m))$ remain to be checked. There are $M - 1$ fill jobs scheduled at all of them. Let $j \in [0..m - 1]$ and $e_j = \{v_{i_1}, v_{i_2}\}, i_1 < i_2$. In edge check segment $[6(n+j), 6(n+j+1))$ there are exactly two non-fill jobs to be scheduled: $J^{i_1}_{n+j}$ and $J^{i_2}_{n+j}$. As mentioned in the previous paragraph they are respectively scheduled at time $6(n + j) + 2c_{i_1}$ and $6(n + j) + 2c_{i_2}$. Since $(c_0, \ldots, c_{n-1}) \in \{0, 1, 2\}^n$ is a 3-coloring and $\{v_{i_1}, v_{i_2}\} = e_j \in E$, we have $c_{i_1} \neq c_{i_2}$ and thus $\sigma(J^{i_1}_{n+j}) \neq \sigma(J^{i_1}_{n+j})$. Therefore at most $M$ jobs are scheduled at any time position.

( $\impliedby$ ) Suppose we have a feasible schedule. For all $0 \leq i \leq n - 1$, let $s_i \in \{0, 1, 2\}$ be such that $6i + 2s_i + 1$ is the starting time of chain $\mathcal{C}_i$ (recall that it can only be an odd time because of the fill jobs defined in Definition 15). We show that $(s_0, \ldots, s_{n-1})$ is a 3-coloring of $G$. By contradiction suppose there is an edge $e_j = \{v_{i_1}, v_{i_2}\} \in E$ such that $s_{i_1} = s_{i_2}$. Then according to Lemma 17 jobs $J^{i_1}_{n+j}$ and $J^{i_2}_{n+j}$ are scheduled at the same time $6(n + j) + 2s_{i_1}$. Thus taking into account the $M - 1$ fill jobs at this time position, there are $M + 1$ jobs scheduled at the same time position, which is greater than the number of machines $M$. Thus the schedule is not feasible, which leads to a contradiction. Thus $(s_0, \ldots, s_{n-1})$ is indeed a 3-coloring of $G$. ◀

This proves that $P|chains(\ell^{ex}_{i,j}), p_j = 1, r_j, d_j|\star$ with $\ell_{max} = 1$ is NP-hard, which concludes the para-NP-completeness proof of the corresponding parameterized problem.

▶ **Theorem 18.** *$P|chains(\ell^{ex}_{i,j}), p_j = 1, r_j, d_j|\star$ is para-NP-complete when parameterized by maximum delay $\ell_{max}$.*

## 2.4 NP-hardness of $P|chains(\ell_{i,j}^{min}), p_j = 1, r_j, d_j|\star$ with $\ell_{max} = 1$

We build an instance of $P|chains(\ell_{i,j}^{min}), p_j = 1, r_j, d_j|\star$ with with $\ell_{max} = 1$. The general idea is to combine the two previously proposed extensions of the basic reduction from Section 2.1: gadget chains from Section 2.2 that dealt with the extra flexibility coming from using minimum delays instead of exact ones, and propagators from Section 2.3 that helped to keep the maximum delay value equal to one. However getting the final product proved to be significantly more technical than with the previous reductions, as propagators and gadget chains could interfere with each other. The reduction will not be detailed in this section: the full description and proof is available in Appendix A.1, as well as an example in Figure 7. Instead we give insight into the major roadblock that we faced and eventually managed to overcome: interference between propagators and gadget chains.

Again the goal was to prove that *in any feasible schedule the edge jobs accurately represent the color choice*. As we have minimum delays we thought about using gadget chains like the reduction in Section 2.2. This required to replicate a situation equivalent to the one displayed in Figure 2 while propagators from other chains were around and could potentially trade places. As we needed propagators at every other time position to keep the maximum delay value at 1, it was not possible to completely isolate each triplet of chains as we did in Section 2.2 when pathwidth $\mu$ was the parameter. So when a triplet of chains was considered by the lemma it had to be guaranteed that the propagators from other chains were fixed and thus could not trade places.

This was made possible by setting the closing time windows of the chain triplets in the reverse order of the color choice time windows. Then, as shown in Figure 7, chains $\mathcal{C}'_0, \mathcal{C}'_{0,1}$, and $\mathcal{C}'_{0,2}$ start at the first part of the color choice segment and end at the last part of the closing segment. Then chains $\mathcal{C}'_1, \mathcal{C}'_{1,1}$, and $\mathcal{C}'_{1,2}$ start at the second part of the color choice segment and end at the second to last part of the closing segment, and so on. This way only propagators of chains $\mathcal{C}'_j, \mathcal{C}'_{j,1}, \mathcal{C}'_{j,2}$ with $j < i$ might interfere. These jobs would be fixed by induction hypothesis and as such could not trade places.

Now that such a property was proved, the correspondence between valid graph 3-colorings and feasible schedules could be established the same way as in our previous reductions.

▶ **Proposition 19.** *$G$ is 3-colorable if and only if there exists a feasible schedule for this instance of $P|chains(\ell_{i,j}^{min}), p_j = 1, r_j, d_j|\star$.*

This proves that $P|chains(\ell_{i,j}^{min}), p_j = 1, r_j, d_j|\star$ with $\ell_{max} = 1$ is NP-hard, which concludes the para-NP-completeness proof of the corresponding parameterized problem.

