


# Twin-Width VIII: Delineation and Win-Wins

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## Abstract

We introduce the notion of delineation. A graph class  $\mathcal{C}$  is said *delineated by twin-width* (or simply, *delineated*) if for every hereditary closure  $\mathcal{D}$  of a subclass of  $\mathcal{C}$ , it holds that  $\mathcal{D}$  has bounded twin-width if and only if  $\mathcal{D}$  is monadically dependent. An effective strengthening of *delineation* for a class  $\mathcal{C}$  implies that tractable FO model checking on  $\mathcal{C}$  is perfectly understood: On hereditary closures of subclasses  $\mathcal{D}$  of  $\mathcal{C}$ , FO model checking on  $\mathcal{D}$  is fixed-parameter tractable (FPT) exactly when  $\mathcal{D}$  has bounded twin-width. Ordered graphs [BGODMSTT, STOC '22] and permutation graphs [BKTW, JACM '22] are effectively delineated, while subcubic graphs are not. On the one hand, we prove that interval graphs, and even, rooted directed path graphs are delineated. On the other hand, we observe or show that segment graphs, directed path graphs (with arbitrarily many roots), and visibility graphs of simple polygons are not delineated.

In an effort to draw the delineation frontier between interval graphs (that are delineated) and axis-parallel two-lengthed segment graphs (that are not), we investigate the twin-width of restricted segment intersection classes. It was known that (triangle-free) pure axis-parallel unit segment graphs have unbounded twin-width [BGKWT, SODA '21]. We show that  $K_{t,t}$ -free segment graphs, and axis-parallel  $H_t$ -free unit segment graphs have bounded twin-width, where  $H_t$  is the half-graph or ladder of height  $t$ . In contrast, axis-parallel  $H_4$ -free two-lengthed segment graphs have unbounded twin-width. We leave as an open question whether unit segment graphs are delineated.

More broadly, we explore which structures (large bicliques, half-graphs, or independent sets) are responsible for making the twin-width large on the main classes of intersection and visibility graphs. Our new results, combined with the FPT algorithm for first-order model checking on graphs given with  $O(1)$ -sequences [BKTW, JACM '22], give rise to a variety of algorithmic win-win arguments. They all fall in the same framework: If  $p$  is an FO definable graph parameter that effectively functionally upperbounds twin-width on a class  $\mathcal{C}$ , then  $p(G) \geq k$  can be decided in FPT time  $f(k) \cdot |V(G)|^{O(1)}$ . For instance, we readily derive FPT algorithms for  $k$ -LADDER on visibility graphs of 1.5D terrains, and  $k$ -INDEPENDENT SET on visibility graphs of simple polygons. This showcases that the theory of twin-width can serve outside of classes of bounded twin-width.

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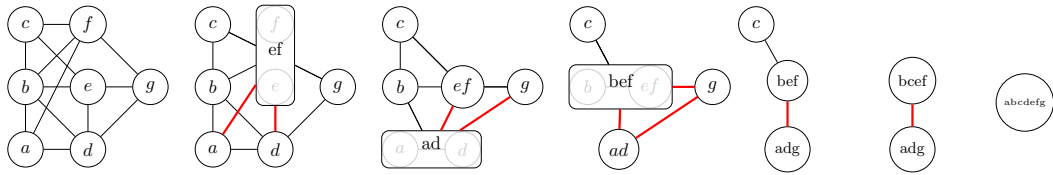


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**1 Introduction**

A *trigraph*  $G$  has a vertex set  $V(G)$ , and two disjoint edge sets, the black edge set  $E(G)$  and the red edge set  $R(G)$ . A (vertex) *contraction* consists of merging two (non-necessarily adjacent) vertices, say,  $u, v$  into a vertex  $w$ , and keeping every existing edge  $wz$  black if and only if  $uz$  and  $vz$  were previously black edges. The other edges incident to  $w$  turn red (if not already), while the rest of the trigraph remains the same. A *contraction sequence* of an  $n$ -vertex (tri)graph  $G$  is a sequence of trigraphs  $G = G_n, \dots, G_1 = K_1$  such that  $G_i$  is obtained from  $G_{i+1}$  by performing one contraction. A *d-sequence* is a contraction sequence wherein all trigraphs have red degree at most  $d$ . The *twin-width* of  $G$ , denoted by  $\text{tww}(G)$ , is the minimum integer  $d$  such that  $G$  admits a  $d$ -sequence. A graph class  $\mathcal{C}$  has then *bounded twin-width* if there is a constant  $t$  such that every graph  $G \in \mathcal{C}$  satisfies  $\text{tww}(G) \leq t$ . See Figure 1 for an illustration of a 2-sequence of a graph.



**Figure 1** A 2-sequence witnesses that the initial 7-vertex graph has twin-width at most 2.

The main algorithmic application of twin-width is that first-order (FO) model checking, that is, deciding if a first-order sentence  $\varphi$  holds in a graph  $G$ , can be decided in fixed-parameter time (FPT)  $f(|\varphi|, d) \cdot |V(G)|$  for some computable function  $f$ , when given a  $d$ -sequence of  $G$  [6]. We recall that there is an ample list of graph classes (or more generally of binary structures, since the definition of twin-width extends to them) of bounded twin-width, including bounded clique-width graphs,  $H$ -minor free graphs, posets with antichains of bounded size, strict subclasses of permutation graphs, map graphs, bounded-degree string graphs [6], as well as  $\Omega(\log n)$ -subdivisions of  $n$ -vertex graphs, and some particular classes of cubic expanders [4]. In contrast, (sub)cubic graphs, interval graphs, triangle-free unit segment graphs, unit disk graphs have unbounded twin-width [4].

The missing element for an FPT FO model-checking algorithm on any class of bounded twin-width is a polynomial-time algorithm and a computable function  $f$ , that given a constant integer bound  $c$  and a graph  $G$ , either finds an  $f(c)$ -sequence for  $G$ , or correctly reports that  $\text{tww}(G) > c$ . The runtime of the algorithm could be  $n^{g(c)}$ , for some function  $g$ . However to get an FPT algorithm in the combined parameter *size of the sentence + bound on the twin-width*, one would further require that the approximation algorithm takes FPT time in  $c$  (now thought of as a parameter), i.e.,  $g(c)n^{O(1)}$ . Such an algorithm exists on ordered graphs (more generally, ordered binary structures) [5], graphs of bounded clique-width, proper minor-closed classes [6], but not on general graphs. Let us observe that exactly computing the twin-width, and even distinguishing between  $\text{tww}(G) = 4$  and  $\text{tww}(G) = 5$ , is NP-complete [3].

**Motivation.** We aim to get around the two main caveats of using twin-width for algorithm design. Namely:

- an FPT (or XP) approximation of twin-width is still missing, and
- a priori *only* classes of bounded twin-width are concerned.

The central theme of this paper is to showcase how to bypass the above caveats using twin-width and to provide a necessary toolbox. First we show that on certain graph classes, bounded twin-width is precisely what renders FO model-checking FPT, and the notion of *delineation* is introduced to this end. Second, we demonstrate how to summon a win-win strategy on important graph classes by means of twin-width, which is reminiscent of the well-known win-win argument based on treewidth.

The main obstacle for computing the twin-width is to get a good vertex ordering. Geometric graph classes of *unbounded* twin-width constitute a diverse and intriguing pool for testing these two avenues: interval graphs, (rooted) directed path graphs, segment graphs, visibility graphs of polygons and terrains. For all these classes, a vertex-ordering procedure either comes naturally or can be worked out and efficiently computed.

**Global strategy.** For our purpose, the following characterization of bounded twin-width will be pivotal.

► **Theorem 1** ([5]). *A class  $\mathcal{C}$  has bounded twin-width if and only if there is an integer  $k$  such that every graph of  $\mathcal{C}$  admits an adjacency matrix without rank- $k$  division, i.e.,  $k$ -division such that every cell has combinatorial rank at least  $k$ .*

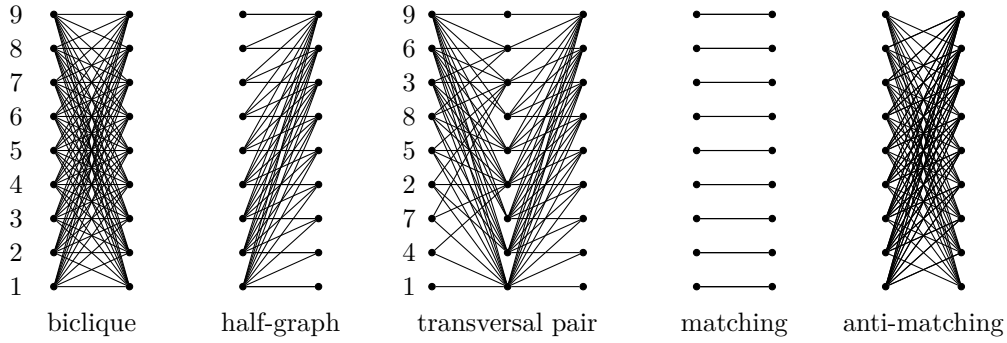
Here, a  $k$ -division of a matrix is a partition of its column (resp. row) set into  $k$  intervals, called *column* (resp. *row*) *parts*, of consecutive columns (resp. rows). A  $k$ -division naturally defines  $k^2$  *cells* (contiguous submatrices) made by the entries at the intersection of a column part with a row part. A *rank- $k$  division* of  $M$  is a  $k$ -division  $\mathcal{D}$  such that each of the  $k^2$  cells has at least  $k$  distinct rows or at least  $k$  distinct columns (that is, combinatorial rank at least  $k$ ). The maximum integer  $k$  such that  $M$  admits a rank- $k$  division is called *grid rank*, and is denoted by  $\text{gr}(M)$ . Theorem 1 is effective: There is a computable function  $f$ , such that, given a vertex ordering along which the adjacency matrix of a graph  $G$  has no rank- $k$  division, one can efficiently find an  $f(k)$ -sequence for  $G$ , witnessing that  $\text{tw}(G) \leq f(k)$ .

