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THERMAL EFFECTS IN ANISOTROPIC POROUS ELASTIC RODS

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This research is concerned with the thermoelastic deformation of a porous anisotropic right cylinder subjected to a thermal field independent of the axial coordinate. The case of a material with a plane of elastic symmetry which contains the axis of cylinder is considered. The solution of the problem is expressed in terms of solutions of some generalized plane strain problems. It is shown that the temperature field produces extension, torsion, and a plane strain. For this kind of anisotropy, an infinitesimal twist produces a variation of volume fraction field. The method is used to study the deformation of an inhomogeneous circular cylinder.

Keywords: Anisotropic solids; Generalized plane strain; Porous elastic bodies; Thermal stresses in rods

INTRODUCTION

There has been much recent interest in the study of porous elastic materials. In [1, 2], Cowin and Nunziato developed a theory of elastic materials with voids. The intended applications of this theory are to elastic bodies with vacuous pores which are distributed throughout material. In [3], Eringen introduced a special class of bodies with microstructure, characterized by an isotropic microdeformation tensor. In this case the material particles undergo a uniform microdilatation. In the absence of microrotations the linear equations which describe the behaviour of an elastic body with this kind of microstructure coincide with the equations of elastic materials with voids established by Nunziato and Cowin in [1, 2]. In what follows we shall refer to this model as a porous elastic continuum. The theory of elastic materials with voids was investigated in various works (see, e.g., [4–9] and the references therein). In recent years there has been some interest in the study of anisotropic porous materials (see, e.g., [10–15]).

In this paper we consider the linear elastostatics of anisotropic porous materials. We study the deformation of an inhomogeneous and anistropic right cylinder subjected to a temperature field which is independent of the axial

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coordinate. In the case of orthotropic porous elastic cylinders, a plane temperature field produces no torsional effect [16]. We also note that if the medium has a plane of elastic symmetry, normal to the axis of cylinder, then a temperature field which is independent of the axial coordinate does not induce a torsion of the cylinder. Moreover, we note that an infinitesimal twist produces no variation of porosity (see, e.g., [16–18]). These aspects led us to investigate the case when the medium has a plane of elastic symmetry which contains the axis of cylinder. In this paper we show that for this kind of anisotropy a temperature field which is independent of the axial coordinate produces torsional effects. Moreover, the torsion of the cylinder induces a variation of volume fraction field. In recent years, research activity on functionally graded materials has stimulated the interest in porous of inhomogeneous elasticity (see, e.g., [19, 20] and the references therein). In this paper the analysis is carried out for a subclass of inhomogeneous materials where the constitutive coefficients are independent of the axial coordinate. The solution of the problem is expressed in terms of solutions of some generalized plane strain problems. In the next section we present the basic equations and the formulation of the boundary value problem. Then we study the generalized isothermal plane strain problem. We introduce four special generalized plane strain problems characterized by external data which depend only on the constitutive coefficients. The solutions of these problems will be used to solve the thermoelastic problem. In the following section we solve the problem of a right cylinder subjected to a plane temperature field and to extension, bending and torsion. We have introduced mechanical loads in order to compare the effects of temperature field with those produced by the resultants which act on the ends. The three-dimensional problem is reduced to the study of some two-dimensional problems. The solution is used to study the deformation of an inhomogeneous circular cylinder subjected to a prescribed thermal field.

STATEMENT OF THE PROBLEM

We consider a body that in its undeformed state occupies the regular region B of Euclidean three-dimensional space and is bounded by the piecewise smooth surface ∂B. The deformation of the body is referred to the undeformed state and a fixed system of rectangular cartesian axes Ox_k. We designate by n_i the components of the outward unit normal of ∂B. We shall employ the usual summation and differentiation conventions: Greek subscripts are understood to range over the integers (1,2) whereas Latin subscripts (unless otherwise specified) to the range (1,2,3); summation over repeated subscripts is implied and subscripts preceded by a comma denote partial differentiation with respect to the corresponding cartesian coordinate. We assume that the region B is occupied by a linearly porous elastic material. Let u_i be the components of the displacement vector field over B. Then, the linear strain measure e_{ij} is defined by

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{1}$$

We denote by t_{ij} the stress tensor and by h_k the equilibrated stress vector. The surface force and the equilibrated surface force at a regular point of ∂B, are given by

$$t_i = t_{ji}n_j, \quad h = h_k n_k$$

respectively.

We stipulate that the region B from here on refers to the interior of a right cylinder of length l with the open cross-section Σ and the lateral boundary S. The rectangular cartesian coordinate frame is supposed to be chosen in such a way that the x_3-axis is parallel to the generators of B and the x_1, x_2-plane contains one of the terminal cross-sections. We denote by Σ_1 and Σ_2, respectively, the cross-section located at $x_3 = 0$ and $x_3 = l$. We assume that the generic cross-section is a simply connected regular region. We denote L the boundary of Σ.