▶ **Theorem 20.** *$1|chains(\ell_{i,j}^{min}), p_j = 1, r_j, d_j|\star$ is para-NP-complete when parameterized by maximum delay $\ell_{max}$.*

## 3 A FPT algorithm with two parameters

In this section we prove that the problem with precedence delays (exact or minimum) $P|prec(\ell_{i,j}), p_i = 1, r_i, d_i|\star$ is fixed-parameter tractable with the couple of parameters $(\ell_{max}, \mu)$. For the sake of readability we detail the minimum delays case.

Let us consider the sorted list $x_k$, $k \in \{0, \dots, K\}$ of the release times and deadlines in non decreasing order. We define a sub-sequence $u_\alpha$, $\alpha \in \{0, \dots, \kappa-1\}$ of this sequence so that two consecutive terms - except the last one - are separated by at least $\ell_{max}$. So we set $u_0 = x_0$, then $u_{\alpha+1} = x_k$ with $k$ the minimum value in $\{1, \dots, K\}$ such that $u_{\alpha+1} - u_\alpha \geq \ell_{max}$. Lastly we set $u_\kappa = x_K$.

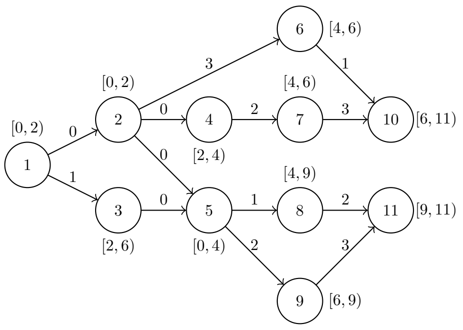


**Figure 4** A precedence graph with minimum delays. Each precedence arc $e = (i, j)$ is labeled by the minimum delay $\ell_{ij}$. Each node $i$ is labelled by its time window $[r_i d_i)$.

Let us consider the instance with minimum delays described in Figure 4.

For this instance we get the sequence $x_0 = 0$, $x_1 = 2$, $x_2 = 4$, $x_3 = 6$, $x_5 = 9$ and $x_6 = 11$ with $K = 6$. The associated pathwidth $\mu = 3$ is reached in the interval $[4, 6)$ crossed by intervals of jobs $3, 6, 7, 8$. We also have $\ell_{max} = 3$, so we get: $u_0 = x_0 = 0$, $u_1 = x_2 = 4$, $u_2 = x_5 = 9$ and $u_3 = x_6 = 11$.

We set $X_0 = \emptyset$ and for any $\alpha \in \{1, \ldots \kappa\}$ we define $X_\alpha = \{i \in \mathcal{T}, [r_i, d_i) \cap [u_{\alpha-1}, u_\alpha) \neq \emptyset\}$ the set of jobs that could be scheduled in interval $[u_{\alpha-1}, u_\alpha)$. The idea of sequence $(u_0, \ldots, u_\kappa)$ is that the number of jobs in each $X_\alpha$ is bounded by $(\mu + 1) \times \ell_{\max}$ (see Lemma 22). We also define $Z_\alpha$, $\alpha \in \{0, \ldots, \kappa\}$ the set of jobs with a deadline not greater than $u_\alpha$, *i.e.* $Z_\alpha = \{i \in \mathcal{T}, d_i \leq u_\alpha\}$. In our example we have: $Z_0 = \emptyset$, $Z_1 = \{1, 2, 4, 5\}$, $Z_2 = Z_1 \cup \{3, 6, 7, 8, 9\}$ and $Z_3 = \mathcal{T}$.

We define a dynamic programming scheme for our problem. The stages of the scheme are $\{0, \ldots, \kappa\}$. For each stage $\alpha \in \{0, \ldots, \kappa\}$ we denote $N_\alpha$ the set of states of stage $\alpha$. A state $s \in N_\alpha$ represents the minimum information from a feasible schedule spanning in $[0, u_\alpha)$ that is necessary to extend this schedule in interval $[u_\alpha, u_\kappa)$.

Hence a state $s \in N_\alpha$ with $\alpha \in \{1, \ldots, \kappa - 1\}$ is a tuple $s = (\beta, Y)$, where:

- $Y \subseteq X_\alpha - Z_\alpha$ is a subset of jobs such that $Y \cup Z_\alpha$ represents the set of jobs scheduled in $[0, u_\alpha)$.
- $\beta$ is a complete schedule (*i.e* a set of jobs with their starting time) of the last $\ell_{max}$ time units before time $u_\alpha$- i.e in interval $[u_\alpha - \ell_{max}, u_\alpha)$. We denote $J(\beta)$ the set of jobs scheduled in $\beta$. $\beta$ is called a *border schedule*. Only such a schedule can influence the earliest starting times of the jobs not scheduled yet.

For $\alpha = \kappa$ we set $N_\kappa = \{s_\kappa\}$ with $s_\kappa = (\bullet, \emptyset)$ where $\bullet$ is an empty schedule. Moreover $X_\kappa - Z_\kappa = \emptyset$ and so $N_\kappa$ can be reduced to only one element. Similarly, we set $N_0 = \{s_0\}$ with $s_0 = (\bullet, \emptyset)$ since $X_0 = \emptyset$ and no job may be executed in interval $[u_0, u_0) = \emptyset$.

As an example let us consider a feasible schedule $\sigma$ pictured by Figure 5 for $m = 2$ identical machines associated to the instance given by Figure 4. The associates states are: $s_0 = (\bullet, \emptyset)$, $s_1 = (\beta_1, \{3\})$, $s_2 = (\beta_2, \emptyset)$ and $s_3 = (\bullet, \emptyset)$.