Suppose that for a graph class  $\mathcal{C}$ , a canonical vertex ordering can be obtained. Either the consequential adjacency matrix has no rank- $k$  division – and we get a favorable contraction sequence by Theorem 1 – or it does have such a division. In the latter case, a large structured object of variable complexity may be found, such as a biclique, a half-graph (or ladder), or even an obstacle to an FPT FO model checking in the form of a *transversal pair of half-graphs* (or *transversal pair*, for short) or some variant of it; see the middle figure in Figure 2, Section 3 for a formal definition, and for why transversal pairs indeed are such obstacles.

**Delineation.** For monotone (i.e., closed under removing vertices and edges) classes, the FPT algorithm of Grohe, Kreutzer, and Siebertz [20] for FO model checking on nowhere dense classes, is complemented by W[1]-hardness on classes that are somewhere dense (i.e., *not* nowhere dense) [13], and even AW[\*]-hardness on classes that are *effectively* somewhere dense [23]. The latter two results imply that, for monotone classes, FO model checking is unlikely to be FPT beyond nowhere dense classes. Thus the classification of monotone classes admitting an FPT FO model checking is complete. However such a classification remains an active line of work for the more general hereditary classes of graphs and binary structures [15, 18, 19]. It is conjectured (see for instance [18, Conjecture 8.2]) that:

► **Conjecture 2.** *For every hereditary class  $\mathcal{C}$  of structures, FO model checking is FPT on  $\mathcal{C}$  if and only if  $\mathcal{C}$  is monadically dependent.*<sup>1</sup>

<sup>1</sup> A model-theoretic notion which roughly says that not every graph  $G$  can be built from a nondeterministic  $O(1)$ -coloring of some  $\mathcal{S} \in \mathcal{C}$  by means of a first-order formula  $\varphi(x, y)$ , in the relations of  $\mathcal{S}$  and the added colors, imposing the edge set of  $G$ ; see Section 2 for a definition.



■ **Figure 2** Biclique, half-graph (or ladder), transversal pair of half-graphs, matching, anti-matching, all of height 9. Bicliques and half-graphs are semi-induced by default. The number next to each leftmost vertex  $v$  of the transversal pair indicates the height of the neighbor of  $v$  in the central column which is not also a neighbor of the vertex just below  $v$ .

If for every hereditary closure  $\mathcal{D}$  of a subclass<sup>2</sup> of  $\mathcal{C}$ ,  $\mathcal{D}$  has bounded twin-width if and only if  $\mathcal{D}$  is monadically dependent, we say that  $\mathcal{C}$  is *delineated by twin-width* (or simply, *delineated*). Although not stated in those terms, permutation graphs were already proven to be delineated [6], as well as ordered graphs [5]. We add interval graphs and rooted directed path graphs (see Section 3 for a definition) to the list of delineated classes. Therefore, for every hereditary subclass of these classes the classification of FPT FO model checking, Conjecture 2, is now provably settled.<sup>3</sup> In contrast, we rule out delineation for directed path graphs (with multiple roots), intersection graphs of pure axis-parallel segments with two distinct lengths, and visibility graphs of simple polygons.

► **Theorem 3.** *Interval graphs, and more generally rooted directed path graphs, are delineated.*

A (variant of a) transversal pair plays the key role to establish Theorem 3. We show that on a class  $\mathcal{C}$ , if a (variant of a) transversal pair can systematically be found as a result of unbounded twin-width, then the classification of FPT FO model checking for hereditary subclasses of  $\mathcal{C}$  is entirely settled by the algorithm on graphs of bounded twin-width [6].

**Twin-width win-wins.** If segment graphs and visibility graphs of simple polygons do not yield in their subfamilies of unbounded twin-width complex enough structures to settle Conjecture 2, unbounded twin-width still imply in those classes that some other graph parameters are unbounded. This gives rise to a win-win approach to compute these parameters. To give a context, we draw a parallel with what happens with treewidth.

The algorithmic use of a parameter like treewidth extends beyond classes wherein treewidth is bounded. Any problem admitting an FPT algorithm parameterized by treewidth (like MSO definable problems [9]), and a trivial answer (such as a systematic YES or a systematic NO) when the treewidth is large, subjects itself to a straightforward win-win argument. This is at the basis of the so-called *bidimensionality* theory [17]. Since a problem like  $k$ -VERTEX COVER admits a  $2^{\text{tw}(G)}n^{O(1)}$ -time algorithm [10] and a systematic NO answer in presence of a, say,  $(2\sqrt{k} + 1) \times (2\sqrt{k} + 1)$  grid minor, one then derives for this problem an FPT algorithm running in time  $2^{O(\sqrt{k})}n^{O(1)}$  in planar graphs.

<sup>2</sup> The reason we do not simply quantify over hereditary subclasses of  $\mathcal{C}$  is to have a notion that is also meaningful when  $\mathcal{C}$  is not hereditary.

<sup>3</sup> We actually need an effective strengthening of delineation that also holds for these classes and will be defined in Section 2.

Let us forget one moment the intermediary role of the grid minor. Efficiently computing a parameter  $p(G)$  – like the *vertex cover number*  $\tau(G)$  – can boil down to establishing an upperbound of the form  $\text{tw}(G) \leq f(p(G))$ .

We explore such upper bounds, and resultant win-wins, with twin-width in place of treewidth. Given two graph parameters  $p, q$ , and a graph class  $\mathcal{C}$ , we will write  $p \sqsubseteq q$  on  $\mathcal{C}$  to signify that there is a computable function  $f$  such that  $\forall G \in \mathcal{C}, p(G) \leq f(q(G))$ . By a similar argument to what was presented in the previous paragraphs, one gets the following.

► **Theorem 4** (informal). *Let  $\mathcal{C}$  be a graph class and  $p$  be a graph invariant such that*

1. *computing  $p$  is FPT in the combined parameter  $p + \text{tw}$  on  $\mathcal{C}$ , and*
2.  *$\text{tw} \sqsubseteq p$  on  $\mathcal{C}$ .*

*Then, computing  $p$  is FPT on  $\mathcal{C}$ .*

First-order logic yields a natural pool of invariants  $p$  that are fixed-parameter tractable with respect to  $p + \text{tw}$  [6]. As a first example of Item 2, we show the following.

► **Theorem 5.** *Biclique-free segment graphs have bounded twin-width. Furthermore, if a geometric representation is given, an  $O(1)$ -sequence of the graph is found in polynomial time.*

A reformulation is that, in segment graphs, twin-width is upperbounded by a function of the largest biclique; or, denoting by  $\beta(G)$  the largest integer  $t$  such that  $G$  admits a biclique  $K_{t,t}$  as a subgraph, it holds that  $\text{tw} \sqsubseteq \beta$  on segment graphs. The corresponding problem  $k$ -BICLIQUE was famously shown W[1]-hard by Lin [25], after its parameterized complexity has been open for over a decade [11]. From Theorems 5 and 19 one rederives<sup>4</sup> that  $k$ -BICLIQUE is FPT on segment graphs given with a geometric representation.

The counterpart of the large grid minor (in treewidth win-wins) is a large rank division in *every* adjacency matrix of the graph (recall Theorem 1). Large twin-width in a class  $\mathcal{C}$  in particular implies a large rank division in the adjacency matrix along a vertex ordering that, at least partially, respects the structure of  $\mathcal{C}$ . In turn, this complex structure – despite being along a canonical order – may help lowerbounding other parameters (like the grid minor was lowerbounding the vertex cover number in our example). We give two such examples, both on classes of visibility graphs.

