myjournal manuscript No.
(will be inserted by the editor)

Chasing Non-diagonal Cycles in a Certain System of Algebras of Operations

Maurizio Brunetti · Luciano A. Lomonaco

Received: date / Accepted: date

Abstract The mod 2 universal Steenrod algebra Q is a non-locally finite homogeneous quadratic algebra closely related to the ordinary mod 2 Steenrod algebra and the Lambda algebra. The algebra Q provides an example of a Koszul algebra which is a direct limit of a family of certain non-Koszul algebras R_k's. In this paper we see how far the several R_k's are to be Koszul by chasing in their cohomology non-trivial cocycles of minimal homological degree.

Mathematics Subject Classification (2010) 16S37, 55S10, 18G15

Keywords Koszul algebras, cohomology of algebras, cohomology operations

1 Introduction

S. Priddy introduced the notion of Koszul algebra in [18] to construct resolutions for the Steenrod algebra and study the universal enveloping algebra of a Lie algebra. Ever since, the theory of Koszul algebras never stopped to attract the interest of people working in several areas of mathematics. Koszul algebras arise in fact in commutative and non-commutative algebraic geometry, representation theory, number theory, combinatorics and algebraic topology. The now nine years old [17] gives a beautiful account on the dramatic impact of Koszul algebras in the theory of quadratic algebras, and literature already offers several attempts to generalize the koszulness condition to non-quadratic algebras (see, for instance, [1] and [8]).

In this paper we deal with homogeneous graded augmented algebras A isomorphic to a quotient of the form $T(V)/J(R)$, where $T(V) = \bigoplus_i T_i$ is the tensor algebra over a $\mathbb{K}$-vector space V with basis $X = \{x_i \mid i \in \mathcal{I}\}$, $\mathcal{I}$ is a subset of $\mathbb{Z}$, $J(R)$ is the two-sided ideal of relations generated by some $R \subset T_2 = V \otimes V$, and the augmentation ϵ acts as follows

$$\epsilon : x_k \in T_1 \longmapsto 0 \in \mathbb{F}_2 \;\; \forall\, k \in \mathbb{Z}, \quad \text{and} \quad \epsilon : 1 \in T_0 = \mathbb{K} \longmapsto 1.$$

M. Brunetti and L. A. Lomonaco
Department of Mathematics and Applications, University of Naples Federico II, via Cintia, I-80126 Naples, Italy.
E-mail: mbrunett@unina.it, lomonaco@unina.it

Once we assign to each monomial of type $x_{i_1}\cdots x_{i_k}$ length k and *internal degree* $i_1+\cdots+i_k$, and suppose R homogeneous with respect to the internal degree, the tensor algebra $T(V)$ and the algebra $A \cong T(V)/J(R)$ become bigraded, and its cohomology

$$H^{i,j,k}(A) = \mathrm{Ext}_A^{i,j,k}(\mathbb{K},\mathbb{K})$$

trigraded, the homological degree being i. The diagonal cohomology

$$D^{j,*}(A) = \oplus H^{j,j,*}(A)$$

is in general a subalgebra of $H(A)$. Among the several (and equivalent when A is finitely generated) definition of koszulness we choose the so-called *diagonal purity*:

Definition 1 A homogeneous quadratic algebra A is said to be *Koszul* if

$$H(A) = D(A).$$

The mod 2 universal Steenrod algebra Q is the algebra generated by $\{y_i \,|\, i \in \mathbb{Z}\}$ subject to the so-called generalized Adem relations:

$$y_{2k-1-n}\, y_k = \sum_j \binom{n-1-j}{j} y_{2k-1-j}\, y_{k+j-n} \quad (k \in \mathbb{Z},\ n \in \mathbb{N}_0). \tag{1.1}$$

The algebra Q first appeared in [16], and it is isomorphic to the algebra of cohomology operations in the category of H_∞-ring spectra (see [7], Ch. 3 and 8). Together with its odd p analogue $Q(p)$, the universal Steenrod algebra Q has been extensively studied, among others, by the authors ([2]-[6], [9]-[10], [12]-[14]). In particular, it has been proved that Q and $Q(p)$ are Koszul algebras in [5] and [2], respectively.

The koszulness of Q has not been worked out by using tools described in [11] and [18], since they are not suitable for non-locally finite algebras like Q. We rather handled a system of locally finite quadratic algebras $\{R_k, \phi_k\}$ proving that

$$Q \cong \varinjlim\{\, R_k, \phi_k \,\}, \quad \text{and} \quad H(Q) \cong \varprojlim\{H(R_k), \phi_k^*\}. \tag{1.2}$$

In fact, as it turned out, $\{R_k, \phi_k\}$ satisfies the Mittag-Leffler condition, and the non-zero elements in $H^{s,t,*}(R_k)$ with $s \neq t$ do not give any contribution to $H(Q)$. In this paper, we show that the algebras R_k's are not Koszul for $k \geq 3$. Hence the algebra Q turns out to be an example of a Koszul algebra which is a direct limit of non-Koszul algebras. Such curious phenomenon makes worthy to search for non-diagonal classes in $H^{s,t,*}(R_k)$ of minimal homological degree, and see how many steps they can be pushed backward in the inverse system (1.2). In a sense that we'll make precise in the next section, such classes appears as soon as possible. In fact we shall prove the following Theorem.