Now assume that $s = (\beta, Y) \in N_\alpha$ with $\alpha \in \{0, \ldots, \kappa\}$. The boolean function ExistSched$(s)$ is set to *true* if and only if there exists a (partial) feasible schedule of jobs from $Y \cup Z_\alpha$ in time interval $[u_0, u_\alpha)$ that ends with schedule $\beta$.

We can now establish the recurrence equation for this function:

1. ExistSched$(s_0) = true$; indeed, $s_0 = (\bullet, \emptyset)$ and $Z_0 = \emptyset$, thus no job has to be scheduled.
2. Let us now consider $\alpha \in \{1, \ldots \kappa\}$. If ExistSched$(s) = true$ then there exists a feasible schedule in $[0, u_\alpha)$ which can be decomposed into a feasible schedule in $[0, u_{\alpha-1})$ associated with a state $s' \in N_{\alpha-1}$ and a schedule in the interval $[u_{\alpha-1}, u_\alpha)$ consistent with $s$ and $s'$. The existence of such a schedule is denoted by the function Sched$(s, s')$.

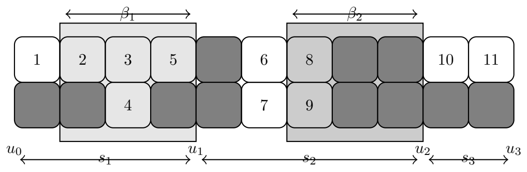


**Figure 5** A feasible schedule $\sigma$ associated with the example given in Figure 4 for $m = 2$ machines.

We now bound the complexity of computing $\text{Sched}(s, s')$ from a tuple of states $(s', s) \in N_{\alpha-1} \times N_\alpha$.

Let $s = (\beta, Y)$ and $s' = (\beta', Y')$. Then boolean $\text{Sched}(s', s)$ is true if and only if there exists a schedule of $Y \cup Z_\alpha - Y' - Z_{\alpha-1}$ in the interval $[u_{\alpha-1}, u_\alpha)$ that is consistent with the border schedule $\beta'$ and ends with the border schedule $\beta$.

▶ **Lemma 21.** *For any $\alpha \in \{1, \ldots, \kappa\}$ and $(s', s) \in N_{\alpha-1} \times N_\alpha$, the time complexity of Sched$(s', s)$ is $\mathcal{O}(\mu^2 \times \ell_{max}^2 \times ((\mu + 1) \times \ell_{max})!)$.*

To prove this lemma, two more technical lemmas are needed. These lemmas bound the total number of schedulable jobs in time interval $[u_{\alpha-1}, u_\alpha)$ for $\alpha \in \{1, \ldots \kappa\}$:

▶ **Lemma 22.** $\forall \alpha \in \{0, \ldots, \kappa\}, |X_\alpha| \leq (\mu + 1) \times \ell_{\max}$.

**Proof.** We simply observe that if $u_{\alpha-1} = x_k$ and $u_\alpha = x_{k'}$ then by construction $k' - k \leq \ell_{max}$. Thus the inequality holds by the definition of $\mu$. ◀

▶ **Lemma 23.** *For any $\alpha \in \{1, \ldots, \kappa - 1\}$, $|N_\alpha| \leq 2^{(\mu+1)\times\ell_{max}} \times (\ell_{max} + 1)^{(\mu+1)\times\ell_{max}}$.*

**Proof.** The total number of schedules from a set $V = J(\beta)$ is bounded by $(\ell_{max} + 1)^{|V|}$. Thus, by Lemma 22, it is bounded by $(\ell_{max} + 1)^{(\mu+1)\times\ell_{max}}$. And because the number of sets $V \subseteq X_\alpha$ is bounded by $2^{|X_\alpha|} \leq 2^{(\mu+1)\times\ell_{max}}$, the lemma holds. ◀

Now we are able to prove Lemma 21:

**Proof.** The problem is to schedule jobs from $S = Y \cup Z_\alpha - (Y' \cup Z_{\alpha-1})$ in the interval $[u_{\alpha-1}, u_\alpha)$ so that the schedule is consistent with the two border schedules $\beta'$ and $\beta$ This can be done in several steps:

1. adjusting the release times of jobs of $S$ with respect to the border schedule $\beta'$: if $j$ is a successor of $i \in J(\beta')$ then $r_j = \max(r_j, \beta'(i) + 1 + \ell_{i,j})$, and propagate to precedence constraints in $S$.
2. adjusting the deadlines of jobs of $S$ with respect to the border schedule $\beta$: if $i$ is a predecessor of $j \in J(\beta)$ then $d_i = \min(d_i, \beta(j) - \ell_{i,j})$, and propagate to precedence constraints in $S$
3. if a contradiction is detected at this step (a job $j$ for which $r_j \geq d_j$), $\text{Sched}(s', s) = \mathit{false}$.
4. Otherwise we can enumerate all active schedules (i.e. schedules in which no job can be scheduled earlier provided the other jobs are not delayed) of $S - J(\beta)$ and verify that one of them spans in $[u_{\alpha-1}, u_\alpha - \ell_{\max})$

The time complexity of the two first steps is $\mathcal{O}(|S|^2)$. For the last step it is known that any active schedule can be generated by list scheduling using a permutation of jobs [18]. Thus the enumeration of active schedules can be done by a brute force algorithm that enumerates all permutations of jobs and then performs a list scheduling algorithm to check whether the schedule spans in the interval $[u_{\alpha-1}, u_\alpha - \ell_{\max})$.