A *simple polygon* is a polygon without holes. Two vertices (more generally, points) of a polygon *see* each other if the line segment defined by these vertices (or points) is entirely contained in the polygon. The following problem is sometimes advertised as *hiding (people) in polygons*, and its solution is called a *hidden set*. It is NP-complete [27], even APX-hard [16], and can be equivalently defined as  $k$ -INDEPENDENT SET in visibility graphs of simple polygons given with a representation.

► **Theorem 6.** *Given a simple polygon  $\mathcal{P}$  and an integer  $k$ , finding  $k$  vertices of  $\mathcal{P}$  pairwise not seeing each other is FPT.*

A key step for proving Theorem 6 is to turn a large rank division in the adjacency matrix along a Hamiltonian path describing the boundary of the polygon into a large independent set. In conclusion: we establish  $\text{tw} \sqsubseteq \alpha$  in visibility graphs of simple polygons (where  $\alpha(G)$  is the independence number of  $G$ ), which immediately implies Theorem 6 thanks to Theorem 4.

<sup>4</sup> This fact can alternatively be obtained via the algorithmic theory of Sparsity [13, 14], and the existence of truly sublinear balanced separators in  $K_{t,t}$ -free segment graphs [24].

In contrast,  $k$ -DOMINATING SET remains  $W[1]$ -hard on visibility graphs of simple polygons [7], thus likely does *not* admit an FPT algorithm. We remark that Hliněný et al. [22] conjectured that FO model checking is FPT on weak visibility graphs of simple polygons additionally parameterized by the independence number. Our proof that  $\text{tw} \sqsubseteq \alpha$  on visibility graphs of simple polygons confirms this conjecture, even for the more general (non-weak) visibility graphs. We observe that the approach would not work with a classic width measure, since none of the three items hold replacing *twin-width* by *clique-width*; this mainly because grids and long paths of consistently ordered half-graphs have bounded twin-width but unbounded clique-width.

A *1.5D terrain* (or here, *terrain* for short) is an  $x$ -monotone polygonal chain in the plane. Two vertices of a terrain *see* each other if the line segment they define entirely lies above the terrain. Let  $\lambda(G)$ , the *ladder index* of  $G$ , be the greatest height of a semi-induced half-graph in  $G$ . A folklore structural property of terrains, often called *Order Claim*, imposes the existence of large half-graphs in a large rank division along the left-right ordering. Thus  $\text{tw} \sqsubseteq \lambda$  in visibility graphs of 1.5D terrains. We conclude:

► **Theorem 7.**  *$k$ -LADDER and  $k$ -BICLIQUE are FPT on visibility graphs of 1.5D terrains given with a geometric representation.*

The full version of this paper is available on arXiv, where all missing proofs can be found.

## 2 Preliminaries

We may denote the set of integers between  $i$  and  $j$  by  $[i, j]$ , and  $[k]$  may be used as a short-hand for  $[1, k]$ .

### 2.1 Graph theory

We use the standard graph-theoretic definitions and notations. We denote by  $V(G)$ , and  $E(G)$ , the vertex set, and the edge set, of a graph  $G$ , and by  $G[S]$  the subgraph of  $G$  induced by  $S \subseteq V(G)$ . When  $A, B \subseteq V(G)$  are two disjoint sets, we denote by  $G[A, B]$  the bipartite graph  $(A, B, \{ab : a \in A, b \in B, ab \in E(G)\})$ . We denote by  $\text{Adj}_{\prec}(G)$  the adjacency matrix of  $G$  along the total order  $\prec$  of  $V(G)$ .

A *biclique* and *half-graph* (or *ladder*) of height  $t$  play a central role in this paper. The formal definition can be found in the long version, and See Figure 2 for illustrations. A bipartite graph  $H$  is *semi-induced* in  $G$  if there are two disjoint  $A, B \subseteq V(G)$  such that  $H$  is isomorphic to  $G[A, B]$ . A graph is  $K_{t,t}$ -free (resp.  $H_t$ -free) if it does not contain  $K_{t,t}$  (resp.  $H_t$ ) as a semi-induced subgraph.

### 2.2 Model checking, interpretations, transductions, and dependence

A relational *signature*  $\sigma$  is a finite set of relation symbols  $R$ , each having a specified arity  $r \in \mathbb{N}$ . A  $\sigma$ -structure  $\mathbf{A}$  is defined by a set  $A$  (the *domain* of  $\mathbf{A}$ ) and a relation  $R^{\mathbf{A}} \subseteq A^r$  for each relation symbol  $R \in \sigma$  with arity  $r$ .

A *binary structure* is a relational structure with symbols of arity at most 2. The syntax and semantics of first-order formulas over  $\sigma$  (or  $\sigma$ -formulas for short), are defined as usual. We recall that a *sentence* is a formula without free variable. Most of the time we will consider  $\sigma$ -structures with  $\sigma$  consisting of a single binary relation symbol  $E$ , and identify them to graphs. But we will also deal with binary structures that are graphs augmented with a total order (called *totally ordered graphs*, or *ordered graphs* for short) and/or some unary relations.



**Interpretations, transductions, and monadic dependence.** Let  $\sigma, \tau$  be relational signatures. A *simple FO interpretation* (here, *FO interpretation* for short)  $\mathbb{I}$  of  $\tau$ -structures in  $\sigma$ -structures consists of the following  $\sigma$ -formulas: a *domain* formula  $\nu(x)$ , and for each relation symbol  $R \in \tau$  of arity  $r$ , a formula  $\varphi_R(x_1, \dots, x_r)$ . If  $\mathbf{A}$  is a  $\sigma$ -structure, the  $\tau$ -structure  $\mathbb{I}(\mathbf{A})$  has domain  $\nu(\mathbf{A}) = \{v \in A : \mathbf{A} \models \nu(v)\}$  and the interpretation of a relation symbol  $R \in \sigma$  of arity  $r$  is  $R^{\mathbb{I}(\mathbf{A})} = \{(v_1, \dots, v_r) \in \nu(\mathbf{A})^r : \mathbf{A} \models \varphi_R(v_1, \dots, v_r)\}$ . If  $\mathcal{C}$  is a class of  $\sigma$ -structures, we set  $\mathbb{I}(\mathcal{C}) = \{\mathbb{I}(\mathbf{A}) : \mathbf{A} \in \mathcal{C}\}$ .

Let  $\sigma \subseteq \sigma^+$  be relational signatures. The  $\sigma$ -*reduct* of a  $\sigma^+$ -structure  $\mathbf{A}$ , denoted by  $\text{reduct}_{\sigma^+ \rightarrow \sigma}(\mathbf{A})$ , is the structure obtained from  $\mathbf{A}$  by deleting all the relations not in  $\sigma$ . A *monadic  $h$ -lift* of a  $\sigma$ -structure  $\mathbf{A}$  is a  $\sigma^+$ -structure  $\mathbf{A}^+$ , where  $\sigma^+$  is the union of  $\sigma$  and a set of  $h$  unary relation symbols, and  $\text{reduct}_{\sigma^+ \rightarrow \sigma}(\mathbf{A}^+) = \mathbf{A}$ .

A *simple non-copying FO transduction* (here, *FO transduction* for short)  $\mathbb{T}$  of  $\tau$ -structures in  $\sigma$ -structures is an interpretation of  $\tau$ -structures in  $\sigma^+$ -structures, where the  $\sigma^+$ -structures are *monadic  $h$ -lifts* of  $\sigma$ -structures for some fixed integer  $h$ . As there are many ways of interpreting the extra unary relations, a transduction (contrary to an interpretation) builds on a given input structure several output structures. If  $\mathcal{C}$  is a class of  $\sigma$ -structures,  $\mathbb{T}(\mathcal{C})$  denotes the class of all the  $\tau$ -structures output on any  $\sigma$ -structure  $\mathbf{A} \in \mathcal{C}$ .

We say that a class  $\mathcal{C}$  *interprets* a class  $\mathcal{D}$  (or that  $\mathcal{D}$  *interprets in*  $\mathcal{C}$ ) if there is an interpretation  $\mathbb{I}$  such that  $\mathcal{D} \subseteq \mathbb{I}(\mathcal{C})$ . Further, a class  $\mathcal{C}$  *efficiently interprets*  $\mathcal{D}$  if additionally a polytime algorithm inputs  $\mathbf{A} \in \mathcal{D}$ , and outputs a structure  $\mathbf{B} \in \mathcal{C}$  such that  $\mathbb{I}(\mathbf{B})$  is isomorphic to  $\mathbf{A}$ . Similarly, we say that a class  $\mathcal{C}$  *transduces* a class  $\mathcal{D}$  (or that  $\mathcal{D}$  *transduces in*  $\mathcal{C}$ ) if there is a transduction  $\mathbb{T}$  such that  $\mathcal{D} \subseteq \mathbb{T}(\mathcal{C})$ . Two classes  $\mathcal{C}$  and  $\mathcal{D}$  are *transduction equivalent* if  $\mathcal{C}$  transduces  $\mathcal{D}$ , and  $\mathcal{D}$  transduces  $\mathcal{C}$ . We will frequently use the fact that one can compose transductions: If  $\mathcal{C}$  transduces  $\mathcal{D}$ , and  $\mathcal{D}$  transduces  $\mathcal{E}$ , then  $\mathcal{C}$  transduces  $\mathcal{E}$ .