Theorem 1 *For all $k \geq 3$, the graded vector $\mathbb{F}_2$-space $\mathrm{Ext}_{R_k}^{3,4,*}(\mathbb{F}_2,\mathbb{F}_2)$ has positive dimension.*

The material is organized as follows. In Section 2, we include some classical results on quadratic algebras, recall the definition of the algebras R_k's and study the properties of a certain $\mathbb{F}_2$-basis picked in each of them. In Section 3, we give the proof of Theorem 1 split in several Lemmas and Propositions. Finally, Section 4 collects some open problems.

Despite the theoretical importance of ordinary and higher Massey products to generate non-diagonal cohomology classes (see [15]), such machinery has not been used here, since in our context it is quite hard to see whether a fixed Massey product is trivial or not.

2 PBW bases and the algebras R_k's

Let A be a graded augmented homogeneous algebra as in Section 1, and $p : T(V) \longrightarrow A$ be the quotient map (of augmented algebras). If

$$X = \{x_i \,|\, i \in \mathcal{I}\} \quad \text{with } \mathcal{I} \subseteq \mathbb{Z}$$

is a set of generators of V, then obviously the elements $a_i = p(x_i)$ generate A_+, the kernel of the augmentation $\epsilon : A \to \mathbb{K}$. For multi-indexes $I = (i_1, \dots, i_n) \in \mathcal{I}^n$ and $J = (j_1, \dots, j_m) \in \mathcal{I}^m$, we write (I, J) for the multi-index $(i_1, \dots, i_n, j_1, \dots, j_m)$, and denote by a_I the monomial $a_{i_1} \cdots a_{i_n}$, the one corresponding to the empty multi-index being 1. Any subset of $\cup \mathcal{I}^k$ is totally ordered by length first, and then by lexicographical order.

To a fixed basis of monomials $\mathcal{B}$ for A, we associate the following set of multi-indexes

$$S_{\mathcal{B}} = \{I \,|\, a_I \in \mathcal{B}\}.$$

Definition 2 A basis of monomials $\mathcal{B}$ for A is a *Poincaré-Birkhoff-Witt (PBW) basis* if the following conditions hold.
1. For any I and J in $S_{\mathcal{B}}$ such that $a_I a_J \neq 0$ and $(I, J) \notin \mathcal{S}_{\mathcal{B}}$, the multi-index of each monomial appearing in the expression of $a_I a_J$ in terms of elements of $\mathcal{B}$ is greater than (I, J).
2. For any $k > 2$ and $(i_1, \dots, i_k) \in S_{\mathcal{B}}$, the multi-indexes $(i_1, \dots, i_j)$ and $(i_{j+1}, \dots, i_k)$ are in $S_{\mathcal{B}}$ for each $j \in \{1, \dots, k-1\}$.
3. If $(i_1, \dots, i_j)$ and $(i_j, \dots, i_k)$ are in $S_{\mathcal{B}}$, then $(i_1, \dots, i_k)$ is in $S_{\mathcal{B}}$ as well.

As proved by the second author in [12], The algebra Q has a PBW linear basis made by 1 and the *admissible monomials*

$$\{y_{i_1} \cdots y_{i_h} \mid i_j \geq 2i_{j+1} \ \text{ for each } j = 1, \dots, h-1\}. \tag{2.1}$$

In [5], the authors introduced a system of locally finite quadratic algebras $\{R_k, \phi_k\}$ defined as follows. The algebra R_k is generated by

$$y_k, y_{k-1}, y_{k-2}, \dots,$$

and its ideal of relations is generated by

$$P_{2i-1-n,i} = y_{2i-1-n} y_i - \sum_j \binom{n-1-j}{j} y_{2i-1-j} y_{i+j-n} \tag{2.2}$$

with $\max\{i, 2i-1\} \leq k$. The map $\phi_k : R_k \longrightarrow R_{k+1}$ of the system maps $y_I \in R_k$ onto the homonymous monomial in R_{k+1}. Such maps are not all inclusions. In fact it can be proved that the dimension of $\operatorname{Ker} \phi_{2h}$ is positive for $h > 0$, for example $\operatorname{Ker} \phi_2$ contains $y_2 y_2 y_1$ which is non-zero in R_2. On the contrary the maps ϕ_{2h+1} are all monomorphisms. For $h > 0$ this comes as a consequence of Lemma 2 below.