At most $m$ jobs are executed at each instant, and the number of iterations is bounded by $|S|$. For each iteration, we must check that all the precedence constraints (with exact or minimum delays) are fulfilled, and thus one execution of this priority list has a complexity bounded by $\mathcal{O}(|S|^2)$.

The total number of permutations is $|S - J(\beta)|!$. Thus the overall complexity is bounded by $\mathcal{O}(|S|^2 \times |S - J(\beta)|!)$. And since $S \subseteq X_\alpha$, by Lemma 22 we get that $|S| \leq (\mu+1) \times \ell_{max}$ and the lemma holds. ◀

Finally we formalize the recurrence equation that yields a FPT algorithm when we have minimum delays: if $s \in N_\alpha$,

$$\text{ExistSched}(s) = \bigvee_{s' \in N_{\alpha-1}} \text{Sched}(s', s) \wedge \text{ExistSched}(s') \qquad (1)$$

▶ **Theorem 24.** *The answer to an instance $\mathcal{I}$ of $P|prec(\ell_{ij}), p_j = 1, r_j, d_j|\star$ (with minimum or exact delays) is "yes" if and only if ExistSched($s_\kappa$) is true. Moreover the time complexity of the computation of ExistSched($s_\kappa$) is $\mathcal{O}(n \times (2\ell_{max}+2)^{2(\mu+1)\times\ell_{max}} \times \mu^2 \times \ell_{max}^2 \times ((\mu+1)\times\ell_{max})!)$.*

**Proof.** If $\text{ExistSched}(s_\kappa) = \mathit{true}$, then a sequence of states $s_0, s_1, \dots s_\kappa$ with $s_\alpha \in N_\alpha$ for $\alpha \in \{0, \dots, \kappa\}$ and $\text{Sched}(s_{\alpha-1}, s_\alpha) = \mathit{true}$ for all $\alpha \in \{1, \dots, \kappa\}$ can be built. And conversely such a sequence induces a feasible schedule.

Now, the number of calls of the function Sched necessary to compute the recurrence equation (1) is proportional to $\sum_{\alpha=1}^{\kappa} |N_{\alpha-1}| \times |N_\alpha|$. By Lemma 23, this value is bounded by $\kappa \times 2^{2(\mu+1)\times\ell_{max}} \times (\ell_{max}+1)^{2(\mu+1)\times\ell_{max}}$. Since $\kappa \leq 2n$, by Lemma 21 we get the theorem.

Notice that for exact delays, the computation of $\text{Sched}(s_{\alpha-1}, s_\alpha)$ is slightly different since the border schedule of $s_{\alpha-1}$ induces the schedule of all successors of these jobs. Similarly the border of $s_\alpha$ induces starting times for predecessors of these jobs, so the first step is to verify the consistency of the job starting times induced by the two border schedules. This can be done in $O((\mu \times \ell_{\max})^2)$. The remaining enumeration concerns the schedule of jobs without predecessors that start after $u_{\alpha-1}$. In the worst case the number of these jobs is still $O(\mu \times \ell_{\max})$ so that the complexity is the same as in the min delays case. ◀

## 4 Conclusion

In this paper we analyzed the parameterized complexity of two scheduling problems with precedence delays, unit processing times and job time windows with respect to two parameters: the pathwidth $\mu$ and the maximum precedence delay $\ell_{\max}$. To the best of our knowledge this is the first hardness result with pathwidth $\mu$ as a parameter and unit processing times, and also the first time that $\ell_{\max}$ is considered as a parameter. Our work raises an open problem with parameter $\ell_{\max}$, namely the single-machine problem $1|chains(\ell_{i,j}), p_j = 1, r_i, d_i|\star$ for which the hardness reductions we developed do not apply. Further work is also underway to extend our FPT algorithm to more general problems, for instance with any processing times jobs or more complex resource constraints.

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## A Appendix

### A.1 Description of the reduction in Section 2.4 and proof of Proposition 19

▶ **Definition 25** (Vertex chain $\mathcal{C}'_i$ with with $\ell_{max} = 1$). *We define $\mathcal{C}'_i$ as a chain of $3(2n+m-2i-1)+deg(v_i)+1$ jobs. These jobs will fulfill two roles:*

- *Propagators $O^i_{j,k}$: job $O^i_{i,0}$ will give the color choice of node $v_i$. The other $3(2n+m-2i)$ jobs $O^i_{j,k}$ ($i \leq j \leq 2n+m-i-1$, $0 \leq k \leq 2$) and $O^i_{2n+m-i-1,0}$ will propagate this color choice along the whole chain while keeping the maximum delay value at $1$. Job $O^i_{j,k}$ will have time window $[6j+2k, 6j+2k+3)$.*
- *Edge jobs $J^i_j$: The $deg(v_i)$ jobs $J^i_{n+j}$ will represent the color choice of node $v_i$ in every edge $e_j$ where node $v_i$ is in ($0 \leq j \leq m-1$). Job $J^i_{n+j}$ will have time window $[6(n+j)+1, 6(n+j)+4)$.*

*In order to define vertex chain $\mathcal{C}'_i$ we segment time into $m+2$ segments: a color choice segment $[0, 6n)$, $m$ edge check segments along $[6n, 6(n+m))$ and a closing segment $[6(n+m), 6(n+2m))$. We describe the chain from left to right:*