The following is a particularly useful fact to bound the twin-width of a class.

► **Theorem 8** ([6]). *Every FO transduction of a class with bounded twin-width has bounded twin-width.*

*Furthermore, given an FO transduction  $\mathbb{T}$  and a class  $\mathcal{C}$  on which  $0(1)$ -sequences can be computed in polynomial time, one can also compute  $O(1)$ -sequences for graphs of  $\mathbb{T}(\mathcal{C})$  in polynomial time.*

We will not need the original definition of monadic dependence; solely the following characterization:

► **Theorem 9** (Baldwin and Shelah [1]).  *$\mathcal{C}$  is monadically dependent if and only if  $\mathcal{C}$  does not transduce the class  $\mathcal{G}$  of all finite graphs.*

Since FO model checking on the class of all graphs is AW[\*]-hard [12], one notices that if  $\mathcal{C}$  efficiently interprets the class of all graphs then FO model checking on  $\mathcal{C}$  is AW[\*]-hard[1, 12]. Conjecture 2 anticipates that every hereditary class of structures not transducing the class of all graphs admits an FPT FO model checking, and no other hereditary class does.

### 2.3 Rank divisions, universal patterns and twin-width

A *division*  $\mathcal{D}$  of a matrix  $M$  is a pair  $(\mathcal{D}^R, \mathcal{D}^C)$ , where  $\mathcal{D}^R$  (resp.  $\mathcal{D}^C$ ) is a partition of the rows (resp. columns) of  $M$  into intervals of consecutive rows (resp. columns). Each element of  $\mathcal{D}^R$  (resp.  $\mathcal{D}^C$ ) is called a *row part* (resp. *column part*). A  *$k$ -division* is a division satisfying  $|\mathcal{D}^R| = |\mathcal{D}^C| = k$ . We often list the row (resp. column) parts of  $\mathcal{D}^R$  (resp.  $\mathcal{D}^C$ )  $R_1, R_2, \dots, R_k$  (resp.  $C_1, C_2, \dots, C_k$ ) when  $R_i$  is just below  $R_{i+1}$  (resp.  $C_j$  is just to the left of  $C_{j+1}$ ). For every pair  $R_i \in \mathcal{D}^R, C_j \in \mathcal{D}^C$ , the (contiguous) submatrix  $R_i \cap C_j$  is called *cell*

or *zone* of  $\mathcal{D}$ , or more precisely, the  $(i, j)$ -cell of  $\mathcal{D}$ . Note that a  $k$ -division defines  $k^2$  zones. We say that a cell, or more generally a matrix, is *empty* or *full 0* if all its entries are 0. The *dividing lines* of  $\mathcal{D}^R = R_1, R_2, \dots$  (resp.  $\mathcal{D}^C = C_1, C_2, \dots$ ) are the strips (of width 2) made by the last row of  $R_i$  and the first row of  $R_{i+1}$  (resp. last column of  $C_j$  and the first column of  $C_{j+1}$ ). A dividing line of  $\mathcal{D}^R$  (resp.  $\mathcal{D}^C$ ) *stabs* a set of rows (resp. of columns) if it intersects it. We may call *regular  $k$ -division* a  $k$ -division where every row part and column part have the same size.

A *rank- $k$  division* of  $M$  is a  $k$ -division  $\mathcal{D}$  such that for every  $R_i \in \mathcal{D}^R$  and  $C_j \in \mathcal{D}^C$  the cell  $R_i \cap C_j$  has at least  $k$  distinct rows or at least  $k$  distinct columns (that is, combinatorial rank at least  $k$ ). By *large rank division*, we informally mean a rank- $k$  division for arbitrarily large values of  $k$ . The maximum integer  $k$  such that  $M$  admits a rank- $k$  division is called *grid rank*, and is denoted by  $\text{gr}(M)$ .

An adjacency matrix  $M$  of a binary structure encodes in any bijective fashion the *atomic type* of every pair of vertices  $(u, v)$  (i.e., the set of atomic propositions the pair  $(u, v)$  satisfies) at position  $(u, v)$  in  $M$ . We denote by  $\text{Adj}_{\prec}(\mathbf{A})$  the adjacency matrix of  $\mathbf{A}$  *along*  $\prec$ , a total order on  $A$ . The *grid rank* of a binary structure  $\mathbf{A}$ , denoted by  $\text{gr}(\mathbf{A})$ , is the least integer  $k$  such that there is a total order  $\prec$  of  $A$  satisfying  $\text{gr}(\text{Adj}_{\prec}(\mathbf{A})) = k$ .

We will not need the original definition of twin-width (presented in the introduction) generalized to binary structures.<sup>5</sup> So we do *not* reproduce it here. Instead we recall that the twin-width and the grid rank of a binary structure are functionally equivalent, and we encourage the reader to think of the twin-width of  $\mathbf{A}$ ,  $\text{tw}(\mathbf{A})$ , simply as its grid rank  $\text{gr}(\mathbf{A})$ .

Instead we give the following useful characterization of bounded twin-width, readily generalizable to classes of other binary structures than graphs. The twin-width of the binary structure is then defined as the twin-width of the unordered matrix  $M$ , denoted by  $\text{tw}(M)$ . The precise value of  $\text{tw}(M)$  is also defined by contraction

► **Theorem 10** ([5]). *There is a computable function  $f : \mathbb{N} \rightarrow \mathbb{N}$  such that for every binary structure  $\mathbf{A}$ , the following two implications hold:*

- *If  $\text{tw}(\mathbf{A}) \leq k$ , then there is a total order  $\prec$  of  $A$  such that  $\text{gr}(\text{Adj}_{\prec}(\mathbf{A})) \leq f(k)$ , and*
- *If there is a total order  $\prec$  of  $A$  such that  $\text{gr}(\text{Adj}_{\prec}(\mathbf{A})) \leq k$ , then  $\text{tw}(\mathbf{A}) \leq f(k)$ .*

*Furthermore there are computable functions  $g, h : \mathbb{N} \rightarrow \mathbb{N}$  and an algorithm running in time  $h(k) \cdot |A|^{O(1)}$  which inputs an adjacency matrix  $\text{Adj}_{\prec}(\mathbf{A})$  without rank- $k$  division and outputs a  $g(k)$ -sequence of  $\mathbf{A}$ .*

► **Theorem 11** (informal version, see [5]). *Twin-width and grid rank are effectively tied.*

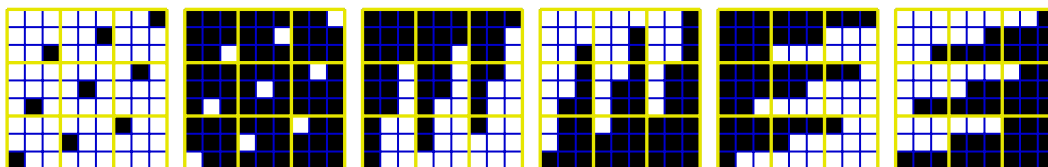
It was shown in a previous paper of the series [5] that highly-structured rank divisions can always be found in large rank divisions. We now make that statement precise. Let  $\mathcal{M}_k(0)$  be the  $k^2 \times k^2$  permutation matrix such that if  $\mathcal{M}_k(0)$  is divided in  $k$  row parts and  $k$  column parts, each of size  $k$ , there is exactly one 1 entry in each cell of the division, and this 1 entry is at position  $(i, j)$  of the  $(j, i)$ -cell; see leftmost matrix in Figure 3. For every positive integer  $k$  and  $s \in \{1, \uparrow, \downarrow, \leftarrow, \rightarrow\}$ , let  $\mathcal{M}_k(s)$  be the  $k^2 \times k^2$  0, 1-matrix defined from  $\mathcal{M}_k(0)$  by doing one of the following operations:

- switching 1 entries and 0 entries, if  $s = 1$ ,
- turning 0 entries into 1 entries if there is a 1 entry somewhere below them, if  $s = \uparrow$ ,
- turning 0 entries into 1 entries if there is a 1 entry somewhere above them, if  $s = \downarrow$ ,
- turning 0 entries into 1 entries if there is a 1 entry somewhere to their right, if  $s = \leftarrow$ ,
- turning 0 entries into 1 entries if there is a 1 entry somewhere to their left, if  $s = \rightarrow$ .

<sup>5</sup> The definition is similar with red edges appearing between the contraction of  $u$  and  $v$ , and vertex  $z$  whenever  $(u, z)$  and  $(v, z)$  have different atomic types. We refer the curious reader to [6].



We call  $\mathcal{M}_k(s)$  a *universal pattern* and  $\{\mathcal{M}_k(s) : k \in \mathbb{N}\}$  a *permutation-universal family*; see Figure 3.