For the rest of the paper we shall adopt a nomenclature resembling the ordinary Steenrod algebra: each $P_{a,b}$ in (2.2) will be called an *Adem relation*, while a monomial $y_{i_1} \cdots y_{i_k}$ (and its multi-index as well) will be said to be *admissible* if $i_h \geq 2i_{h+1}$.

To recall the recursive procedure explained in [5] to define a basis $\mathcal{B}_k$ for R_k, and reproduced almost verbatim below, we gives the following Definition.

Definition 3 A non-admissible monomial $y_{i_1} y_{i_2} \cdots y_{i_\ell}$ in R_k is said to be *exceptional* if $i_{h+1} > (k+1)/2$ for any non-admissible couple (i_h, i_{h+1}).

By Definition 3, it immediately follows that R_k contains exceptional monomials if and only if $k \geq 2$.

In $\mathcal{B}_k$ we first include 1 and the set $\mathcal{C}_k$ of admissible monomials. Fixed a length l and an internal degree m such that $R_k^{l,m} \setminus \operatorname{Span}(\mathcal{C}_k^{l,m})$ is not empty, we pick in $\mathcal{B}_k$ an exceptional monomial c_1 with maximal label in $R_k^{l,m}$. This is possible since, by Lemma 3.1 in [5], the number of monomials in $R_k^{l,m}$ is finite. We now iterate the procedure. If

$$R_k^{l,m} \setminus \operatorname{Span}(\mathcal{C}_k^{l,m} \cup \{c_1, ..., c_{t-1}\})$$

is not empty, then we pick the monomial c_t of maximal label out, and put it in $\mathcal{B}_k^{l,m}$. Monomials in $\mathcal{C}_k^{l,m} \cup \{c_1, ..., c_t\}$ are independent by construction, hence we get a basis for $R_k^{l,m}$ in a finite number of steps. Finally $\mathcal{B}_k = \cup \mathcal{B}_k^{l,m}$.

Since the main tool in the proof of Theorem 1 consists in finding suitable exceptional monomials not belonging to $\mathcal{B}_k$, it is worthy to examine some properties of such basis.

Proposition 1 *Let $k > 2$. For any exceptional monomial $y_i y_j y_\ell$ in R_k not belonging to $\mathcal{B}_k$, the monomial $y_i y_j$ is exceptional, while $y_j y_\ell$ is admissible and involved in at least one Adem relation.*

Proof A priori, the only other possibility for an exceptional $y_i y_j y_\ell$ to be written as sum of monomials of greater label is that $y_i y_j$ is admissible and involved in at least one Adem relation, and $y_j y_\ell$ is exceptional. This case, however, cannot occur, since in any non-trivial relation

$$y_i y_j y_\ell = \sum y_{i_h} y_{j_h} y_{\ell_h},$$

all the ℓ_h's should be necessarily equal to ℓ, and there exists at least one summand with $y_{i_h} y_{j_h}$ neither admissible or exceptional.

Lemma 1 *Let $k > 1$. In each relation in R_k involving both admissible monomials and exceptional monomials, the minimal label is exceptional.*

Proof Consider in $R_k^{l,m}$ a relation

$$y_{I_1} + \cdots + y_{I_s} + y_{J_1} + \cdots + y_{J_t} = 0,$$

where the y_I's are admissible, the y_J's are exceptional, and none of them can be cancelled out.

If the integer $\bar{k}$ is sufficiently large, we may assume that in $R_{\bar{k}}$ there are all the generalized Adem relations required to write each

$$\phi_{\bar{k}-1} \circ \cdots \circ \phi_k(y_{J_h})$$

as a sum $c_1^h + \cdots + c_{t_h}^h$, where $c_j^h \in \mathcal{C}_{\bar{k}}$ for any $j = 1, \ldots, t_h$. Monomials in this sum have a label greater than J_h, since in every generalized Adem relation the minimal label belongs to the non-admissible monomial. By definition $\mathcal{C}_{\bar{k}} \subseteq \mathcal{B}_{\bar{k}}$, hence every $\phi_{\bar{k}-1} \circ \cdots \circ \phi_k(y_{I_i})$ in the equality

$$\sum_{i=1}^{s} \phi_{\bar{k}-1} \circ \cdots \circ \phi_k(y_{I_i}) + \sum_{h=1}^{t} (c_1^h + \cdots + c_{t_h}^h) = 0$$

has to be cancelled out. This is only possible if, for each i, there exists a label J_h less than I_i.

Proposition 2 *The basis $\mathcal{B}_k$ satisfies Condition 1 and 2 of Definition 2.*

Proof Consider two monomials $y_I = y_{i_1} \cdots y_{i_s}$ and $y_J = y_{j_1} \cdots y_{j_t}$ in $\mathcal{B}_k$ such that $y_I y_J$ is not zero and not belonging to $\mathcal{B}_k$.