Throughout this discussion we assume that the material has a plane of elastic symmetry which contains the axis of cylinder. Let x_2Ox_3 be the plane of elastic symmetry. In this case the constitutive equations of a thermoelastic solid are

$$\begin{aligned}
t_{11} &= C_{11}e_{11} + C_{12}e_{22} + C_{13}e_{33} + 2C_{14}e_{23} + \beta_1\varphi - b_1T \\
t_{22} &= C_{12}e_{11} + C_{22}e_{22} + C_{23}e_{33} + 2C_{24}e_{23} + \beta_2\varphi - b_2T \\
t_{33} &= C_{13}e_{11} + C_{23}e_{22} + C_{33}e_{33} + 2C_{34}e_{23} + \beta_3\varphi - b_3T \\
t_{23} &= C_{14}e_{11} + C_{24}e_{22} + C_{34}e_{33} + 2C_{44}e_{23} + \beta_4\varphi - b_4T \\
t_{31} &= 2C_{55}e_{13} + 2C_{56}e_{12}, \quad t_{12} = 2C_{56}e_{13} + 2C_{66}e_{12} \\
h_1 &= \alpha_1\varphi_{,1}, \quad h_2 = \alpha_2\varphi_{,2} + \alpha_4\varphi_{,3}, \quad h_3 = \alpha_4\varphi_{,2} + \alpha_3\varphi_{,3} \\
\gamma &= -\beta_1e_{11} - \beta_2e_{22} - \beta_3e_{33} - 2\beta_4e_{23} - \xi\varphi + mT
\end{aligned} \tag{2}$$

where φ is the volume fraction field, T is the temperature measured from the absolute temperature of the natural state, g is the intrinsic equilibrated body force, and $C_{mn}(m, n = 1, 2, \ldots, 6)$, β_j, α_k, b_k, $(k = 1, 2, 3, 4), m$ and ξ are constitutive coefficients. We assume that T is a prescribed function, independent of the axial coordinate,

$$T = f(x_1, x_2), \quad (x_1, x_2) \in \Sigma_1 \tag{3}$$

In the absence of body loads, the equations of equilibrium are

$$t_{jk,j} = 0, \quad h_{j,j} + g = 0 \tag{4}$$

We suppose that the cylinder is free from lateral loading. On the lateral surface of the cylinder we have the conditions

$$t_{\alpha i}n_\alpha = 0, \quad h_\alpha n_\alpha = 0 \quad \text{on } S \tag{5}$$

We assume that the cylinder B is subjected to extension, bending and torsion. We introduce the mechanical loads in order to compare the effects of temperature field with those produced by the resultants which act on the ends. Let $\mathbf{R} = (0, 0, R_3)$

and $\mathbf{M} = (M_1, M_2, M_3)$ be prescribed vectors representing the resultant force and the resultant moment about O of the tractions acting on Σ_1. On Σ_2 there are tractions applied so as to satisfy the equilibrium conditions of the body. Consequently, for $x_3 = 0$ we have the conditions

$$\int_{\Sigma} t_{\alpha 3} da = 0 \tag{6}$$

$$\int_{\Sigma_1} t_{33} da = -R_3, \quad \int_{\Sigma_1} x_\alpha t_{33} da = \varepsilon_{\alpha\beta 3} M_\beta, \quad \int_{\Sigma_1} \varepsilon_{\alpha\beta 3} x_\alpha t_{\beta 3} da = -M_3 \tag{7}$$

where ε_{ijk} is the alternating symbol. We note that there is no contribution of the equilibrated surface force in the resultant force and resultant moment (cf. [21]).

The problem consists in finding of the functions u_i and φ of class $C^2(B) \cap C^1(\overline{B})$ which satisfy the equations (1), (2) and (4) on B, the conditions (5) on the lateral surface S, and the conditions (6) and (7) on the end Σ_1.

We consider inhomogeneous materials for which the constitutive coefficients are independent of the axial coordinate,

$$\begin{aligned} C_{rs} &= \widetilde{C}_{rs}(x_1, x_2), \quad \alpha_k = \tilde{\alpha}_k(x_1, x_2), \quad \beta_j = \tilde{\beta}_j(x_1, x_2) \\ b_k &= \hat{b}_k(x_1, x_2), \quad m = \widehat{m}(x_1, x_2), \quad \xi = \tilde{\xi}(x_1, x_2) \end{aligned} \tag{8}$$

$(x_1, x_2) \in \Sigma$, $(r, s = 1, 2, \ldots, 6; k = 1, 2, 3, 4)$. We suppose that the constitutive coefficients and temperature field are prescribed functions of class C^∞, and that the domain Σ is C^∞-smooth.

By an admissible state on B we mean an ordered array of functions $s = (u_i, \varphi, e_{ij}, t_{ij}, h_i, g)$ with the properties $u_i, \varphi \in C^2(B) \cap C^1(\overline{B})$, $e_{ij} \in C^1(B) \cap C^0(\overline{B})$, $e_{ij} = e_{ji}$, $t_{ji}, h_i \in C^1(B) \cap C^0(\overline{B})$, $t_{jk,j}, h_{j,j}$, $g \in C^0(\overline{B})$, $t_{ij} = t_{ji}$. We say that $s = (u_i, \varphi, e_{ij}, t_{ij}, h_i, g)$ is an elastic state on B if s is an admissible state that satisfies the equations (1)–(3) on B. We define the elastic potential $W(s)$ associated to s by

$$\begin{aligned} 2W(s) = {} & C_{11} e_{11}^2 + 2C_{12} e_{11} e_{22} + 2C_{13} e_{11} e_{33} + 4C_{14} e_{11} e_{23} \\ & + 2\beta_1 e_{11} \varphi + C_{22} e_{22}^2 + 2C_{23} e_{22} e_{33} + 4C_{24} e_{22} e_{23} + 2\beta_2 e_{22} \varphi \\ & + C_{33} e_{33}^2 + 4C_{34} e_{23} e_{33} + 2\beta_3 \varphi e_{33} + 4C_{44} e_{23}^2 + 4\beta_4 e_{23} \varphi \\ & + 4C_{55} e_{13}^2 + 8C_{56} e_{12} e_{13} + 4C_{66} e_{12}^2 + \alpha_1 (\varphi_{,1})^2 + \alpha_2 (\varphi_{,2})^2 \\ & + 2\alpha_4 \varphi_{,2} \varphi_{,3} + \alpha_3 (\varphi_{,3})^2 + \xi \varphi^2 \end{aligned} \tag{9}$$