■ **Figure 3** The six universal patterns with  $k = 3$ . The black cells always represent 1 entries, and white cells, 0 entries. From left to right,  $\mathcal{M}_3(0)$ ,  $\mathcal{M}_3(1)$ ,  $\mathcal{M}_3(\uparrow)$ ,  $\mathcal{M}_3(\downarrow)$ ,  $\mathcal{M}_3(\leftarrow)$ , and  $\mathcal{M}_3(\rightarrow)$ . We always adopt the convention that the matrix entry at position  $(1, 1)$  is the bottom-left one.

It was shown that, taking the adjacency matrix of a graph  $G$  along some order, either yields a matrix with bounded grid rank, and Theorem 1 effectively gives a favorable contraction sequence of  $G$ , or yields a matrix with huge grid rank, wherein a large universal pattern can be extracted:

► **Theorem 12** ([5]). *Given  $M$  an adjacency matrix of an  $n$ -vertex graph  $G$ , and an integer  $k$ , there is an  $f(k)n^{O(1)}$ -time algorithm which either returns*

- $\mathcal{M}_k(s)$  as an off-diagonal submatrix of  $M$ , for some  $s \in \{0, 1, \uparrow, \downarrow, \leftarrow, \rightarrow\}$ ,
- or a contraction sequence certifying that  $\text{tw}(G) \leq g(k)$ .

where  $f$  and  $g$  are computable functions.

Here, an *off-diagonal* submatrix of a square matrix is entirely contained strictly above the diagonal, or entirely contained strictly below it. In particular, its row indices and column indices are disjoint.

### 3 Delineation: intersection graphs of trees and paths

In this section we present a tool for showing that a class  $\mathcal{D}$  is delineated, and explore the delineation of intersection graphs of trees and paths, i.e., certain (subclasses of) chordal graphs. Our proofs of (effective) delineation will follow the same path. Either an  $O(1)$ -sequence of the graph is found (bounded twin-width) or an arbitrarily large semi-induced generalized transversal pair is detected. We shall see that the latter implies monadic independence (hence, in particular, unbounded twin-width).

A *generalized transversal pair of half-graphs* consists of  $3 + \ell$  sets  $A = \{a_{i,j} : i, j \in [t]\}$ ,  $B_0 = \{b_{i,j}^0 : i, j \in [t]\}, \dots, B_\ell = \{b_{i,j}^\ell : i, j \in [t]\}$ , and  $C = \{c_{i,j} : i, j \in [t]\}$  such that there is an edge between  $a_{i,j}$  and  $b_{i',j'}^k$  if and only if  $(i, j) \leq_{\text{lex}} (i', j')$ , for  $k \in [\ell]$  there is an edge between  $b_{i,j}^{k-1}$  and  $b_{i',j'}^k$  if and only if  $(i, j) = (i', j')$  and there is an edge between  $b_{i,j}^\ell$  and  $c_{i',j'}$  if and only if  $(j, i) \leq_{\text{lex}} (j', i')$ , where  $\leq_{\text{lex}}$  denotes the lexicographic (left-right) order. We denote this graph by  $T_{t,\ell}$ , and a *semi-induced*  $T_{t,\ell}$  is such a graph with possibly some extra edges between two sets  $X, Y \in \{A, B_0, \dots, B_\ell, C\}$  with no predefined edges. Note that  $A \cup B_0$  and  $B_\ell \cup C$  both induce a half-graph, but the “order” these two half-graphs put on the endpoints of the paths  $(b_{i,j}^0, \dots, b_{i,j}^\ell)$  is different. We define  $T_k := T_{k,0}$  and we call  $T_k$  a transversal pair (of half-graphs); see middle of Figure 2.

► **Lemma 13.** *Let  $\ell$  be a fixed non-negative integer. Let  $\mathcal{C}$  be a hereditary class containing a semi-induced generalized transversal pair of half-graphs  $T_{n,\ell}$ , for every positive integer  $n$ . Then  $\mathcal{C}$  is monadically independent.*

**Proof.** It is folklore that the class  $\mathcal{M}_b$  of all totally ordered bipartite matchings is monadically independent (see for instance [5, 8]). By *totally ordered bipartite matching*, we mean two sets  $X, Y$  of same cardinality, with a total order over  $X \cup Y$  such that  $X$  and  $Y$  are each an interval along that order, and a matching between  $X$  and  $Y$ . We shall just argue that  $\mathcal{M}_b$  transduces in  $\mathcal{C}$ . We first show the lemma when  $\ell = 0$ , that is,  $\mathcal{C}$  contains a semi-induced  $T_{n,0} = T_n$  for every  $n$ .

Let  $(G = (X, Y, E(G)), \prec)$  be any member of  $\mathcal{M}_b$ . Let  $x_1 \prec x_2 \prec \dots \prec x_n$  be the elements of  $X$ , and  $y_1 \prec y_2 \prec \dots \prec y_n$ , the elements of  $Y$ . Finally let  $\pi$  be the permutation such that, for every  $i \in [n]$ ,  $x_i y_j \in E(G)$  if and only if  $j = \pi(i)$ .

Let  $(A, B, C)$  be the tripartition of a semi-induced  $T_n$  in  $\mathcal{C}$ . The transduction  $\mathsf{T}$  guesses the tripartition  $(A, B, C)$  with 3 corresponding unary relations. Eventually  $(X, Y)$  will be a subset of  $(A, B)$ . We interpret a total order on  $A \cup B$  by

$$\begin{aligned} x \prec y \equiv & (A(x) \wedge B(y)) \vee (x \neq y \wedge A(x) \wedge A(y) \wedge \forall z (B(z) \wedge E(x, z)) \rightarrow E(y, z)) \\ & \vee (x \neq y \wedge B(x) \wedge B(y) \wedge \forall z (C(z) \wedge E(x, z)) \rightarrow E(y, z)). \end{aligned}$$

We then interpret a matching between  $A$  and  $B$  by  $\varphi(x, y) \equiv A(x) \wedge B(y) \wedge E(x, y) \wedge \forall z (z \prec x \rightarrow \neg E(z, y))$ . Observe that  $\varphi$  and  $\prec$  define a universal structure for totally ordered bipartite matchings on  $2n$  vertices.

In particular, a fourth unary relation can guess the domain ( $X \subseteq A, Y \subseteq B$ ), by picking the rows and columns of the biadjacency matrix  $\text{Adj}_{\prec}(A, B, \{ab : T_n \models \varphi(a, b)\})$  corresponding to the 1 entries, which, in the regular  $n$ -division falls in the  $(i, \pi(i))$ -cells with  $i \in [n]$ . Thus  $\mathsf{T}(T_n)$  outputs  $(G, \prec)$ .

We now deal with the general case by reducing it to  $\ell = 0$ . For that, we transduce a semi-induced  $T_n$  in a semi-induced  $T_{n,\ell}$ . The transduction is imply based on the definition of generalized transversal pairs. It uses  $3 + \ell$  unary relations  $A, B_0, \dots, B_\ell, C$ , redefines the domain as  $A \cup B_0 \cup C$ , keeps the edges between  $A$  and  $B_0$ , and adds an edge between  $x \in B_0$  and  $y \in C$  if and only if there is a path from  $x$  to  $y$  going through  $B_1, B_2, \dots, B_\ell$ , in this order. All of this is easily expressible in first-order logic.  $\blacktriangleleft$

From Lemma 13, one can easily deduce the following.

► **Lemma 14.** *Let  $\ell$  be a fixed non-negative integer. Let  $f : \mathbb{N} \rightarrow \mathbb{N}$  be any computable function, and  $\mathcal{C}$  be a graph class. If for every natural  $k$  and  $G \in \mathcal{C}$ , either  $G$  admits an  $f(k)$ -sequence or  $G$  has a semi-induced generalized transversal pair  $T_{k,\ell}$ , then  $\mathcal{C}$  is delineated.*

*Furthermore, if the contraction sequence can be found in time  $g(k) \cdot |V(G)|^{O(1)}$  for some computable function  $g$ , then  $\mathcal{C}$  is effectively delineated.*

By Theorem 1, the  $f(k)$ -sequence of  $G$  in Lemma 14 can be replaced by an adjacency matrix of  $G$  of grid rank at most  $f(k)$ .

Showing that a class  $\mathcal{D}$  is effectively delineated establishes that, as far as efficient (that is, FPT) FO model checking is concerned, twin-width gives a complete picture of what happens on  $\mathcal{D}$ . Indeed it is unlikely that a monadically independent class admits an FPT algorithm for FO model checking (see Section 2.2). Trivially, every class with bounded twin-width is delineated, and every class where  $O(1)$ -sequences can be found in polynomial time (see [4]) is effectively delineated. We now list some non-trivial examples of (effectively) delineated classes.

