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Robust Fault Detection using Zonotopic Parameter Estimation

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Abstract: This paper addresses the system identification problem, as well as its application to robust fault detection, considering parametric uncertainty and using zonotopes. As a result, a Zonotopic Recursive Least Squares (ZRLS) estimator is proposed and compared with the Set-membership (SM) approach when applied to fault detection, taking as a reference the minimum detectable fault generated in the worst-case. To illustrate the effectiveness of the proposed robust parameter estimation and fault detection methodologies, a quadruple tank process is employed.

Keywords: zonotopes, least squares, set-membership, fault detection, minimum detectable fault

1. INTRODUCTION

Models are the basis of fault diagnosis (FD) approaches based on the concept of analytical redundancy. These models are generally affected by uncertainty that comes from the measurements discretization process, external disturbances, or inevitable modelling errors. Therefore, defining adaptive thresholds for enclosing the uncertainty is essential to achieve robust decisions. Basically, FD schemes suggest a strategy focused on comparing the measured outputs from the system with the estimated outputs provided by the faultless model at each time step (Blanke et al., 2006). When inconsistencies are perceived, the existence of faults will be demonstrated. In essence, a significant trade-off between robustness to disturbances and sensitivity to faults must be assured (Combastel et al., 2008).

Depending on the way of determining the bounds of uncertainty, two kinds of approaches are mainly found in the literature: probabilistic and deterministic. Deterministic methods (worst-case), have been widely reported in the literature, due to the fact that assuming bounded uncertainty is much easier to satisfy in real applications. Diverse geometrical sets have been employed to propagate the uncertainty from the parameter estimation point of view, for instance: ellipsoids (Kurzhanski, 1997), polytopes (Mo and Norton, 1990) and zonotopes.

In particular, zonotopes provide a less conservative model compared to using ellipsoids and are more efficient and fast in terms of computational complexity compared to polytopes. This is due to the fact that zonotopes are closed under Minkowski sum and linear maps and even its intersections can be efficiently over-approximated (Althoff and Rath, 2021). In the case of model-based fault detection, several works using zonotopes have been reported and grouped in two kinds of algorithms: Interval Predictors (IO) and Set-membership (SM) (Combastel et al., 2008; Blesa et al., 2012).

The fact that IO employs interval hulls for enclosing the parametric uncertainty may lead to an over-approximation in the uncertainty sets. This paper proposes a Zonotopic Recursive Least Squares (ZRLS)-based approach for fault detection. In this sense, ZRLS guarantees a robust convergence towards the set that encloses all the parameter vectors consistent with the model structure and the noise bounded by minimizing the weighted Frobenius norm of its covariance. Likewise, ZRLS is compared to the well-known SM approach taking into consideration the minimum detectable fault in the worst-case. It is worth pointing out that both ZRLS and SM formulation are different taking into account a conceptual point of view. However, their mathematical interpretations provide equivalent results.

2. PRELIMINARIES AND PROBLEM STATEMENT

2.1 Background on Zonotopes

Definition 1. (McMullen, 1971) Zonotopes are represented as $\langle c, R\rangle$ with the center $c \in \mathbb{R}^{mx1}$ and the generator matrix $R \in \mathbb{R}^{mxp}$. Zonotopes are particular forms of polytopes defined as the linear image of the unit hypercube $[-1, +1]^p \subset \mathbb{R}^p$ such that $\langle c, R\rangle = \{c + R\psi : \|\psi\|_\infty \leq 1\}$.

Definition 2. (Combastel, 2003) $\langle R\rangle = \langle 0, R\rangle$ is denominated centered zonotope. Any permutation of the columns of R, the zonotope remains invariant.

Definition 3. (Kühn, 1998) Given $\mathcal{Z} = \langle c, R\rangle$, the interval hull $rs(R) \subset \mathbb{R}^{mxm}$ is the smallest aligned box where $\mathcal{Z}$ can be enclosed; that is, $\langle c, R\rangle \subset \langle c, rs(R)\rangle$. $rs(R)$ is a diagonal matrix determined by the sum of the rows of R.

Definition 4. (Combastel, 2005) Given $\mathcal{Z} = \langle c, R\rangle$, the reduction operator is represented as Red_q where q specifies the maximum number of columns of Red_q that holds the inclusion property $\langle c, R\rangle \subset \langle c, Red_q\rangle$.