If $y_I y_J$ is exceptional, then it is a sum of admissible monomials and exceptional monomials of greater label by construction of $\mathcal{B}_k$. Otherwise y_I and y_J are both admissible and the Adem relation P_{i_s, j_1} is available in R_k. In this case we can argue as in the proof of Theorem 3.1 in [19], ch. 1. We start to apply a generalized Adem relation to $y_{i_s} y_{j_1}$. Thus we get $y_I y_J$ as a sum of monomials of greater label and smaller *moment*. Applying again a suitable generalized Adem relation to every non-admissible pair which comes out, we eventually succeed in writing $y_I y_J$ as sum of admissibles and possibly exceptionals of greater label.

Condition 2 is immediately verified for admissible monomials. When a monomial is instead exceptional, validity of Condition 2 comes quite easily from Lemma 1.

Proposition 3 *For $k > 2$ the algebra R_k does not admit any PBW basis.*

Proof Let $\mathcal{B}$ be any basis of R_k containing only monomials. By definition of R_k, it follows that $\mathcal{B}$ contains every monomial of length 2 which is admissible or exceptional. Let now r be the integral part of $(k+1)/2$. If r is even, consider in R_k the monomial $y_{2r-1} y_{r+1}$ and $y_{r+1} y_{r/2}$. Both are in $\mathcal{B}$, since the former is exceptional, and the latter is admissible; but $y_{2r-1} y_{r+1} y_{r/2}$ does not belong to $\mathcal{B}$, in fact

$$y_{2r-1}(y_{r+1} y_{r/2}) = y_{2r-1}(y_r y_{1+r/2}) = 0 \cdot y_{1+r/2} = 0.$$

Thus $\mathcal{B}$ does not satisfy Condition 3 of Definition 2. When r is odd, $\mathcal{B}$ is not PBW for the same reason: one could check that

$$y_{2r-1} y_{r+2} y_{(r-1)/2} = y_{2r-1} y_r y_{(r+3)/2} = 0.$$

Lemma 2 *For $\ell > 0$, the algebra $R_{2\ell+2}$ is isomorphic to the free product $R_{2\ell+1} \sqcup \mathbb{F}_2[y_{2\ell+2}]$, i.e. the algebra freely generated by $R_{2\ell+1}$ and $\mathbb{F}_2[y_{2\ell+2}]$.*

Proof Since the multi-index $(2\ell+2, 2\ell+2)$ is exceptional in $R_{2\ell+2}$, and $\dim R^{s,s(2\ell+2)}_{2\ell+2}$ is 1, necessarily $y^s_{2\ell+2}$ belongs to $\mathcal{B}_{2\ell+2}$ for every $s \geq 0$. Hence $\mathbb{F}_2[y_{2\ell+2}]$ is a subalgebra of $R_{2\ell+2}$. The statement now follows from the fact that no Adem relation available in $R_{2\ell+2}$ involves $y_{2\ell+2}$.

Proposition 4 *The algebras R_k's are all Koszul for $k \leq 2$.*

Proof For $k \leq 1$, there are no exceptional monomials, hence $\mathcal{B}_k$ is made only by admissibles. This in particular implies that $\mathcal{B}_k$ is a PBW basis, and the existence of a PBW basis is a sufficient condition for a locally finite algebra to be Koszul. In fact, the argument along the proof of Theorem 5.2 in [18] can be adapted to any locally finite algebra. (see also Theorem 3.1 in [17]). Let now $k = 2$. By Lemma 2, R_2 is isomorphic to the free product of two Koszul algebras, hence it is itself Koszul (see, for instance Proposition 1.1 in [17], ch. 3).

Our chase of non-diagonal trivial classes in $H^{i,i+j}(R_k)$ for $k > 2$ and $j > 0$ starts with the following Proposition.

Proposition 5 $H^{1,1+j}(R_k) = 0$ *and* $H^{2,2+j}(R_k) = 0$ *for all $j > 0$ and for all k.*

Proof This is essentially Corollary 5.3 in [17], ch. 1. Note that the algebras R_k are all quadratic and 1-generated in the sense of [17], p. 6.

Proposition 5 implies that the minimal possible homological degree for non-diagonal cohomology classes of R_k is 3, and Theorem 1 precisely says that for $k \geq 3$ such minimum is actually achieved. In other words neither of the algebras R_k's for $k \geq 3$ is 4-Koszul in the sense of [17], p. 29.

We end this section by summarizing some algebraic features of the Adem relations repeatedly used in the next section. The omitted proof just relies on the arithmetics of the indices and the mod 2 binomial coefficients in (2.2).