We consider two elastic states $s^{(\alpha)} = (u_i^{(\alpha)}, \varphi^{(\alpha)}, t_{ij}^{(\alpha)}, h_i^{(\alpha)}, g^{(\alpha)})$, $(\alpha = 1, 2)$, on B and introduce the notations

$$\begin{aligned} 2W(s^{(\alpha)}, s^{(\beta)}) = {} & C_{11} e_{11}^{(\alpha)} e_{11}^{(\beta)} + C_{12} (e_{11}^{(\alpha)} e_{22}^{(\beta)} + e_{11}^{(\beta)} e_{22}^{(\alpha)}) \\ & + C_{13} (e_{11}^{(\alpha)} e_{33}^{(\beta)} + e_{11}^{(\beta)} e_{33}^{(\alpha)}) + 2C_{14} (e_{11}^{(\alpha)} e_{23}^{(\beta)} + e_{11}^{(\beta)} e_{23}^{(\alpha)}) \\ & + \beta_1 (e_{11}^{(\alpha)} \varphi^{(\beta)} + e_{11}^{(\beta)} \varphi^{(\alpha)}) + C_{22} e_{22}^{(\alpha)} e_{22}^{(\beta)} + C_{23} (e_{22}^{(\alpha)} e_{33}^{(\beta)} \\ & + e_{22}^{(\beta)} e_{33}^{(\alpha)}) + 2C_{24} (e_{22}^{(\alpha)} e_{23}^{(\beta)} + e_{22}^{(\beta)} e_{23}^{(\alpha)}) + \beta_2 (e_{22}^{(\alpha)} \varphi^{(\beta)} + e_{22}^{(\beta)} \varphi^{(\alpha)}) \end{aligned}$$

$$
\begin{aligned}
&+ C_{33}e_{33}^{(\alpha)}e_{33}^{(\beta)} + 2C_{34}(e_{23}^{(\alpha)}e_{33}^{(\beta)} + e_{23}^{(\beta)}e_{33}^{(\alpha)}) + \beta_3(\varphi^{(\alpha)}e_{33}^{(\beta)} \\
&+ \varphi^{(\beta)}e_{33}^{(\alpha)}) + 4C_{44}e_{23}^{(\alpha)}e_{23}^{(\beta)} + 2\beta_4(e_{23}^{(\alpha)}\varphi^{(\beta)} + e_{23}^{(\beta)}\varphi^{(\alpha)}) \\
&+ 4C_{55}e_{13}^{(\alpha)}e_{13}^{(\beta)} + 4C_{56}(e_{12}^{(\alpha)}e_{13}^{(\beta)} + e_{12}^{(\beta)}e_{13}^{(\alpha)}) + 4C_{66}e_{12}^{(\alpha)}e_{12}^{(\beta)} \\
&+ \alpha_1\varphi_{,1}^{(\alpha)}\varphi_{,1}^{(\beta)} + \alpha_2\varphi_{,2}^{(\alpha)}\varphi_{,2}^{(\beta)} + \alpha_4(\varphi_{,2}^{(\alpha)}\varphi_{,3}^{(\beta)} + \varphi_{,2}^{(\beta)}\varphi_{,3}^{(\alpha)}) \\
&+ \alpha_3\varphi_{,3}^{(\alpha)}\varphi_{,3}^{(\beta)} + \xi\varphi^{(\alpha)}\varphi^{(\beta)} \qquad (10)
\end{aligned}
$$

It is easy to see that

$$2W(s^{(\alpha)}, s^{(\beta)}) = t_{ij}^{(\alpha)}e_{ij}^{(\beta)} + h_k^{(\alpha)}\varphi_{,k}^{(\beta)} - g^{(\alpha)}\varphi^{(\beta)} \qquad (11)$$

If we take into account the equations (1), (4), (11) and the divergence theorem, then we find that

$$2\int_B W(s^{(\alpha)}, s^{(\beta)})dv = \int_{\partial B}(t_{ji}^{(\alpha)}u_i^{(\beta)} + h_j^{(\alpha)}\varphi^{(\beta)})n_j da \qquad (12)$$

In view of (9) and (10) we obtain

$$W(s^{(\alpha)}, s^{(\beta)}) = W(s^{(\beta)}, s^{(\alpha)}), \quad W(s, s) = W(s) \qquad (13)$$

Clearly, (11) and (12) imply the relation

$$\int_{\partial B}(t_{ji}^{(\alpha)}u_i^{(\beta)} + h_j^{(\alpha)}\varphi^{(\beta)})n_j da = \int_{\partial B}(t_{ji}^{(\beta)}u_i^{(\alpha)} + h_j^{(\beta)}\varphi^{(\alpha)})n_j da \qquad (14)$$

GENERALIZED PLANE STRAIN PROBLEM

Throughout this section we consider the isothermal theory ($T = 0$) and assume that a body force $\mathbf{f}$ and an extrinsic equilibrated force p are given on B. We suppose that on the lateral surface S are prescribed the surface force $\tilde{\mathbf{t}}$ and the equilibrated surface force $\tilde{h}$. We consider that $\mathbf{f}, p, \tilde{\mathbf{t}}$ and $\tilde{h}$ are all independent of the axial coordinate, and that $\mathbf{f}$ and $\tilde{\mathbf{t}}$ are parallel to the x_1, x_2-plane. Let $s = (u_i, \varphi, e_{ij}, t_{ij}, h_j, g)$ be an admissible state on B. We say that s is an isothermal state of generalized plane strain if

$$u_j = u_j(x_1, x_2), \quad \varphi = \varphi(x_1, x_2), \quad T = 0, (x_1, x_2) \in \Sigma_1 \qquad (15)$$