► **Theorem 15** ([5, 6, 21] + this paper). *The following classes of binary structures are effectively delineated: permutation graphs [6], and even, circle graphs [21], ordered graphs [5], interval graphs, and even, rooted directed path graphs.*

The proof of Theorem 15 relies on finding a good vertex ordering  $\prec$  for interval graphs or rooted directed path graphs so that for any graph  $G$  which is an interval or a rooted directed path graph,  $\text{Adj}_\prec(G)$  already has small grid rank or  $G$  contains a semi-induced generalized transversal pair  $T_{k,\ell}$ . Then Lemma 14 is applicable, especially for the last two classes, thus implies Theorem 15.

On the contrary, the class of subcubic graphs is *not* delineated. Indeed the whole class is monadically dependent (see for instance [26]), even monadically stable, but has unbounded twin-width [4]. We will see that the classes of segment graphs (even with some further restrictions) and visibility graphs of simple polygons are also *not* delineated. In some sense, what we do is to reduce to the easy case of subcubic graphs.

It is known [4, 8] that classes of bounded twin-width have exponential growth. Thus by the contrapositive, classes of super-exponential growth, like the following ones, have unbounded twin-width.

► **Theorem 16** ([4]). *The following classes have unbounded twin-width:*

- *the class  $\mathcal{G}_{\leq 3}$  of every subcubic graph;*
- *the class  $\mathcal{B}_{\leq 3}$  of every bipartite subcubic graph;*
- *the 2-subdivision of every biclique  $K_{n,n}$ .*

► **Lemma 17.** *If  $\mathcal{C}$  admits a subclass which is transduction equivalent to  $\mathcal{G}_{\leq 3}$  or to  $\mathcal{B}_{\leq 3}$ , then  $\mathcal{C}$  is not delineated.*

In what follows, we sketch the key ideas for settling the last piece toward Theorem 15, stated below.

► **Proposition 18.** *There exist a computable function  $f : \mathbb{N} \rightarrow \mathbb{N}$  such that the following holds. For any interval graph, or rooted directed path graph  $G$ , there exists a vertex ordering  $\prec$  on  $V(G)$  such that for every natural  $k$ , either  $\text{Adj}_\prec(G)$  had grid rank at most  $f(k)$  or  $G$  has a semi-induced generalized transversal pair  $T_{k,\ell}$  for some  $\ell$ .*

**The class of interval graphs is delineated, proof idea.** Let  $G$  be an interval graph and  $\mathcal{I}_G = \{I_v : v \in V(G)\}$  be an interval representation of  $G$ , where the interval  $I_v$  is of the form  $[\ell_v, r_v]$  for some integers  $1 \leq \ell_v \leq r_v$ . We further assume some minimality on the representation  $\mathcal{I}$ , i.e., if  $\ell_u < \ell_w$  for vertices  $u, w \in V(G)$ , there exists a vertex  $v \in V(G)$  such that  $\ell_u \leq r_v < \ell_w$ .

Let  $\mathcal{C}$  be a hereditary class of interval graphs of unbounded twin-width. For each graph  $G \in \mathcal{C}$  with an interval representation  $\mathcal{I}$ , we associate a total order  $\prec$ , following a lexicographic order on  $\mathcal{I}$ . For any integer  $t$ , and any interval graph  $G \in \mathcal{C}$  of sufficiently large twin-width we use a large rank division of the adjacency matrix  $\text{Adj}_\prec(G)$  to find a semi-induced transversal pair  $T_t$  and obtain delineation of interval graphs by Lemma 13. To find  $T_t$  using the rank division we extract two groups  $\{A_1, \dots, A_{f(t)}\}$  and  $\{B_1, \dots, B_{f(t)}\}$  of vertex disjoint blocks from the rank division of  $A_\prec(G)$  such that  $A_1 \prec \dots \prec A_{f(t)} \prec B_1 \prec \dots \prec B_{f(t)}$ . We can now assign an interval  $I_i$  to each block  $A_i$  and an interval  $J_i$  to each block  $B_i$  containing all respective start points. After some cleaning we can assume that these intervals are disjoint. By picking out appropriate selections  $a_1, \dots, a_{t^2}$ ,  $a_i \in A_i$  and  $b_1, \dots, b_{t^2}$ ,  $b_i \in B_i$  we can force the  $a_i$ 's and  $b_i$ 's to form a half-graph which induce any order we wish on the  $a_i$ 's using the large rank of each cell. Furthermore, using the minimality assumption, and the order and disjointness of intervals  $I_i$  we can find  $c_1, \dots, c_\ell$  forming a half-graph with the  $a_i$ 's inducing the natural order on the  $a_i$ 's. Appropriate selections of  $a_i, b_i$  and  $c_i$  will therefore yield a transversal pair.

**The class of rooted directed paths is delineated, proof idea.** *Directed path graphs* are the intersection graphs of directed paths of an oriented tree. In other words, there is a collection  $\{P_v : v \in V(G)\}$  consisting directed paths of an oriented tree  $T$  such that  $(u, w) \in E(G)$  if and only if  $V(P_u) \cap V(P_w) \neq \emptyset$ . If  $T$  in a tree model of  $G$  is an out-tree, we say that  $G$  is a *rooted directed path graph*. For  $v \in V(G)$  we denote by  $\text{high}(v)$  and  $\text{low}(v)$  the nodes of  $P_v$  that are closest and furthest from the root, respectively. Notice they are not necessarily distinct. We extend this notation to sets of vertices by defining  $\text{high}(X) = \{\text{high}(v) : v \in X\}$  and  $\text{low}(X) = \{\text{low}(v) : v \in X\}$  for  $X \subseteq V(G)$ .

Since interval graphs can be visualized as the intersection graph of subpath of a directed path, they form a subclass of rooted directed path graphs.

In general, directed path graphs are not delineated. This can be observed by subdividing the edges of a subcubic bipartite graph  $G$  and making a clique of the newly added vertices to generate a directed path graph  $G'$  that encodes  $G$ , and then applying Lemma 17 to the family of all directed path graphs constructed this way. Since this class is chordal, this also implies that chordal graphs, and even split graphs, are not delineated. On the positive side, we show that rooted directed path graphs are delineated.

We start by extracting the vertices of a rooted directed path graph  $G$  which consist of two collections associated with row and column parts of a rank- $f(t)$  division of  $\text{Adj}_{\prec}(G)$  as an off-diagonal submatrix: we may assume  $\mathcal{A} = \{A_i : i \in [f(t)/2]\}$  be the first  $f(t)/2$  parts of the row division and  $\mathcal{B} = \{B_i : i \in [f(t)/2]\}$  to be the last  $f(t)/2$  parts of the column division. Then, for each  $A_i$  and  $B_i$  we take vertices  $a_i, b_i$  to represent the sets, respectively, define  $A^\circ$  to contain all  $a_i$  and  $B^\circ$  to contain all  $b_i$ . The goal is to use  $A^\circ$  and  $B^\circ$  to distinguish between two cases in the proof.

We first observe that there is a directed path  $P$  of  $T$  containing all  $\text{high}(u)$  where  $u$  is a vertex defining some adjacency between sets of  $\mathcal{A}$  and  $\mathcal{B}$  and that  $P$  defines an order  $\prec_P$  on both  $A^\circ$  and  $B^\circ$ . We denote by  $p(u)$  the node in  $V(P_u) \cap V(P)$  that is closer to  $\text{low}(u)$  and say that  $u \preceq_P v$  if and only if  $p(u) \preceq_T p(v)$ . From this point, we prove a series of claims to show that, to organize large parts of  $\mathcal{A}$  and  $\mathcal{B}$  in a desirable way, we can focus on organizing large parts of  $A^\circ$  and  $B^\circ$ .

The easier case is when both  $A^\circ$  and  $B^\circ$  contain sufficiently large strictly increasing chains with respect to  $\prec_P$ . Since  $\prec_P$  may not agree with  $\prec$ , we apply the Erdős-Szekeres theorem to extract a large monotone sequence of both chains, and keep only the vertices appearing in those sequences in  $A^\circ$  and  $B^\circ$ . We then use those sequences to define, for each  $A_i$  associated with a vertex in the new  $A^\circ$ , an exclusive subpath  $I_i$  of  $P$  that contains  $p(a)$  for every  $a \in A_i$ , and do the same for each  $B_i$ . This is done by observing that no  $p(a)$  can be “very far away” from  $p(a_i)$  with respect to the monotone sequence. With these subpaths, we construct an interval graph and then solve this case as in the proof of delineation for this class.

If only  $A^\circ$ , for instance, contains a large strictly increasing chain with respect to  $\prec_P$  then there must be a node  $p \in P$  on which a large subset of  $\{p(b_i) : b_i \in B^\circ\}$  is concentrated. Although we can, to some extent, predict the behavior of the paths associated with vertices in  $B^\circ$  after they leave  $P$  through  $p$ , we cannot use a minimality assumption on the tree model to find in  $G$  vertices distinguishing each of the parts of  $\mathcal{B}$  associated with vertices of  $B^\circ$ . This is the crucial difference that makes finding a semi-induced  $T_{t,2}$  in this configuration much harder than in the first one.