Definition 5. (Combastel, 2015) Given $\mathcal{Z} = \langle c, R\rangle$ and $W \in \mathbb{R}^{mxm}$, $W = W^T \succ 0$ is a weighting matrix, the F_W-radius of $\mathcal{Z}$ is defined by computing the weighted Frobenius norm of R, such that $\|\langle c, R\rangle\|_{F,W} = \sqrt{Tr(R^T W R)}$ with $W = I_n$.

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Definition 6. (Combastel, 2015) Given $\mathcal{Z} = \langle c, R\rangle$, the covariation of $\mathcal{Z}$ is determined by the expression $Cov[\mathcal{Z}] = RR^T$.

Definition 7. (Alamo et al., 2005) Given $\mathcal{Z} = \langle c, R\rangle$, the strip $\mathcal{S} = \{x \in \mathbb{R}^{\eta_x} : |Hx - y| \leq F\}$ and $\lambda \in \mathbb{R}^{\eta_x}$, the intersection between the zonotope and the strip is defined as $\mathcal{Z} \cap \mathcal{S} = \langle \hat{c}(\lambda), \hat{R}(\lambda)\rangle$, where $\hat{c}(\lambda) = c + \lambda(y - Hc)$ and $\hat{R}(\lambda) = [(I - \lambda H)R, \lambda F]$.

Property 1. The following arithmetic set properties are satisfied by zonotopes, representing the Minkowski sum and the linear image as $\oplus$ and $\odot$, respectively. Likewise, M is a matrix with appropriate dimension, such that:

$$\langle c_1, R_1\rangle \oplus \langle c_2, R_2\rangle = \langle c_1 + c_2, [R_1, R_2]\rangle \tag{1a}$$
$$M \odot \langle c, R\rangle = \langle Mc, MR\rangle \tag{1b}$$

Property 2. Given x, A, B and C matrices of appropriate dimension we have:

$$Tr(A) = Tr(A^T) \tag{2a}$$
$$\frac{\partial Tr(Ax^TB)}{\partial x} = A^TB^T \tag{2b}$$
$$\frac{\partial Tr(AxBx^TC)}{\partial x} = Bx^TCA + B^Tx^TA^TC^T \tag{2c}$$

2.2 Problem statement

Let us consider a linear and uncertain discrete-time system modelled in regression form as follows:

$$\begin{aligned} &y_k = H_k\theta_k + \upsilon_k, \ \forall k \in \mathbb{N} \\ &\upsilon_k \in \langle 0, F_k\rangle \\ &\theta_0 \in \langle c_0, R_0\rangle \subset \mathbb{R}^{\eta_\theta} \end{aligned} \tag{3}$$

where $y_k \in \mathbb{R}$ denotes the available measurement, $H_k \in \mathbb{R}^{1x\eta_\theta}$ represents the regressor vector, $\theta_k \in \mathbb{R}^{\eta_\theta x1}$ is the unknown parameter vector to be estimated, and $\upsilon_k \in \mathbb{R}$ denotes the uncertainty associated with the measurement, that is bounded by a centered zonotope. In addition, the initial parameter vector θ_0 belongs to a zonotope which can be large depending on the lack of knowledge of the system. Then, the feasible parameter set, at each time step is approximated by a zonotopic set consistent with the measurements and the bounded-error model,

$$\hat{\theta}_{k+1} \in \langle c_{k+1}, R_{k+1}\rangle \tag{4}$$

such that c_{k+1} is the center and R_{k+1} is the generator matrix of the parameter bounding zonotope.

3. ZONOTOPIC PARAMETER ESTIMATOR

3.1 Estimator structure

Recursive estimator in prediction-type is defined as:

$$\begin{aligned} \hat{\theta}_{k+1} &= \hat{\theta}_k + K_k(y_k - H_k\hat{\theta}_k - \upsilon_k) \\ &= (I_n - K_kH_k)\hat{\theta}_k + K_ky_k - K_k\upsilon_k \end{aligned} \tag{5}$$

and provides the update estimate when one more observation is added. The prediction parameter $\hat{\theta}_{k+1} \in \mathbb{R}^{\eta_\theta x1}$ is calculated as a function of the previous estimate $\hat{\theta}_k$ plus a correction term which is proportional to the difference between the available measurement (y_k) and the quantity ($H_k\hat{\theta}_k - \upsilon_k$) for $k \geq 0$. $K_k \in \mathbb{R}^{\eta_\theta}$ is the correction gain vector.