The relations (15), (1) and (3) show that the functions e_{ij}, t_{ij}, h_i and g are independent of the axial coordinate. The constitutive equations reduce to

$$
\begin{aligned}
t_{11} &= C_{11}u_{1,2} + C_{12}u_{2,2} + C_{14}u_{3,2} + \beta_1\varphi \\
t_{22} &= C_{12}u_{1,1} + C_{22}u_{2,2} + C_{24}u_{3,2} + \beta_2\varphi \\
t_{33} &= C_{13}u_{1,1} + C_{23}u_{2,2} + C_{34}u_{3,2} + \beta_3\varphi \\
t_{23} &= C_{14}u_{1,1} + C_{24}u_{2,2} + C_{44}u_{3,2} + \beta_4\varphi \\
t_{31} &= C_{55}u_{3,1} + C_{56}(u_{1,2} + u_{2,1})
\end{aligned}
$$

$$t_{12} = C_{56}u_{3,1} + C_{66}(u_{1,2} + u_{2,1})$$
$$h_1 = \alpha_1\varphi_{,1}, \quad h_2 = \alpha_2\varphi_{,2}, \quad h_3 = \alpha_4\varphi_2$$
$$g = -\beta_1 u_{1,1} - \beta_2 u_{2,2} - \beta_4 u_{3,2} - \xi\varphi \tag{16}$$

In the case of a generalized plane strain the equations of equilibrium are

$$t_{\alpha i,\alpha} + f_i = 0, \quad h_{\alpha,\alpha} + g + p = 0 \quad \text{on } \Sigma_1 \tag{17}$$

The conditions on the lateral surface S are given by

$$t_{\alpha i}n_\alpha = \tilde{t}_i, \quad h_\alpha n_\alpha = \tilde{h} \quad \text{on } L \tag{18}$$

We suppose that f_i, p, $\tilde{t}_i$ and $\tilde{h}$ are prescribed functions of class C^∞. The generalized plane strain problem consists in finding the functions u_i and φ on Σ_1 which satisfy the equations (16) and (17) on Σ_1 and the conditions (18) on L. We note that the plane strain of isotropic porous elastic solids has been investigated in [21].

In the generalized plane strain the elastic potential is

$$\begin{aligned} 2W_0 = {} & C_{11}e_{11}^2 + 2C_{12}e_{11}e_{22} + 4C_{14}e_{11}e_{23} + 2\beta_1 e_{11}\varphi \\ & + C_{22}e_{22}^2 + 4C_{24}e_{22}e_{23} + 2\beta_2 e_{22}\varphi + 4C_{44}e_{23}^2 + 4\beta_4 e_{23}\varphi \\ & + 4C_{55}e_{13}^2 + 8C_{56}e_{12}e_{13} + 4C_{66}e_{12}^2 + \alpha_1(\varphi_{,1})^2 + \alpha_2(\varphi_{,2})^2 + \xi\varphi^2 \end{aligned} \tag{19}$$

Let

$$u_\alpha^0 = c_\alpha + \varepsilon_{3\beta\alpha}c_4 x_\beta, \quad u_3^0 = c_3, \quad \varphi^0 = 0$$

where c_k, $(k = 1, 2, 3, 4)$ are arbitrary constants. Then, $\omega^0 = (u_1^0, u_2^0, u_3^0, \varphi^0)$ is called a plane rigid vector field.

Theorem 1. *If the elastic potential W_0 is a positive definite quadratic form, then any two solutions of the generalized plane strain problem* (15)–(18) *are equal modulo a plane rigid vector field.*

Proof. In view of (16), (17) and (19), we get

$$2W_0 = t_{i\alpha}u_{i,\alpha} + h_\alpha\varphi_{,\alpha} - g\varphi = (t_{\alpha i}u_i + h_\alpha\varphi)_{,\alpha} + f_i u_i + p\varphi \tag{20}$$

With the help of the divergence theorem we obtain

$$2\int_{\Sigma_1} W_0 da = \int_L (t_{\alpha i}u_i + h_\alpha)n_\alpha ds + \int_{\Sigma_1}(f_i u_i + p\varphi)da \tag{21}$$

We assume that the boundary value problem (16)–(18) has two solutions (u_i', φ') and (u_i'', φ''). We denote $u_i^* = u_i' - u_i''$, $\varphi^* = \varphi' - \varphi''$. The functions u_i^* and φ^* satisfy the generalized plane strain problem corresponding to null body loads and null tractions on L. Thus, from (21) we find that

$$\int_{\Sigma_1} W^* da = 0 \tag{22}$$

where W^* is the elastic potential corresponding to u_i^* and φ^*. Since W^* is positive definite, we find that $u^*_{\beta,\alpha} + u^*_{\alpha,\beta} = 0$, $u^*_{3,\alpha} = 0$ and $\varphi^* = 0$. Clearly, $(u_i^*, u_2^*, u_3^*, \varphi^*)$ is a plane rigid vector field. □

We suppose for the remainder of this paper that the elastic potential is positive definite. Following Fichera [22] we can prove the following result.

Theorem 2. *The boundary value problem* (16)–(18) *has solutions belonging to* $C^\infty(\Sigma_1)$ *if and only if the relations*

$$\int_{\Sigma_1} f_i da + \int_L \tilde{t}_i ds = 0, \quad \int_{\Sigma_1} \varepsilon_{3\alpha\beta} x_\alpha f_\beta da + \int_L \varepsilon_{3\alpha\beta} x_\alpha \tilde{t}_\beta ds = 0 \tag{23}$$

hold.