Proposition 6 *i) No Adem relation* (2.2) *contains monomials of type $y_{2s}y_s$ among its summands.*
ii) $P_{2s-1,s} = y_{2s-1}y_s$.
iii) $P_{2s-1-2^n,s} = y_{2s-1-2^n}y_s - y_{2s-1}y_{s+2^n}$ *for all $n \geq 0$.*

Obviously, the Adem relations in Parts ii) and iii) of Proposition 6 are available in R_k when $k \geq \max\{s, 2s-1\}$.

3 Proof of Theorem 1

For every Adem relation $P_{a,b}$ available in R_k, we shall often denote by $F_{(a,b)}$ its admissible part. In other words,

$$P_{a,b} = y_a y_b - F_{(a,b)}.$$

The element in the cobar construction corresponding to $y_I \in \mathcal{B}_k \setminus \{1\}$ will be denoted by α_I.

Proposition 7 *For every $h \geq 0$ the graded $\mathbb{F}_2$-vector space $H^{3,4,*}(R_{8h+3})$ is non-zero.*

Proof Consider the cochain $\beta(8h+3)$ equal to the sum

$$\alpha_{8h+3}\,\alpha_{(4h+3,2h+1)}\,\alpha_{h+1} + \alpha_{(8h+3,4h+3)}\,\alpha_{2h+1}\,\alpha_{h+1} + \alpha_{8h+3}\,\alpha_{4h+2}\,\alpha_{(2h+2,h+1)}.$$

We have

$$\delta\alpha_{(4h+3,2h+1)} = \alpha_{4h+3}\,\alpha_{2h+1} + \alpha_{4h+2}\,\alpha_{2h+2}.$$

Furthermore $\delta\alpha_{(8h+3,4h+3)}$ is simply equal to $\alpha_{8h+3}\,\alpha_{4h+3}$, since $y_{8h+3}\,y_{4h+3}$ is exceptional, and

$$\delta\alpha_{(2h+2,h+1)} = \alpha_{2h+2}\,\alpha_{h+1} \quad \text{by Proposition 6, i).}$$

It immediately follows that $\beta(8h+3)$ is a cocycle. To see that $\beta(8h+3)$ is not a coboundary, note that for any cochain γ of homological degree 2 and length 4, the monomial $\alpha_{8h+3}\,\alpha_{(4h+3,2h+1)}\,\alpha_{h+1}$ never appears as top term among the summands of $\delta\gamma$, since $y_{8h+3}\,y_{4h+3}\,y_{2h+1} = 0$ (see the proof of Proposition 3), and $y_{4h+3}\cdot(y_{2h+1}\,y_{h+1})$ is null by Proposition 6, ii).

To prove the non-koszulness of the algebras R_{8h+5} for $h \geq 0$, we need some Lemmas. Moreover, the case $k = 5$ has to be managed separately.

Lemma 3 *The exceptional monomials of length 3 not belonging to $\mathcal{B}_5$ are*

$$y_s y_5 y_{2t-1} \qquad \textit{for } t \leq 1 \textit{ and } s \leq 4t+1,$$

and

$$y_s y_4 y_{2t-1} \qquad \textit{for } t \leq 1 \textit{ and } s \leq 4t-1.$$

Proof By Lemma 1, an exceptional monomial not in $\mathcal{B}_5$ has either the form $y_a y_5 y_b$ or $y_a y_4 y_b$.The only way to write them as sum of admissibles and exceptionals of greater label is to have them involved in the equality

$$F_{(s,c)} y_3 = y_s F_{(c,3)} \qquad \text{for } s \leq 2c-1. \tag{3.1}$$

Once we write each non-admissible monomials in (3.1) as sum of admissibles and exceptionals, it is not hard to show that, the monomial of lowest multi-index will be $y_s y_5 y_{c-2}$ if c is odd, and $y_s y_4 y_{c-1}$ if c is even.

Proposition 8 *There exists a non-trivial class in $H^{3,4,*}(R_5)$.*

Proof Since $\delta\alpha_{(5,1)} = \alpha_5\alpha_1 + \alpha_3\alpha_3$, the element

$$\beta(5) = \alpha_5\alpha_{(5,1)}\alpha_3 + \alpha_{(5,5)}\alpha_1\alpha_3 + \alpha_5\alpha_3\alpha_{(5,1)} + \alpha_5\alpha_{(3,5)}\alpha_1$$

is a cocycle. By Lemma 3, $y_5 y_5 y_1$ does not belong to $\mathcal{B}_5$, and $y_5 y_1 y_3$ is neither admissible nor exceptional. This implies that $\beta(5)$ is not a coboundary.