Proposition 1. Considering the system (3) with deterministic uncertainties, the zonotopic parameter estimator is recursively defined by $\hat{\theta}_{k+1} \in \langle c_{k+1}(K_k), R_{k+1}(K_k)\rangle$ as:

$$\begin{aligned} c_{k+1} &= (I_n - K_kH_k)c_k + K_ky_k \\ R_{k+1} &= [(I_n - K_kH_k)\bar{R}_k, -K_kF_k] \end{aligned} \tag{6}$$

where $\bar{R}_k = Red_q$, and $\hat{\theta}_k \in \langle c_k, R_k\rangle \subset \langle c_k, \bar{R}_k\rangle$.

Proof. Assuming $\hat{\theta}_k \in \langle c_k, R_k\rangle$ and the operator Red_q holds $\hat{\theta}_k \in \langle c_k, \bar{R}_k\rangle$. Since υ_k from (3) is bounded by F_k and taking into consideration (5), it results in:

$$\begin{aligned} \hat{\theta}_{k+1}(K) \in \langle c_{k+1}(K), R_{k+1}(K)\rangle = ((I - K_kH_k)\odot \\ \langle c_k, R_k\rangle) \oplus (K_k \odot \langle y_k, 0\rangle) \oplus (-K_k \odot \langle 0, F_k\rangle) \end{aligned} \tag{7}$$

By using zonotope properties (1a) and (1b) to the above equation, c_{k+1} and R_{k+1} are obtained as in (6). Therefore, the proof have been completed.

3.2 Optimal gain estimation

The optimal criterion to compute K_k is based on minimizing the F_W-radius from R_{k+1} in (6), where F_W-radius is a dimension criterion of a zonotope. Combastel (2015) demonstrates that minimizing the F_W-radius is equivalent to minimize a weighted function of its covariance (see Definition 6). Then, according to Definition 5, the F_W-radius leads to calculate $\|R_{k+1}\|_{F,W}$.

Theorem 1. Given R_{k+1} as in (6), the optimal estimator gain ($\tilde{K}_k$) is determined by minimizing the cost function $J_W = Tr(WP_{k+1}) \implies \tilde{K}_k = arg\,min_K \|R_{k+1}\|^2_{F,W}$, such that:

$$\tilde{K}_k = LS^{-1} \tag{8a}$$
$$L = \bar{P}_kH_k^T, \quad S = H_k\bar{P}_kH_k^T + Q_{\upsilon,k} \tag{8b}$$

with $\bar{P}_k = \bar{R}_k\bar{R}_k^T = Cov[Red_q]$ and $Q_{\upsilon,k} = F_kF_k^T$. It is worth emphasizing that $\tilde{K}_k$ is independent of the weighting matrix W.

Proof. From (6) and taking into account Definition 6, P_{k+1} is derived as:

$$P_{k+1} = (I_n - K_kH_k)\bar{P}_k(I_n - K_kH_k)^T + K_kQ_{\upsilon,k}K_k^T \tag{9}$$

So, replacing P_{k+1} in J_W leads to:

$$\begin{aligned} J_W = Tr(W(I_n - K_kH_k)\bar{P}_k(I_n - K_kH_k)^T + \dots \\ \dots + WK_kQ_{\upsilon,k}K_k^T) \end{aligned} \tag{10}$$

Then, applying suitable manipulations using (2a), and taking advantage that P and W are symmetric $J_W = Tr(W\bar{P}_k - 2WLK_k^T + WK_kSK_k^T)$ is obtained, with $L = \bar{P}_kH_k^T$ and $S = H_k\bar{P}_kH_k^T + Q_{\upsilon,k}$, such that S is equally symmetric. Finally, computing $\frac{\partial J_W}{\partial K_k} = 0$ through (2b) and (2c), and solving for K_k result in $2WL = 2SK_k^TW$ which yields to (8).

The proposed method is based on the RLS estimator. At each sample instant with the available measurement, a zonotopic set $\langle c_{k+1}, R_{k+1}\rangle$ that satisfies the F_W-radius minimization (see Theorem 1) is estimated from the error covariance matrix P_{k+1}. The predicted set preserves the inclusion property $\hat{\theta}_k \in \langle c_k, R_k \rangle$ defining the error at instant k as $\hat{\theta}_k - c_k \in \langle 0, R_k \rangle$. Analogously, the error at $k+1$ time is deduced by $\hat{\theta}_{k+1} - c_{k+1} \in \langle 0, R_{k+1} \rangle$.