In the next section we will have occasion to use four generalized plane strain problems $P^{(k)}$, $(k = 1, 2, 3, 4)$. We denote by $u_i^{(k)}$ and $\varphi^{(k)}$, respectively, the displacement vector and the volume fraction field in the problem $P^{(k)}$, $(k = 1, 2, 3, 4)$. The problems $P^{(k)}$, $(k = 1, 2, 3, 4)$, are characterized by the equations of equilibrium

$$t^{(k)}_{\alpha i,\alpha} + f_i^{(k)} = 0, \quad h^{(k)}_{\alpha,\alpha} + g^{(k)} + p^{(k)} = 0 \tag{24}$$

the constitutive equations

$$\begin{aligned}
t^{(k)}_{11} &= C_{11}u^{(k)}_{1,1} + C_{12}u^{(k)}_{2,2} + C_{14}u^{(k)}_{3,2} + \beta_1\varphi^{(k)} \\
t^{(k)}_{22} &= C_{12}u^{(k)}_{1,1} + C_{22}u^{(k)}_{2,2} + C_{24}u^{(k)}_{3,2} + \beta_2\varphi^{(k)} \\
t^{(k)}_{33} &= C_{13}u^{(k)}_{1,1} + C_{23}u^{(k)}_{2,2} + C_{34}u^{(k)}_{3,2} + \beta_3\varphi^{(k)} \\
t^{(k)}_{23} &= C_{14}u^{(k)}_{1,1} + C_{24}u^{(k)}_{2,2} + C_{44}u^{(k)}_{3,2} + \beta_4\varphi^{(k)} \\
t^{(k)}_{31} &= C_{55}u^{(k)}_{3,1} + C_{56}(u^{(k)}_{1,2} + u^{(k)}_{2,1}) \\
t^{(k)}_{12} &= C_{56}u^{(k)}_{3,1} + C_{66}(u^{(k)}_{1,2} + u^{(k)}_{2,1}) \\
g &= -\beta_1 u^{(k)}_{1,1} - \beta_2 u^{(k)}_{2,2} - \beta_4 u^{(k)}_{3,2} - \xi\varphi^{(k)} \\
h^{(k)}_1 &= \alpha_1\varphi^{(k)}_{,1}, \quad h^{(k)}_2 = \alpha_2\varphi^{(k)}_{,2}, \quad h^{(k)}_3 = \alpha_4\varphi^{(k)}_{,2}
\end{aligned} \tag{25}$$

on Σ_1 and the boundary conditions

$$t^{(k)}_{\alpha i} n_\alpha = \tilde{t}^{(k)}_i, \quad h^{(k)}_\alpha n_\alpha = \tilde{h}^{(k)} \tag{26}$$

on L, where

$$
\begin{aligned}
&f_1^{(1)} = (C_{13}x_1)_{,1}, \quad f_2^{(1)} = (C_{23}x_1)_{,2}, \quad f_3^{(1)} = (C_{34}x_1)_{,2}\\
&f_1^{(2)} = (C_{13}x_2)_{,1}, \quad f_2^{(2)} = (C_{23}x_2)_{,2}, \quad f_3^{(2)} = (C_{34}x_2)_{,2}\\
&f_1^{(3)} = C_{13,1}, \quad f_2^{(3)} = C_{23,2}, \quad f_3^{(3)} = C_{34,2}\\
&f_1^{(4)} = (C_{14}x_1)_{,1} - (C_{56}x_2)_{,2}, \quad f_2^{(4)} = (C_{24}x_1)_{,2} - (C_{56}x_2)_{,1}\\
&f_3^{(4)} = (C_{44}x_1)_{,2} - (C_{55}x_2)_{,1}\\
&p^{(1)} = -\beta_3 x_1, \quad p^{(2)} = -\beta_3 x_2, \quad p^{(3)} = -\beta_3, \quad p^{(4)} = -\beta_4 x_1\\
&\tilde{t}_1^{(1)} = -C_{13}x_1 n_1, \quad \tilde{t}_2^{(1)} = -C_{23}x_1 n_2, \quad \tilde{t}_3^{(1)} = -C_{34}x_1 n_2\\
&t_1^{(2)} = -C_{13}x_2 n_1, \quad \tilde{t}_2^{(2)} = -C_{23}x_2 n_2, \quad \tilde{t}_3^{(2)} = -C_{34}x_2 n_2\\
&\tilde{t}_1^{(3)} = -C_{13}n_1, \tilde{t}_2^{(3)} = -C_{23}n_2, \quad \tilde{t}_3^{(3)} = -C_{34}n_2\\
&\tilde{t}_1^{(4)} = C_{56}x_2 n_2 - C_{14}x_1 n_1, \quad \tilde{t}_2^{(4)} = C_{56}x_2 n_1 - C_{24}x_1 n_2\\
&\tilde{t}_3^{(4)} = C_{55}x_2 n_1 - C_{44}x_1 n_2, \tilde{h}^{(k)} = 0, \quad (k = 1, 2, 3, 4)
\end{aligned}
\tag{27}
$$

It is a simple matter to see that the necessary and sufficient conditions (23) for the existence of solution are satisfied for each problem $P^{(k)}$, $(k = 1, 2, 3, 4)$. We note that the solutions of the problems $P^{(k)}$ depend only on the constitutive coefficients and the domain Σ_1.