Lemma 4 *Let $h > 0$. The monomial $y_{8h} y_{4h+6} y_{2h}$ in R_{8h+5} is not in $\mathcal{B}_{8h+5}$.*

Proof Consider the relation

$$F_{(8h,4h+2)}y_{2h+4} = y_{8h}F_{(4h+2,2h+4)}. \tag{3.2}$$

When we write the polynomial on the first side as sum of admissibles and exceptional, the lowest label is $J = (8h+2, 4h+4, 2h)$. On the right side, we find

$$y_{8h}y_{4h+7}y_{2h-1} + y_{8h}y_{4h+6}y_{2h} + y_{8h}y_{4h+5}y_{2h+1}.$$

The third monomial is also equal to

$$F_{(8h,4h+3)}y_{2h+3},$$

whose lowest multi-index is in any case bigger than J. Replacing $y_{8h}y_{4h+5}y_{2h+1}$ by $F_{(8h,4h+3)}y_{2h+3}$ in (3.2), we succeed in writing $y_{8h}y_{4h+6}y_{2h}$ as sum of admissibles and exceptionals of greater label.

Lemma 5 *The only Adem relation* (2.2) *containing* $y_{4h+6}y_{2h+2}$ *among its summands is* $P_{4h+4,2h+4}$. *The monomial* $y_{4h+6}y_{2h+0}$ *is instead involved in just two Adem relations, namely* $P_{4h+2,2h+4}$ *and* $P_{4h,2h+6}$.

Proof Among the Adem relations $P_{i,j}$ of internal degree $6h+6$ that possibly involve $y_{4h+6}y_{2h+2}$, the one with maximal label (i, j) is $P_{4h+4,2h+4}$, which actually contains $y_{4h+6}y_{2h+2}$ among its summands. Note now that

$$F_{(4h+4-t,2h+4+t)} = \sum_j \binom{3t+2-j}{j} y_{4h+7+2t-j}\, y_{2h+1-2t+j},$$

and the binomial coefficient corresponding to $j = 2t+1$ is non-zero only for $t = 0$. A similar argument proves the second part of the statement.

Proposition 9 *For* $h > 0$, *the* $\mathbb{F}_2$*-graded vector space* $H^{3,4,*}(R_{8h+5})$ *is non-zero.*

Proof We have seen by Lemma 4 that $y_{8h}y_{4h+6}y_{2h} \in R_{8h+5}$ is not in $\mathcal{B}_{8h+5}$, and the monomial $y_{4h+6}y_{2h}y_{h+2}$ is neither admissible nor exceptional. This implies that the cochain

$$\begin{aligned}\beta(8h+5) = {} & \alpha_{8h}\,\alpha_{(4h+6,2h)}\,\alpha_{h+2} + \alpha_{(8h,4h+6)}\,\alpha_{2h}\,\alpha_{h+2} \\ & +\alpha_{8h}\,\alpha_{4h+2}\,\alpha_{(2h+4,h+2)} + \alpha_{(8h,4h)}\,\alpha_{2h+6}\,\alpha_{h+2}\end{aligned}$$

is not a coboundary. To see that $\beta(8h+5)$ is a cocycle use the fact that

$$\delta\alpha_{(4h+6,2h)} = \alpha_{4h+6}\,\alpha_{2h} + \alpha_{4h+2}\,\alpha_{2h+4} + \alpha_{4h}\,\alpha_{2h+6}$$

that follows from Lemma 5.

Proposition 10 *For every* $h \geq 0$ *the graded* $\mathbb{F}_2$*-vector space* $H^{3,4,*}(R_{8h+7})$ *is non-zero.*

Proof Note first that $y_{8h+4}\, y_{4h+6}\, y_{2h+2}$ does not belong to $\mathcal{B}_{8h+7}$. It is in fact the lowest monomial in the equality

$$y_{8h+4} F_{(4h+4,2h+4)} = F_{(8h+4,4h+4)} y_{2h+4}.$$

This proves that the cochain

$$\begin{aligned}\beta(8h+7) = {} & \alpha_{8h+4}\, \alpha_{(4h+6,2h+2)}\, \alpha_{h+2} \\ & + \alpha_{(8h+4,4h+6)}\, \alpha_{2h+2}\, \alpha_{h+2} + \alpha_{8h+4}\, \alpha_{4h+4}\, \alpha_{(2h+4,h+2)}\end{aligned}$$

is not a coboundary.

To prove that $\beta(8h+7)$ represents a non-trivial class in in $\mathrm{Ext}^{3,4}_{R_{8h+7}}(\mathbb{F}_2, \mathbb{F}_2)$ is essential to know that

$$\delta\alpha_{(4h+6,2h+2)} = \alpha_{4h+6}\alpha_{2h+2} + \alpha_{4h+4}\alpha_{2h+4}, \quad \text{and} \quad \delta\alpha_{(2h+4,h+2)} = \alpha_{2h+4}\, \alpha_{h+2}$$

which come from Lemma 5 and Proposition 6 i), respectively.