4. SET-MEMBERSHIP PARAMETER ESTIMATOR

Taking into account the proposed system in (3), the SM approach aims to determine the feasible parameter set (FPS) consistent with the N input-output samples in the parameter space. Thus, the FPS is defined as:

$$\mathrm{FPS} = \{\theta_k \in x_0 : y_k - F_k \leq H_k \theta_k \leq y_k + F_k, \forall k \in \mathbb{N}\} \tag{11}$$

Several methods in the literature are reported to compute inner or outer regions that approximate the FPS (Andrew, 2002). The zonotope-based approach proposed by Alamo (Alamo et al., 2005) to estimate uncertain parameters is applied in this work.

4.1 SM estimator structure

Proposition 2. Given the dynamic modelled by the deterministic approach (3), $\mathcal{Z}^{sm}_{k+1} \in \langle c_{k+1}(\lambda_k), R_{k+1}(\lambda_k) \rangle$ is the outer-approximated feasible parameter zonotopic set conditioned by each measurement component y_k. The center and the segment matrix are c_{k+1} and R_{k+1}, respectively, deriving in:

$$\begin{aligned} c_{k+1} &= c_k + \lambda_k (y_k - H_k c_k) \\ R_{k+1} &= [(I_n - \lambda_k H_k) R_k, -\lambda_k F_k] \end{aligned} \tag{12}$$

Proof. Recalling again the regression model (3), the region with each measurement vector component, defined as a strip at k instant is expressed as follows

$$\mathcal{S}^s_k = \left\{ \theta_k \in \mathbb{R}^{\eta_\theta} : |H_k \theta_k - y_k| \leq F_k \right\} \tag{13}$$

Following Definition 7 according to Alamo et al. (2005), the $\mathcal{Z}^{sm}$ (12) that satisfies $\mathcal{Z}^{sm}_{k+1} \supseteq \mathcal{Z}^{sm}_k \cap \mathcal{S}^s_k$ is obtained.

4.2 Optimal zonotope segment size

Based on (12), $\mathcal{Z}^{sm}_{k+1}$ depends on λ_k at each iteration. Several procedures focused on decreasing the size of the set, such as, segments, volume, or P-radius minimization are reported (Le et al., 2013). In this paper, the optimal $\tilde{\lambda}_k$ is calculated using Theorem 2.

Theorem 2. Considering the $\mathcal{Z}^{sm}_{k+1}$ structure (12), the optimal $\tilde{\lambda}_k$ is determined by minimizing the segment size of the zonotope, that implies computing the Frobenius norm of the generator matrix. Therefore, $\tilde{\lambda}_k = arg\,min_\lambda \|R_{k+1}\|^2_F$ such that:

$$\tilde{\lambda}_k = R_k R_k^T H_k^T (F_k F_k^T + H_k R_k R_k^T H_k^T)^{-1} \tag{14}$$

Proof. The proof is explicitly carried out in the work presented by (Alamo et al., 2005).

5. FAULT DETECTION USING ZRLS APPROACH

Considering the dynamic parameterized as in (3), and including the fault effects it yields:

$$y_k = H_k \theta_k + f^H_k \theta_k + H_k f^\theta_k + f^y_k + F_k \upsilon_k, \forall k \in \mathbb{N} \tag{15}$$

where f^H_k and f^y_k indicate the sensor fault signals added to the regressor and the output, and f^θ_k represents the parametric fault signal. To apply and compare ZRLS and SM approaches for fault detection, two different tests, called direct and inverse, (Blesa et al., 2012) are used, respectively. In particular, ZRLS relies on the consistency test of the residual whereas SM checks the intersection between the outer-approximated zonotope $\mathcal{Z}^{sm}$ and the strip consistent with the available measurements $\mathcal{S}^s$. In this sense, the residual generation problem in ZRLS is reformulated in terms of the strip for the sake of characterizing the minimum detectable fault. Closely related to the sensitivity against faults, the minimum detectable fault is equivalent to the magnitude of the fault able to produce an inconsistency; that is, to bring the parameters out of their bounds. Hence, defining large bounds can provide undetected faults whereas false alarms can occur if tight bounds are selected. In this paper, only parametric faults (f^θ_k) are considered.