THERMOELASTIC DEFORMATION OF THE ROD

In this section we study the problem of deformation of the considered cylinder when the temperature field has the form (2). We seek the solution in the form

$$
\begin{aligned}
&u_\alpha = -\frac{1}{2}a_\alpha x_3^2 - a_4 \varepsilon_{3\alpha\beta} x_\beta x_3 + \sum_{k=1}^{4} a_k u_\alpha^{(k)} + w_\alpha(x_1, x_2)\\
&u_3 = (a_1 x_1 + a_2 x_2 + a_3)x_3 + \sum_{k=1}^{4} a_k u_3^{(k)} + w_3(x_1, x_2)\\
&\varphi = \sum_{k=1}^{4} a_k \varphi^{(k)} + \psi(x_1, x_2)
\end{aligned}
\tag{28}
$$

where $(u_i^{(k)}, \varphi^{(k)})$ is the solution of the problem $P^{(k)}$, $(k = 1, 2, 3, 4)$, w_i and ψ are unknown functions of the variables x_1 and x_2, and a_k are unknown constants. From (1)–(3) and (28) we obtain

$$
t_{11} = C_{13}(a_1 x_1 + a_2 x_2 + a_3) + C_{14} a_4 x_1 + \sum_{k=1}^{4} a_k t_{11}^{(k)} + s_{11}
$$

$$
\begin{aligned}
t_{22} &= C_{23}(a_1x_1 + a_2x_2 + a_3) + C_{24}a_4x_1 + \sum_{k=1}^{4} a_k t_{22}^{(k)} + s_{22} \\
t_{33} &= C_{33}(a_1x_1 + a_2x_2 + a_3) + C_{34}a_4x_1 + \sum_{k=1}^{4} a_k t_{33}^{(k)} + s_{33} \\
t_{23} &= C_{34}(a_1x_1 + a_2x_2 + a_3) + C_{44}a_4x_1 + \sum_{k=1}^{4} a_k t_{23}^{(k)} + s_{23}
\end{aligned} \tag{29}
$$

$$
t_{31} = -C_{55}a_4x_2 + \sum_{k=1}^{4} a_k t_{31}^{(k)} + s_{31}, \quad t_{12} = -C_{56}a_4x_2 + \sum_{k=1}^{4} a_k t_{12}^{(k)} + s_{12}
$$

$$
\begin{aligned}
h_j = \sum_{k=1}^{4} a_k h_j^{(k)} + \sigma_j, \quad g = &-\beta_3(a_1x_1 + a_2x_2 + a_3) - \beta_4 a_4 x_1 \\
&+ \sum_{k=1}^{4} a_k^{(k)} g^{(k)} + \gamma
\end{aligned}
$$

where $t_{ij}^{(k)}$, $h_j^{(k)}$ and $g^{(k)}$, $(k = 1, 2, 3, 4)$, are given by (25) and we have used the notations

$$
\begin{aligned}
s_{11} &= C_{11}w_{1,1} + C_{12}w_{2,2} + C_{14}w_{3,2} + \beta_1\psi - b_1 f \\
s_{22} &= C_{12}w_{1,1} + C_{22}w_{2,2} + C_{24}w_{3,2} + \beta_2\psi - b_2 f \\
s_{33} &= C_{13}w_{1,1} + C_{23}w_{2,2} + C_{34}w_{3,2} + \beta_3\psi - b_3 f \\
s_{23} &= C_{14}w_{1,1} + C_{24}w_{2,2} + C_{44}w_{3,2} + \beta_4\psi - b_4 f \\
s_{31} &= C_{55}w_{3,1} + C_{56}(w_{1,2} + w_{2,1}), \quad s_{12} = C_{56}w_{3,1} + C_{66}(w_{1,2} + w_{2,1}) \\
\sigma_1 &= \alpha_1\psi_{,1}, \quad \sigma_2 = \alpha_2\psi_{,2}, \quad \sigma_3 = \alpha_4\psi_{,2} \\
\gamma &= -\beta_1 w_{1,1} - \beta_2 w_{2,2} - \beta_4 w_{3,2} - \xi\psi + mf
\end{aligned} \tag{30}
$$

In view of (24), (27) and (29), the equations of equilibrium (4) reduce to

$$
s_{\alpha i,\alpha} = 0, \quad \sigma_{\alpha,\alpha} + \gamma = 0 \quad \text{on } \Sigma_1 \tag{31}
$$

By using (26), (27) and (29) we find that the boundary conditions (5) take the form

$$
s_{\alpha i} n_\alpha = 0, \quad \sigma_\alpha n_\alpha = 0 \quad \text{on } L \tag{32}
$$

We conclude that the functions w_i and φ satisfy the equations of a thermoelastic generalized plane strain corresponding to the temperature $T = f$. If we use the last equation of equilibrium, then from (4) and the divergence theorem we find that

$$
\int_\Sigma t_{3\alpha} da = \int_\Sigma [(x_\alpha t_{\beta 3})_{,\beta} + x_\alpha t_{33,3}] da = \int_\Sigma x_\alpha t_{33,3} da + \int_L x_\alpha t_{\beta 3} n_\beta ds
$$

We conclude that the conditions (6) are satisfied on the basis of relations (29) and the conditions (5). It follows from (7) and (29) that the constants a_k, $(k = 1, 2, 3, 4)$, satisfy the system

$$\sum_{k=1}^{4} D_{\alpha k} a_k = \varepsilon_{3\alpha\beta}(M_\beta + M_\beta^*), \quad \sum_{k=1}^{4} D_{3k} a_k = -R_3 - R_3^*, \sum_{k=1}^{4} D_{4k} a_k = -M_3 - M_3^* \tag{33}$$