Proposition 11 *For every $h > 0$, $H^{3,4,*}(R_{8h+1})$ is non-trivial.*

Proof In the relation

$$F_{(8h,4h+1)} y_{2h+2} = y_{8h} F_{(4h+1,2h+2)}$$

the monomial $y_{8h} y_{4h+3} y_{2h}$ can be written as sum of admissible and exceptional monomials of greater label, hence it is not in $\mathcal{B}_{8h+1}$. This in particular implies that the cochain

$$\begin{aligned}\beta(8h+1) = {} & \alpha_{8h}\, \alpha_{(4h+3,2h)}\, \alpha_{h+1} + \alpha_{(8h,4h+3)}\, \alpha_{2h}\, \alpha_{h+1} \\ & + \alpha_{8h}\, \alpha_{4h+1}\, \alpha_{(2h+2,h+1)} + \alpha_{(8h,4h)}\, \alpha_{2h+3}\, \alpha_{h+1},\end{aligned}$$

is not a coboundary. To see that $\delta\beta(8h+1)$ is zero, use the fact that

$$\delta\alpha_{(2s,s)} = \alpha_{2s}\, \alpha_s, \quad \text{and} \quad \delta\alpha_{(2s+3,s)} = \alpha_{2s+1}\, \alpha_{s+2} + \alpha_{2s}\, \alpha_{s+3} \tag{3.3}$$

for all $s \in \mathbb{Z}$; you also need that $\delta\alpha_{(8h,4h+3)} = \alpha_{8h}\, \alpha_{4h+3}$ since $y_{8h}\, y_{4h+3}$ is exceptional in R_{8h+1}.

So far, we have proved that $H(R_k)$ has at least a non-trivial cohomology class in homological degree 3 and length 4 for any odd $k \geq 3$. Therefore the proof of Theorem 1 ends with Proposition 12 below. Following [17], p. 25, in the next statement we denote by $A \sqcap B$ the direct sum of two graded algebras A and B. Here and in Section 4 we write $\tilde{\beta}$ to denote the cohomology class represented by the cochain β.

Proposition 12 *For any $\ell > 0$, $H(R_{2\ell})$ is isomorphic to*

$$H(R_{2\ell-1}) \sqcap \Lambda[\tilde{\alpha}_{2\ell}],$$

where $\Lambda[\tilde{\alpha}_{2\ell}]$ is the exterior algebra on the class in $H^{1,1,2\ell}(R_{2\ell})$ represented by the dual of $y_{2\ell}$.

Proof Immediate from Lemma 2 and Proposition 1.1 in [17], ch. 3.

4 Open problems

The cohomology of the algebras R_k's for $k \geq 3$ is far from being fully understood. Recursion in defining $\mathcal{B}_k$ prevents us from stating results in the vein of Lemma 3 for bigger k's and larger lengths, it also inhibits any effort to give a complete description of the cohomology rings, and leaves many questions open.

For instance, we could ask whether the non-diagonal part of $H(R_k)$ is nilpotent. The answer is surely negative for $k = 3$ and $k = 5$, since the cochains

$$\xi(3) = \alpha_{(1,3)}\alpha_0\alpha_1 + \alpha_1\alpha_{(3,0)}\alpha_1 + \alpha_1\alpha_1\alpha_{(2,1)}$$

and

$$\xi(5) = \alpha_{(1,5)}\alpha_1\alpha_1\alpha_1 + \alpha_1\alpha_{(5,1)}\alpha_1\alpha_1 + \alpha_1\alpha_3\alpha_{(3,1)}\alpha_1 + \alpha_1\alpha_3\alpha_1\alpha_2\alpha_{(2,1)}$$

represent torsion-free classes in $H^{3,4}(R_3)$ and in $H^{4,5}(R_5)$ respectively. If follows in particular, that $H^{3n,4n}(R_3)$ and $H^{4n,5n}(R_5)$ are non-zero for any $n \geq 0$. On the other hand, classes $\tilde{\beta}(k) \in H^{3,4}(R_k)$ have nilpotency degree 2. For instance, the cochain $\beta(3)^2$ represents 0 in $H^{6,8}(R_3)$, being hit by

$$\begin{array}{lll} \alpha_3\alpha_{(3,1)}\alpha_{(1,3,3)}\alpha_1\alpha_1 & +\alpha_{(3,3)}\alpha_1\alpha_{(1,3,3)}\alpha_1\alpha_1 & +\alpha_3\alpha_{(3,1)}\alpha_{(1,3)}\alpha_{(3,1)}\alpha_1 \\ +\alpha_{(3,3)}\alpha_1\alpha_{(1,3)}\alpha_{(3,1)}\alpha_1 & +\alpha_3\alpha_{(3,1)}\alpha_{(1,3)}\alpha_2\alpha_{(2,1)} & +\alpha_{(3,3)}\alpha_1\alpha_{(1,3)}\alpha_2\alpha_{(2,1)} \\ +\alpha_3\alpha_2\alpha_{(2,1,3)}\alpha_{(3,1)}\alpha_1 & +\alpha_3\alpha_2\alpha_{(2,1,3,3)}\alpha_1\alpha_1 & +\alpha_3\alpha_2\alpha_{(2,1,3)}\alpha_2\alpha_{(2,1)}. \end{array}$$

Another open question concerns the existence of a maximal homological degree for the generators of the cohomology algebras. In any case, for any $k > 2$ we could provide examples of classes in $H^{4,5}(R_k)$ that are not product of elements of lower homological degree.