**(1)** *Color choice segment $[0, 6n)$*
- *Set the first job $O^i_{i,0}$ of $\mathcal{C}'_i$ in time window $[6i, 6(i+1))$. Then add a unit-time minimum delay.*
- *Add a job then a unit-time minimum delay, and do this $3(n-i)-1$ times in a row. These jobs are named $O^i_{i,1}$, $O^i_{i,2}$, $O^i_{i+1,0}$, $\dots$, $O^i_{n,2}$.*

**(2)** *Edge check segment $[6(n+j), 6(n+j+1))$, $0 \leq j \leq m-1$*
*Let edge $e_j = \{v_{i_1}, v_{i_2}\}, i_1 < i_2$. This segment will check if the vertices $v_{i_1}$ and $v_{i_2}$ have different colors.*
**a.** *Vertex chain $\mathcal{C}'_i$ with $i \notin \{i_1, i_2\}$*
- *Add job $O^i_{n+j,0}$ then a unit-time minimum delay then job $O^i_{n+j,1}$ then a unit-time minimum delay then job $O^i_{n+j,2}$ then a unit-time minimum delay.*

**b.** *Vertex chain $\mathcal{C}'_i$ with $i = i_1$ or $i = i_2$*
- *Add job $O^i_{n+j,0}$ then a minimum delay of length zero then job $J^i_{n+j}$ then a minimum delay of length zero then job $O^i_{n+j,1}$ then a unit-time minimum delay then job $O^i_{n+j,2}$ then a unit-time minimum delay.*

**(3)** *Closing segment $[6(n+m), 6(n+2m))$*
- *Add a job then a unit-time minimum delay, and do this $3(n-i-1)$ times in a row. These jobs are named $O^i_{n+m,0}$, $O^i_{n+m,1}$, $O^i_{n+m,2}$, $O^i_{n+m+1,0}$, $\dots$, $O^i_{2n+m-(i+2),2}$.*
- *Add the last job $O^i_{2n+m-(i+1),0}$ of $\mathcal{C}_i$ with time window $[6(2n+m-(i+1)), 6(2n+m-(i+1))+3)$.*

▶ **Definition 26** (Gadget chain $\mathcal{C}'_{i,1}$ (resp. $\mathcal{C}'_{i,2}$)).
- *Set the first job $O^{i,1}_{i,0}$ (resp. $O^{i,2}_{i,0}$) in time window $[6i, 6i+3)$.*
- *Add a unit-time minimum delay then a job, and do this $3(2n+m-2i-1)-1$ times in a row. These jobs are named $O^{i,1}_{i,1}$, $O^{i,1}_{i,2}$, $O^{i,1}_{i+1,0}$, $\dots$, $O^{i,1}_{2n+m-(i+1),0}$ (resp. $O^{i,2}_{i,1}$, $O^{i,2}_{i,2}$, $O^{i,2}_{i+1,0}$, $\dots$, $O^{i,2}_{2n+m-(i+1),0}$).*

▶ **Definition 27** (Fill jobs).
**(1)** *Color choice segment $[0, 6n)$*
*Let $i \in [0, n-1]$. In time segment $[6i, 6(i+1))$:*
- *At time $6i$: set $M-1-2i$ fill jobs.*
- *At time $6i+1$: set $M-1-i$ fill jobs.*
- *At time $6i+2$: set $M-2-2i$ fill jobs.*

**(2)** *Edge check segments* $[6n, 6(n+m))$
*Let* $j \in [0, m-1]$*. In time segment* $[6(n+j), 6(n+j+1))$*:*
- *At time* $6(n+j)+1$*: set* $M-n-1$ *fill jobs.*
- *At time* $6(n+j)+3$*: set* $M-n-1$ *fill jobs.*

**(3)** *Closing segment* $[6(n+m), 6(2n+m))$
*Let* $i \in [0, n-1]$*. In time segment* $[6(n+2m-(i+1)), 6(n+2m-i))$*:*
- *At time* $6(n+2m-(i+1))$*: set* $M-2-2i$ *fill jobs.*
- *At time* $6(n+2m-(i+1))+1$*: set* $M-1-i$ *fill jobs.*
- *At time* $6(n+2m-(i+1))+2$*: set* $M-1-2i$ *fill jobs.*

▶ **Lemma 28.** *Let* $0 \leq i \leq n-1$*. In any feasible schedule, if a chain starts at time* $6i+k$ *with* $k \in \{0,1,2\}$*, then all jobs* $J$ *in this chain are scheduled at time* $r(J)+k$ *or later, where* $r(J)$ *is the release date of job* $J$*.*

**Proof.** First the result is proved for vertex chains $\mathcal{C}'_i$. Suppose we have a feasible schedule where vertex chain $\mathcal{C}'_i$ starts at time $6i+l$ with $l \in \{0,1,2\}$.
- Propagators $O^i_{j,k}$: by Definition 25 there is always either a unit-time minimum delay or a job $J^i_j$ between two consecutive jobs $O^i_{j,k}$ in vertex chain $\mathcal{C}_i$. Thus if the first job $O^i_{i,0}$ is scheduled at time $6i+l = r(O^i_{i,0})+l$ (or later), then we know that the next propagator $O^i_{i,1}$ is scheduled at time $(6i+l)+2 = r(O^i_{i,1})+l$ or later, and so on. By induction on the couple $(j,k)$ with $i \leq j \leq n+m-1$ and $0 \leq k \leq 2$, we get that all the jobs $O^i_{j,k}$ in vertex chain $\mathcal{C}_i$ are scheduled at time $(6i+l)+2\times(3(j-i)+k) = 6j+2k+l = r(O^i_{j,k})+l$ or later.
- Edge jobs $J^i_j$: let $j_0 < j_1 < \ldots < j_{deg(v_i)-1}$ be the indices of the edges $e_j$ such that $v_i \in e_j$ (if there are any). Let $p \in [0, deg(v_i)-1]$. According to Definition 25 job $J^i_{j_p}$ is scheduled right before job $O^i_{j_p,0}$ with a minimum delay of length zero between them. Therefore according to our previous point about jobs $O^i_{j,k}$, job $J^i_{j_p}$ is scheduled at time $(r(O^i_{j_p,0})+l)+1 = r(J^i_{j_p})+l$ or later.