Here, the constants D_{mn}, $(m, n = 1, 2, 3, 4)$, M_i^* and R_3^* are given by

$$\begin{aligned}
D_{\alpha\beta} &= \int_{\Sigma_1} x_\alpha (C_{33} x_\beta + t_{33}^{(\beta)}) da, \quad D_{\alpha 3} = \int_{\Sigma_1} x_\alpha (C_{33} + t_{33}^{(3)}) da \\
D_{\alpha 4} &= \int_{\Sigma_1} x_\alpha (C_{34} x_1 + t_{33}^{(4)}) da, \quad D_{3\alpha} = \int_{\Sigma_1} (C_{33} x_\alpha + t_{33}^{(\alpha)}) da, \\
D_{33} &= \int_{\Sigma_1} (C_{33} + t_{33}^{(3)}) da, \quad D_{34} = \int_{\Sigma_1} (C_{34} x_1 + t_{33}^{(4)}) da \\
D_{4\alpha} &= \int_{\Sigma_1} [x_1 (C_{34} x_\alpha + t_{23}^{(\alpha)}) - x_2 t_{31}^{(\alpha)}] da \\
D_{43} &= \int_{\Sigma_1} [x_1 (C_{34} + t_{23}^{(3)}) - x_2 t_{31}^{(3)}] da \\
D_{44} &= \int_{\Sigma_1} [x_1 (C_{44} x_1 + t_{23}^{(4)}) - x_2 (t_{31}^{(4)} - C_{55} x_2)] da \\
R_3^* &= \int_{\Sigma_1} s_{33} da, \quad M_\alpha^* = \int_{\Sigma_1} \varepsilon_{3\alpha\beta} x_\beta s_{33} da, \quad M_3^* = \int_{\Sigma_1} (x_1 s_{23} - x_2 s_{13}) da
\end{aligned} \tag{34}$$

As in classical elasticity [23] we can prove that the positive definiteness of the elastic potential implies that

$$\det(D_{mn}) > 0 \tag{35}$$

Moreover, the relation (14) leads to

$$D_{mn} = D_{nm}, \quad (m, n = 1, 2, 3, 4) \tag{36}$$

It follows from (35) that the system (33) determines the constants a_1, a_2, a_3 and a_4.

We conclude that the solution of the problem is given by (28), where $(u_i^{(k)}, \varphi^{(k)})$, $(k = 1, 2, 3, 4)$, are the solutions of the isothermal generalized plane strain problems $P^{(k)}$. (w_i, φ) is the solution of the thermoelastic generalized plane strain problem (30)–(32) and the constants a_k, $(k = 1, 2, 3, 4)$, are given by (33). As in the classical thermoelasticity, the solution is expressed in terms of solutions of some two-dimensional problems.

We note that the plane temperature (3) produces the following effects: (a) a generalized plane strain parallel to the x_1, x_2-plane, characterized by the functions w_i and ψ; (b) an extension of the cylinder due to the force R_3^*; (c) a bending due to the moments M_1^* and M_2^*; (d) a torsion of the cylinder due to the moment M_3^*. We also note that in this case the torsion of the cylinder produces a variation of volume fraction field. If the medium has a plane of elastic symmetry, normal to the axis of cylinder, then the temperature field (3) does not produce torsional effects.

APPLICATION

Let us consider a circular cylinder defined by $B = \{x : x_1^2 + x_2^2 < a^2, 0 < x_3 < l\}$, $(a > 0)$. We assume that B is occupied by an inhomogeneous material characterized by constitutive coefficients of the form

$$\begin{aligned} C_{pq} &= C_{pq}^* e^{-\kappa r}, \quad \alpha_k = \alpha_k^* e^{-\kappa r}, \quad \beta_k = \beta_k^* e^{-\kappa r} \\ b_k &= b_k^* e^{-\kappa r}, \quad m = m^* e^{-\kappa r}, \quad \xi = \xi^* e^{-\kappa r}, \quad \kappa > 0 \end{aligned} \tag{37}$$

where C_{pq}^*, $(p, q = 1, 2, \ldots, 6)$, α_k^*, β_k^*, b_k^*, $(k = 1, 2, 3, 4)$, m^*, ξ^* and κ are prescribed constants, and $r = (x_1^2 + x_2^2)^{1/2}$. This kind of inhomogeneity has been intensively studied in the literature (see, e.g., Lomakin [24]). We assume that the cylinder is subjected to the temperature field

$$T = T^* \tag{38}$$

where T^* is a prescribed constant. We suppose that the mechanical loads are absent, i.e., $R_3 = 0$, $M_j = 0$. First, we study the thermoelastic generalized plane strain problem (30)–(32) corresponding to the temperature field (38). We seek the solution of this problem in the form

$$w_1 = \lambda_1 x_1 T^*, \quad w_2 = \lambda_2 x_2 T^*, \quad w_3 = \lambda_3 x_2 T^*, \quad \psi = \nu T^* \tag{39}$$

where λ_j and ν are unknown constants. From (30) and (39) we obtain

$$\begin{aligned} s_{11} &= (C_{11}\lambda_1 + C_{12}\lambda_2 + C_{14}\lambda_3 + \beta_1 \nu - b_1)T^* \\ s_{22} &= (C_{12}\lambda_1 + C_{22}\lambda_2 + C_{24}\lambda_3 + \beta_2 \nu - b_2)T^* \\ s_{33} &= (C_{13}\lambda_1 + C_{23}\lambda_2 + C_{34}\lambda_3 + \beta_3 \nu - b_3)T^* \\ s_{23} &= (C_{14}\lambda_1 + C_{24}\lambda_2 + C_{44}\lambda_3 + \beta_4 \nu - b_4)T^* \\ s_{31} &= s_{12} = 0, \quad \sigma_j = 0 \\ \gamma &= (m - \beta_1\lambda_1 - \beta_2\lambda_2 - \beta_4\lambda_3 - \xi\nu)T^* \end{aligned} \tag{40}$$

