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The Fractal Structure of the Universal Steenrod Algebra: An Invariant-theoretic Description

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Abstract

As recently observed by the second author, the mod 2 universal Steenrod algebra $\mathcal{Q}$ has a fractal structure given by a system of nested subalgebras $\mathcal{Q}_s$, for $s \in \mathbb{N}$, each isomorphic to $\mathcal{Q}$. In the present paper we provide an alternative presentation of the subalgebras $\mathcal{Q}_s$ through suitable derivations δ_s, and give an invariant-theoretic description of them.

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1 Preliminaries

The mod 2 universal Steenrod algebra $\mathcal{Q}$ is the $\mathbb{F}_2$-algebra generated by x_k, $k \in \mathbb{Z}$, together with $1 \in \mathbb{F}_2$, subject to the so-called *generalized Adem relations*:

$$R(k,n) = x_{2k-1-n}x_k + \sum_j \binom{n-1-j}{j} x_{2k-1-j}\, x_{k-n+j}, \tag{1.1}$$

for $k \in \mathbb{Z}$ and $n \in \mathbb{N}_0$.

First appeared in [16], such algebra is isomorphic to the algebra of cohomology operations in the category of H_∞-ring spectra (see [7], Ch. 3 and 8). Together with its odd p analogue $Q(p)$, the universal Steenrod algebra Q has been extensively studied, among others, by the authors ([1]-[6], [8]-[10], [11]-[14]).

Let $\lambda : \mathcal{Q} \to \mathcal{Q}$ be the algebra homomorphism defined by

$$\lambda(1) = 1 \quad \text{and} \quad \lambda(x_h) = x_{2h-1}.$$

Its s-th iterated map λ^s maps x_h onto $x_{2^s(h-1)+1}$. In [8], the second author proved that λ is a monomorphism of algebras. Furthermore, the subalgebras $\mathcal{Q}_s = \lambda^s(\mathcal{Q})$ have the following presentation:

$$\mathcal{Q}_s = \langle \ \{x_{2^s h+1}\}_{h\in\mathbb{Z}} \quad | \quad R(2^s t + 1, 2^s n) = 0 \ \rangle. \tag{1.2}$$

2 A derivation on $\mathcal{Q}_s$

Let $d : \mathcal{Q} \to \mathcal{Q}$ be the derivation given by $d(x_k) = x_{k-1}$.

In [11], Lomonaco proved that $\mathcal{Q}$ is isomorphic to the algebra generated by the set $1 \cup \{x_k\}_{k\in\mathbb{Z}}$ with relations $d^n(x_{2k-1}x_k) = 0$ for $n \in \mathbb{N}_0$, where d^0 is the identity, and

$$d^n = \underbrace{d \circ \cdots \circ d}_{n \text{ times}} \qquad \text{for } n > 0.$$

To compute the action of some particular d^n's on monomials of length 2, we need the following Lemma.

Lemma 2.1. *Let p be any prime. For any non-negative integers a, b and s, the following congruential identity holds:*

$$\binom{p^s a}{p^s b} \equiv \binom{a}{b} \bmod p.$$

Proof. Once you write a and b as $\sum_{i=0}^m a_i p^i$ and $\sum_{i=0}^m b_i p^i$, $(0 \le a_i, b_i < p)$ respectively, then

$$\binom{b}{a} \equiv \prod_{i=0}^{m} \binom{b_i}{a_i} \quad \mathrm{mod}\, p \tag{2.1}$$

(see [18], I 2.6). From (2.1), where, as usual, $\binom{0}{0} = 1$, our Lemma follows quite easily. □

Proposition 2.2. *For any $(h,k) \in \mathbb{Z} \times \mathbb{Z}$ and $(s,n) \in \mathbb{N}_0 \times \mathbb{N}_0$,*

$$d^{2^s n}(x_h x_k) = \sum_{j=0}^{n} \binom{n}{j} x_{h-2^s j}\, x_{k-2^s(n-j)}. \tag{2.2}$$

Proof. Being d a derivation, a straightforward argument shows that

$$d^{\ell}(x_h x_k) = \sum_{j=0}^{\ell} \binom{\ell}{j} x_{h-j}\; x_{k-\ell+j}. \tag{2.3}$$

When $\ell = 2^s n$, Equation (2.3) becomes

$$d^{2^s n}(x_h x_k) = \sum_{l=0}^{2^s n} \binom{2^s n}{l} x_{h-l}\; x_{k-2^s n+l}. \tag{2.4}$$

By Equation (2.1), it also follows that

$$\binom{2^s n}{l} \equiv 0 \bmod 2 \quad \text{for any } l \not\equiv 0 \bmod 2^s.$$