5.1 Description of the FD test

Algorithm 1 is used for fault detection as an extension of ZRLS approach. The predicted zonotopic set (6) is employed to compute the zonotope support strip which is defined as the region between two hyperplanes, such that:

$$\mathcal{S}^z_k = \left\{ \theta_k \in \langle c_k, R_k \rangle : q^{z_{low}}_k \leq H_k \theta_k \leq q^{z_{upp}}_k \right\} \tag{16}$$

where $q^{z_{low}}_k$ and $q^{z_{upp}}_k$ constitute the minimum and the maximum bounds of the zonotope support strip, which are calculated as in (17).

$$\begin{aligned} q^{z_{low}}_k &= H_k c_k - \|R_k H_k\|_1 \\ q^{z_{upp}}_k &= H_k c_k + \|R_k H_k\|_1 \end{aligned} \tag{17}$$

Algorithm 1 Fault detection based on ZRLS approach

$k \leftarrow 0$
$\langle c_k, R_k \rangle \leftarrow \theta_0$
$\bar{R}_0 \leftarrow Red_q(R)$, q is determined according to Definition 4.
$\bar{P}_0 \leftarrow \bar{R}_0 \bar{R}_0^T$
$Q_\upsilon \leftarrow F F^T$
while $k \leftarrow N$ **do**
  Obtain input-output u_k, y_k and build regressor H_k.
  Compute the optimal $\tilde{K}_k$ using (8).
  Compute c_{k+1} and R_{k+1} utilizing (6).
  Compute P_{k+1} using (9).
  Compute $\mathcal{S}^z_k$ employing (17).
  if $\mathcal{S}^z_k \cap \mathcal{S}^s_k = \emptyset$ **then**
    $Fault \leftarrow true$
  end if
  $k \leftarrow k+1$
end while

On the other hand, from the strip consistent with the observations $\mathcal{S}^s_k$ (see (18)), the minimum and the maximum values $q^{s_{low}}_k$ and $q^{s_{upp}}_k$, respectively, are determined using (19).

$$\mathcal{S}^s_k = \left\{ \theta_k \in \mathbb{R}^{\eta_\theta} : |H_k \theta_k - y_k| \leq F_k \right\} \tag{18}$$

$$\begin{aligned} q_k^{s_{low}} &= y_k - F_k \\ q_k^{s_{upp}} &= y_k + F_k \end{aligned} \tag{19}$$

Then, the inconsistency or fault is proven if and only if $\mathcal{S}_k^z \cap \mathcal{S}_k^s = \emptyset$ or complying the following inequalities:

$$\begin{aligned} q_k^{z_{low}} &> q_k^{s_{upp}} \\ q_k^{z_{upp}} &< q_k^{s_{low}} \end{aligned} \tag{20}$$

5.2 Minimum parametric fault magnitude

The minimum parametric fault magnitude f_k^θ is computed using (20) and assuming that no other faults are present (Gertler, 1998). Considering $\theta_k = \langle c_k, R_k \rangle$ and replacing $q_k^{z_{upp}}$ from (17) and $q_k^{s_{low}}$ from (19) into expression (20) it follows:

$$H_k c_k + \|R_k H_k\|_1 < y_k - F_k \tag{21}$$

Depending on the position of the strip $\mathcal{S}_k^s$, the fault magnitude must overcome the uncertainty for generating an inconsistency. In the worst-case scenario for the parameter vector, $\mathcal{S}_k^s$ is located on the border of $\mathcal{S}_k^z$; that is, $H_k c_k + \|R_k H_k\|_1 = 2\|R_k H_k\|_1$ is deduced. In this sense, $\mathcal{S}_k^z$ must change the parameter value at least twice to achieve the opposite threshold. Using (15), and taking into consideration the fault parametric effect only, the magnitude of f_k^θ that will always be detected is calculated as follows:

$$H_{k,i} f_k^{\theta,i} > 2\|R_k H_k\|_1 + 2F_k \tag{22}$$

where i is the component of the regressor H_k associated with the i-th parametric fault.

6. FAULT DETECTION USING SM APPROACH

6.1 Description of the FD test

FD methods grounded on SM approach check the intersection between the $\mathcal{Z}_k^{sm}$ and the strip $\mathcal{S}_k^s$ at each k iteration. If the intersection is empty, i.e., $\mathcal{Z}_k^{sm} \cap \mathcal{S}_k^s = \emptyset$ the existence of faults can be demonstrated. Otherwise, the method predicts a new zonotopic set approximating the resulting intersection if consistency test is proved. Algorithm 2 is used for fault detection. Analogously, the zonotope support strip is defined as:

Algorithm 2 Fault detection based on SM approach

$k \leftarrow 0$
$\mathcal{Z}_{k+1}^{sm} \leftarrow \theta_0$
while $k \leftarrow N$ **do**
  Obtain input-output u_k, y_k and build regressor H_k.
  Compute the strip $\mathcal{S}_k^s$ according to (13).
  if $\mathcal{S}_k^{sm} \cap \mathcal{S}_k^s = \emptyset$ **then**
    $Fault \leftarrow true$
  end if
  Compute $\mathcal{Z}_{k+1}^{sm} \supseteq \mathcal{Z}_k^{sm} \cap \mathcal{S}_k^s$ using (12) and (14).
  $k \leftarrow k + 1$
end while

$$\mathcal{S}_k^{sm} = \left\{ \theta_k \in \langle c_k, R_k \rangle : q_k^{sm_{low}} \le H_k \theta_k \le q_k^{sm_{upp}} \right\} \tag{23}$$

where $q_k^{sm_{low}}$ and $q_k^{sm_{upp}}$ are defined as the minimum and the maximum values of the zonotope support strip, respectively, which are determined as follows:

$$\begin{aligned} q_k^{sm_{low}} &= H_k c_k - \|R_k H_k\|_1 \\ q_k^{sm_{upp}} &= H_k c_k + \|R_k H_k\|_1 \end{aligned} \tag{24}$$

In the same way, $q_k^{s_{low}}$ and $q_k^{s_{upp}}$ are the minimum and maximum bounds of the strip $\mathcal{S}_k^s = \left\{ \theta_k \in \mathbb{R}^{\eta_\theta} : |H_k \theta_k - y_k| \le F_k \right\}$, expressed by:

$$\begin{aligned} q_k^{s_{low}} &= y_k - F_k \\ q_k^{s_{upp}} &= y_k + F_k \end{aligned} \tag{25}$$

Finally, the appearance of the fault is demonstrated when $\mathcal{S}_k^{sm} \cap \mathcal{S}_k^s = \emptyset$ or satisfying the conditions below:

$$\begin{aligned} q_k^{sm_{low}} &> q_k^{s_{upp}} \\ q_k^{sm_{upp}} &< q_k^{s_{low}} \end{aligned} \tag{26}$$

6.2 Minimum parametric fault magnitude f^θ

Following a similar procedure such as ZRLS, the minimum fault magnitude f^θ is obtained employing inequalities (26). Then, substituting $q_k^{sm_{upp}}$ from (24), and $q_k^{s_{low}}$ from (25) results in:

$$H_k c_k + \|R_k H_k\|_1 < y_k - F_k \tag{27}$$

Then, taking into account the worst-case for the parameter vector, a fault able to provide an effect greater than (28) will always be detected

$$H_{k,i} f_k^{\theta,i} > 2\|R_k H_k\|_1 + 2F_k \tag{28}$$

where i is the component of the regressor H_k associated with the i-th parametric fault.

6.3 Comparative analysis

In order to compare both ZRLS and SM approaches when applied to FD, a methodology based on performing a consistency test has been proposed. This procedure aims to check the intersection defined by the strip consistent with the measurements and the predicted zonotope or the zonotope support strip (see Algorithms 1 and 2). Specifically, the strip $\mathcal{S}^s$ is determined considering the bound F_k of the noise, according to (18) and (13), and the zonotope support strip is obtained by means of an optimization problem that results in (17) and (24) for ZRLS and SM, respectively. In addition, the minimum detectable fault has been considered for analysing the behaviour of both approaches. As can be noted, the expression to compute the minimum detectable parametric fault in the worst-case scenario is equivalent in both methods according to (22) and (28). Since the minimum detectable fault depends on the amplitude of the generator matrix R, therefore the convergence of both methods is similar. However, a slightly greater conservativeness degree is provided by ZRLS. Another interesting point to discuss is the sensitivity of the system to faults or external disturbances. In this sense, for detecting parametric faults in systems as (15), the sensitivity is characterized by the magnitude of the additive noise v_k, the amplitude of the predicted set and the criterion to determine the optimal observer gain. Roughly speaking, the noise is associated with the measurements or the sensor accuracy levels. Depending on the unknown degree of the noise, an appropriate threshold must be selected for F to avoid false alarms or missed detections. In practice, higher noise values lead to low sensitivity against faults and, therefore faults could be undetected; whereas lower noise magnitudes could induce false alarms. On the other hand, the amplitude of the zonotopic set can affect the sensitivity considering the minimum detectable fault in the worst-case (22) and (28) for both methods. Finally, concerning the influence of the observer gain on the sensitivity, in this work the gain was selected to guarantee parameter estimation convergence. However, the observer gain could be selected for optimizing fault detection instead of estimation accuracy. In (Pourasghar et al., 2019), the optimal gain observer is designed to achieve better performance in fault detection scenarios, but from the state estimations point of view.