## 4 Win-wins via twin-width: segment graphs and visibility graphs

A graph parameter  $p$  is said *FO definable* if there is a function that inputs a positive integer  $k$  and outputs a first-order sentence  $\varphi_k$  such that for every graph  $G$ ,  $p(G) = k$  if and only if  $G \models \varphi_k$ . It is further *effectively FO definable* if an algorithm realizes that function and takes time  $f(k)$  for some computable function  $f$ .

We say that a parameter  $q$  is  *$p$ -bounded on class  $\mathcal{C}$* , denoted by  $q \sqsubseteq p$  on  $\mathcal{C}$  or  $q \sqsubseteq^{\mathcal{C}} p$ , if there is a non-decreasing function  $f$  such that for every graph  $G \in \mathcal{C}$ ,  $q(G) \leq f(p(G))$ . We say that twin-width is *effectively  $p$ -bounded on  $\mathcal{C}$* , denoted by  $\text{tw} \sqsubseteq_{\text{eff}} p$  on  $\mathcal{C}$  or  $\text{tw} \sqsubseteq_{\text{eff}}^{\mathcal{C}} p$ , if further there is an algorithm that outputs a  $g(p(G))$ -sequence for every graph  $G \in \mathcal{C}$  in time  $h(p(G)) \cdot |V(G)|^{O(1)}$  for some computable functions  $g, h$ .

The following reduces the task of showing that an FO definable parameter  $p$  is FPT on  $\mathcal{C}$  to showing that  $\text{tw} \sqsubseteq_{\text{eff}}^{\mathcal{C}} p$  holds.

► **Theorem 19.** *Let  $p$  be an effectively FO definable parameter, and  $\mathcal{C}$  a class such that  $\text{tw} \sqsubseteq_{\text{eff}}^{\mathcal{C}} p$ . Then  $p(G) \geq k$  for  $G \in \mathcal{C}$  can be decided in FPT time  $f(p(G)) \cdot |V(G)|^{O(1)}$  for some computable function  $f$ .*

### 4.1 Segment graphs

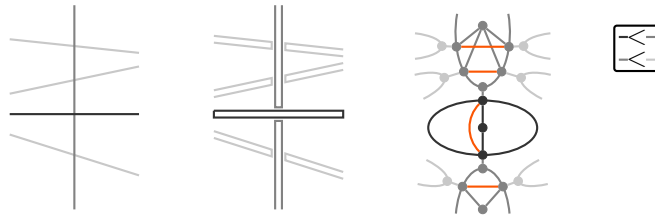
Pure (hence triangle-free) axis-parallel unit segment graphs were shown to have unbounded twin-width [4], by constructing a family of such graphs with super-exponential growth. This family contains arbitrarily large bicliques (see [4, Figure 4]). We will show that bicliques are necessary to make the twin-width large, even when we lift the requirements that the segments are axis-parallel and unit.

**Techniques to show Theorem 5.** We first FO transduce  $K_{t,t}$ -free segment graphs from a class  $\mathcal{F}$  of 2-edge-colored graphs that, we next show, has bounded twin-width. Once this is done, it follows from Theorem 8 that  $K_{t,t}$ -free segment graphs have twin-width at most  $h(t)$  and  $h(t)$ -sequence can be obtained from Theorem 8.

To capture a suitable graph class  $\mathcal{F}$ , imagine a representation of  $K_{t,t}$ -free segment graph  $G$  which uses thin rectangles in place of segments. Using bounded degeneracy of  $K_{t,t}$ -free segment graphs [24], each rectangle can fragment into at most  $d + 1$  sub-rectangles at places *stabbed* by a rectangle preceding it in the  $d$ -degeneracy ordering. This fragmented representation preserves the adjacency of  $G$  in the form of local adjacency between sub-rectangles, and the former can be restored by FO transduction from the latter. It can be naturally translated to a superposition of two graphs over the same vertex set, namely a plane graph  $P = (V, E)$  and a matching graph  $H = (V, M)$  which is *nicely aligned* with respect to a packing  $\mathcal{C}$  of facial cycles partitioning  $V(P)$ , see Figure 4.

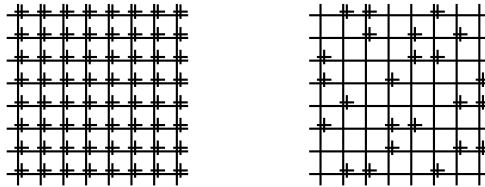
Our task then boils down to showing that a plane graph  $P$  admits a vertex ordering  $\prec$  which circularly orders each the facial cycle of  $\mathcal{C}$  so that  $(P, \prec)$  has bounded twin-width. It turns out that a variant of BFS discovery order works, with two important features: The edges are considered in the cyclic order during the exploration phase of a vertex  $v$ , and the vertices of a facial cycle  $C \in \mathcal{C}$  are processed in batch, when ordered in the cyclic order around the face.

In the previous theorem, one cannot relax the  $K_{t,t}$ -freeness assumption to  $H_t$ -freeness. Let  $B_n$  be the graph obtained from the 2-subdivision of a biclique  $K_{n,n}$  by adding back the edges of the original biclique. The left part of Figure 5 shows that, for every  $n \in \mathbb{N}$ , the graph  $B_n$  is realizable with axis-parallel segments of two different lengths. Note however that  $B_n$  has no semi-induced  $H_4$ , and that  $\lim_{n \rightarrow \infty} \text{tw}(B_n) = \infty$ .



■ **Figure 4** Left: Original segment representation. Center: Thin rectangle stabbed by its predecessor in the degeneracy ordering. Right: Plane graph with a nicely aligned matching, preserving the information of the original segment graph.

To establish the latter claim, one can for instance “remove” the edges of the biclique by means of an FO transduction, and invoke Theorem 8 and the third item of Theorem 16. The transduction first marks the long horizontal segments by unary relation  $U_1$  (color 1), and the long vertical segments, by unary relation  $U_2$  (color 2), and interpret the new edges as  $\varphi(x, y) \equiv E(x, y) \wedge \neg(U_1(x) \wedge U_2(y)) \wedge \neg(U_1(y) \wedge U_2(x))$ .



■ **Figure 5** Left: An axis-parallel  $H_4$ -free two-lengthed segment graph realizing  $B_n$  (here drawn with  $n = 8$ ), whose twin-width grows with  $n$ . Right: Axis-parallel segment graphs are transduction equivalent to  $\mathcal{B}_{\leq 3}$ , thus not delineated.

Similarly the 2-subdivision of any subcubic bipartite graph, augmented with the biclique between its two partite sets, is realizable with axis-parallel segments of two different lengths (see right-hand side of Figure 5). Indeed those graphs – let us denote by  $\mathcal{C}$  the class they form – are induced subgraphs of some  $B_n$ . We claim (see right of Figure 5 and long version, for a proof) that  $\mathcal{C}$  and  $\mathcal{B}_{\leq 3}$  are transduction equivalent, and by Lemma 17, two-lengthed axis-parallel segments are not delineated.

In the construction of Figure 5, we use two different lengths for the segments. We show that with a unique length (unit segments), axis-parallel  $H_t$ -freeness implies bounded twin-width.

► **Theorem 20.** *Axis-parallel  $H_t$ -free unit segment graphs have bounded twin-width.*

Actually we prove a stronger statement than the previous theorem, where segments are not necessarily unit, but the ratio between the largest and the smallest lengths is bounded. Again it shows that the fact that this ratio is unbounded in Figure 5 (left) is unavoidable.

**Techniques to show Theorem 20.** We face again the challenging task of finding a “good” linear order on objects from a two-dimensional space. We place a virtual grid whose cells are of size  $1 \times 1$ , and cut the segments along this grid, adding some junction vertices in between the cut pieces corresponding to the same segment. We first prove by FO transduction that if this new graph has bounded twin-width, then the original segment graph has bounded twin-width. For the newly built graph, a natural order consists of locally enumerating the



segments counter-clockwise according to where they cross the grid, and globally enumerating the cells of the grid row by row. Note that the dimension of the grid cells imposes that every segment crosses the grid.

The crux is then to argue that the circular order along the boundary of a cell yields adjacency matrices with bounded grid rank. Somewhat surprisingly this part leverages the same argument as we will later use for  $H_t$ -free visibility graphs of terrains; a forbidden pattern like the Order Claim.

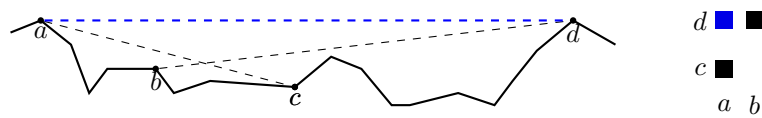
## 4.2 Visibility graphs

We first show that visibility graphs of terrains without arbitrarily large ladders have bounded twin-width.