Thus the non-zero coefficients in the sum 2.4 possibly occur when $l = 2^s j$ for some $0 \le j \le n$. Hence

$$d^{2^s n}(x_h x_k) = \sum_{j=0}^{n} \binom{2^s n}{2^s j} x_{h-2^s j}\; x_{k-2^s n+2^s j}.$$

We now invoke Lemma 2.1 for $p = 2$ to end the proof. □

For any $s \in \mathbb{N}_0$ we set $\delta_s = d^{2^s}$. We get

$$\delta_s(x_h) = x_{h-2^s} = d^{2^s}(x_h), \tag{2.5}$$

and, according to Proposition 2.2 for $n = 1$,

$$\delta_s(x_h x_k) = x_{h-2^s} x_k + x_h x_{k-2^s} = \delta_s(x_h) x_k + x_h \delta_s(x_k). \tag{2.6}$$

Equation (2.6) shows that δ_s is another derivation on $\mathcal{Q}$. Its n-iterated δ_s^n acts as follows.

$$\delta_s^n : x_h \in \mathcal{Q} \longmapsto x_{h-2^s n} \in \mathcal{Q}, \tag{2.7}$$

and

$$\delta_s^n(x_h x_k) = d^{2^s n}(x_h x_k) = \sum_{j=0}^{n} \binom{n}{j} x_{h-2^s j}\; x_{k-2^s (n-j)}, \tag{2.8}$$

by Proposition 2.2.

Equations (2.5) and (2.6) tell us that the restriction of δ_s to $\mathcal{Q}_s$ yields to a derivation on $\mathcal{Q}_s$. Such restriction (that, abusing notation, will be again denoted by δ_s) allows us to get a new presentation of the algebra $\mathcal{Q}_s$.

Proposition 2.3. *For any $(s, n) \in \mathbb{N} \times \mathbb{N}_0$, the following diagram commutes*

$$\begin{array}{ccc} \mathcal{Q} & \xrightarrow{d^n} & \mathcal{Q} \\ \downarrow{\scriptstyle \lambda^s} & & \downarrow{\scriptstyle \lambda^s} \\ \mathcal{Q}_s & \xrightarrow{\delta_s^n} & \mathcal{Q}_s \end{array}$$

Proof. The statement is trivial for $n = 0$. When $n = 1$, note that $\lambda^s \circ d$ and $\delta_s \circ \lambda^s$ both map the monomial $x_{i_1} \cdots x_{i_m}$ onto

$$x_{2^s(i_1-2)+1} x_{2^s(i_2-1)+1} \cdots x_{2^s(i_m-1)+1} + \cdots + x_{2^s(i_1-1)+1} x_{2^s(i_2-1)+1} \cdots x_{2^s(i_m-2)+1},$$

hence $\lambda^s \circ d = \delta_s \circ \lambda^s$. We now use induction on n:

$$\lambda^s \circ d^n = \lambda^s \circ d \circ d^{n-1} = \delta_s \circ \lambda^s \circ d^{n-1} = \delta_s \circ \delta_s^{n-1} \circ \lambda^s.$$

□

Theorem 2.4. *A full set of generating relations for* $\mathcal{Q}_s$ *is*

$$\{ P_{n,t} \mid (n,t) \in \mathbb{N}_0 \times \mathbb{Z} \},$$

where $P_{n,t}$ *is the polynomial on the right side of Equation* (2.8) *obtained by setting* $h = 2^{s+1}t + 1$ *and* $k = 2^s t + 1$.

Proof. It has been proved in [11] that the set

$$\{ P'_{\ell,t} \mid (\ell,t) \in \mathbb{N}_0 \times \mathbb{Z} \},$$

of polynomials on the right side of Equation (2.3) obtained by setting $h = 2t - 1$ and $k = t$ is a set of generating relations for $\mathcal{Q}$.

Since λ^s is a monomorphism (see [8]), a set of generating relations in Q_s is

$$\{ \lambda^s(P'_{\ell,t}) \mid (\ell,t) \in \mathbb{N}_0 \times \mathbb{Z} \},$$

We now use Proposition 2.3:

$$\lambda^s(P'_{\ell,t}) = \lambda^s(d^\ell(x_{2t-1}x_t)) = \delta_s^\ell(\lambda^s(x_{2t-1}x_t)) = P_{\ell,t-1}.$$