7. SIMULATION RESULTS

A quadruple-tank process (Johansson, 2000) is considered to illustrate the proposed methodology. Due to space limitation, only the first tank is used for simulations. Thus, the mathematical model in continuous-time is obtained using mass-balance relations and Bernoulli's law as follows:

$$\frac{dh_{1,t}}{dt} = -\frac{\alpha_1}{A_1}\sqrt{2gh_{1,t}} + \frac{\alpha_3}{A_3}\sqrt{2gh_{3,t}} + \frac{\gamma_1\kappa_1}{A_1}\nu_{1,t} \tag{29}$$

where:

- A_i is the cross-section of tank i; $A_1 = A_3 = 28cm^2$.
- h_i is the water level of tank i.
- α_i is the cross-section area of the outlet pipe i; $\alpha_1 = \alpha_3 = 0.071cm^2$.
- γ_i depends on the i valve position (degree of opening); $\gamma_1 = 0.7$.
- $g = 981cm/s^2$ is the acceleration due to gravity.
- $\kappa_i\nu_i$ is the flow through of pump i; $\kappa = 3.33cm^3/Vs$.

On the other hand, the operating level range from both tanks are: $h_1 \in [2, 11]cm$ and $h_3 \in [1, 15]cm$.

7.1 Performing ZRLS and SM approaches for parameter identification

In a fault-free environment, $N = 140$ samples are stored to perform both approaches. Then, for determining the model in the regression form, equation (29) is discretized using the Euler method with sampling time $T_s = 1s$. Finally, making some manipulations and assuming α_1 and α_3 as unknown, (29) can be rewritten as:

$$\begin{aligned} y_k &= h_{1,k} - h_{1,k-1} - \frac{\gamma_1\kappa_1}{A_1}\nu_{1,k-1} \\ H_k &= \left[-\frac{1}{A_1}\sqrt{2gh_{1,k-1}},\ \frac{1}{A_3}\sqrt{2gh_{3,k-1}}\right] \\ \theta_k &= [\alpha_1, \alpha_3]^T \end{aligned} \tag{30}$$

where $h_{1,k}$ is corrupted by some noise υ which is bounded by $F = 0.044$. In addition, the predicted segment matrix R in ZRLS (6) is computed without using the reduction operator Red_q to avoid over-approximations and for achieving better accuracy in the comparison. Hence, the term $-K_kF_k$ will introduce one column at each iteration. Taking into account the initial conditions which are defined in (31), estimation results are shown in Figures 1-2. It is worth empathizing that confidence regions assuming the worst-case are determined employing Definition 3.

$$\begin{aligned} \text{ZRLS}: \ \langle c_0, R_0\rangle &= \langle [0, 0]^T, 2I_2\rangle,\ P_0 = 4I_2,\ F = 0.044 \\ \text{SM}: \ \langle c_0, R_0\rangle &= \langle [0, 0]^T, 2I_2\rangle,\ F = 0.044 \end{aligned} \tag{31}$$

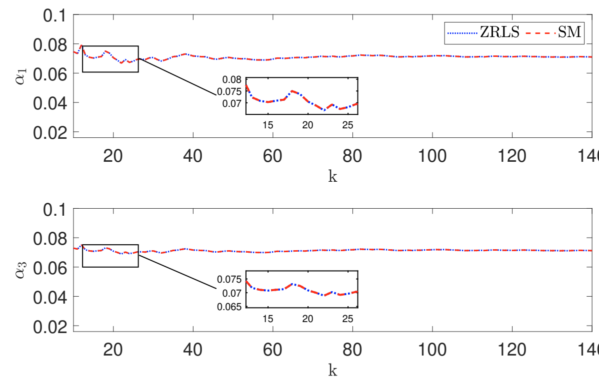

Fig. 1. Evolution of the estimated center into the parameter space for ZRLS and SM approaches.