A final question concerns how the "resistance" of each non-diagonal class in the inverse system (1.2) is related to its degrees and k. For example the class $\tilde{\beta}(8h+3)$ can be pulled back just once, i.e. it is not in $\mathrm{Im}(\phi^*_{8h+4} \circ \phi^*_{8h+3})$; $\tilde{\beta}(8h + 1)$ and $\tilde{\beta}(8h + 7)$ resist for 3 steps, while the cycle $\tilde{\beta}(8h + 5)$ can be pulled back 5 times when $h > 0$.

Acknowledgements

This research has been carried out as part of "Programma STAR", financially supported by UniNA and Compagnia di San Paolo.

References

1. R. Berger, *Koszulity for nonquadratic algebras.* J. Algebra **239** (2001), no. 2, 705–734.
2. M. Brunetti, A. Ciampella, *A Priddy-type Koszulness criterion for non-locally finite algebras*, Colloq. Math. **109** (2007), no. 2, 179–192.
3. M. Brunetti, A. Ciampella, L. A. Lomonaco, *An embedding for the E_2-term of the Adams spectral sequence at odd primes*, Acta Math. Sin. (Engl. Ser.) **22** (2006), no. 6, 1657–1666.
4. M. Brunetti, A. Ciampella, L. A. Lomonaco, *Homology and cohomology operations in terms of differential operators*, Bull. Lond. Math. Soc. **42** (2010), no. 1, 53–63.

5. M. Brunetti, A. Ciampella, L. A. Lomonaco, *The cohomology of the universal Steenrod algebra*, Manuscripta Math., **118** (2005), no. 3, 271–282.
6. M. Brunetti, L. A. Lomonaco, *An embedding for the E_2-term of the Adams spectral sequence*, Ricerche Mat. **54** (2005), no. 1, 185200.
7. R. R. Bruner, J. P. May, J. E. McClure and M. Steinberger, *H_∞ ring spectra and their applications.* Lecture Notes in Math., **1176**, Springer, Berlin, 1986.
8. T. Cassidy and B. Shelton, *Generalizing the notion of Koszul algebra*, Math. Z. **260** (2008), no. 1, 93–114.
9. A. Ciampella, L. A. Lomonaco, *Homological computations in the universal Steenrod algebra*, Fund. Math. **183** (2004), no. 3, 245–252.
10. A. Ciampella, L. A. Lomonaco, *The universal Steenrod algebra at odd primes*, Comm. Algebra **32** (2004) no. 7, 2589–2607.
11. C. Löfwall, *On the subalgebra generated by the one-dimensional elements in the Yoneda Ext-algebra.* Algebra, algebraic topology and their interactions (Stockholm, 1983), 291–338, Lecture Notes in Math., **1183**, Springer, Berlin, (1986).
12. L. A. Lomonaco, *A basis of admissible monomials for the universal Steenrod algebra*, Ricerche Mat. **40** (1991), 137–147.
13. L. A. Lomonaco, *A phenomenon of reciprocity in the universal Steenrod algebra*, Trans. Amer. Math. Soc. **330** (1992), no. 2, 813–821.
14. L. A. Lomonaco, *The diagonal cohomology of the universal Steenrod algebra*, J. Pure Appl. Algebra **121** (1997), no. 3, 315–323.
15. J. P. May, *The cohomology of augmented algebras and generalized Massey products for DGA-algebras* Trans. Amer. Math. Soc. **122** (1966), 334–340.
16. J. P. May, *A general algebraic approach to Steenrod operations.* The Steenrod algebra and its applications. Lecture Notes in Math., **168**, Springer, Berlin, 153–231, (1970).
17. A. Polishchuk and L. Positselski, *Quadratic algebras.* University Lecture Series, 37. American Mathematical Society, Providence, RI, (2005).
18. S. B. Priddy, *Koszul resolutions.* Trans. Amer. Math. Soc. **152** (1970), 39–60.
19. N. E. Steenrod and D. B. A. Epstein, *Cohomology operations.* Annals of Mathematics Studies, No. 50 Princeton University Press, Princeton, N.J. (1962).