In what follows we assume that the constants λ_j and ν satisfy the equations

$$\begin{aligned} C_{11}^*\lambda_1 + C_{12}^*\lambda_2 + C_{14}^*\lambda_3 + \beta_1^*\nu &= b_1^* \\ C_{12}^*\lambda_1 + C_{22}^*\lambda_2 + C_{24}^*\lambda_3 + \beta_2^*\nu &= b_2^* \\ C_{14}^*\lambda_1 + C_{24}^*\lambda_2 + C_{44}^*\lambda_3 + \beta_4^*\nu &= b_4^* \\ \beta_1^*\lambda_1 + \beta_2^*\lambda_2 + \beta_4^*\lambda_3 + \xi^*\nu &= m^* \end{aligned} \tag{41}$$

The positive definiteness of the elastic potential implies that the determinant of the system (41) is different from zero, so that Eqs. (41) uniquely determine the constants λ_j and ν. By (37), (40) and (41) we find that

$$s_{\alpha\beta} = 0, \quad s_{\alpha 3} = 0, \quad \gamma = 0, \quad \sigma_\alpha = 0 \tag{42}$$

Clearly, Eqs. (31) and the boundary conditions (32) are identically satisfied. Thus, the solution of the boundary value problem (30)–(32) is given by (39) where the constants λ_i and ν are determined by (41).

Next we investigated the generalized plane strain problems $P^{(k)}$, $(k = 1, 2, 3, 4)$. Let us prove that the solution of the problem $P^{(3)}$ is given by

$$u_1^{(3)} = \mu_1 x_1, \quad u_2^{(3)} = \mu_2 x_2, \quad \mu_3^{(3)} = \mu_3 x_2, \quad \varphi^{(3)} = \zeta \tag{43}$$

where μ_i and ζ are constants determined by the system

$$\begin{aligned} C_{11}^*\mu_1 + C_{12}^*\mu_2 + C_{14}^*\mu_3 + \beta_1^*\zeta &= -C_{13}^* \\ C_{12}^*\mu_1 + C_{22}^*\mu_2 + C_{24}^*\mu_3 + \beta_2^*\zeta &= -C_{23}^* \\ C_{14}^*\mu_1 + C_{24}^*\mu_2 + C_{44}^*\mu_3 + \beta_4^*\zeta &= -C_{34}^* \\ \beta_1^*\mu_1 + \beta_2^*\mu_2 + \beta_4^*\mu_3 + \xi^*\zeta &= -\beta_3^* \end{aligned} \tag{44}$$

In view of (25), (37), (43) and (44) we find that

$$\begin{aligned} &t_{11}^{(3)} = -C_{13}, \quad t_{22}^{(3)} = -C_{23}, \quad t_{33}^{(3)} = C_{13}\mu_1 + C_{23}\mu_2 + C_{34}\mu_3 + \beta_3\zeta \\ &t_{23}^{(3)} = -C_{34}, \quad t_{31}^{(3)} = 0, \quad t_{12}^{(3)} = 0, \quad h_i^{(3)} = 0, \quad g^{(3)} = \beta_3 \end{aligned} \tag{45}$$

It is a simple matter to see that the functions $t_{ij}^{(3)}$, $h_j^{(3)}$ and $g^{(3)}$ given by (45) satisfy the equations of equilibrium and the boundary conditions which characterize the problem $P^{(3)}$. From (34), (37), (42) and (45) we get

$$\begin{aligned} &D_{\alpha 3} = 0, \quad D_{43} = 0 \\ &D_{33} = 2\pi\kappa^{-2}(C_{33}^* + C_{13}^*\mu_1 + C_{23}^*\mu_2 + C_{34}^*\mu_3 + \beta_3^*\zeta)[1 - e^{-\kappa a}(1 + \kappa a)] \\ &R_3^* = 2\pi T^*\kappa^{-2}(C_{13}^*\lambda_1 + C_{23}^*\lambda_2 + C_{34}^*\lambda_3 + \beta_3^*\nu - b_3^*)[1 - (1 + \kappa a)e^{-\kappa a}], \quad M_i^* = 0 \end{aligned} \tag{46}$$

It follows from (36) and (46) that

$$D_{3\alpha} = 0, \quad D_{34} = 0 \tag{47}$$

In view of (46) and (47), the system (33) reduces to

$$D_{\alpha\beta}a_\beta + D_{\alpha 4}a_4 = 0, \quad D_{33}a_3 = -R_3^*, \quad D_{4\beta}a_\beta + D_{44}a_4 = 0$$

The solution of this system is

$$a_1 = a_2 = a_4 = 0, \quad a_3 = -R_3^*/D_{33} \tag{48}$$

where R_3^* and D_{33} are given by (46). It follows from (28), (39), (43) and (48) that the solution of the problem is

$$\begin{aligned} &u_1 = (a_3\mu_1 + \lambda_1 T^*)x_1, \; u_2 = (a_3\mu_2 + \lambda_2 T^*)x_2 \\ &u_3 = a_3 x_3 + (a_3\mu_3 + \lambda_3 T^*)x_2, \; \varphi = a_3\zeta + \nu T^* \end{aligned}$$

We note that the influence of material inhomogeneity on the behaviour of the cylinder is reflected by the relations (46) and (48). We conclude that the temperature field T^* produces a plane deformation parallel to the x_1, x_2-plane, an extension of the cylinder and a uniform variation of volume fraction field.

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