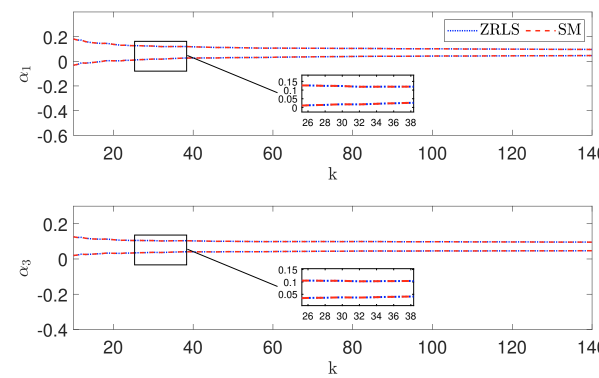

Fig. 2. Evolution of the confidence regions assuming the worst-case for ZRLS and SM approaches.

7.2 FD results

In this comparative assessment, faults in the estimated parameters α_1 and α_3 taking into account the minimum detectable fault in the worst-case are introduced. The expressions that characterize the minimum detectable parametric faults are defined in (22) and (28) for the ZRLS and SM methods, respectively. The resulting values are shown in Table 1 as well as the evolution of the minimum detectable fault during 140 samples is depicted in Figure 3 for parameter α_1. As it can be appreciated and due to the fact that generator matrices are similar, the minimum detectable fault obtained by performing both methods converges towards similar values. The proposed fault scenario consists in affecting the parameter α_1 at $k = 70$ instant by a slightly greater value than the minimum detectable; that is, $f^{\alpha_1} = 0.05\,cm^2$. Figure 4 shows the result of fault detection when the introduced fault is greater than the minimum detectable. As it can be seen, the strips do not intersect with the zonotopes, or specifically, the zonotope support strips. Hence, because of the

Table 1. Minimum detectable parametric faults (f^{α_1}) and (f^{α_3}) in the worst-case scenario.

	ZRLS	**SM**
f^{α_1}	$0.0416\,cm^2$	$0.0413\,cm^2$
f^{α_3}	$0.1287\,cm^2$	$0.1269\,cm^2$

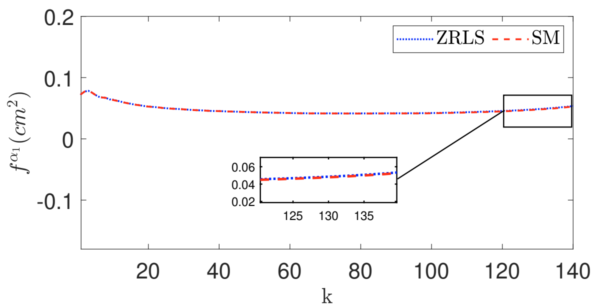

Fig. 3. Evolution of the minimum detectable parametric fault (f^{α_1}) in the worst-case.

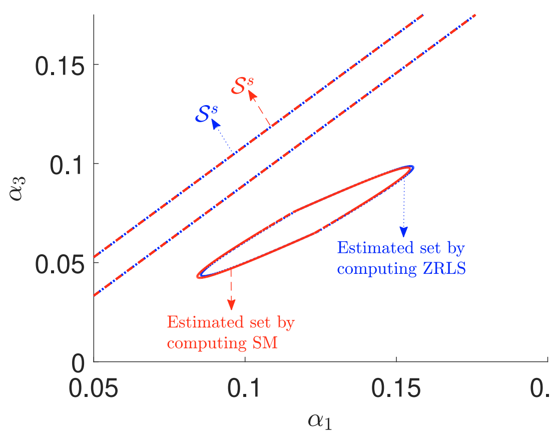

Fig. 4. Detected fault by performing ZRLS and SM approaches at instant $k = 70$. The fault added has a magnitude equivalent to $f^{\alpha_1} = 0.05\,cm^2$.

inconsistency, the fault is detected. The strip consistent with the measurements $\mathcal{S}^s$, the zonotope support strip $\mathcal{S}^z$ and the estimated set by computing ZRLS are plotted with blue colour; while $\mathcal{S}^s$, $\mathcal{S}^{sm}$ and the $\mathcal{Z}^{sm}$ obtained by executing SM are represented in red colour. It is worth noting that for faults bigger than the minimum detectable, the detection is always guaranteed.

8. CONCLUSIONS

This paper has addressed the problem of robust fault diagnosis using a zonotopic parameter estimation approach. First, system identification problem considering parametric uncertainty and using zonotopes has been investigated. Then, its application to robust fault detection is addressed. As a result, a Zonotopic Recursive Least Squares (ZRLS) estimator is proposed and compared in fault detection with the set-membership (SM) approach, taking as a reference the minimum detectable fault in the worst-case. The proposed estimation and fault detection methodologies have satisfactorily been assessed using a quadruple tank process.

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