► **Theorem 21.**  *$H_t$ -free visibility graphs of 1.5D terrains have bounded twin-width.*

Rather naturally, we choose the order  $\prec$  along the boundary of the terrain. Due to the Order Claim (see Lemma 22 and Figure 6) the obtained adjacency matrices exclude a pattern (right of Figure 6) that, combined with  $H_t$ -freeness, prevents large universal patterns. Hence we conclude by Theorem 12.

► **Lemma 22** (Order Claim [2]). *If  $a \prec b \prec c \prec d$ ,  $a$  see  $c$ , and  $b$  see  $d$ , then  $a$  and  $d$  also see each other.*



■ **Figure 6** Left: The Order Claim. The dashed black edges imply the dashed blue edge. Right: In the thus ordered adjacency matrix, the 1 entries at  $(a, c)$  and  $(b, d)$  implies the 1 entry at  $(a, d)$ .

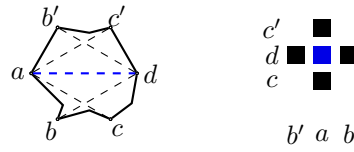
In stark contrast, we can exhibit a subclass of visibility graphs of simple polygons whose hereditary closure has unbounded twin-width but is monadically dependent, and even monadically stable. This transduction is more involved than the previous ones, so we give full details (in the long version). Finally, we show that the twin-width of simple polygons is bounded by a function of their independence number  $\alpha$ .

► **Theorem 23.** *Twin-width is  $\alpha$ -bounded in visibility graphs of simple polygons, and effectively  $\alpha$ -bounded if a geometric representation is given.*

**Proof (Sketch).** Let  $\mathcal{P}$  be a simple polygon, and  $G$  its visibility graph. We identify a vertex of  $G$  with its corresponding geometric vertex of  $\mathcal{P}$ . Let  $\prec$  be the total order whose successor relation is a Hamiltonian path of the boundary of  $\mathcal{P}$ . Visibility graphs of simple polygons satisfy the double-X property: If  $b' \prec a \prec b \prec c \prec d \prec c'$ , and  $ac, bd, ac', db'$  are all in  $E(G)$ , then  $ad$  is also an edge of  $G$  (see Figure 7).

This excludes that the complement of a(n arbitrary) permutation is realized by the adjacency matrix of  $G$  ordered along  $\prec$ . That is, the second universal pattern in Figure 3 would not appear in  $\text{Adj}_{\prec}(G)$ .

We now upperbound the size of a universal pattern in  $\text{Adj}_{\prec}(G)$  (among the other five patterns) in terms of  $\alpha(G)$ , and conclude by Theorem 12. We will actually not need the universal pattern in its whole, but simply a decreasing subsequence of it. (This is made



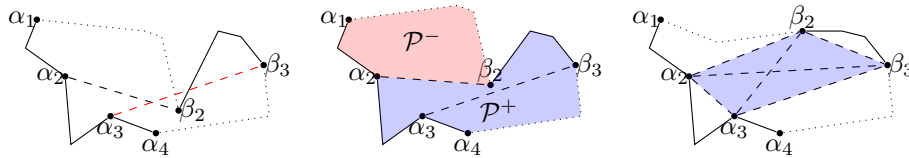
■ **Figure 7** Left: The double-X property. Right: What it implies in the adjacency matrix ordered along the boundary of the polygon; the four 1 entries in black force the central one in blue.

formal in the next paragraph, where we extract a large anti-diagonal induced matching or half-graph.) This is convenient since we can thus apply Ramsey’s theorem while keeping the “complexity” of the initial structure.

Let  $p = \text{Ram}(\text{Ram}(4, \alpha(G)), \alpha(G))$ , where  $\text{Ram}(s, t)$  is the function of Ramsey’s theorem which enforces a monochromatic clique on  $s$  or  $t$  vertices in a 2-edge-colored complete graph on  $\text{Ram}(s, t)$  vertices. Note that if the twin-width of  $G$  is larger than a certain function of  $p$ , we can find in each of the five allowed universal patterns  $2p$  vertices of  $G$ :  $a_1 \prec a_2 \prec \dots \prec a_{p-1} \prec a_p \prec b_p \prec b_{p-1} \prec \dots \prec b_2 \prec b_1$  such that  $a_i b_j \in E(G)$  if and only if  $i = j$  (resp.  $i \leq j$ , resp.  $i \geq j$ ). We denote  $\{a_1, \dots, a_p\}$  (resp.  $\{b_1, \dots, b_p\}$ ) by  $A$  (resp.  $B$ ). We now work toward finding a contradiction.

Let  $A' \subseteq A$  induce a clique in  $G$  with  $|A'| = \text{Ram}(\alpha(G), 4)$ . Let  $B'$  be the vertices of  $B$  with the same index as a vertex of  $A'$ , and let  $B'' \subseteq B'$  induce a clique in  $G$  of size 4. Finally let  $A''$  be the vertices in  $A'$  (or  $A$  for that matter) with same index as a vertex in  $B''$ . We relabel the eight vertices of  $A'' \cup B''$  by  $\alpha_1 \prec \alpha_2 \prec \alpha_3 \prec \alpha_4 \prec \beta_4 \prec \beta_3 \prec \beta_2 \prec \beta_1$ .

First observe that, since they form a clique,  $\alpha_1, \alpha_2, \alpha_3, \alpha_4$  are in convex position. For  $\alpha_2 \beta_2$  and  $\alpha_3 \beta_3$  to be in  $E(G)$ , the vertices  $\beta_2$  and  $\beta_3$  have to be in the convex (possibly infinite) region delimited by the line segment  $\alpha_2 \alpha_3$ , the ray starting at  $\alpha_2$  and passing through  $\alpha_1$ , and the ray starting at  $\alpha_3$  and passing through  $\alpha_4$ . Since  $\beta_2$  comes after  $\beta_3$  in the boundary order, the quadrangle  $\alpha_2 \alpha_3 \beta_3 \beta_2$  has to be non self-intersecting (otherwise  $\alpha_2 \beta_2$  and  $\alpha_3 \beta_3$  cannot both be edges, see left of Figure 8). We now claim that  $\alpha_2 \alpha_3 \beta_3 \beta_2$  is a convex quadrangle. Assume for the sake of contradiction that  $\beta_2$  is in the interior of the triangle  $\alpha_2 \alpha_3 \beta_3$  (this is without loss of generality). As  $\alpha_2 \beta_2$  is an edge of  $G$ , the line segment  $\alpha_2 \beta_2$  cuts  $\mathcal{P}$  into two simple polygons:  $\mathcal{P}^-$  containing  $\alpha_1$ , and  $\mathcal{P}^+$  containing  $\alpha_3$ . Observe that no line segment starting at  $\beta_3$  and fully contained in  $\mathcal{P}$  can intersect  $\mathcal{P}^- \setminus \{\beta_2\}$ . Indeed, since  $\beta_2$  is in the interior of  $\alpha_2 \alpha_3 \beta_3$ , the ray starting at  $\beta_3$  and passing through  $\beta_2$  remains entirely within  $\mathcal{P}^+$ . However  $\alpha_1$  is in  $\mathcal{P}^-$ . Therefore  $\beta_3$  and  $\beta_1$  cannot see each other; a contradiction (see middle of Figure 8).



■ **Figure 8** Left: If  $\alpha_2 \alpha_3 \beta_3 \beta_2$  is self-intersecting, at least one edge of  $\alpha_2 \beta_2, \alpha_3 \beta_3$  (here  $\alpha_3 \beta_3$ ) is missing from  $G$ . Center: If  $\beta_2$  lies in the interior of  $\alpha_2 \alpha_3 \beta_3$ , vertices  $\beta_1$  (in  $\mathcal{P}^+$ ) and  $\beta_3$  cannot see each other without blocking the edge  $\alpha_2 \beta_2$ . Right: If  $\alpha_2 \alpha_3 \beta_3 \beta_2$  is in convex position (in this order), then both  $\alpha_2 \beta_3$  and  $\alpha_3 \beta_2$  are edges; another contradiction.

Since the four sides of the convex, non self-intersecting quadrangle  $\alpha_2 \alpha_3 \beta_3 \beta_2$  are edges of  $G$ , the two diagonals  $\alpha_2 \beta_3$  and  $\alpha_3 \beta_2$  are also edges (since  $\mathcal{P}$  cannot intersect the interior of  $\alpha_2 \alpha_3 \beta_3 \beta_2$ , see right of Figure 8); a contradiction to the induced matching or half-graph in between  $A$  and  $B$ . ◀

Combined with the FO model checking algorithm in [6], generalizes a conjecture of Hliněný, Pokrývka, and Roy [22], and shows in particular that  $k$ -INDEPENDENT SET is FPT on visibility graphs of simple polygons.

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