Gadget chain $\mathcal{C}'_{i,1}$ (resp. $\mathcal{C}'_{i,2}$) only features propagators. By Definition 26 there is always a unit-time minimum delay between two consecutive jobs $O^{i,1}_{j,k}$ (resp. $O^{i,2}_{j,k}$), so the result can be proven the same way as in the first item of the proof for vertex chains. ◀

▶ **Lemma 29.** *Let* $0 \leq i \leq n-1$*. In any feasible schedule, if a chain starts at time* $6i+k$ *with* $k \in \{0,1,2\}$*, then all jobs* $J$ *in this chain have to be scheduled at time* $r(J)+k$*, where* $r(J)$ *is the release date of job* $J$*.*

**Proof.** We prove by induction on $i \in [0, n-1)$ that for all $0 \leq j \leq i$ exactly one chain starts at each time $6j$, $6j+1$, $6j+2$ and all jobs $J$ in a chain $\mathcal{C}'_j$, $\mathcal{C}'_{j,1}$, $\mathcal{C}'_{j,2}$ that starts at time $6j+k$ have to be scheduled at time $r(J)+k$.
- According to Definition 27 on time windows $[0,6)$ and $[6(2n+m-1), 6(2n+m))$, the chain triplet $\mathcal{C}'_0$, $\mathcal{C}'_{0,1}$, $\mathcal{C}'_{0,2}$ is in the situation described in *Figure* 6. Thus at least one chain must start at time 2 which means by Lemma 28 that this chain has to end at time $6(2n+m-1)+2$. Then time $6(2n+m-1)+2$ is blocked, so by the same lemma another chain cannot start at time 2. Thus the two other chains have to start at the two remaining time positions 0 and 1, one per chain. By Lemma 28 the chain that starts at time 1 ends at time $6(2n+m-1)+1$ (or later but the only other time position possible $6(2n+m-1)+2$ is already blocked). So time position $6(2n+m-1)+1$ is now blocked, which forces the chain that starts at time 0 to end at time $6(2n+m-1)$.

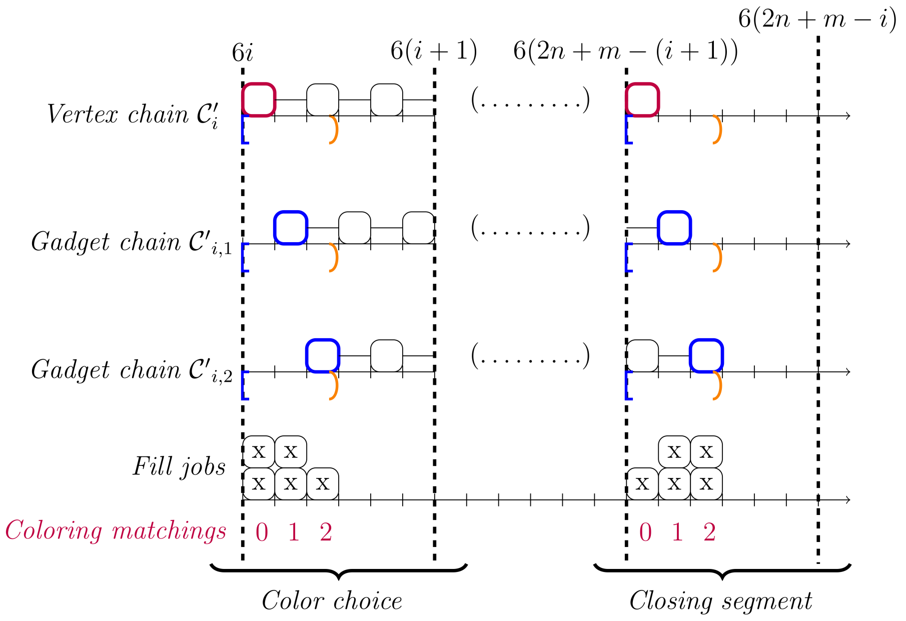

**Figure 6** A toy situation with three machines and the three chains related to a node in the $P|chains(\ell_{i,j}^{min}), p_j = 1, r_j, d_j|\star$ reduction. In any feasible schedule featuring these chains, for $k \in \{0,1,2\}$ exactly one chain starts at time $6i+k$, and then this chain has to end at time $6(n+2m-(i+1))+k$.

- Let $i \in [0, n-1)$. Assume the induction hypothesis to be true for chain triplets of index $j$ with $0 \le j \le i-1$. By Definition 25 we know that only these chain triplets have propagators that might interfere in time windows $[6i, 6(i+1))$ and $[6(2n+m-(i+1)), 6(2n+m-i))$. By induction hypothesis we know that these propagators are fixed, and we deduce that the number of fixed jobs (propagators from other chains plus fill jobs) at the relevant time positions is the following:

**(1)** Color choice segment, part $[6i, 6(i+1))$:
  - At time $6i$: $M-1-2i$ fill jobs and $2i$ propagators from other chains which add up to $M-1$ jobs.
  - At time $6i+1$: set $M-1-i$ fill jobs and $i$ propagators from other chains which add up to $M-1$ jobs.
  - At time $6i+2$: set $M-2-2i$ fill jobs and $2i$ propagators from other chains which add up to $M-2$ jobs.