□

3 Invariant-theoretic description

Let $\Gamma_n = \mathbb{F}_2[Q_{n,0}^{\pm 1}, Q_{n,1}, \ldots, Q_{n,n-1}]$ be the Dickson algebra with the Euler class $Q_{n,0}$ inverted. The generators $Q_{n,i}$ can be defined inductively in terms of elements in $\Delta_s = \mathbb{F}_2[v_1^{\pm 1}, \ldots v_s^{\pm 1}]$, which is also a ring of invariants (see [15] and [17]). In particular,

$$Q_{2,0} = v_1^2 v_2, \qquad Q_{2,1} = v_1^2 + v_1 v_2. \tag{3.1}$$

In [11], Lomonaco proved that $\mathcal{Q}$ is isomorphic to $\Delta/(\Gamma_2)$, where Δ is the algebra obtained by taking the graded vector space $\bigoplus_{s \geq 0} \Delta_s$ (here $\Delta_0 = \mathbb{F}_2$), and endowing it with the following multiplication

$$\mu : v_1^{i_1} \cdots v_h^{i_h} \otimes v_1^{j_1} \cdots v_k^{j_k} \in \Delta_h \otimes \Delta_k \longmapsto v_1^{i_1} \cdots v_h^{i_h} v_{h+1}^{j_1} \cdots v_{h+k}^{j_k} \in \Delta_{h+k}.$$

An isomorphism is given by

$$f : x_{i_1} \cdots x_{i_n} \in \mathcal{Q} \longmapsto [v_1^{i_1-1} \cdots v_n^{i_n-1}] \in \Delta/(\Gamma_2),$$

where $[v]$ stands for the coset represented by $v \in \Delta$. One of the key points is that f maps the polynomial $P'_{n,t}$ (which is 0 in $\mathcal{Q}$) onto

$$\left[\sum_{i=0}^{n} \binom{n}{i} v_1^{2t-2-i} \, v_2^{t-n+i-1} \right]$$

which is represented by $Q_{2,0}^{t-n-1} Q_{2,1}^{n} \in \Gamma_2$.

As explicitly shown in [15], the following diagram

$$\begin{array}{ccc} \mathcal{Q} & \xrightarrow{\lambda} & \mathcal{Q} \\ \downarrow f & & \downarrow f \\ \Delta/(\Gamma_2) & \xrightarrow{\tilde{\psi}} & \Delta/(\Gamma_2) \end{array} \tag{3.2}$$

is commutative. In diagram (3.2), $\tilde{\psi}$ is induced on the quotient by

$$\psi : v_1^{i_1} \cdots v_h^{i_h} \in \Delta \longmapsto v_1^{2i_1} \cdots v_h^{2i_h} \in \Delta.$$

It follows that $f \circ \lambda^s$ is equal to $\tilde{\psi}^s \circ f$ for any positive integer s.

Our aim is to identify $f(\mathcal{Q}_s)$ inside $\Delta/(\Gamma_2)$. The element

$$(f \circ \lambda^s)(d^n(x_{2h-1}x_h)) = (\psi^s \circ f)(d^n(x_{2h-1}x_h))$$

is represented by $Q_{2,0}^{2^s(h-n-1)} Q_{2,1}^{2^s n}$. Further,

$$(f \circ \lambda^s)(d^n(x_{2h-1}x_h)) = f(\delta_s^n(\lambda^s(x_{2h-1}x_h))) = f(\delta_s^n(x_{2^{s+1}(h-1)+1}x_{2^s(h-1)+1})).$$

Set

$$\Delta^s = \bigoplus_{k\geq 0} \Delta_k^s = \bigoplus_{k\geq 0} \mathbb{F}_2[v_1^{\pm 2^s}, v_2^{\pm 2^s}, \ldots, v_k^{\pm 2^s}], \qquad \Gamma_2^s = \mathbb{F}_2[Q_{2,0}^{\pm 2^s}, Q_{2,1}^{2^s}],$$

and note that $\psi^s(Q_{2,0}) = Q_{2,0}^{2^s}$ and $\psi^s(Q_{2,1}) = Q_{2,1}^{2^s}$ (it immediately follows from (3.1)).

Theorem 3.1. $\mathcal{Q}_s \cong \Delta^s/(\Gamma_2^s)$.

Proof. Since λ^s is a monomorphism, by the commutativity of the diagram (3.2), $\tilde{\psi}^s$ is a monomorphism as well, and $Im(\tilde{\psi}^s) = \Delta^s/(\Gamma_2^s)$. So $\tilde{\psi}^s$ sets an isomorphism between $\Delta/(\Gamma_2)$ and $\Delta^s/(\Gamma_2^s)$. By the commutative diagram, $\mathcal{Q}_s \cong Im(\lambda^s) \cong Im(\tilde{\psi}^s) = \Delta^s/(\Gamma_2^s)$. □

We could reword Theorem 3.1 by saying that the map f establishes a correspondence between the descending chain of subalgebras

$$\mathcal{Q} = \mathcal{Q}_0 \supset \mathcal{Q}_1 \supset \cdots \supset \mathcal{Q}_{s-1} \supset \mathcal{Q}_s \supset \ldots$$

and the chain

$$\Delta/(\Gamma_2) \supset \Delta^1/(\Gamma_2^1) \supset \cdots \supset \Delta^{s-1}/(\Gamma_2^{s-1}) \supset \Delta^s/(\Gamma_2^s) \supset \cdots.$$

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