**(2)** Closing segment, part $[6(n+2m-(i+1)), 6(n+2m-i))$:
  - At time $6(n+2m-(i+1))$: set $M-2-2i$ fill jobs and $2i$ propagators from other chains which add up to $M-2$ jobs.
  - At time $6(n+2m-(i+1))+1$: set $M-1-i$ fill jobs and $i$ propagators from other chains which add up to $M-1$ jobs.
  - At time $6(n+2m-(i+1))+2$: set $M-1-2i$ fill jobs and $2i$ propagators from other chains which add up to $M-1$ jobs.

Thus we are again in the situation described in Figure 6 and we can prove the result for the triplet $\mathcal{C}'_i$, $\mathcal{C}'_{i,1}$, $\mathcal{C}'_{i,2}$ the same way as in the initialization.

This concludes the proof of the lemma for all chains. ◀

Now we are able to prove Proposition 19:

**Proof.** ( $\Longrightarrow$ ) Suppose we have $(c_0, \ldots, c_{n-1}) \in \{0,1,2\}^n$ a 3-coloring of $G$ where vertex $v_i$ has color $c_i$. We propose a schedule $\sigma$ where for all $0 \leq i \leq n-1$, chain $\mathcal{C}'_i$ starts at time $6i + c_i$ and gadget chains $\mathcal{C}'_{i,1}, \mathcal{C}'_{i,2}$ start in the two remaining time positions $6i + l_1, 6i + l_2$ in $[3i, 3(i+1))$ (with $l_1 \neq l_2$). Plus we require all delays to match their minimum value.

Then, according to Definition 25, Definition 26 and going from left to right as we did in the proof of Lemma 28, we know that propagators $O^i_{j,k}$, $O^{i,1}_{j,k}$ and $O^{i,2}_{j,k}$ are respectively scheduled at times $6j + 2k + c_i$, $6j + 2k + l_1$ and $6j + 2k + l_2$. In the same way we know that in every edge $e_j \in E$ where node $v_i$ appears, edge job $J^i_{n+j}$ of vertex chain $\mathcal{C}'_i$ is scheduled at time $6(n+j) + 1 + c_i$. Thus for all jobs $J$ in our proposed schedule, if its chain starts at time $6i + l$ with $l \in \{0,1,2\}$ then it is scheduled at time $r(J) + l$.

We show that there are never more than $M = 2n + 1$ jobs scheduled at any time position. According to the previous paragraph we can infer that for every chain triplet two propagators are scheduled at every even time position and one propagator at every odd time position in time segment $[6i, 6(2n + m - (i+1)) + 3)$. Recall that only the chain triplets of index $j \leq i$ are present in the two time segments $[0, 6n)$, $6(n + 2m - (i+1))$ related to node $v_i$. With Definition 27 we count the number of propagators plus the number of fill jobs at every time position and show that it is always no more than $M - 1$:

**(1)** Color choice segment $[0, 6n)$
Let $i \in [0, n-1]$. In time segment $[6i, 6(i+1))$:
- At time $6i$: $M - 1 - 2i$ fill jobs and $2i$ propagators which add up to $M - 1$ jobs.
- At time $6i + 1$: set $M - 1 - i$ fill jobs and $i$ propagators which add up to $M - 1$ jobs.
- At time $6i + 2$: set $M - 2 - 2i$ fill jobs and $2i$ propagators which add up to $M - 2$ jobs.
- At times $6i + 3$, $6i + 4$, $6i + 5$: respectively $i$, $2i$, $i$ propagators.

**(2)** Edge check segments $[6n, 6(n+m))$
Let $j \in [0, m-1]$. In time segment $[6(n+j), 6(n+j+1))$:
- At time $6(n+j) + 1$: $M - n - 1$ fill jobs and $n$ propagators which add up to $M - 1$ jobs.
- At time $6(n+j) + 3$: $M - n - 1$ fill jobs and $n$ propagators which add up to $M - 1$ jobs.
- At times $6(n+j)$, $6(n+j) + 2$, $6(n+j) + 4$, $6(n+j) + 5$: respectively $2n$, $2n$, $2n$, $i$ propagators.

**(3)** Closing segment $[6(n+m), 6(2n+m))$
Let $i \in [0, n-1]$. In time segment $[6(n + 2m - (i+1)), 6(n + 2m - i))$:
- At time $6(n + 2m - (i+1))$: set $M - 2 - 2i$ fill jobs and $2i$ propagators which add up to $M - 2$ jobs.
- At time $6(n + 2m - (i+1)) + 1$: set $M - 1 - i$ fill jobs and $i$ propagators which add up to $M - 1$ jobs.
- At time $6(n + 2m - (i+1)) + 2$: set $M - 1 - 2i$ fill jobs and $2i$ propagators which add up to $M - 1$ jobs.
- At times $6(n + 2m - (i+1)) + 3$, $6(n + 2m - (i+1)) + 4$, $6(n + 2m - (i+1)) + 5$: respectively $i$, $2i$, $i$, propagators.

Thus only the two edge jobs $J^{i_1}_{n+j}$, $J^{i_2}_{n+j}$ from an edge $e_j = \{v_{i_1}, v_{i_2}\}$ could invalidate the schedule if both jobs were scheduled at the same time. This would mean that $6(n+j)+1+c_{i_1} = 6(n+j) + 1 + c_{i_2}$ and thus $c_{i_1} = c_{i_2}$, which is impossible since we started from a valid 3-coloring. Therefore at most $2n + 1 = M$ jobs are scheduled at any time position.

( $\Longleftarrow$ ) Suppose we have a feasible schedule. For all $0 \leq i \leq n-1$, let $s_i \in \{0,1,2\}$ be such that $6i + s_i$ is the starting time of chain $\mathcal{C}'_i$ (recall that it can only be an odd time because of the fill jobs defined in Definition 15). We show that $(s_0, \ldots, s_{n-1})$ is a 3-coloring of $G$. By contradiction suppose there is an edge $e_j = \{v_{i_1}, v_{i_2}\} \in E$ such that $s_{i_1} = s_{i_2}$. Then according to Lemma 29 jobs $J^{i_1}_{n+j}$ and $J^{i_2}_{n+j}$ are scheduled at the same time $6(n+j)+s_{i_1}$. However according to Definition 27 and Lemma 29 fill jobs and propagators add up to $M-1$ in all three positions $6(n+j)+1$, $6(n+j)+2$, $6(n+j)+3$.

Thus adding both edge jobs there are $M+1$ jobs scheduled at one of these three time positions, which would make the schedule not feasible. This leads to a contradiction. Thus $(s_0,\ldots,s_{n-1})$ is indeed a 3-coloring of $G$. ◀

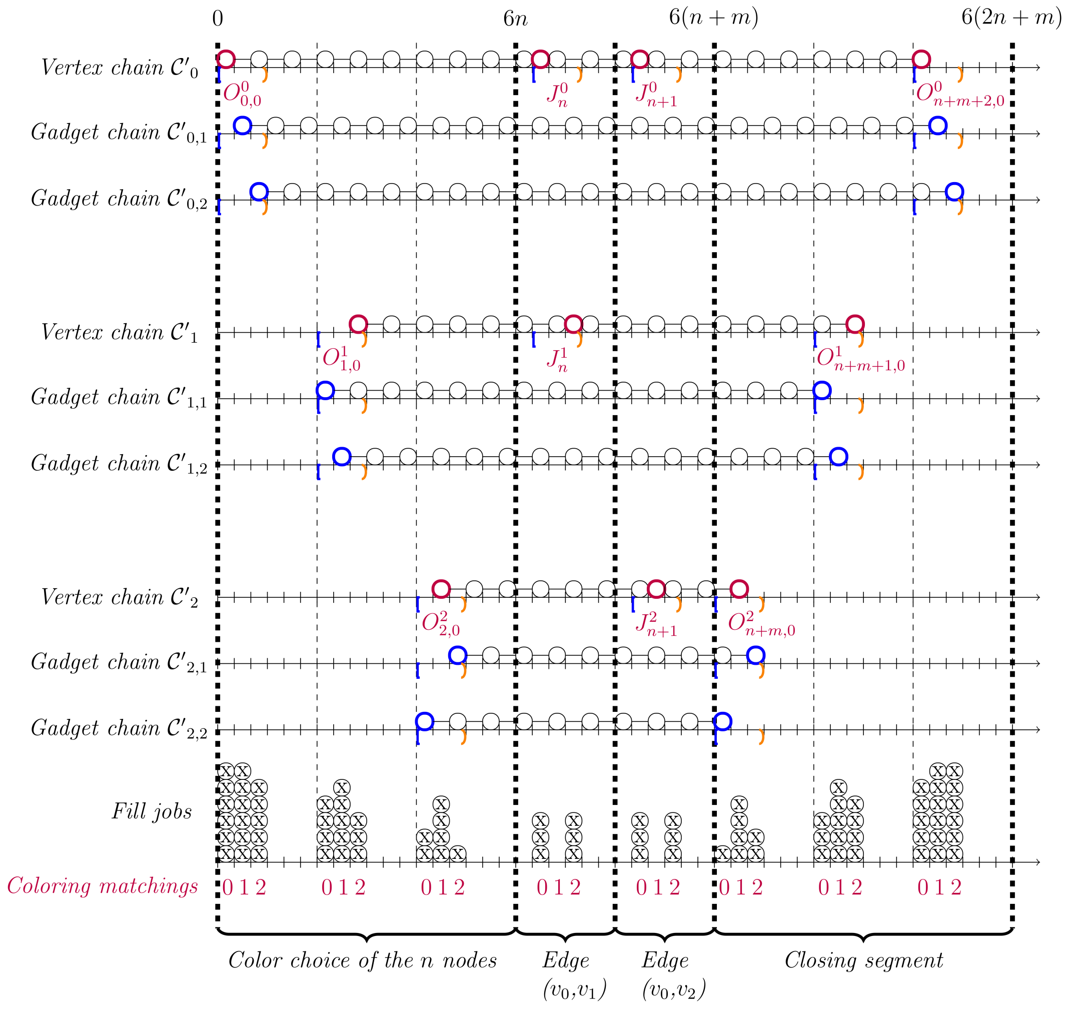

**Figure 7** An instance of $P|chains(\ell^{min}_{i,j}), p_j = 1, r_j, d_j|\star$ with $M = 2n+1 = 7$ machines representing a graph coloring. We have $G = (V,E)$ with $V = \{v_0, v_1, v_2\}$ and $E = (\{v_0,v_1\},\{v_0,v_2\})$. This schedule corresponds to the coloring $(0,2,1)$.